# Stable Sheaves on a Smooth Quadric Surface with Linear Hilbert Bipolynomials 

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#### Abstract

We investigate the moduli spaces of stable sheaves on a smooth quadric surface with linear Hilbert bipolynomial in some special cases and describe their geometry in terms of the locally free resolution of the sheaves.


## 1. Introduction

Throughout the paper, our base field is $\mathbb{C}$, the field of complex numbers.

By the work of Simpson [1], we can consider the moduli space of semistable sheaves on a smooth projective variety $X$ with a fixed Hilbert polynomial, which is itself a projective variety, and the moduli space has been studied quite intensively in the last decade for the case with linear Hilbert polynomial over projective spaces [2-5]. Our interest is on the moduli space over a smooth quadric surface.

Let $Q$ be a smooth quadric surface in $\mathbb{P}^{3}$ and let $\mathbf{M}_{Q}(\mu, \chi)$ be the moduli space of semistable sheaves on $Q$ with linear Hilbert polynomial $\chi(m)=\mu m+\chi$ with respect to the ample line bundle $\mathcal{O}_{Q}(1,1)$. Unlike the case of projective spaces, this moduli space is not irreducible in general. Indeed, for a purely 1-dimensional sheaf $\mathscr{F}$ on $Q$, we can define a linear Hilbert bipolynomial $\chi_{\mathscr{F}}(x, y)$ such that

$$
\begin{equation*}
\chi(\mathscr{F}(x, y))=\chi_{\mathscr{F}}(x, y) \tag{1}
\end{equation*}
$$

for all $(x, y) \in \mathbb{Z}^{\oplus 2}$. Then we can consider, due to [6], the moduli space $\mathbf{M}(m, n, t)$ of semistable sheaves on $Q$ with linear Hilbert bipolynomial $\chi(x, y)=m x+n y+t$. The moduli space is a projective variety with a Zariski open subset $\mathbf{M}^{\circ}(m, n, t)$ consisting of stable ones, with dimension $2 m n+1$ and the open set is nonempty if one of $m$ or $n$ is nonzero (see Proposition 7).

By its definition we have a natural decomposition

$$
\begin{equation*}
\mathbf{M}_{\mathrm{Q}}(m+n, t)=\coprod_{0 \leq a \leq m+n} \mathbf{M}(a, m+n-a, t) \tag{2}
\end{equation*}
$$

Thus, the moduli $\mathbf{M}(m, n, t)$ is an irreducible component of Simpson's moduli space because the bidegree function is locally constant.

If $\mathscr{F}$ is a stable sheaf in $\mathbf{M}(m, n, t)$, then its schematic support $C_{\mathscr{F}}$ is a curve of bidegree $(n, m)$ on $Q$ and so a general sheaf is a line bundle over a smooth subcurve. Thus, the moduli space can be considered as an analogue of the universal line bundle $\mathscr{P} i c_{(n, m)}^{d}$ of some fixed degree $d$ over the family of the bidegree $(n, m)$-curves in $Q$.

Now, some simple observations lead us to consider only $\mathbf{M}(m, n, t)$ with $0 \leq t \leq \operatorname{gcd}(m, n)$ due to proper twists. For small $m$ or $n$, the moduli space is very simple. Indeed, $\mathbf{M}(n, 0, t)$ is isomorphic to $\mathbb{P}^{n}$ if $t=n$ and is empty otherwise. If $m$ or $n$ is equal to 1 , say $m=1$, then it is isomorphic to $\mathbb{P}^{2 n+1}$. These descriptions are quite simple from the definition of stability condition and so the first nontrivial case happens when $(m, n)=(2,2)$. The main result of this paper is to describe the moduli spaces $\mathbf{M}(2,2, t)$ with $t=1,2$.

Theorem 1. For $\mathbf{M}_{t}=\mathbf{M}(2,2, t)$, one obtains the following:
(1) $\mathbf{M}_{1}$ is isomorphic to $\mathscr{P} i c_{(2,2)}^{1}$ and it is rational;
(2) $\mathbf{M}_{2}$ is birational to $\mathscr{P} c_{(2,2)}^{2}$ and it is unirational with degree 4.

In fact, we explicitly describe the sheaves in each moduli space in terms of their locally free resolution. Indeed, a sheaf $\mathscr{F}$ is in $\mathbf{M}_{1}$ if and only if it admits a resolution

$$
\begin{align*}
0 & \longrightarrow \mathcal{O}_{\mathrm{Q}}(-2,-1) \oplus \mathcal{O}_{\mathrm{Q}}(-1,-2) \\
& \longrightarrow \mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1) \longrightarrow \mathscr{F} \longrightarrow 0 \tag{3}
\end{align*}
$$

where the degeneracy locus of the first map is the support of $\mathscr{F}$. It enables us to identify $\mathbf{M}_{1}$ with $\mathscr{P} i c_{(2,2)}^{1}$ and show its rationality.

For $\mathbf{M}_{2}$, the situation is a bit more complicated; we can classify the sheaves in $\mathbf{M}_{2}$ up to 3 types, in terms of the short exact sequences they admit, and express the moduli as the union of 3 subschemes

$$
\begin{equation*}
\mathbf{M}_{2}=\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C} . \tag{4}
\end{equation*}
$$

In particular, we can show that every sheaf in $\mathbf{M}_{2}$ is globally generated, from which we obtain a resolution that they admit:

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{\mathrm{Q}}(-1,-1)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathrm{Q}}^{\oplus 2} \longrightarrow \mathscr{F} \longrightarrow 0 \tag{5}
\end{equation*}
$$

We investigate the property of the subvarieties and the relationship between them. We also construct a map from $\mathbf{M}_{1}$ to $\mathbf{M}_{2}$, which is generically 4 to 1 and thus we obtain that $\mathbf{M}_{2}$ is unirational of degree 4. We leave the rationality question of $\mathbf{M}_{2}$ as a conjecture.

## 2. Preliminaries

Let $Q$ be a smooth quadric surface isomorphic to $\mathbb{P} V_{1} \times \mathbb{P} V_{2}$ for 2-dimensional vector spaces $V_{1}$ and $V_{2}$, and then it is embedded into $\mathbb{P}^{3} \cong \mathbb{P} V$ by the Segre map where $V=V_{1} \otimes V_{2}$. If we denote by $f_{1}, f_{2}$ the two projections from $Q$ to each factor, then we will denote $f_{1}^{*} \mathcal{O}_{\mathbb{P}^{1}}(a) \otimes f_{2}^{*} \mathcal{O}_{\mathbb{P}^{1}}(b)$ simply by $\mathcal{O}_{Q}(a, b)$. We also denote $\mathscr{E} \otimes \mathscr{O}_{Q}(a, b)$ by $\mathscr{E}(a, b)$ for a coherent sheaf $\mathscr{E}$ on $Q$ and in particular the canonical sheaf $\omega_{\mathrm{Q}}$ of $Q$ is $\mathcal{O}_{\mathrm{Q}}(-2,-2)$.

Proposition 2. For a purely 1-dimensional sheaf $\mathscr{F}$ on $Q$, there is a bipolynomial $\chi_{\mathscr{F}}(x, y) \in \mathbb{Q}[x, y]$ of degree 1 such that

$$
\begin{equation*}
\chi(\mathscr{F}(u, v))=\chi_{\mathscr{F}}(u, v) \tag{6}
\end{equation*}
$$

for all $(u, v) \in \mathbb{Z}^{\oplus 2}$.
Proof. Let us assume that $m t+c$ is the Hilbert polynomial of $\mathscr{F}$ with respect to the ample line bundle $\mathcal{O}_{\mathrm{Q}}(1,1)$. Let us take any $D \in\left|\mathcal{O}_{\mathrm{Q}}(0,1)\right|, T \in\left|\mathcal{O}_{\mathrm{Q}}(1,0)\right|$, and a smooth conic $C \in\left|\mathcal{O}_{Q}(1,1)\right|$ such that neither $D, T$, nor $C$ is contained in the 1-dimensional reduced curve $\operatorname{Supp}(\mathscr{F})$.

The curves $D, T$, and $C$ induce maps $j_{D}: \mathscr{F}(t, t) \rightarrow$ $\mathscr{F}(t, t+1), j_{T}: \mathscr{F}(t, t) \rightarrow \mathscr{F}(t+1, t)$, and $j_{C}: \mathscr{F}(t, t) \rightarrow$ $\mathscr{F}(t+1, t+1)$. Since neither $D$ nor $T$ is contained in the 1 -dimensional reduced curve $\operatorname{Supp}(\mathscr{F})$, we have $j_{D} \neq 0$ and
$j_{T} \neq 0$. Since $\mathscr{F}$ is pure, we obtain that $j_{D}, j_{T}$, and $j_{C}$ are injective. Thus, there are exact sequences

$$
\begin{align*}
0 & \longrightarrow \mathscr{F}(t, t) \longrightarrow \mathscr{F}(t, t+1) \longrightarrow \mathscr{F}(t, t+1) \otimes \mathcal{O}_{D}  \tag{7}\\
& \longrightarrow 0 \\
0 & \longrightarrow \mathscr{F}(t, t) \longrightarrow \mathscr{F}(t+1, t) \longrightarrow \mathscr{F}(t+1, t) \otimes \mathcal{O}_{T}  \tag{8}\\
& \longrightarrow 0 \\
0 & \longrightarrow \mathscr{F}(t, t) \longrightarrow \mathscr{F}(t+1, t+1) \\
& \longrightarrow \mathscr{F}(t+1, t+1) \otimes \mathcal{O}_{C} \longrightarrow 0 \tag{9}
\end{align*}
$$

Let us set $a:=h^{0}\left(\mathscr{F}(t, t+1) \otimes \mathcal{O}_{D}\right)$ and $b:=h^{0}(\mathscr{F}(t+1, t) \otimes$ $\left.\mathcal{O}_{T}\right)$. The sheaves $\mathscr{F}(t, t+1) \otimes \mathcal{O}_{D}, \mathscr{F}(t+1, t) \otimes \mathcal{O}_{T}$, and $\mathscr{F}(t+$ $1, t+1) \otimes \mathcal{O}_{C}$ have finite supports and thus the dimensions of their cohomology $H^{0}(Q,-)$ do not change even if we twist them by a line bundle on $Q$. From (9), we get $a+b=h^{0}(\mathscr{F}(t+$ $\left.1, t+1) \otimes \mathcal{O}_{C}\right)=m$.

We claim that $\chi(\mathscr{F}(u, v))=a v+b u+c$ for all $(u, v) \in \mathbb{Z}^{\oplus 2}$. If $u=v$, then the claim is true. Now assume that $u \neq v$, say $u>$ $v$. We use $u-v$ exact sequences like (8) with $\mathscr{F}(c, 0)$ instead of $\mathscr{F}$ with $0 \leq c<u-v$ to get $\chi(\mathscr{F}(u, v))=\chi(v, v)+(u-v) b$.

Definition 3. One defines the Hilbert bipolynomial $\chi_{\mathscr{F}}(x, y) \in \mathbb{Q}[x, y]$ of $\mathscr{F}$ to be a linear bipolynomial such that

$$
\begin{equation*}
\chi_{\mathscr{F}}(x, y)=\chi\left(\mathscr{F} \otimes \mathcal{O}_{\mathrm{Q}}(x, y)\right) . \tag{10}
\end{equation*}
$$

In particular, the Hilbert polynomial of $\mathscr{F}$ with respect to $\mathcal{O}_{Q}(1,1)$ is defined to be $\chi_{\mathscr{F}}(t)=\chi_{\mathscr{F}}(t, t)$.

We are mainly interested in the case when $\chi_{\mathscr{F}}(x, y)$ is a linear function, that is, $\chi_{\mathscr{F}}(x, y)=m x+n y+t$ for some $(m, n, t) \in \mathbb{Z}^{\oplus 3}$.

Definition 4. Let $\mathscr{F}$ be a pure sheaf of dimension 1 on $Q$ with $\chi_{\mathscr{F}}(x, y)=m x+n y+t$. The $p$-slope of $\mathscr{F}$ is defined to be $p(\mathscr{F})=t /(m+n) . \mathscr{F}$ is called semistable (stable) with respect to the ample line bundle $\mathcal{O}_{Q}(1,1)$ if
(1) $\mathscr{F}$ does not have any 0 -dimensional torsion,
(2) for any proper subsheaf $\mathscr{F}^{\prime}$, one has

$$
\begin{equation*}
p\left(\mathscr{F}^{\prime}\right)=\frac{t^{\prime}}{m^{\prime}+n^{\prime}} \leq(<) \frac{t}{m+n}=p(\mathscr{F}) \tag{11}
\end{equation*}
$$

where $\chi_{\mathscr{F}^{\prime}}(x, y)=m^{\prime} x+n^{\prime} y+t^{\prime}$.
For every semistable 1 -dimensional sheaf $\mathscr{F}$ with $\chi_{\mathscr{F}}(x, y)=m x+n y+t$, let us define $C_{\mathscr{F}}:=\operatorname{Supp}(\mathscr{F})$ to be its scheme-theoretic support and then we have $C_{\mathscr{F}} \in\left|\mathcal{O}_{Q}(n, m)\right|$. We often use slope stability and slope semistability instead of Gieseker stability or Gieseker semistability just to simplify the notation; they should be the same because the support is 1-dimensional, and from $m t+\chi$ and $m^{\prime} t+\chi^{\prime}$, the inequality for Hilbert and slopes $\chi / m$ the same.

Definition 5. Let $\mathbf{M}(m, n, t)$ be the moduli space of semistable sheaves on $Q$ with linear Hilbert bipolynomial $\chi(x, y)=m x+$ $n y+t$.

We can define $\mathbf{M}(m, n, t)$ in a different way as a subvariety of $\mathbf{M}_{\mathrm{Q}, \mathbb{P}^{3}}(m+n, t)$, the moduli space of semistable sheaves on $\mathbb{P}^{3}$ with linear Hilbert polynomial $\chi(x)=m x+t$, which are $\mathcal{O}_{Q^{-}}$-sheaves. To be precise, if $\mathscr{F}$ is $\mathcal{O}_{Q^{-}}$-sheaf, then all of its $\mathcal{O}_{\mathbb{P}^{3}}$ subsheaves are also $\mathcal{O}_{Q}$-sheaves. It implies that the notions of $p$-stability and $\mu$-stability of $\mathscr{F}$ are the same and thus $\mathbf{M}_{\mathrm{Q}, \mathbb{P}^{3}}(m+n, t)$ may be defined without using $\mathbb{P}^{3}$. Moreover, the sheaf with linear Hilbert bipolynomial $\chi(x, y)=a x+b y+$ $c$ has Hilbert polynomial $\chi(x)=(a+b) x+c$ with respect to $\mathcal{O}_{Q}(1,1)$ and thus we have a natural decomposition

$$
\begin{equation*}
\mathbf{M}_{\mathrm{Q}, \mathbb{P}^{3}}(m+n, t)=\coprod_{0 \leq a \leq m+n} \mathbf{M}(a, m+n-a, t) . \tag{12}
\end{equation*}
$$

In particular, $\mathbf{M}(m, n, t)$ is a subvariety of $\mathbf{M}_{Q, \mathbb{P}^{3}}(m+n, t)$.
Remark 6. Let $\mathscr{F}$ be any purely 1-dimensional coherent sheaf on $\mathbb{P}^{3}$ with Hilbert polynomial $m x+t i$. Assume that $\mathscr{F}$ is not semistable and let

$$
\begin{equation*}
0=\mathscr{F}_{0} \subset \mathscr{F}_{1} \subset \cdots \subset \mathscr{F}_{k}=\mathscr{F} \tag{13}
\end{equation*}
$$

be the Harder-Narasimhan filtration of $\mathscr{F}$ (see page 55 in [1]). If $\mathscr{F}$ is an $\mathscr{O}_{X}$-module, then each $\mathscr{F}_{i}$ is an $\mathscr{O}_{X}$-module because it is a subsheaf of $\mathscr{F}$. Thus the Harder-Narasimhan filtration of $\mathscr{F}$ as an $\mathcal{O}_{\mathbb{P}^{n}}$-sheaf is the same as the one as an $\mathcal{O}_{X}$-sheaf.

Proposition 7. The moduli $\mathbf{M}(m, n, t)$ is a projective and irreducible scheme. If $m n>0$, then $\mathbf{M}^{\circ}(m, n, t)$ is a Zariski dense and open subset of $\mathbf{M}(m, n, t)$ with dimension $2 m n+1$.

Proof. The first assertion follows verbatim from the proof of Proposition 2.3 and Theorem 3.1 in [6], only when the assertion in Lemma 3.3 over $Q$ holds. But it holds, using Castelnuovo-Mumford criterion with the Serre duality

$$
\begin{align*}
H^{1}\left(\mathscr{E} x t^{1}(\mathscr{F}, \mathscr{F}(j, j))\right) & \cong \operatorname{Ext}^{2}(\mathscr{F}, \mathscr{F}(j, j)) \\
& \cong \operatorname{Hom}(\mathscr{F}(j, j), \mathscr{F}(-2,-2))^{\vee}=0 \tag{14}
\end{align*}
$$

for $\mathscr{F} \in \mathbf{M}(m, n, t)$ and $j \geq-1$.
For the second assertion, let us consider a map

$$
\begin{equation*}
\mathbf{M}\left(m^{\prime}, n^{\prime}, t^{\prime}\right) \times \mathbf{M}\left(m^{\prime \prime}, n^{\prime \prime}, t^{\prime \prime}\right) \longrightarrow M(m, n, t) \tag{15}
\end{equation*}
$$

defined by sending $\left(\mathscr{F}^{\prime}, \mathscr{F}^{\prime \prime}\right)$ to $\mathscr{F}^{\prime} \oplus \mathscr{F}^{\prime \prime}$, where $m=m^{\prime}+m^{\prime \prime}$ and $n=n^{\prime}+n^{\prime \prime}$. Then the dimension of the image of this map is at least $2 m n-2 m^{\prime} n^{\prime}-2 m^{\prime \prime} n^{\prime \prime}-1$ and it is at least 1 if $m n>0$. In other words, general sheaf in $\mathbf{M}(m, n, t)$ is stable.

For any pure sheaf $\mathscr{F}$ on $Q$ with Hilbert bipolynomial $\chi_{\mathscr{F}}(x, y)=m x+n y+t$, let us define

$$
\begin{equation*}
\mathscr{F}^{D}:=\mathscr{E} x t_{\mathrm{Q}}^{1}\left(\mathscr{F}, \omega_{\mathrm{Q}}\right) \tag{16}
\end{equation*}
$$

to be the Grothendieck dual of $\mathscr{F}$. Since $\mathscr{F}$ is pure, the natural $\operatorname{map} \varphi_{\mathscr{F}}: \mathscr{F} \rightarrow \mathscr{F}^{D D}$ is injective. Since the support of $\mathscr{F}$ is $1-$ dimensional, $\varphi_{\mathscr{F}}$ is bijective as in Remark 4 of [4]. Moreover, the support of $\mathscr{F}^{D}$ is also 1-dimensional and so $\chi_{\mathscr{F} D}(x, y)$ is also linear. By the Serre duality, we have

$$
\begin{equation*}
H^{i}\left(\mathscr{F}^{D}(c, d)\right) \cong H^{i}\left((\mathscr{F}(-c,-d))^{D}\right) \cong H^{1-i}(\mathscr{F}(-c,-d))^{\vee} \tag{17}
\end{equation*}
$$

for $i \in\{0,1\}$ and, in particular, we have

$$
\begin{equation*}
\chi_{\mathscr{F} D}(x, y)=-\chi_{\mathscr{F}}(-x,-y)=m x+n y-t . \tag{18}
\end{equation*}
$$

Lemma 8. There is an isomorphism

$$
\begin{equation*}
\mathbf{M}(m, n, t) \longrightarrow \mathbf{M}(m, n,-t) \tag{19}
\end{equation*}
$$

sending $\mathscr{F}$ to $\mathscr{F}^{D}$.
Note also that $\chi_{\mathscr{F}(d, e)}(x, y)=m x+n y+t+(m d+n e)$. Since the map

$$
\begin{equation*}
\mathbf{M}(m, n, t) \longrightarrow \mathbf{M}(m, n, t+m d+n e), \tag{20}
\end{equation*}
$$

defined by $\mathscr{F} \mapsto \mathscr{F}(d, e)$, is an isomorphism, so we may assume that $0<t \leq \operatorname{gcd}(m, n)$.

Lemma 9. For a not necessarily integral curve $C$ in $\left|\mathcal{O}_{Q}(n, m)\right|$, the sheaf $\mathcal{O}_{C}$ is semistable. IfC is integral, then $\mathcal{O}_{C}$ is stable.

Proof. We have the following sequence:

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{Q}(-n,-m) \longrightarrow \mathcal{O}_{Q} \longrightarrow \mathcal{O}_{C} \longrightarrow 0 \tag{21}
\end{equation*}
$$

In particular, we have $\chi_{\sigma_{C}}(x, y)=m x+n y+(m+n-m n)$ and so $p\left(\mathcal{O}_{C}\right)=1-1 /(1 / m+1 / n)$. If $C$ is integral, then $\mathcal{O}_{C}$ is stable since every line bundle on an integral curve is stable. In general, $\mathcal{O}_{C}$ is semistable. Otherwise, there exists a semistable quotient sheaf $\mathcal{O}_{C} \rightarrow \mathscr{F} \rightarrow 0$ such that the Hilbert bipolynomial $\chi_{\mathscr{F}}(x, y)=m^{\prime} x+n^{\prime} y+t^{\prime}$ satisfies $m^{\prime}+n^{\prime}<m+n$ and $p(\mathscr{F})<p\left(\mathscr{O}_{C}\right)$. By induction, we get that $\mathscr{O}_{C^{\prime}}$ with $C^{\prime}:=C_{\mathscr{F}}$ is semistable and thus we have

$$
\begin{equation*}
p\left(\mathcal{O}_{C^{\prime}}\right) \leq p(\mathscr{F})<p\left(\mathcal{O}_{C}\right) \tag{22}
\end{equation*}
$$

This is absurd since $p\left(\mathcal{O}_{C}\right)$ is a decreasing function on $m$ and $n$.

Let us assume that $m=0$, that is, $\operatorname{Hilb}_{\mathrm{Q}}(n y+t)$ with $0<$ $t \leq n$.

Proposition 10. One has

$$
\mathbf{M}(0, n, t) \cong \begin{cases}\left(\mathbb{P}^{1}\right)^{[n]} \cong \mathbb{P}^{n} & \text { if } t=n  \tag{23}\\ \emptyset & \text { if } 0<t<n .\end{cases}
$$

In fact, each point in $\operatorname{Hilb}_{\mathrm{Q}}(n y+n)$ corresponds to an equivalence class $\left[\mathcal{O}_{L_{1}} \oplus \cdots \oplus \mathcal{O}_{L_{n}}\right.$ ], where $L_{i}$ is a line in $\left|\mathcal{O}_{\mathrm{Q}}(1,0)\right|$.

Proof. Let us assume that $t=n$ and let us choose $L \in$ $\left|\mathcal{O}_{Q}(n, 0)\right|$ and then it fits into

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{\mathrm{Q}}(-n, 0) \longrightarrow \mathcal{O}_{\mathrm{Q}} \longrightarrow \mathcal{O}_{L} \longrightarrow 0 \tag{24}
\end{equation*}
$$

Thus we have

$$
\begin{align*}
\chi_{\widehat{O}_{L}}(x, y) & =\chi_{\widehat{\vartheta}_{Q}}(x, y)-\chi_{\widehat{\vartheta}_{Q}(-n, 0)}(x, y) \\
& =(x+1)(y+1)-(x-n+1)(y+1)  \tag{25}\\
& =n y+n .
\end{align*}
$$

Clearly, $\mathcal{O}_{L}$ is stable. For a line $L \in\left|\mathcal{O}_{Q}(1,0)\right|$, we have

$$
\begin{equation*}
\chi_{\mathscr{O}_{2 L}}(x, y)=\chi_{\mathcal{O}_{L} \oplus \mathscr{O}_{L}}(x, y)=2 y+2 . \tag{26}
\end{equation*}
$$

From the sequence for $L$, we have

$$
\begin{align*}
& 0 \longrightarrow \operatorname{Hom}\left(\mathcal{O}_{L}, \mathcal{O}_{L}\right) \longrightarrow \operatorname{Hom}\left(\mathcal{O}_{\mathrm{Q}}, \mathcal{O}_{L}\right) \\
& \quad \xrightarrow{f} \operatorname{Hom}\left(\mathcal{O}_{\mathrm{Q}}(-1,0), \mathcal{O}_{L}\right) \longrightarrow \operatorname{Ext}^{1}\left(\mathcal{O}_{L}, \mathcal{O}_{L}\right) \longrightarrow 0 \tag{27}
\end{align*}
$$

and the map $f$ is a zero map. Thus, there exists a nontrivial extension of $\mathcal{O}_{L}$ by $\mathcal{O}_{L}$ and it is $\mathcal{O}_{2 L}$. In particular, $\mathcal{O}_{L}^{\oplus 2}$ and $\mathcal{O}_{2 L}$ represent the same point in $\operatorname{Hilb}_{\mathrm{Q}}(2 y+2)$. In general, $\mathcal{O}_{L}^{\oplus k}$ and $\mathcal{O}_{k L}$ with $k \geq 1$ represent the same point in $\mathbf{M}(0, k, k)$. Thus, $\mathscr{O}_{L}$ with $L \in\left|\mathcal{O}_{\mathrm{Q}}(n, 0)\right|$ is strictly semistable if and only if $n \geq 2$. Conversely, let us choose a semistable sheaf $\mathscr{F}$ with $\chi_{\mathscr{F}}(x, y)=n y+n$. In particular, the schematic support $L=$ $\operatorname{Supp}(\mathscr{F})$ of $\mathscr{F}$ is in $\left|\mathcal{O}_{Q}(n, 0)\right|$. Since $\chi(\mathscr{F})=n>0$, there exists a nontrivial morphism $\mathcal{O}_{\mathrm{Q}} \rightarrow \mathscr{F}$ and it induces an injection $\mathcal{O}_{L_{1}} \rightarrow \mathscr{F}$, where $L_{1}$ is a subscheme of $L$. Here we have $L_{1} \in\left|\hat{O}_{\mathrm{Q}}(s, 0)\right|$ for some $s \leq n$ and so $\chi_{L_{1}}(x, y)=s x+$ $s$. Thus, the quotient $\mathscr{G}=\mathscr{F} / \mathcal{O}_{L_{1}}$ is a semistable sheaf with $\chi_{\mathscr{G}}(x, y)=(n-s) y+(n-s)$. By induction, we have $[\mathscr{G}]=\left[\mathcal{O}_{L_{2}}\right]$ with $L_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(n-s, 0)\right|$. In particular, $\mathscr{F}$ is an extension of $\mathcal{O}_{L_{2}}$ by $\mathcal{O}_{L_{1}}$ with $L_{1}+L_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(n, 0)\right|$ and thus $\mathscr{F}$ is equivalent to $\mathcal{O}_{L_{1}} \oplus \mathcal{O}_{L_{2}}$.

Now, let us assume that $0<t<n$ and fix $\mathscr{F} \in \mathbf{M}(0, n, t)$ with $C:=C_{\mathscr{F}} \in\left|\mathcal{O}_{C}(n, 0)\right|$. Since $\chi(\mathscr{F})=t>0$, there is a nonzero map $f: \mathcal{O}_{\mathrm{Q}} \rightarrow \mathscr{F}$. Since $\mathscr{F}$ is an $O_{\mathrm{C}}$-sheaf, $f$ induces a nonzero map $h: \mathcal{O}_{C} \rightarrow \mathscr{F}$. Since $\mathcal{O}_{C}$ has slope $1>t / n$ and it is semistable, we get a contradiction. Alternatively, as in Lemma 4.10 of [6], we may first take the schematic support $T \subseteq C$ of $\operatorname{Im}(h)$ and then use an injective $\operatorname{map} \mathcal{O}_{T} \rightarrow \mathscr{F}$ with $\mathcal{O}_{T} \in\left|\mathcal{O}_{\mathrm{Q}}\left(n^{\prime}, 0\right)\right|$ with $1 \leq n^{\prime}<n$, and thus we have $\mu\left(\mathcal{O}_{T}\right)=1$.

For the case of $m=1$, that is, $\chi_{\mathscr{F}}(x, y)=x+n y+t$, it is enough to check the case $t=1$ since $\operatorname{gcd}(1, n)=1$.

Proposition 11. $\mathbf{M}(1, n, 1)$ consists of $\mathcal{O}_{C}$ with $C \in\left|\mathcal{O}_{Q}(n, 1)\right|$. In particular, one has $\mathbf{M}(1, n, 1) \cong \mathbb{P}^{2 n+1}$.

Proof. From the sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{Q}(-n,-1) \longrightarrow \mathcal{O}_{Q} \longrightarrow \mathcal{O}_{C} \longrightarrow 0 \tag{28}
\end{equation*}
$$

we have $\chi_{\mathcal{O}_{C}}(x, y)=x+n y+1$ and $\mathcal{O}_{C}$ is semistable by Lemma 9 . Conversely, let $\mathscr{F}$ be a semistable sheaf with
$\chi_{\mathscr{F}}(x, y)=x+n y+1$ and so $C:=C_{\mathscr{F}}$ is a curve in $\left|\mathcal{O}_{Q}(n, 1)\right|$. Since we have $\chi(\mathscr{F})=1$, there exists a nonzero map $\mathcal{O}_{\mathrm{Q}} \rightarrow$ $\mathscr{F}$ and it induces a nonzero map $h: \mathcal{O}_{C} \rightarrow \mathscr{F}$. Note that $\operatorname{Im}(h)$ has no 0 -dimensional torsion since $\mathscr{F}$ is semistable. Since $\mathcal{O}_{C}$ is also semistable, we have

$$
\begin{equation*}
p\left(\mathcal{O}_{\mathrm{C}}\right) \leq p(\operatorname{Im}(h)) \leq p(\mathscr{F}) . \tag{29}
\end{equation*}
$$

The map $h$ factors through an injection $\mathcal{O}_{D} \hookrightarrow \mathscr{F}$, where $D$ is a curve contained in $C$. If $D$ is properly contained in $C$, we have $p\left(\mathcal{O}_{D}\right)>p(\mathscr{F})$ contradicting the semistability of $\mathscr{F}$ and thus we have $D=C$; that is, $h$ is an isomorphism from $\mathcal{O}_{C}$ to its image. Since $\mathscr{O}_{C}$ and $\mathscr{F}$ have the same Hilbert polynomial, we have $\mathscr{F} \cong \mathcal{O}_{C}$.

## 3. Hilbert Bipolynomial $2 x+2 y+1$

For the moduli space of semistable sheaves with linear Hilbert bipolynomial $2 x+2 y+t$, it is enough to investigate the case when $t=1,2$. Let us denote the moduli space $\mathbf{M}(2,2, t)$ by $\mathbf{M}_{t}$.

Proposition 12. The moduli space $\mathbf{M}_{1}$ consists of the unique nontrivial extensions $\mathscr{F}$ of $\mathcal{O}_{P}$ by $\mathcal{O}_{C}$ for each curve $C \in$ $\left|\mathcal{O}_{\mathrm{Q}}(2,2)\right|$ and a point $P \in C$, and one also has $h^{0}(\mathscr{F})=1$.

Proof. Since $\chi(\mathscr{F})=1$, there is a nonzero map $\mathcal{O}_{\mathrm{Q}} \rightarrow \mathscr{F}$, inducing a nonzero map $h: \mathcal{O}_{C} \rightarrow \mathscr{F}$, where $C:=C_{\mathscr{F}} \in$ $\left|\mathcal{O}_{\mathrm{Q}}(2,2)\right|$. Since $\chi_{\mathcal{O}_{C}}(x, y)=2 x+2 y$, we have $p\left(\mathcal{O}_{C}\right)=$ $0<1 / 4=p(\mathscr{F})$. The map $h$ factors through an injection $\mathcal{O}_{D} \hookrightarrow \mathscr{F}$, where $D$ is a curve contained in $C$. If $D$ is properly contained in $C$, we have $p\left(\mathcal{O}_{D}\right)>p(\mathscr{F})$ contradicting to the semistability of $\mathscr{F}$ and thus we have $D=C$; that is, $h$ is an isomorphism from $\mathcal{O}_{C}$ to its image, that is, we have

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{C} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0 \tag{30}
\end{equation*}
$$

where $\chi_{\mathscr{G}}(x, y)=1$. In particular, we have $\mathscr{G} \cong \mathcal{O}_{P}$, the skyscraper sheaf supported on a point $P \in C$. Since $\mathscr{F}$ has no 0 -dimensional torsion, the sequence does not split. Note that $\operatorname{Ext}^{1}\left(\mathcal{O}_{P}, \mathcal{O}_{\mathrm{Q}}\right) \cong H^{1}\left(\mathcal{O}_{P}\right)^{\vee}=0$, and thus from the sequence of $C$ we have

$$
\begin{align*}
0 & \operatorname{Ext}^{1}\left(\mathcal{O}_{P}, \mathcal{O}_{C}\right) \longrightarrow \operatorname{Ext}^{2}\left(\mathcal{O}_{P}, \mathcal{O}_{\mathrm{Q}}(-2,-2)\right) \\
& \stackrel{s}{\longrightarrow} \operatorname{Ext}^{2}\left(\mathcal{O}_{P}, \mathcal{O}_{\mathrm{Q}}\right) . \tag{31}
\end{align*}
$$

Here, the map $s$ is the transpose of $\operatorname{Hom}\left(\mathcal{O}_{\mathrm{Q}}(2,2), \mathscr{O}_{P}\right) \rightarrow$ $\operatorname{Hom}\left(\mathcal{O}_{\mathrm{Q}}, \mathcal{O}_{P}\right)$ which is given by the multiplication by the defining equation of $C$. Since $P$ is a point on $C$, the map $s$ is a zero map. In particular, the dimension of $\operatorname{Ext}^{1}\left(\mathcal{O}_{P}, \mathcal{O}_{C}\right)$ is 1 and so $\mathscr{F}$ corresponds to a unique nontrivial extension

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{\mathrm{C}} \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_{P} \longrightarrow 0 \tag{32}
\end{equation*}
$$

From the sequence (32), we have $h^{0}(\mathscr{F}) \leq 2$ and that $h^{0}(\mathscr{F})=$ 1 if and only if no injective map $\mathcal{O}_{C} \rightarrow \mathscr{F}$ is an isomorphism at $P$. This is certainly true if $\mathscr{F}$ is not locally free of rank 1 at $P$. Note that $\mathscr{F}$ is a line bundle at each point of $C \backslash\{P\}$ and thus it is sufficient to prove $h^{0}(\mathscr{F})=1$ when $\mathscr{F}$ is a line bundle on
the curve $C$. In this case the nonexistence of a section of $\mathscr{F}$ that does not vanish at $P$ is equivalent to the nonsplitting of (32). Thus, we have $h^{0}(\mathscr{F})=0$ and so the point $P$ is uniquely determined by $\mathscr{F}$.

Conversely, let us assume that $\mathscr{F}$ is a nontrivial extension of $\mathcal{O}_{P}$ by $\mathscr{O}_{C}$, where $P$ is a point on $C$. If $\mathscr{F}$ is not semistable, then there exists a subsheaf $\mathscr{K} \subset \mathscr{F}$ with $p(\mathscr{K})>p(\mathscr{F})=1 / 4$ and so we have $\chi_{\mathscr{K}}(x, y)=m^{\prime} x+n^{\prime} y+t^{\prime}$ with $\left(m^{\prime}, n^{\prime}\right) \leq(2,2)$ and $t^{\prime} \geq 1$. If the composite $s: \mathscr{K} \rightarrow \mathscr{F} \rightarrow \mathcal{O}_{P}$ is a zero map, then we have an injection $\mathscr{K} \hookrightarrow \mathcal{O}_{\mathrm{C}}$, contradicting the semistability of $\mathcal{O}_{C}$. Thus, the composite is surjective and so we have the following diagram:


Here, $\mathscr{K}^{\prime}$ is the kernel of the map $s$ and $\mathscr{H}$ is the quotient $\mathscr{F} / \mathscr{K}$. Since $\chi_{\mathscr{K}^{\prime}}(x, y)=m^{\prime} x+n^{\prime} y+\left(t^{\prime}-1\right)$ and $\mathcal{O}_{C}$ is semistable, we have $t^{\prime}=1$ and thus $\chi_{\mathscr{H}}(x, y)=\left(2-m^{\prime}\right) x+$ $\left(2-n^{\prime}\right) y$ with no constant term. Since $\mathscr{H}$ is the quotient of $\mathcal{O}_{C}$, it must be $\mathcal{O}_{T}$ for some curve $T$ contained in $C$. But no such curves have the Hilbert polynomials with no constant term. Hence $\mathscr{F}$ is semistable.

Remark 13. There is no strictly semistable sheaf in $\mathbf{M}_{1}$. Let us assume the existence of a polystable sheaf $\mathscr{F}=\mathscr{F}_{1} \oplus \cdots \oplus \mathscr{F}_{s}$ with $s \geq 2$. We have $\chi(\mathscr{F})=1=\chi\left(\mathscr{F}_{1}\right)+\cdots+\chi\left(\mathscr{F}_{s}\right)$. If we let $\chi_{\mathscr{F}_{i}}(x, y)=a_{i} x+b_{i} y+c_{i}$, then we have

$$
\begin{equation*}
c_{1}+\cdots+c_{s}=1, \quad \frac{c_{i}}{a_{i}+b_{i}}=\frac{1}{2}, \quad \forall i . \tag{34}
\end{equation*}
$$

It implies that $c_{i}>0$ for all $i$ and thus we have $s=1$, a contradiction.

Proposition 14. A sheaf $\mathscr{F}$ is in $\mathbf{M}_{1}$ if and only if it admits the following resolution:

$$
\begin{equation*}
0 \longrightarrow \mathscr{A} \xrightarrow{\varphi} \mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1) \xrightarrow{\psi} \mathscr{F} \longrightarrow 0 \tag{35}
\end{equation*}
$$

where $\mathscr{A}:=\mathcal{O}_{\mathrm{Q}}(-2,-1) \oplus \mathcal{O}_{\mathrm{Q}}(-1,-2)$ and $\varphi=\left(\begin{array}{ll}h_{1} & l_{1} \\ h_{2} & l_{2}\end{array}\right)$. Here, $f:=h_{1} l_{2}-h_{2} l_{1}$ is a defining equation of $C_{\mathscr{F}}$.

Proof. Note that $h^{0}(\mathscr{F})=1$ and so $h^{1}(\mathscr{F})=0$. If $\mathscr{F}$ admits the sequence (32), then $\mathscr{F}$ is globally generated outside $P$ and so is $\mathscr{F}(1,1)$. Take any $A \in\left|\mathcal{O}_{Q}(1,1)\right|$ which is not contained
in $C$ and with $P \notin A$. The multiplication by an equation of $A$ gives an exact sequence

$$
\begin{equation*}
\left.0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}(1,1) \longrightarrow \mathscr{F}(1,1)\right|_{A} \longrightarrow 0 \tag{36}
\end{equation*}
$$

where $\operatorname{deg}\left(\left.\mathscr{F}(1,1)\right|_{A}\right)=\operatorname{deg}(A \cap C)=4$. Thus we have $h^{1}(\mathscr{F}(1,1))=0$ and $h^{0}(\mathscr{F}(1,1))=5$. Together with the exact sequence (32) tensored by $\mathcal{O}_{Q}(1,1)$, we obtain that $\mathscr{F}(1,1)$ is globally generated at $P$ and so we have a surjection

$$
\begin{equation*}
\psi: \mathcal{O}_{Q} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1) \longrightarrow \mathscr{F} \longrightarrow 0 \tag{37}
\end{equation*}
$$

Let us set $\mathscr{H}:=\operatorname{ker}(\psi)$ and then $\mathscr{H}$ is a torsion-free sheaf of rank 2 on $Q$ with $c_{1}=(-3,-3)$. By Theorem 19.9 in [7], the sheaf $\mathscr{H}$ is locally free. Note that $\chi_{\mathscr{H}(1,2)}(x, y)=2 x y+3 x+$ $y+1$. Thus, we have $h^{0}(\mathscr{H}(1,2))>0$ and so we have an exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{\mathrm{Q}}(a, b) \longrightarrow \mathscr{H}(1,2) \longrightarrow \mathscr{I}_{Z}(-1-a, 1-b) \longrightarrow 0 \tag{38}
\end{equation*}
$$

where $Z$ is a 0 -dimensional subscheme of $Q$ and $(a, b) \in$ $\{(0,0),(0,1)\}$. If $(a, b)=(0,1)$, then we have $\chi_{\mathcal{I}_{z}(-1,0)}(x, y)=$ $x y+x+1$ and it is absurd since $\chi_{\mathcal{O}_{Q}(-1,0)}(x, y)=x y+x$. Thus, we have $(a, b)=(0,0)$ and $Z=\emptyset$. Since $\operatorname{Ext}^{1}\left(\mathcal{O}_{\mathrm{Q}}(-1,1), \mathcal{O}_{\mathrm{Q}}\right)=$ 0 , we have $\mathscr{H}(1,2) \cong \mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,1)$ and the sequence (35). Note that the map $\varphi: \mathscr{H} \rightarrow \mathcal{O}_{Q} \oplus \mathcal{O}_{Q}(-1,-1)$ is given by $\binom{h_{1} l_{1}}{h_{2} l_{2}}$, where $f:=h_{1} l_{2}-h_{2} l_{1}$ is a defining equation of $C=C_{\mathscr{F}}$.

The converse is trivial.
Remark 15. Using the proof of Lemma 5.3 in [2], we can obtain the same assertion of Proposition 14. Similarly, we also obtain that $\mathscr{F}(1,0)$ is globally generated and so a surjection $\varphi^{\prime}: \mathcal{O}_{Q} \oplus \mathcal{O}_{Q}(-1,0) \rightarrow \mathscr{F}$. In this case, $\operatorname{ker}\left(\varphi^{\prime}\right)$ is no longer a direct sum of two line bundles.

Let us define a vector space $W$ to be

$$
\begin{equation*}
W:=\operatorname{Hom}\left(\mathscr{A}, \mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1)\right) \tag{39}
\end{equation*}
$$

and $W^{0} \subset W$ to be the set of $\varphi \in W$ such that $h_{1} l_{2}-h_{2} l_{1} \neq 0$. Then we have a surjective morphism

$$
\begin{equation*}
\pi: W^{0} \longrightarrow M_{1} \tag{40}
\end{equation*}
$$

Let us choose $\varphi_{1}, \varphi_{2} \in W^{0}$ with $\pi\left(\varphi_{1}\right)=\pi\left(\varphi_{2}\right)$; that is, we have the following diagram:

$$
\begin{align*}
& 0 \longrightarrow \mathscr{A} \xrightarrow{\varphi_{1}} \mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1) \xrightarrow{\psi_{1}} \mathscr{F} \longrightarrow 0 \\
& \downarrow f  \tag{41}\\
& 0 \longrightarrow \mathscr{A} \xrightarrow{\varphi_{2}} \mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1) \xrightarrow{\psi_{2}} \mathscr{F} \longrightarrow 0,
\end{align*}
$$

where $f$ is an isomorphism. Since $\operatorname{Ext}^{1}\left(\mathcal{O}_{Q} \oplus\right.$ $\left.\mathcal{O}_{\mathrm{Q}}(-1,-1), \mathscr{A}\right)=0$, we have a $\operatorname{map} f_{1} \in \operatorname{End}\left(\mathcal{O}_{\mathrm{Q}} \oplus\right.$ $\left.\mathcal{O}_{\mathrm{Q}}(-1,-1)\right)$ associated with $f$. Note that $f_{1}$ is given by $\left(\begin{array}{cc}a & 0 \\ z & b\end{array}\right)$, where $a, b \in \mathbb{C}^{\times}$and $z \in H^{0}\left(\mathcal{O}_{\mathrm{Q}}(1,1)\right)$. Similarly, we have
a map $f_{2}: \mathscr{A} \rightarrow \mathscr{A}$ which is $\left(\begin{array}{cc}c_{1} & 0 \\ 0 & c_{2}\end{array}\right)$, where $c_{1}, c_{2} \in \mathbb{C}^{\times}$. In particular, we have

$$
\varphi_{2}=\left(\begin{array}{cc}
c_{1} & 0  \tag{42}\\
0 & c_{2}
\end{array}\right)^{-1} \varphi_{1}\left(\begin{array}{ll}
a & 0 \\
z & b
\end{array}\right) .
$$

In this equation, we can assume that $c_{1}=1$. In other words, $\pi\left(\varphi_{1}\right)=\pi\left(\varphi_{2}\right)$ if and only if $\varphi_{1}$ and $\varphi_{2}$ are in the same orbit in $W^{0}$ under the action by

$$
\begin{align*}
\mathbf{G}:= & \frac{\left(\operatorname{Aut}(\mathscr{A}) \times \operatorname{Aut}\left(\mathcal{O}_{\mathrm{Q}} \oplus \mathcal{O}_{\mathrm{Q}}(-1,-1)\right)\right)}{\mathbb{C}^{\times}} \\
= & \left\{\left.\left(\left(\begin{array}{ll}
1 & 0 \\
0 & c
\end{array}\right),\left(\begin{array}{ll}
a & 0 \\
z & b
\end{array}\right)\right) \right\rvert\, a, b, c \in \mathbb{C}^{\times}\right.  \tag{43}\\
& \left.z \in H^{0}\left(\mathcal{O}_{\mathrm{Q}}(1,1)\right)\right\} .
\end{align*}
$$

Theorem 16. $\pi: W^{0} \rightarrow \mathbf{M}_{1}$ is a geometric quotient map by the action of $\mathbf{G}$. In particular, one has $\mathbf{M}_{1} \cong W^{0} / \mathbf{G}$ and so $\mathbf{M}_{1}$ is isomorphic to $\mathscr{P i c}_{(2,2)}^{1}$.

Proof. To get the assertion it suffices to prove that it has local sections as in Lemma 5.1 and Theorem 5.5 in [3].

Since every element of $\mathbf{M}_{1}$ is stable, $\mathbf{M}_{1}$ has a universal family $\mathbb{H}_{1}$ on $\mathbf{M}_{1} \times \mathbb{P}^{2}$ (see page 180 of [8] or Theorem 4.6 .5 of [9]). Since every semistable sheaf with bipolinomial $2 x+2 y+5$ is of the form $\mathscr{F}(1,1)$ for a unique $\mathscr{F} \in \mathbf{M}_{1}$, we also have a universal family $\mathbb{H}_{5}$ on $\mathbf{M}_{5} \times \mathbb{P}^{2}$ with $\mathscr{F}(1,1)$ as the fibre. Since $h^{0}(\mathscr{F})=1$ and $h^{1}(\mathscr{F})=0$ for all $\mathscr{F} \in \mathbf{M}_{1}$, the base change theorem gives that $u_{*}\left(\mathbb{H}_{1}\right)$ is a line bundle on $\mathbf{M}_{1}$, where $u: \mathbf{M}_{1} \times \mathbb{P}^{2} \rightarrow M_{1}$ is the first projection. Since $h^{0}(\mathscr{F}(1,1))=5$ and $h^{1}(\mathscr{F}(1,1))=0$ for all $\mathscr{F} \in \mathbf{M}_{1}$, the base change theorem gives that $v_{*}\left(\mathbb{H}_{5}\right)$ is a vector bundle of rank 5 on $\mathbf{M}_{1}$ by identifying $\mathbf{M}_{5}$ with $\mathbf{M}_{1}$, where $v: \mathbf{M}_{5} \times \mathbb{P}^{2} \rightarrow M_{5}$. For a fixed $\mathscr{F} \in \mathbf{M}_{1}$ and a matrix $\varphi \in \pi^{-1}(\mathscr{F})$, let us write $\varphi=\binom{h_{1} l_{1}}{h_{2} l_{2}}$, where $f:=h_{1} l_{2}-h_{2} l_{1}$ is a defining equation of $C_{\mathscr{F}}$. Take an open neighborhood $U$ of $\mathscr{F}$ in $\mathbf{M}_{1}$ over which $u_{*}\left(\mathbb{H}_{1}\right)$ and $v_{*}\left(\mathbb{H}_{5}\right)$ are trivial. The matrix $\varphi$ was constructed starting with a section $\sigma$ of $\mathscr{F}(1,1)$ which spans $\mathscr{F}(1,1)$ together with the twist $\sigma^{\prime}$ of a nonzero section of $\mathscr{F}$. Since $\left.u_{*}\left(\mathbb{W}_{1}\right)\right|_{U}$ and $\left.v_{*}\left(\mathbb{H}_{5}\right)\right|_{U}$ are trivial, there are maps $e_{1}: \mathcal{O}_{U} \rightarrow \mathcal{O}_{U}$ and $e_{2}: \mathscr{O}_{U} \rightarrow \mathcal{O}_{U}^{\oplus 5}$ with $e_{1}(\mathscr{F})=\sigma^{\prime}$ and $e_{2}(\mathscr{F})=\sigma$. Since $\sigma^{\prime}$ and $\sigma$ span $\mathscr{F}$, there is a neighborhood $V$ of $\mathscr{F}$ in $U$ such that the sections $e_{1}(\mathscr{G})$ and $e_{2}(\mathscr{G})$ span every $\mathscr{G} \in V$. The construction of $\varphi$ gives that $e_{1}$ and $e_{2}$ induce a section of $\pi$ in a neighborhood of $\varphi$ whose image by $\pi$ is $V$.

As an automatic consequence, we obtain that $\mathbf{M}_{1}$ is irreducible and unirational. In fact, we can prove more.

Theorem 17. $\mathrm{M}_{1}$ is rational.
Proof. Let $\Delta \subset Q \times Q$ be the diagonal and denote its ideal sheaf by $\mathscr{J}_{\Delta}$. Denoting by $p_{1}$ and $p_{2}$ the projection from $Q \times Q$ to each factor, let us define a sheaf $\mathscr{U}$ to be $\left.p_{1}^{*} \mathcal{O}_{Q}(2,2)\right) \boxtimes \mathscr{F}_{\Delta}$ on $Q \times Q$. For each point $P \in Q$, we have $\left.\mathscr{U}\right|_{\mathrm{Q} \times\{P\}} \cong \mathscr{J}_{P}(2,2)$. Thus, we have $h^{1}\left(\left.\mathscr{U}\right|_{\mathrm{Q} \times\{P\}}\right)=0$ and so $p_{2 *} \mathscr{U}$ is a vector bundle
of rank 8 on $Q$ since $h^{0}\left(\left.\mathscr{U}\right|_{\mathrm{Q} \times\{P\}}\right)=8$. Let us consider the projective bundle

$$
\begin{equation*}
\mathscr{Z}=\mathbb{P}\left(p_{2 *} \mathscr{U}\right) \longrightarrow Q . \tag{44}
\end{equation*}
$$

By its definition, the fibre of $\mathscr{Z}$ over a point $P \in Q$ is the set of curves of type $(2,2)$ on $Q$ passing through $P$ and so there is a natural map from $\mathscr{Z}$ to $\mathbb{P} H^{0}\left(\mathcal{O}_{\mathrm{Q}}(2,2)\right) \cong \mathbb{P}^{8}$. In other words, $\mathscr{Z}$ is the universal curve of type $(2,2)$ on $Q$ and it is isomorphic to $\mathbf{M}_{1}$. Since $\mathscr{Z}$ is locally trivial over $Q$, it is rational.

## 4. Hilbert Bipolynomial $2 x+2 y+2$

Lemma 18. Any sheaf $\mathscr{F} \in \mathbf{M}_{2}$ admits one of the following types:
(A) $0 \rightarrow \mathcal{O}_{C} \rightarrow \mathscr{F} \rightarrow \eta \rightarrow 0$, where $\eta$ is a skyscraper $\mathcal{O}_{C}$-sheaf with degree 2,
(B) $0 \rightarrow \mathcal{O}_{T_{1}} \rightarrow \mathscr{F} \rightarrow \mathcal{O}_{T_{2}} \rightarrow 0$ with $T_{1}, T_{2} \in$ $\left|\mathcal{O}_{Q}(1,1)\right|$,
(C) $0 \rightarrow \mathcal{O}_{T_{1}} \rightarrow \mathscr{F} \rightarrow \mathcal{O}_{T_{2}} \rightarrow 0$, where $T_{1} \in\left|\mathcal{O}_{\mathrm{Q}}(a, b)\right|$ and $T_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(2-a, 2-b)\right|$ with $(a, b) \in\{(1,2),(2,1)\}$.

Proof. Since $\chi(\mathscr{F})=2$, we have $h^{0}(\mathscr{F}) \geq 2$. Thus, there exists a nonzero map $\mathscr{O}_{\mathrm{Q}} \rightarrow \mathscr{F}$ and it induces a nonzero map $h$ : $\mathcal{O}_{C} \rightarrow \mathscr{F}$, where $C:=C_{\mathscr{F}} \in\left|\mathcal{O}_{Q}(2,2)\right|$. The map $h$ factors through an injection $\mathcal{O}_{T_{1}} \hookrightarrow \mathscr{F}$ where $T_{1}$ is a curve contained in $C$ :


If we have $T_{1}=C$, that is the map $h$ is an isomorphism from $\mathcal{O}_{C}$ to its image in $\mathscr{F}$, then its cokernel $\mathscr{H}$ is the skyscraper sheaf supported on two points, say $P_{1}, P_{2} \in C$. Thus we have the sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{C} \longrightarrow \mathscr{F} \longrightarrow \eta \longrightarrow 0 \tag{46}
\end{equation*}
$$

Let us assume that $T_{1}$ is properly contained in $C$ and then we obtain that $T_{1}$ has bidegree $(1,1),(1,2)$, or $(2,1)$ since $p\left(\mathcal{O}_{T_{1}}\right) \leq p(\mathscr{F})=1 / 2$ and $\mathscr{F}$ is semistable. Let $T_{2} \subset Q$ be the only curve such that $T_{1}+T_{2}=C$. Let $\mathscr{H}^{\prime}$ be the quotient of $\mathscr{H}$ by its torsion $\tau$, that is, $\mathscr{H}^{\prime}:=\mathscr{H}^{D D}$.

First, assume $T_{1} \in\left|\mathcal{O}_{Q}(1,1)\right|$ and so we have $\chi_{\mathscr{H}^{\prime}}(x, y)=$ $x+y+1-\operatorname{deg}(\tau)$. Since $\mathscr{F}$ is semistable, we get $\tau=0$. Since every quotient of $\mathscr{F}$ has the slope at least $1 / 2$, the same is true for $\mathscr{H}$. Thus, $\mathscr{H}$ is semistable and Proposition 11 gives $\mathscr{H} \cong$ $\mathcal{O}_{T_{2}}$.

Now, without loss of generality, let us assume that $T_{1} \in$ $\left|\mathcal{O}_{\mathrm{Q}}(1,2)\right|$, that is, $\chi_{\mathcal{O}_{T_{1}}}(x, y)=2 x+y+1$ and so we have $\chi_{\mathscr{H}}(x, y)=y+1$. If $\mathscr{H}$ has 0 -dimensional torsion $\mathscr{T}$ with length $k \geq 1$, then the quotient $\mathscr{H} / \mathscr{T}$ is a quotient of $\mathscr{F}$ with
the $p$-slope $1-k \leq 0$, contradicting the semistability of $\mathscr{F}$. Thus $\mathscr{H}$ has no 0 -dimensional torsion and so we have $\mathscr{H} \cong$ $\mathcal{O}_{T_{2}}$ for a curve $T_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(1,0)\right|$ with $C=T_{1}+T_{2}$.

## Corollary 19. Every sheaf in $\mathbf{M}_{2}$ is globally generated.

Proof. Let us take $\mathscr{F} \in \mathbf{M}_{2}$ and then there is no nonzero map $\mathscr{F} \rightarrow \mathcal{O}_{\mathrm{C}} \cong \omega_{\mathrm{C}}$ since $\mathscr{F}$ is semistable. Thus we have $h^{1}(\mathscr{F})=$ 0 and so $h^{0}(\mathscr{F})=2$. It is clear that $\mathscr{F}$ of types (B) and (C) is globally generated and so we may assume that $\mathscr{F}$ is of type (A), but neither of (B) nor of (C).

Let $\mathscr{H} \subseteq \mathscr{F}$ be the image of the evaluation map $H^{0}(\mathscr{F}) \otimes$ $\widehat{O}_{\mathrm{Q}} \rightarrow \mathscr{F}$ and then $\mathscr{H}$ is pure. Assume that $\mathscr{H} \neq \mathscr{F}$. Since $\mathscr{F}$ is of type (A), it is globally generated outside at most two points of $C_{\text {red }}$. In particular, we have $\chi_{\mathscr{H}}(x, y)=2 x+2 y+c$ with $c \leq 1$ and $\operatorname{deg}(\mathscr{F} / \mathscr{H})=2$. Since $h^{0}(\mathscr{H})=h^{0}(\mathscr{F})=2$, we have $h^{1}(\mathscr{H})=2-c$. Note that every nonzero section of $\mathscr{H}$ vanishes at finitely many points since $\mathscr{F}$ is neither of types (B) nor (C). Since $h^{0}\left(\mathcal{O}_{\mathrm{C}}\right)<h^{0}(\mathscr{H})$, we have $\mathscr{H} \neq \mathcal{O}_{\mathrm{C}}$ and $c=1$. A nonzero section of $\mathscr{H}$ induces an exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{C} \longrightarrow \mathscr{H} \longrightarrow \mathscr{G} \longrightarrow 0 \tag{47}
\end{equation*}
$$

where $\mathscr{G} \cong \mathcal{O}_{P}$ for some $P \in C_{\text {red }}$. Since $\mathscr{H}$ is pure, this exact sequence does not split. As in the proof of Proposition 12, we get a contradiction. Thus, we have $\mathscr{H}=\mathscr{F}$ and so $\mathscr{F}$ is globally generated.

Lemma 20. $\mathscr{F}$ is a sheaf in $\mathbf{M}_{2}$ if and only if it admits a sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{\mathrm{Q}}(-1,-1)^{\oplus 2} \xrightarrow{\varphi} \mathcal{O}_{\mathrm{Q}}^{\oplus 2} \longrightarrow \mathscr{F} \longrightarrow 0 \tag{48}
\end{equation*}
$$

where $\varphi=\left(\begin{array}{c}z_{11} \\ z_{21} \\ z_{22}\end{array}\right), z_{i j} \in H^{0}\left(\mathcal{O}_{Q}(1,1)\right)$ such that $f:=z_{11} z_{22}-$ $z_{12} z_{21}$ is a defining equation of $C_{\mathscr{F}}$.

Proof. Let $\mathscr{F} \in \mathbf{M}_{2}$ be a sheaf of type (A) and then it is globally generated by Corollary 19. Since $h^{0}(\mathscr{F})=2$, we have a surjection

$$
\begin{equation*}
\psi: \mathcal{O}_{Q}^{\oplus 2} \longrightarrow \mathscr{F} \longrightarrow 0 \tag{49}
\end{equation*}
$$

Let us set $\mathscr{H}:=\operatorname{ker}(\psi)$ and then it is a torsion-free sheaf of rank 2 on $Q$ with $c_{1}=(-2,-2)$. By Theorem 19.9 in [7], $\mathscr{H}$ is locally free. Note that $h^{0}(\mathscr{H}(1,1))=2$. From the sequence

$$
\begin{equation*}
0 \longrightarrow \mathscr{H}(1,0) \longrightarrow \mathcal{O}_{Q}(1,0)^{\oplus 2} \longrightarrow \mathscr{F}(1,0) \longrightarrow 0 \tag{50}
\end{equation*}
$$

we obtain that the map $H^{0}\left(\mathcal{O}_{\mathrm{Q}}(1,0)^{\oplus 2}\right) \rightarrow H^{0}(\mathscr{F}(1,0))$ is an isomorphism and so $h^{1}(\mathscr{H}(1,0))=0$. Similarly, we have $h^{1}(\mathscr{H}(0,1))=0$ and $h^{2}(\mathscr{H})=h^{1}(\mathscr{F})=0$. By Remark 2.3 in [10], we obtain that $\mathscr{H}(1,1)$ is globally generated. Since $c_{1}(\mathscr{H}(1,1))=0$ or $h^{0}(\mathscr{H}(1,1))=2$, we have $\mathscr{H} \cong$ $\mathcal{O}_{\mathrm{Q}}(-1,-1)^{\oplus 2}$ and the resolution (48). The cases of the other types also work verbatim.

Definition 21. Let us define a subscheme $\mathfrak{A} \subset \mathbf{M}_{2}$ as follows:

$$
\begin{gather*}
\mathfrak{A}:=\left\{\mathscr{F} \in \mathbf{M}_{2} \mid \mathscr{F}\right. \text { admits a nontrivial } \\
\text { extension of type }(\mathrm{A})\} . \tag{51}
\end{gather*}
$$

Similarly, we define $\mathfrak{B}$ and $\mathfrak{C}$ for the semistable sheaves of types (B) and (C), respectively. In particular, we have

$$
\begin{equation*}
\mathbf{M}_{2}=\mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C} . \tag{52}
\end{equation*}
$$

Lemma 22. The sheaves $\mathscr{F}$ of type $(B)$ are strictly semistable. In particular, they are contained in $\mathfrak{B}$.

Proof. It is enough to check the semistability of $\mathscr{F}$. Let $\mathscr{K}$ be a subsheaf of $\mathscr{F}$ with $p(\mathscr{K})>1 / 2$ and the quotient sheaf $\mathscr{H}:=\mathscr{F} / \mathscr{K}$. If the composite map $s: \mathscr{K} \hookrightarrow \mathscr{F} \rightarrow \mathcal{O}_{T_{2}}$ is a zero map, then $\mathscr{K}$ is a subsheaf of $\mathcal{O}_{T_{1}}$, contradicting the semistability of $\mathcal{O}_{T_{1}}$. The sheaf $\operatorname{Im}(s)$ is a subsheaf of $\mathcal{O}_{T_{2}}$ and so we have $p(\operatorname{Im}(s)) \leq 1 / 2$. Similarly, the sheaf $\operatorname{ker}(s)$ is a subsheaf of $\mathcal{O}_{T_{1}}$ and so $p(\operatorname{ker}(s)) \leq 1 / 2$. From the exact sequence

$$
\begin{equation*}
0 \longrightarrow \operatorname{ker}(s) \longrightarrow \mathscr{K} \xrightarrow{s} \operatorname{Im}(s) \longrightarrow 0 \tag{53}
\end{equation*}
$$

we have $p(\mathscr{K}) \leq 1 / 2$, a contradiction.
Let us denote by $\partial \mathbf{M}_{2}$ the closed subscheme of $\mathbf{M}_{2}$, consisting of the strictly semistable sheaves.

Corollary 23. One has

$$
\begin{equation*}
\partial \mathbf{M}_{2}=\mathfrak{B} \cong \frac{\left(\mathbb{P}^{3} \times \mathbb{P}^{3}\right)}{\mathbb{S}_{2}} \tag{54}
\end{equation*}
$$

where $\mathfrak{S}_{2}$ is the permutation group of order 2. In particular, $\partial \mathbf{M}_{2}$ is a rational variety.

Proof. Obviously, we have $\mathfrak{B} \subset \partial \mathbf{M}_{2}$. Let $\mathscr{F}$ be a strictly semistable sheaf and so it has a proper quotient sheaf $\mathscr{H}$ with $p(\mathscr{H})=1 / 2$. From the semistability of $\mathscr{F}, \mathscr{H}$ has no 0 -dimensional torsion. From the equality $p(\mathscr{F})=p(\mathscr{H})$, we obtain that $\mathscr{H}$ is also semistable. Since $p(\mathscr{H})=1 / 2$, the Hilbert bipolynomial of $\mathscr{H}$ is either $2 x+1,2 y+1$ or $x+y+1$. The first 2 cases cannot happen due to Proposition 10. Thus, we have $\chi_{\mathscr{H}}(x, y)=x+y+1$ and so $\mathscr{H} \cong \mathcal{O}_{T_{2}}$ with $T_{2} \in\left|\mathcal{O}_{Q}(1,1)\right|$ and $T_{2} \subset C_{\mathscr{F}}$ by Proposition 11. If $\mathscr{K}^{2}$ is the kernel of the quotient map $\mathscr{H} \rightarrow \mathscr{H}$, then its p-slope is again $1 / 2$ and so $\mathscr{K}$ is semistable. Similarly as before, we have $\mathscr{K} \cong \mathcal{O}_{T_{1}}$ with $T_{1} \in\left|\mathcal{O}_{Q}(1,1)\right|$ and $C_{\mathscr{F}}=T_{1}+T_{2}$. Hence, we have $\mathscr{F} \in \mathfrak{B}$.

Let $\mathscr{F}$ be a sheaf of type (B), that is, it corresponds to a pair of two curves $\left\{T_{1}, T_{2}\right\}$. Let us assume that $\mathscr{F}$ admits another sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{T_{3}} \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_{T_{4}} \longrightarrow 0 \tag{55}
\end{equation*}
$$

with $T_{3}, T_{4} \in\left|\mathcal{O}_{\mathrm{Q}}(1,1)\right|$. Note that $\mathcal{O}_{T_{i}}$ is stable for all $i$. Thus, the composite map $s: \mathcal{O}_{T_{3}} \rightarrow \mathscr{F} \rightarrow \mathcal{O}_{T_{2}}$ is either a zero map or an isomorphism. In the former case, we have $\mathcal{O}_{T_{3}} \cong \mathcal{O}_{T_{1}}$ and so $\mathcal{O}_{T_{4}} \cong \mathcal{O}_{T_{2}}$. In the latter case, we have $\mathcal{O}_{T_{3}} \cong \mathcal{O}_{T_{2}}$ and $\mathcal{O}_{T_{4}} \cong \mathcal{O}_{T_{1}}$. Hence, the class of a strictly semistable sheaf $\mathscr{F}$ corresponds to a uniquely determined pair of two curves in $\left|\mathcal{O}_{Q}(1,1)\right|$ and we have $\mathfrak{B} \cong\left(\mathbb{P}^{3} \times \mathbb{P}^{3}\right) / \mathbb{S}_{2}$. The second assertion follows from the fact that any symmetric product $S^{d}\left(\mathbb{P}^{N}\right)$ of any projective space is a rational variety (see Theorems 4.2.8 and 4.2.8 ${ }^{\prime}$ in page 137 of [11, 12]).

Lemma 24. For two curves $T_{1}, T_{2} \in\left|\mathcal{O}_{Q}(1,1)\right|$, one has

$$
\operatorname{dim} \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right)= \begin{cases}3, & \text { if } T_{1}=T_{2}  \tag{56}\\ 2, & \text { if } T_{1} \neq T_{2}\end{cases}
$$

Proof. Note that we have

$$
\begin{equation*}
\operatorname{Ext}^{2}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}(-1,-1)\right)=H^{0}\left(\mathcal{O}_{T_{2}}(-1,-1)\right)^{\vee}=0 \tag{57}
\end{equation*}
$$

Thus, if we apply the functor $\operatorname{Hom}\left(\mathcal{O}_{T_{2}},-\right)$ to the sequence of $T_{1}$, we obtain

$$
\begin{align*}
0 & \longrightarrow \operatorname{Hom}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right) \longrightarrow \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}(-1,-1)\right) \\
& \longrightarrow \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}\right) \longrightarrow \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right) \longrightarrow 0 \tag{58}
\end{align*}
$$

We also have

$$
\begin{align*}
& \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}(-1,-1)\right) \cong H^{1}\left(\mathcal{O}_{T_{2}}(-1,-1)\right)^{\vee} \cong H^{0}\left(\mathcal{O}_{T_{2}}\right) \\
& \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}\right) \cong H^{1}\left(\mathcal{O}_{T_{2}}(-2,-2)\right)^{\vee} \cong H^{0}\left(\mathcal{O}_{T_{2}}(1,1)\right) \tag{59}
\end{align*}
$$

and so their dimensions are 1 and 3 , respectively. As $\mathcal{O}_{Q^{-}}$ sheaves, we have

$$
h^{0}\left(\mathscr{H} \operatorname{om}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right)\right)= \begin{cases}1, & \text { if } T_{1}=T_{2}  \tag{60}\\ 0, & \text { otherwise }\end{cases}
$$

for example, because $T_{1}$ and $T_{2}$ are reduced, and so the assertion is derived.

Lemma 25. The sheaves $\mathscr{F}$ of type (C), but not of type (B), are stable. In particular, the sheaves of type ( $C$ ) are semistable.

Proof. As before let us assume the existence of a proper subsheaf $\mathscr{K}$ of $\mathscr{F}$ with $p(\mathscr{K}) \geq 1 / 2$ and the quotient sheaf $\mathscr{H}:=\mathscr{F} / \mathscr{K}$. Since the composite $s: \mathscr{K} \hookrightarrow \mathscr{F}^{2} \rightarrow \mathcal{O}_{T_{2}}$ is not a zero map, thus we have $\operatorname{Im}(s) \cong \mathcal{O}_{T_{2}}(-Z)$ for a $0-$ dimensional subscheme $Z$ of $T_{2}$ with length $k$. In particular, its Hilbert bipolynomial is $y+1-k$. If we let $\chi_{\mathscr{K}}(x, y)=$ $m^{\prime} x+n^{\prime} y+t^{\prime}$, then we have $p(\mathscr{K})=t^{\prime} /\left(m^{\prime}+n^{\prime}\right) \geq 1 / 2$. In particular, we have $t^{\prime} \geq 1$. If we define $\mathscr{K}^{\prime}$ to be the kernel of the map $s$, then it is a subsheaf of $\mathcal{O}_{T_{1}}$ and thus we have $p\left(\mathscr{K}^{\prime}\right)=\left(t^{\prime}-1+k\right) /\left(m^{\prime}+n^{\prime}-1\right) \leq 1 / 3$. Combining the two inequalities, we have $k=0$ and so the map $s$ is surjective. Thus, we have $\mathscr{H} \cong \mathcal{O}_{T_{1}} / \mathscr{K}^{\prime}$. Note also that $t^{\prime}$ can be either 1 or 2 . If $t^{\prime}=2$, then we have $m^{\prime}=n^{\prime}=2$ and so $\chi_{\mathscr{K}^{\prime}}(x, y)=2 x+y+1=\chi_{\Theta_{T_{1}}}(x, y)$. In particular, we have $\mathscr{H}=0$ and so $\mathscr{K} \cong \mathscr{F}$, a contradiction. Now, assume $t^{\prime}=1$ and so $m^{\prime}+n^{\prime} \leq 2$. In particular, $\mathscr{H}$ is not a $0-$ dimensional sheaf. Moreover, $\mathscr{H}$ is a quotient sheaf of $\mathcal{O}_{T_{1}}$ with constant term 1 and so we have $\mathscr{H} \cong \mathcal{O}_{T_{3}}$ with $T_{3} \subset T_{1}$ and $T_{3} \in\left|\mathcal{O}_{\mathrm{Q}}(1,1)\right|$. For example, if $T_{3}=T_{1}$, then we have $\mathscr{K}^{\prime}=0$ and it contradicts the nontriviality of the extension $\mathscr{F}$. Thus, $\mathscr{F}$ also admits the sequence

$$
\begin{equation*}
0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_{T_{3}} \longrightarrow 0 \tag{61}
\end{equation*}
$$

where $\chi_{\mathscr{K}}(x, y)=x+y+1$. Since $\mathscr{K}^{\prime}$ is a subsheaf of $\mathcal{O}_{T_{1}}$ with $\chi_{\mathscr{K}^{\prime}}(x, y)=x$, we have $\mathscr{K}^{\prime} \cong \mathcal{O}_{T_{4}}(-1,0)$, where $T_{4}$ is a subcurve of $T_{1}$ such that $T_{1}=T_{3}+T_{4}$. Thus, $\mathscr{K}$ is an extension of $\mathcal{O}_{T_{2}}$ by $\mathcal{O}_{T_{4}}(-1,0)$. It is nontrivial, otherwise we would have $\mathcal{O}_{T_{2}}$ as a direct factor of $\mathscr{F}$. Since there exists such a unique extension $\mathcal{O}_{T_{2}+T_{4}}, \mathscr{F}$ admits an extension of $\mathcal{O}_{T_{3}}$ by $\mathcal{O}_{T_{2}+T_{4}}$ :


It implies that $\mathscr{F}$ is of type (B).
Lemma 26. Let $\mathscr{F}$ be a line bundle on a reduced curve $C \in$ $\left|\mathcal{O}_{\mathrm{Q}}(2,2)\right|$ with degree 2.
(1) $\mathscr{F}$ is semistable if and only if one has:
(a) $\operatorname{deg}\left(\left.\mathscr{F}\right|_{T}\right) \geq 1$ for all subcurves $T$ of $C$ in $\left|\mathcal{O}_{\mathrm{Q}}(a, b)\right|$ with $(0,0) \leq(a, b) \leftrightarrows(1,1)$,
(b) $\operatorname{deg}\left(\left.\mathscr{F}\right|_{A}\right) \geq 0$ for each smooth subcurve $A$ of $C$ in $\left|\mathcal{O}_{\mathrm{Q}}(a, b)\right|$ with $(1,1) \leq(a, b) \lesseqgtr(2,2)$.
(2) $\mathscr{F}$ is stable if and only if $\operatorname{deg}\left(\left.\mathscr{F}\right|_{T}\right) \geq 1$ for all subcurves $T$ of $C$ in $\left|\mathscr{O}_{\mathrm{Q}}(u, v)\right|$ with $(0,0) \leq(u, v) \leq(2,2)$.

Proof. In both parts, the "only if" part is obvious. Assume that $\mathscr{F}$ is not stable (resp., semistable) and take a proper subsheaf $\mathscr{H}$ of $\mathscr{F}$ with $p(\mathscr{H}) \geq p(\mathscr{F})$ (resp. $p(\mathscr{H})>p(\mathscr{F})$ ). Taking a saturation of $\mathscr{H}$ in $\mathscr{F}$, we may assume that $\mathscr{G}:=\mathscr{F} / \mathscr{H}$ is a pure sheaf. Call $A$ the scheme support of $\mathscr{H}$ and $T$ the scheme support of $\mathscr{G}$. The definition of scheme support of a purely 1 -dimensional sheaf gives $A+T=C$ as effective divisors. Thus $T$ has one of the types in the assertion. Since $C$ is reduced and $\mathscr{F}$ is a line bundle on $C$, the support of $\mathscr{G}$ must be a proper subcurve $T$ of $C$. If $T$ does not have a type of $\mathcal{O}_{\mathrm{Q}}(1,2)$ or $\mathcal{O}_{\mathrm{Q}}(2,1)$, then we are done. But the case of $T$ having such types is excluded using the argument in the proof of Lemma 18.

Lemma 27. One has $\mathfrak{B} \cap \mathfrak{C} \neq \emptyset$.
Proof. Let us set $B=B^{\prime}+T_{2}$ with $B^{\prime} \in\left|\mathcal{O}_{Q}(0,1)\right|$ and $T_{2} \in\left|\mathcal{O}_{Q}(1,0)\right|$, and set $A \in\left|\mathcal{O}_{Q}(1,1)\right|$ to be smooth. For any extension $\mathscr{F} \in \mathfrak{B}$ of $\mathcal{O}_{B}$ by $O_{A}$, for example, $\mathscr{F}=\mathcal{O}_{A} \oplus \mathcal{O}_{B}$, let $\mathscr{H}$ be the kernel of the composition $\mathscr{F} \rightarrow \mathcal{O}_{B} \rightarrow \mathcal{O}_{T_{2}}$ and
then $\mathscr{H}$ is a pure sheaf with $T_{1}:=A+B^{\prime}$ as its scheme support and has Hilbert bipolynomial $\chi_{\mathscr{H}}=\chi_{\Theta_{T_{1}}}$. Note that it has $\mathcal{O}_{A}$ as its subsheaf.

To prove $\mathscr{H} \cong \mathcal{O}_{T_{1}}$, it is sufficient to prove that $\mathscr{H}$ is semistable. Suppose $\mathscr{H}$ is not semistable and take a proper saturated stable subsheaf $\mathscr{G} \subset \mathscr{H}$ with $\chi_{\mathscr{G}}=a x+b y+$ c. Its scheme support is contained in $T_{1}$ and it is of type $(b, a)$. Without loss of generality, let us assume that $a \leq b$. First, assume $(a, b)=(1,2)$. In this case, we would have $c \geq 2$ because $p(\mathscr{G})>p(\mathscr{H})$ and so we have $h^{0}(\mathscr{H}) \geq 2$, contradicting the fact that $h^{0}(\mathscr{F})=2$ and that $\mathscr{F}$ is globally generated. Assume $a=b=1$. The map $\mathscr{G} \rightarrow \mathscr{H}$ on $A \backslash T_{2}$ must be just the inclusion $\mathcal{O}_{A} \rightarrow \mathscr{H}$, because $\left.\mathscr{H}\right|_{A \backslash T_{2}}$ is a line bundle. Thus either we have $\mathscr{G}=\mathcal{O}_{A}$ or $\mathcal{O}_{A}$ is not saturated in $\mathscr{H}$. Hence the saturation $\mathscr{A}$ of $\mathcal{O}_{A}$ in $\mathscr{F}$ has slope greater than $1 / 2$, contradicting the semistability of $\mathscr{F}$. Now assume $a=0$ and $b=1$, that is, $C_{\mathscr{G}}=B^{\prime}$. Since $B^{\prime}$ is smooth, $\mathscr{G}$ is a line bundle on $B^{\prime}$. If its degree $d$ is at least 1 , then $\mathscr{G}$ contradicts the semistability of $\mathscr{F}$. If $d \leq 0$, then we have $p(\mathscr{G})<p(\mathscr{H})$, a contradiction. Hence $\mathscr{F}$ is also contained in $\mathfrak{C}$.

Lemma 28. For $T_{1} \in\left|\mathcal{O}_{Q}(1,2)\right|$ and $T_{2} \in\left|\mathcal{O}_{Q}(1,0)\right|$, one has

$$
\begin{equation*}
\operatorname{dim} \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right)=2 \tag{63}
\end{equation*}
$$

Proof. Applying the functor $\operatorname{Hom}\left(\mathcal{O}_{T_{2}},-\right)$ to the sequence of $T_{1}$, we obtain

$$
\begin{align*}
0 & \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{Q}\right) \longrightarrow \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right)  \tag{64}\\
& \operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{Q}(-1,-2)\right) \longrightarrow 0
\end{align*}
$$

since we have

$$
\begin{align*}
\operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{Q}(-1,-2)\right) & \cong H^{1}\left(\mathcal{O}_{T_{2}}(-1,0)\right)^{\vee} \\
& \cong H^{1}\left(\mathcal{O}_{T_{2}}\right)^{\vee} \cong H^{0}\left(\mathcal{O}_{T_{2}}(-2)\right)=0 \tag{65}
\end{align*}
$$

and similarly $\operatorname{Ext}^{2}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}\right)=0$. Note also that $\operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}\right) \cong H^{0}\left(\mathcal{O}_{T_{2}}\right)$ and $\operatorname{Ext}^{2}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{\mathrm{Q}}(-1,-2)\right) \cong$ $H^{0}\left(\mathcal{O}_{T_{2}}\right)^{\vee}$. Thus we have the assertion.

Remark 29. When $T_{1}$ and $T_{2}$ meet transversally at two points, say $P_{1}$ and $P_{2}$, then $\operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right)$ is the global sheaf of a sheaf with support on $P_{1}$ and $P_{2}$ with one copy of $\mathbb{C}$ on each point $P_{1}, P_{2}$,

$$
\begin{equation*}
\operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right) \cong \mathbb{C}_{P_{1}} \oplus \mathbb{C}_{P_{2}} \tag{66}
\end{equation*}
$$

for the following reason.
Let $R$ be a regular local ring of dimension 2 and take $x, y$ generators of its maximal ideal. All Ext ${ }^{i}$ groups are with respect to $R$. Since $R /(y)$ is Gorenstein, so the duality gives $\operatorname{Ext}_{R}^{1}(R /(y), R) \cong R /(y)$ and $\operatorname{Ext}_{R}^{i}(R /(y), R)=0$ for all $i \neq 1$. From the exact sequence

$$
\begin{equation*}
0 \longrightarrow R \xrightarrow{u} R \longrightarrow \frac{R}{(x)} \longrightarrow 0 \tag{67}
\end{equation*}
$$

in which $u$ is the multiplication by $x$, we get that $\operatorname{Ext}_{R}^{1}(R /(y), R /(x))$ is the cokernel of the multiplication by $x$ in $R /(y) \rightarrow \mathrm{R} /(y)$; that is, we have $\operatorname{Ext}_{R}^{1}(R /(y), R /(x))=\mathbb{C}$. The same is true for extensions of $\mathcal{O}_{B}$ by $\mathcal{O}_{A}$ when $A$ and $B$ are transversal.

Lemma 30. Let $\mathscr{F}$ be a sheaf of type (A) with no 0dimensional torsion. Then $\mathscr{F}$ is semistable unless it admits the sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{T_{2}} \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_{T_{1}} \longrightarrow 0 \tag{68}
\end{equation*}
$$

where $T_{1} \in\left|\mathcal{O}_{\mathrm{Q}}(a, b)\right|$ and $T_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(2-a, 2-b)\right|$ with $(a, b) \in$ $\{(1,2),(2,1)\}$.

Proof. Let $\mathscr{K}$ be a subsheaf $\mathscr{F}$ with maximal $p$-slope $p(\mathscr{K})>$ $1 / 2$ and so the quotient sheaf $\mathscr{H}:=\mathscr{F} / \mathscr{K}$ has no 0 dimensional torsion. Let us set $\chi_{\mathscr{K}}(x, y)=m^{\prime} x+n^{\prime} y+t^{\prime}$ with $t^{\prime} \geq 1$ and $(0,0) \lesseqgtr\left(m^{\prime}, n^{\prime}\right)$. If the composite $s: \mathscr{K} \hookrightarrow \mathscr{F}^{2} \rightarrow$ $\eta$ is a zero map, then $\mathscr{K}$ is a subsheaf destabilizing $\mathcal{O}_{C}$, a contradiction. If $s$ is not surjective, for instance, $\operatorname{Im}(s)=\mathscr{O}_{P} \varsubsetneqq$ $\eta$, then $\operatorname{ker}(s)$ is a subsheaf of $\mathcal{O}_{C}$ with Hilbert bipolynomial $m^{\prime} x+n^{\prime} y+t^{\prime}-1$. Thus we have $t^{\prime}=1$ and the quotient $\mathscr{H}^{\prime}:=$ $\mathcal{O}_{C} / \mathscr{K}^{\prime}$ has Hilbert bipolynomial with zero constant term. Since $\mathscr{H}^{\prime}$ has no 0-dimensional torsion, we have $\mathscr{H}^{\prime} \cong \mathcal{O}_{D}$ for a curve $D$ contained in $C$. But the Hilbert bipolynomial of $\mathcal{O}_{D}$ has nonzero constant term, a contradiction. Thus the map $s$ is surjective. Following the same argument before, we obtain that $t^{\prime}=1$ and $m^{\prime}+n^{\prime} \leq 1$. Without loss of generality, let us assume that $\left(m^{\prime}, n^{\prime}\right)=(0,1)$. Then we have $\chi_{\mathscr{H}}(x, y)=$ $2 x+y+1$ and thus we have $\mathscr{H} \cong \mathcal{O}_{T_{1}}$, where $T_{1}$ is a curve contained in $C_{\mathscr{F}}$ and $T_{1} \in\left|\mathcal{O}_{\mathrm{Q}}(1,2)\right|$. Since $\mathscr{K}$ is a subsheaf of $\mathscr{F}$ with $\chi_{\mathscr{K}}(x, y)=y+1$, we have $\mathscr{K} \cong \mathcal{O}_{T_{2}}$ since $\mathscr{K}$ has no 0 dimensional torsion. Thus $\mathscr{F}$ fits into the sequence (68).

Remark 31. Applying the functor $\operatorname{Hom}(\eta,-)$ to the sequence of $C \in\left|\mathcal{O}_{Q}(2,2)\right|$, we obtain

$$
\begin{align*}
& 0 \longrightarrow \operatorname{Ext}^{1}\left(\eta, \mathcal{O}_{C}\right) \longrightarrow \operatorname{Ext}^{2}\left(\eta, \mathcal{O}_{\mathrm{Q}}(-2,-2)\right) \\
& \xrightarrow{f} \operatorname{Ext}^{2}\left(\eta, \mathcal{O}_{\mathrm{Q}}\right) \tag{69}
\end{align*}
$$

Since the map $f$ is the dual of the map $\operatorname{Hom}\left(\mathcal{O}_{\mathrm{Q}}(2,2), \eta\right) \rightarrow$ $\operatorname{Hom}\left(\mathcal{O}_{\mathrm{Q}}, \eta\right)$ given by the multiplication by the defining equation of $C$, the map $f$ is a zero map and thus we have $\operatorname{Ext}^{1}\left(\eta, \mathcal{O}_{\mathrm{C}}\right) \cong H^{0}(\eta)^{\vee}$. In particular its dimension is 2 .

Lemma 32. Let $\mathscr{F}$ be a sheaf of type (B) fitting into an exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathcal{O}_{T_{1}} \longrightarrow \mathscr{F} \longrightarrow \mathcal{O}_{T_{2}} \longrightarrow 0 \tag{70}
\end{equation*}
$$

with $T_{1}, T_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(1,1)\right|$. Then $\mathscr{F}$ is of type $(A)$ if and only if $T_{1}$ and $T_{2}$ have no common components; that is, $C_{\mathscr{F}}$ has no multiple component.

Proof. If $T_{1}$ and $T_{2}$ have a common component, say $T$, then $\mathscr{F}$ has rank 2 at the general point of $T$ and thus $\mathscr{F}$ is not of type (A).

Conversely, assume that $T_{1} \cap T_{2}$ is finite. Since we have $h^{1}\left(\mathcal{O}_{T_{1}}\right)=0$, the sequence (70) implies that $h^{0}(\mathscr{F})=2$ and $\mathscr{F}$ is globally generated. Let $\sigma$ be a general section of $\mathscr{F}$ and then it does not vanish at the general point of any of the components of $C_{\mathscr{F}}$. Since $C_{\mathscr{F}}$ is reduced, $\sigma$ induces an injective map ${ }^{{ }_{C}}{ }_{\mathscr{F}} \hookrightarrow \mathscr{F}$ and thus $\mathscr{F}$ has type (A).

Lemma 33. Let $\mathscr{F}$ be a sheaf of type (C).
(1) If $T_{2}$ is not a component of $T_{1}$, then $\mathscr{F}$ is of type ( $A$ ).
(2) If $T_{2}$ is a double component of $C_{\mathscr{F}}$, that is, $T_{2} \subset T_{1}$, then it is not of type $(A)$.

Proof. Let us assume that $T_{2} \in\left|\mathcal{O}_{\mathrm{Q}}(1,0)\right|$.
(1) Since $T_{2}$ is not a component of $T_{1}, \mathscr{F}$ is a line bundle on $C=C_{\mathscr{F}}$ outside finitely many points of $C$. Moreover, it is not an $\mathcal{O}_{T_{1}}$-sheaf. Note that $\mathscr{F}$ is globally generated since $\mathcal{O}_{T_{1}}$ and $\mathcal{O}_{T_{2}}$ are globally generated with $h^{1}\left(\mathcal{O}_{T_{1}}\right)=0$. Thus, a general section of $\mathscr{F}$ does not vanish at a general point of $T_{2}$ and so it does not induce an injection $\mathcal{O}_{T_{1}} \hookrightarrow \mathscr{F}$. Hence, $\mathscr{F}$ fits into some sequence (46).
(2) Let us set $C=2 T_{2}+T_{3}$ and $T_{1}:=T_{2}+T_{3}$, where $T_{3} \in\left|\mathcal{O}_{Q}(0,2)\right|$. Let $\Gamma$ be the projectivisation of $\operatorname{Ext}^{1}\left(\mathcal{O}_{T_{2}}, \mathcal{O}_{T_{1}}\right)$ and in particular we have $\Gamma \cong \mathbb{P}^{1}$ by Lemma 28. We also know from Lemma 25 that any $e \in \Gamma$ gives a semistable sheaf. Such a sheaf has rank 2 at the points of $T_{2} \backslash\left(T_{2} \cap T_{3}\right)$ and, in particular, it is not a line bundle over its support at a general point of $T_{2}$. Thus, it never fits into an exact sequence (46). Otherwise it would be locally free of rank 1 at each point of the support of $T_{3}$ but not in $T_{2}$.

In general, the question whether the variety $\mathscr{P} i c_{(m, n)}^{d}$. We observed that $\mathbf{M}_{1}$ is rational and so is $\mathscr{P} i c_{(2,2)}^{1}$. Below we give a partial answer to this question in the case of $\mathscr{P} c_{(2,2)}^{2}$.

Theorem 34. $\mathbf{M}_{2}$ is unirational with degree 4.
Proof. Let us fix a smooth curve $C$ of bidegree $(2,2)$ in $Q$ and a point $P \in C$ to consider a sheaf $\mathcal{O}_{C}(P) \in \mathbf{M}_{1}$. If $\mathbb{T}_{P}$ is the tangent plane of $Q$ at $P$, then we have $\mathbb{T}_{P} \cap C=\left\{2 P, Q_{1}, Q_{2}\right\}$ for some points $Q_{1}, Q_{2}$ on $Q$ since $\operatorname{deg}(C)=4$. It defines a rational map

$$
\begin{equation*}
\Phi: \mathbf{M}_{1} \cdots \mathbf{M}_{2}, \tag{71}
\end{equation*}
$$

sending $\mathcal{O}_{C}(P)$ to $\mathcal{O}_{C}\left(Q_{1}+Q_{2}\right)$. Note that $\mathcal{O}_{C}\left(Q_{1}+Q_{2}\right)=$ $\mathcal{O}_{C}(1,1)(-2 P)$. We claim that the map $\Phi$ is generically 4 to 1 and so the assertion follows.

Let $U$ (resp., $V$ ) be the dense open subset of $\mathbf{M}_{1}$ (resp., $\mathscr{A} \subset \mathbf{M}_{2}$ ) formed by the sheaves $\mathscr{F}$ such that $C_{\mathscr{F}}$ is smooth. Each element of $U$ (resp., $V$ ) is uniquely determined by a smooth $C \in\left|\mathcal{O}_{Q}(2,2)\right|$ and a degree one (resp., degree two) line bundle on $C$. By Riemann-Roch, each degree one line bundle on $C$ is associated with a unique $P \in C$. Then the map $\Phi$ sends $\mathcal{O}_{C}(P)$ to $\mathscr{R}:=\mathcal{O}_{C}(1,1)(-2 P)$. Fix any degree two
line bundle $\mathscr{M}$ on $C$. Since we are in characteristic zero, there are exactly four line bundles $\mathscr{A}$ on $C$ such that $\mathscr{A}^{\otimes 2} \cong \mathcal{O}_{C}$. Hence, for each $\mathscr{R} \in \operatorname{Pic}^{2}(C)$ there are exactly 4 points $P \in C$ such that $\mathscr{R} \cong \mathcal{O}_{C}(1,1)(-2 P)$. Hence, $\Phi$ is dominant and the preimage of each element of $V$ has cardinality 4 .

We did not succeed in getting any smaller degree of unirationality of $\mathbf{M}_{2}$ as of now, and we left the rationality question as a conjecture.

Conjecture 35. $\mathrm{M}_{2}$ is rational.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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