Annali di Matematica pura ed applicata
(IV), Vol. CLXXV (1998), pp. 375-393

Stable Sheaves on Reduced Projective Curves (*).

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Summary. – *Let X be a reduced projective curve; assume X either irreducible or stable. Here we study some geometric properties of stable vector bundles and stable torsion free sheaves on X (essentially, the existence of stable objects with a prescribed order of stability).*

Introduction.

Let X be a singular projective curve. We like the study of vector bundles on X. If X is not smooth during their study torsion free sheaves appear as a fundamental and very natural technical tool. Thus torsion free sheaves deserve to be studied, although in this paper most of the main results will be proved for locally free sheaves and the non locally free ones will appear only during the proofs. In section one and two we will assume X reduced and irreducible. In the third (and last section) X will be any stable curve of arithmetic genus $g \geq 2$, i.e. any connected nodal curve X with $\mathrm{Aut}(X)$ finite. We will give partial extensions to the case of singular curves of several results recently proved for smooth ones. The key word (see [T], [RT], [B2] and [BR]) is the so-called Lange's conjecture, corcerning the existence of exact sequences of stable vector bundles with suitable numerical invariants. Our best results on this topic are Theorems 1.9, 1.15 and 3.1. We will study also the geometric properties of a general vector bundle on X (see Theorem 1.8). Another aim of this paper is the introduction of some new numerical invariants for vector bundles and torsion free sheaves. These invariants capture only part of the informations given by some more classical invariants. We will study both the classical and the new invariants, but the reader will appreciate how easier is to work with the new ones and obtain inductive proofs (see in particular Definitions 2.1, 2.2, 2.3 and Theorems 2.4 and 2.5). A key point of most proofs and the main reason for the introduction of these new numerical invariants is to control the «singularities» of the torsion free sheaves appearing during the proofs, i.e. the isomorphism types of the completion at each $P \in \mathrm{Sing}(X)$ of the stalks of these sheaves.

(*) Entrata in Redazione l'8 maggio 1998.
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The author was partially supported by MURST and GNSAGA of CNR (Italy).

1. – Existence of exact sequences of stable sheaves.

At the beginning of this section we will introduce some definitions, notations and conventions which will be used in the entire paper. We work over an algebraically closed base field $\boldsymbol{K}$. In the first and third section of this paper we will assume $\operatorname{char}(\boldsymbol{K}) = 0$. Let Z be a reduced and irreducible curve with $q := p_a(Z) \geq 2$. For every torsion free sheaf E on Z, let $\mu(E) := \deg(E)/\operatorname{rank}(E)$ be its slope. For all integers r, d with $r > 0$ $M(Z; r, d)$ will denote the moduli scheme of rank r stable torsion free sheaves on Z with rank r and degree d. If Z has only planar singularities (and in particular if Z has only ordinary nodes as singularities), then $M(Z; r, d)_{\mathrm{red}}$ is an irreducible variety of dimension $r^2(q-1)+1$ ([Re]). Let Y be a reduced projective curve and F a torsion free sheaf on Y. For every $P \in \operatorname{Sing}(Y)$, let $F_{P^\wedge}$ be the formal completion of the $\boldsymbol{O}_{Y,P}$-module corresponding to F. Fix $P \in \operatorname{Sing}(Y)$ and let M be a rank r torsion free $\boldsymbol{O}_{Y,P^\wedge}$-module. As in [Co], def. 2.2.3, let $l(M)$ be the minimal integer t such that M contains a free $\boldsymbol{O}_{Y,P^\wedge}$-module N of rank r with $\dim_{\boldsymbol{K}}(M/N) = t$. Hence $l(M) = 0$ if and only if M is free. For every rank r torsion free sheaf F on Y it is important to consider the integer $l(F) := \sum_{P \in \operatorname{Sing}(Y)} l(F_{P^\wedge})$, where $F_{P^\wedge}$ is the torsion free $\boldsymbol{O}_{Y,P^\wedge}$-module of rank r induced by F (see e.g. [Co], Ch. III, for its use). Hence $l(F) = 0$ if and only if F is locally free. We will call the family of torsion free modules $\{F_{P^\wedge}\}_{P \in \operatorname{Sing}(Y)}$ the formal singularity type of F or the formal singularity type of F along $\operatorname{Sing}(Y)$ and $F_{P^\wedge}$ the formal singularity type of F at P. Notice that $l(F) = l(G)$ for any two torsion free sheaves with the same formal singularity type. Hence if $F^\wedge := \{F_{P^\wedge}\}_{P \in \operatorname{Sing}(Y)}$ is a formal singularity type, we will set $l(F^\wedge) := \sum_{P \in \operatorname{Sing}(Y)} l(F_{P^\wedge})$. If Y is irreducible, r and d are integers with $r > 0$ and $\{F_{P^\wedge}\}_{P \in \operatorname{Sing}(Y)}$ is an ordered set of rank r torsion free modules over the completions $\boldsymbol{O}_{Y,P^\wedge}$ of the local rings $\boldsymbol{O}_{Y,P}$, $M(Y; r, d, \{F_{P^\wedge}\}_{P \in \operatorname{Sing}(Y)})$ will denote the subset of $M(Y; r, d)_{\mathrm{red}}$ parametrizing the stable torsion free sheaves with $\{F_{P^\wedge}\}_{P \in \operatorname{Sing}(Y)}$ as formal singularity type. $M(Y; r, d, \emptyset)$ or $U(Y; r, d)$ will denote the Zariski open subset of $M(Y; r, d)$ parametrizing the locally free sheaves. If Y is irreducible, then $U(Y; r, d)$ is irreducible ([N], Remark at p. 167). Let Y be an arbitrary reduced projective curve, $P \in Y_{\mathrm{reg}}$ and F a torsion free sheaf. For any $Q \in Y$, $\boldsymbol{K}_Q$ will denote the skyscraper sheaf on Y with Q as support and $h^0(Y, \boldsymbol{K}_Q) = 1$. Call $F|\{P\}$ the fiber of F over P. Hence if F has rank r at P, $F|\{P\}$ is a r-dimensional vector space over $\boldsymbol{K}$. Fix a linear surjective map $f(P)$: $F|\{P\} \to \boldsymbol{K}$, or, equivalently, a surjection f: $F \to \boldsymbol{K}_P$. $\operatorname{Ker}(f)$ is a torsion free sheaf on Y which has the same formal completion as F at each point of $\operatorname{Sing}(F)$, i.e. F and $\operatorname{Ker}(f)$ have the same formal singularity type. Following [Ma] and [B1] we will say that $\operatorname{Ker}(f)$ is obtained from F making a negative elementary transformation supported by Q and that F is obtained from $\operatorname{Ker}(f)$ making a positive elementary transformation supported by Q. Indeed, since $Q \in Y_{\mathrm{reg}}$, the local to global spectral sequence for the Ext-functors shows that F is uniquely determined by the choice of $\operatorname{Ker}(f)$ and of a linear surjective map from $(\operatorname{Ker}(f)|\{Q\})^*$ to $\boldsymbol{K}$. More generally, we may take an integer k with $0 < k < r$ and a linear surjective map $h(P)$: $F|\{P\}\boldsymbol{K}^{\oplus k}$, i.e. a surjection h: $F \to \boldsymbol{K}_P^{\oplus k}$. $\operatorname{Ker}(h)$ is a locally free sheaf with $\deg(\operatorname{Ker}(h)) = \deg(F) - k$. We will say that $\operatorname{Ker}(h)$ is obtained by a negative elementary transformation associated to the k-dimensional quotient vector space $(\operatorname{Ker}(h(P)))^*$ of $(F|\{P\})^*$. $\operatorname{Ker}(h)$ may be obtained from F making k suitable negative elementary transformations supported by P. F and $\mathrm{Ker}(h)$ have the same formal singularity type along $\mathrm{Sing}(Y)$.

From now on, in this section we assume $\mathrm{char}(\boldsymbol{K}) = 0$. Let X be an integral projective curve. Here we study the existence of exact sequences of stable torsion free sheaves on X with prescribed numerical invariants and with prescribed formal singularity types. Fix integers r, k with $r > k$. For every $P \in \mathrm{Sing}(X)$ a triple $(H_P^\wedge, E_P^\wedge, G_P^\wedge)$ of $\boldsymbol{O}_{X, P^\wedge}$-modules is said to be compatible and with respect to the integers r, k if all these modules are torsion free, $H_P^\wedge$ is a submodule of $E_P^\wedge$, $E_P^\wedge / H_P^\wedge \cong G_P^\wedge$, $\mathrm{rank}(H_P^\wedge) = k$ and $\mathrm{rank}(G_P^\wedge) = r - k$. Notice that $\mathrm{rank}(E_P^\wedge) = r$ and that $E_P^\wedge / H_P^\wedge$ has no torsion. A compatible formal singularity data along $\mathrm{Sing}(X)$ for the integers r, k is a set of compatible triples $\{(H_P^\wedge, E_P^\wedge, G_P^\wedge)\}_{P \in \mathrm{Sing}(X)}$ with respect to the integers r, k. Now fix integers a, b with $a/k < b/(r-k)$ (resp. $a/k \leqslant b/(r-k)$) and a compatible formal singularity data, $\Pi := \{(H_P^\wedge, E_P^\wedge, G_P^\wedge)\}_{P \in \mathrm{Sing}(X)}$, along $\mathrm{Sing}(X)$. We will say that $\$(X; r, k, a, b, \Pi)$ (resp. $\$\$(X; r, k, a, b, \Pi)$) is true if there is an exact sequence of torsion free sheaves on X:

$$0 \to H \to E \to G \to 0 \tag{1}$$

with $\mathrm{rank}(H) = k$, $\deg(H) = a$, $\mathrm{rank}(E) = r$, $\deg(G) = b$, H with formal singularity type $\{H_P^\wedge\}_{P \in \mathrm{Sing}(X)}$, E with formal singularity type $\{E_P^\wedge\}_{P \in \mathrm{Sing}(X)}$, G with formal singularity type $\{G_P^\wedge\}_{P \in \mathrm{Sing}(X)}$, and H, E, G stable (resp. semistable). We will say that $\$(X; r, k, \Pi)$ (resp. $\$\$(X; r, k, \Pi)$) is true if for all integers a, b with $a/k < b/(r-k)$ (resp. $a/k \leqslant b/(r-k)$) the assertion $\$(X; r, k, a, b, \Pi)$ (resp. $\$\$(X; r, k, a, b, \Pi)$) is true. If Π is the compatible formal singularity data of free modules we will write $\$(X; r, k, a, b, \emptyset)$ (resp. $\$\$(X; r, k, a, b, \emptyset)$, resp. $\$(X; r, k, \emptyset)$, resp. $\$\$(X; r, k, \emptyset)$) instead of $\$(X; r, k, a, b, \Pi)$ (resp. $\$\$(X; r, k, a, b, \Pi)$, resp. $\$(X; r, k, \Pi)$, resp. $\$\$(X; r, k, \Pi)$).

REMARK 1.1. – Let X be an integral projective curve and F a torsion free sheaf on X. Fix non-negative integers α, β and an ordered set σ of $\alpha + \beta$ signs, say $\sigma := (+, +, -, -, +, \ldots)$ with α $+$'s and β $-$'s. Take a sheaf G obtained from F making α positive elementary transformations and β negative elementary transformations in the order prescribed by σ. There is an irreducible variety parametrizing all sheaves «near G» obtained from F making α positive and β negative elementary transformations in the order prescribed by σ. This is true both if we impose that all the elementary transformations are supported by a fixed point $Q \in X_{\mathrm{reg}}$ or if we allow that they are made on arbitrary points of X_{reg}. This remark is completely obvious if the $\alpha + \beta$ elementary transformations are supported by $\alpha + \beta$ distinct points of X_{reg}. Hence if $\mathrm{char}(\boldsymbol{K}) = 0$ the union of the proof of [B4], Th. 0.1, and of [B4], Remark 3.1, shows that F is the flat limit of a flat family of stable torsion free sheaves all of them with the same formal singularity type along $\mathrm{Sing}(X)$ as F.

For reader's sake we extract from [RT], proof of in the middle of p. 7, the following key lemma; we reproduce here the proof given in [RT], just adapting it to our more general set-up.

LEMMA 1.2. – *Let X be an integral projective curve and $Q \in X_{\mathrm{reg}}$. Set $g := p_a(X)$ and assume $g \geqslant 2$ (resp. $g \geqslant 1$). Fix integers r, k with $r > k > 0$ and assume the existence of an exact sequence* (1) *of torsion free sheaves on X with* $\mathrm{rank}(H) = k$, $\mathrm{rank}(E) = r$

and H, E, G stable (resp. semistable). Call $\{H^\wedge\}$ (resp. $E^\wedge$, resp. $\{G^\wedge\}$) the formal singularity type of H (resp. E, resp. G) along $\mathrm{Sing}(X)$ *and let $\Pi :=$ $:= \{(H_P^\wedge, E_P^\wedge, G_P^\wedge)\}_{P \in \mathrm{Sing}(X)}$ be the associated compatible formal singularity data. Then there exists an exact sequence of torsion free sheaves on X:*

$$0 \to H' \to E' \to G' \to 0 \tag{2}$$

with H', E' and G' stable (resp. semistable), H' obtained from H making a general negative elementary transformation supported by Q (hence with formal singularity type $\{H^\wedge\}$, $\mathrm{rank}(H') = k$ *and* $\deg(H') = \deg(H) - 1$), *G' obtained from G making a general positive elementary transformation supported by Q (hence with formal singularity type $\{G^\wedge\}$,* $\mathrm{rank}(G') = r - k$ *and* $\deg(G') = \deg(G) + 1$), *and with E' with formal singularity type $\{E^\wedge\}$, E' obtained from E making first a general negative elementary transformation supported by Q and then a general positive elementary transformation supported by Q. In particular if \$$(X; r, k, a, b, \Pi)$ (resp. \$\$$(X; r, k, a, b, \Pi)$) is true, then \$$(X; r, k, a-1, b+1, \Pi)$ (resp. \$\$$(X; r, k, a-1, b+1, \Pi)$) is true.*

PROOF. – Let $\{E'_u\}_{u \in T}$ be the family of all torsion free sheaves obtained from E making first a negative elementary transformation corresponding to a surjection $\alpha(u)\colon E|\{Q\} \to K$ with $\alpha(u)(H|\{Q\}) \neq 0$ (e.g. a general negative elementary transformation supported by Q) and then a positive elementary transformation supported by Q not corresponding to a point of $H|\{Q\}$, i.e. inducing a positive elementary transformation of G (e.g. a general positive elementary transformation supported by Q); here T is an irreducible affine variety. For every $u \in T$ we have an exact sequence

$$0 \to H'_u \to E'_u \to G'_u \to 0 \tag{3}$$

with H'_u obtained from H making a negative elementary transformation supported by Q and G'_u obtained from G_u making a positive elementary transformation supported by Q. Instead of making first a sufficiently general negative elementary transformation and then a sufficiently general elementary transformation, both supported by Q, one could first make a negative elementary transformation supported by Q and then exactly its inverse; in this way one would obtain again E and the exact sequence (1). Hence by the first part of Remark 1.1 E is a flat limit of the family $\{E'_u\}_{u \in T}$. By the openness of stability (resp. semistability) if E is stable (resp. semistable), then for a general $u \in T$ the sheaf E'_u is stable (resp. semistable). We fix one such general $u \in T$. For simplicity we assume E and hence E'_u stable; the semistable case is exactly the same if we assume also E simple and hence E'_u simple for general u. By Remark 1.1 the sheaf H'_u (resp. G'_u) is a flat limit of a flat family $\{A_m\}_{m \in S}$ (resp. $\{B_m\}_{m \in S}$) of stable torsion free sheaves with singularity type $\{H^\wedge\}$ ((resp. $\{G^\wedge\}$) and S integral smooth affine curve. Since H'_u and each A_m (resp. G'_u and each B_m) have the same singularity type along $\mathrm{Sing}(X)$, we have $h^0(X, \mathrm{Ext}^1(G'_u, H'_u)) = h^0(X, \mathrm{Ext}^1(B_m, A_m))$ for every $m \in S$. Since E'_u is simple, we have $h^0(X, \mathrm{Hom}(G'_u, H'_u)) = 0$. Hence by semicontinuity for a general $m \in S$, say for $m \in S'$ with S' open dense in S, we have $h^0(X, \mathrm{Hom}(B_m, H_m)) = 0$. Hence by the local to global spectral sequence for the Ext-functors, we obtain $\dim(\mathrm{Ext}^1(X, G'_u, H'_u)) = \dim(\mathrm{Ext}^1(X, B_m, A_m))$ for every $m \in S'$. Hence by the theory of the Relative Ext (see [BPS] or [L2]) there is a vector bundle V of rank

$\dim(\mathrm{Ext}^1(X, B_m, A_m))$ over a smooth curve S'' containing S' and a point, o (say $\{o\} = S''\backslash S'$), corresponding to the extension (3) and giving over $m \in S'$ all the extensions of B_m by A_m and over o all the extensions of G_u' by H_u'. Since V is irreducible, E is stable and stability is an open condition, we obtain that a general $x \in V$ corresponds to an extension (2) in which all the sheaves H', E' and G' are stable. ■

REMARK 1.3. – By Lemma 1.2 to prove $\$(X, r, k, \Pi)$ (resp. $\$\$(X, r, k, \Pi)$) on an integral curve X and for fixed integers r, k and a compatible triple of formal singularities types Π and all integers a, b with $a/k < b/(r-k)$ (resp. $a/k \leq b/(r-k)$), it is sufficient to prove $\$(X, r, k, a_{\max}, b_{\min}, \Pi)$ (resp. $\$(X, r, k, a, b, \Pi)$) where $a_{\max}$ and $b_{\min}$ are the unique integers with $a_{\max}/k < b_{\min}/(r-k)$ and $(a_{\max}+1)/k \geq (b_{\min}-1)/(r-k)$ (resp. $a_{\max}/k \leq b_{\min}/(r-k)$ and $(a_{\max}+1)/k > (b_{\min}-1)/(r-k)$). Notice that $b_{\min}/(r-k) - a_{\max}/k \leq 1/k + 1/(r-k)$ (resp. $b_{\min}/(r-k) - a_{\max}/k < 1/k + 1/(r-k)$).

The following result was claimed in [T] as [T], Th. 0.2, only for nodal curves. The same proof works verbatim for curves with smoothable singularities, not just the nodal ones. Hence we have the following result.

THEOREM 1.4. – *Let X be an integral projective curve with $g := p_a(X) \geq 2$ and only smoothable singularities. Then for all integers r, k with $r > k > 0$ the assertion $\$\$(X; r, k, \emptyset)$ is true.*

REMARK 1.5. – We claim that $\$\$(X; r, k, \emptyset)$ is true for every integral curve X with $p_a(X) = 1$, i.e. for every smooth elliptic curve and for every rational curve with a unique singular point which is either an ordinary node or an ordinary cusp. By [BR], Prop. 1.6, $\$\$(Z; r, k, \Pi)$ is true for every smooth elliptic curve. Hence we may apply the closed property of semistability in flat families of curves as in the proof of [T], Th. 0.2, and obtain the result for a singular curve X.

Here is an easy case in which $\$\$(X; r, k, a, b, \Pi)$ is true for any integral curve and any compatible formal singularity data Π.

PROPOSITION 1.6. – *Let X be an integral projective curve with $g := p_a(x) \geq 1$. For all integers r, k, t, x with $r > k > 0$, $x \geq 0$ and every compatible formal singularity data Π the assertion $\$\$(X; r, k, kt - x, t(r-k) + x, \Pi)$ is true.*

PROOF. – Fix $Q \in X_{\mathrm{reg}}$. Twisting an exact sequence (1) by the line bundle $\mathcal{O}_X(-tQ)$ we reduce to the case $t = 0$. First assume $x = 0$. Let $(H^\wedge, E^\wedge, G^\wedge)$ be the formal singularity data Π. We take as H (resp. G) any semistable bundle of degree 0 with $H^\wedge$ (resp. $G^\wedge$) as formal singularity type along $\mathrm{Sing}(X)$. Since $\mathrm{Ext}^1(X, G, H) \cong H^1(X, \mathrm{Hom}(G, H)) \oplus H^0(X, \mathrm{Ext}^1(G, H))$ by the local to global spectral sequence for the Ext-functor, there is a spectral sequence (1) with E with formal singularity type $E^\wedge$. Since G and H are semistable and with the same slope, E is semistable, proving the case $x = 0$. If $x > 0$ we apply x times Lemma 1.2 and Remark 1.3. ■

Here is an easy case in which $\$(X; r, k, a, b, \Pi)$ is true for any integral curve and any compatible formal singularity data Π.

PROPOSITION 1.7. – *Let X be an integral projective curve with $g := p_a(X) \geq 2$. For all* *integers r, k, t, x with $r > k > 0$, $x \geq 0$ and every compatible formal singularity data Π the assertion* $\$(X; r, k, kt - x, t(r-k) + x + 1, \Pi)$ *is true.*

PROOF. – As in the proof of Proposition 1.6 it is sufficient to prove the case $t = x = 0$. If $\Pi = (H^\wedge, E^\wedge, G^\wedge)$ we take a stable sheaf $H \in M(X; k, 0, H^\wedge)$ and a stable sheaf $G \in M(X; r-k, 1, G^\wedge)$. Since $h^1(X, \mathrm{Hom}(G, H)) \neq 0$ by Riemann-Roch, we may take a non-splitted extension (1) with E with formal singularity type $E^\wedge$. We claim that any such E is stable. We have $\deg(E) = 1$. Assume the existence of a proper subsheaf of F with $\mu(F) \geq \mu(E) = 1/r$. Hence $\deg(F) > 0$. Since (1) does not split the map $u\colon F \to G$ induced by (1) cannot be an isomorphism. Since G is stable and $\mu(G) = 1/(r-k)$, we have $u = 0$. Hence F is a subsheaf of H and thus $\deg(F) \leq 0$, contradiction. ■

THEOREM 1.8. – *Let X be an integral projective curve with only planar singularities and $\pi\colon Z \to X$ the normalization. Set $g := p_a(X)$ and $g'' := p_a(Z)$. Assume $g'' \geq 2$. Fix integers r, d, k with $r > k > 0$. Let E be a general element of $U(X; r, d)$. Then we have:*

(i) *$\pi^*(E)$ is stable;*

(ii) *for every subsheaf A of E with* $\mathrm{rank}(A) = k$ *we have* $\mu(A) \leq (\deg(E) - \deg(A)/(r-k) - (g''-1)$ *with strict inequality unless $g = g''$, i.e. X is smooth.*

PROOF. – Notice that π is finite, $\pi_*(\boldsymbol{O}_Z)/\boldsymbol{O}_X$ is supported by the finite set $\mathrm{Sing}(X)$, and $\pi_*\pi^*(B) \cong B \otimes \pi_*(\boldsymbol{O}_Z)/\boldsymbol{O}_X$ for every vector bundle B on X (projection formula). Thus the natural map $H^1(X, \mathrm{End}(E)) \to H^1(X, \mathrm{End}(E) \otimes \pi_*(\boldsymbol{O}_X)) \cong H^1(Z, \mathrm{End}((\pi^*(E)))$ is surjective. Hence every small deformation of $\pi^*(E)$ is induced by a small deformation of E. If $g'' \geq 2$ a general deformation of $\pi^*(E)$ is stable ([NR], Prop. 6.2, or [Hi], Cor. 2.2). Hence we have part (i). Furthermore, by the generality of E we may assume that $\pi^*(E)$ is a general stable bundle on Z with rank r and degree d. To check part (ii) we may assume A saturated in E, i.e. we may assume E/A torsion free. Set $M := \pi^*(E/A)/\mathrm{Tors}(\pi^*(E/A))$. Since the functor π^* is right exact, we have a surjection, f, from $\pi^*(E)$ to the rank $r-k$ vector bundle M. By [L1], Satz 2.2, or [Hi], 4.5, we have $\mu(\mathrm{Ker}(f)) \leq \mu(M) + (g''-1)$, i.e. $(-\deg(M)/k \leq \deg(M)/(r-k) + (g''-1)$. We have $\deg(M) \leq \deg(E/A)$ with strict inequality unless E/A is locally free ([Co], Prop. 3.2.4, part 2). Hence it is sufficient to show that if X is not smooth and E is general, either E/A is not locally free or there is a quotient sheaf, U of $\pi^*(E)$ with $\mathrm{rank}(U) = r-k$ and $\deg(U) < \deg(E/A)$. Thus we may assume that E/A and A are locally free and that $\deg(A)$ is maximal among the degrees of the rank k subsheaves of E. Since E is stable, it is simple and hence $H^0(X, \mathrm{Hom}(E/A, A)) = 0$. Since E/A and E are locally free, we have $\deg(\mathrm{Hom}(E/A, A)) = k(r-k)(\mu(A) - \mu(E/A))$. Hence by Riemann-Roch we have $h^1(X, \mathrm{Hom}(E/A, A)) = k(r-k)(\mu(E/A) - \mu(A) + g - 1)$. Since A (resp. E/A) is locally free, it is the flat limit of a family of stable vector bundles (see [NR], Prop. 6.2 in char 0, or [Hi], Cor. 2.2, for the case of a smooth curve, and [N], Remark at p. 167, and the introduction of [Re] for the general case). Hence, it is easy to make sense of the assertion that A (resp. E/A) depends on at most $\dim(U(X; k, \deg(A))$ (resp. $\dim(U(X; r-k, \deg(E/A)))$ parameters, i.e. on at most $k^2(g-1)+1$ (resp. $(r-k)^2(g-1)+1$) parameters. Notice that proportional extensions of E/A by A induce isomorphic vector bundles as middle term of an extension and that $k^2(g-1)+1+$

$+(r-k)^2(g-1)+1+2k(r-k)(g-1)-1=r^2(g-1)=\dim(U(X;\,r,\,d))-1$. Hence if $g>g''$ and $\mu(E/A)-\mu(A)=g''-1$ we obtain a contradiction. ∎

Now we may prove the following result.

THEOREM 1.9. – *Let X be an integral projective curve with only planar singularities. Assume that the genus, g'', of the normalization of X is at least 3. Then for all integers r, k with $r>k>0$ Property $\$(X;\,r,\,k,\,\emptyset)$ is true.*

PROOF. – Using Theorem 1.8 and the assumption $g''\geqslant 3$ we may repeat verbatim (just using g'' instead of the arithmetic genus g) the proofs of Claims 1 and 2 on pages 7 and 8 of [RT] and obtain that $\$(X;\,r,\,k,\,a,\,b,\,\emptyset)$ is true for all integers a, b with $0<k(a+b)-ra\leqslant r$ and hence that $\$(X;\,r,\,k,\,\emptyset)$ is true by Lemma 1.2 and Remark 1.3. ∎

Now we will extend Theorems 1.8 and 1.9 to the case of non-locally free sheaves. We will use the following trivial lemma.

LEMMA 1.10. – *Let X be an integral projective curve, E a* rank *r vector bundle on X and $F^{\wedge}:=\{F_P^{\wedge}\}_{P\in \mathrm{Sing}(X)}$ a formal singularity type for rank r torsion free sheaves. Then there is a rank r torsion free sheaf F with $F^{\wedge}$ as formal singularity type along* $\mathrm{Sing}(X)$ *and an inclusion $i\colon E\to F$ with* $\mathrm{Supp}(F/i(E))\subseteq \mathrm{Sing}(X)$ *and* $h^0(X,\,F/i(E))=l(F^{\wedge})$.

From Lemma 1.10 and the definition of Lange invariant s_k (see for instance the beginning of section 2) we obtain at once the following corollary.

COROLLARY 1.11. – Let X be an integral projective curve, E a rank r vector bundle on X and $F^{\wedge}:=\{F_P^{\wedge}\}_{P\in \mathrm{Sing}(X)}$ a formal singularity type for rank r torsion free sheaves. Fix an integer k with $r>k>0$. Then there is a rank r torsion free sheaf F with $F^{\wedge}$ as formal singularity type along $\mathrm{Sing}(X)$ and with $s_k(F)\geqslant s_k(E)-(r-k)\,l(F^{\wedge})$.

Notice that for every exact sequence (1) of torsion free sheaves on X we have $l(E)\leqslant l(H)+l(G)$. If X is Gorenstein at P, then all torsion free $\boldsymbol{O}_{X,P}$-modules and all torsion free $\boldsymbol{O}_{X,P^{\wedge}}$-modules are reflexive. Furthermore, we have $\deg(\mathrm{Hom}(F,\,wX))=-\deg(F)+(2g-2)((\mathrm{rank}(F))$ for every torsion free sheaf F (see [Co], Prop. 3.1.6, part 2, for the non Gorenstein case). Hence taking duals, Lemma 1.10 and Corollary 1.11 give the following results.

LEMMA 1.12. – *Let X be an integral locally Gorenstein projective curve, E a rank r vector bundle on X and $F^{\wedge}:=\{F_{P^{\wedge}}\}_{P\in \mathrm{Sing}(X)}$ a formal singularity type for rank r torsion free sheaves. Then there is a rank r torsion free sheaf F with $F^{\wedge}$ as formal singularity type along* $\mathrm{Sing}(X)$ *and an inclusion $i\colon F\to E$ with* $\mathrm{Supp}((E/i(F)))\subseteq \mathrm{Sing}(X)$ *and $h^0(X,\,E/i(F))=l(F^{*\wedge})$.*

COROLLARY 1.13. – *Let X be an integral locally Gorenstein projective curve, E a rank r vector bundle on X and $F^{\wedge}:=\{F_{P^{\wedge}}\}_{P\in \mathrm{Sing}(X)}$ a formal singularity type for* *rank r torsion free sheaves. Fix an integer k with $r > k > 0$. Then there is a rank r torsion free sheaf F with $F^\wedge$ as formal singularity type along* Sing(X) *and with* $s_k(F) \geqslant s_k(E) - k(l(F^{*\wedge}))$.

REMARK 1.14. – Let X be an integral projective curve. Set $:= p_a(X)$. Recall that the functor $\mathrm{Hom}(-, \omega_X)$ induces a biduality of the category of all torsion free sheaves on X, i.e. that $\mathrm{Hom}(\mathrm{Hom}(F, \omega_X), \omega_X) \cong F$ for every torsion free F, and that $\deg(\mathrm{Hom}(F, \omega_X)) = -\deg(F) + (2g-2)(\mathrm{rank}(F))$. Hence if X is locally Gorenstein the dual of a stable torsion free sheaf is stable and we have $\mu(F^*) = -\mu(F)$, $\mu(E/F) = \mu(E) - \mu(F)$ if F is saturated in E and the usual numerical formulas are true even in for non locally free sheaves.

THEOREM 1.15. – *Let X be an integral projective curve with only planar singularities. Let g'' be the genus of its normalization. Fix integers r, k with $r > k > 0$. Fix a compatible singularity data $(H^\wedge, E^\wedge, G^\wedge)$ for the integers r, k and set $z := \max\{kl(H^\wedge), (r-k)\, l(G^\wedge)\}$. Assume $g'' \geqslant 3 + z$. Fix integers a, b with $a/k < b/(r-k)$. Then $\$(X; r, k, a, b, (H^\wedge, E^\wedge, G^\wedge))$ is true.*

PROOF. – First we will show the existence of an exact sequence (1) with H, E, G stable, H with formal singularity type $H^\wedge$ and G with formal singularity type $G^\wedge$. At the end of the proof we will show that we may find such exact sequence (1) with the additional property that E has formal singularity type $E^\wedge$. By 1.2 and 1.3 it is sufficient to prove the case $a/k \geqslant b/(r-k) - 1/k - 1/(r-k)$. By 1.11 and 1.13 we may find $H \in M(X; k, a, H^\wedge)$, $G \in M(X; r-k, b, G^\wedge)$ such that for all integers i, j with $0 < i < k$ and $1 < j < r-k$ we have $s_i(H) \geqslant g'' - 1 - kl(H^\wedge)$ and $s_j(G) \geqslant g'' - 1 - (r-k)\, l(G^\wedge)$. We fix any exact sequence (1) with as extremal sheaves the sheaves H and G just chosen. We assume that E is not stable and take a saturated subsheaf F of E with $\mu(F) \geqslant \mu(E)$. Taking rank(F) minimal we may assume F stable. Since H is stable and $\mu(H) < \mu(E)$, F is not contained in H. Hence we have a non-zero map $f\colon F \to G$. Set $r'' := \mathrm{rank}(\mathrm{Im}(f))$. Since $m(\mathrm{Im}(f)) \leqslant \mu(G) - (1 - (r''/(r-k))(g'' - 1 - (r-k)\, l(G^\wedge))$ by the assumption on the Lange invariants of G and $(1 - (r''/(r-k))(g'' - 1 - (r-k)\, l(G^\wedge)) < 0$ by the assumption on g'', we may repeat the proof of [RT], proof of Claim 1 on pages 7 and 8, and obtain $r'' = r - k$. Using Remark 1.14 we may apply the dual proof as on [RT], proof of Claim 2 on page 8, and obtain $\mathrm{Ker}(f) = 0$. Hence F is a subsheaf of G. Since $\mu(F) \geqslant \mu(E) \geqslant \mu(G) - 1/(r-k)$ and $\mathrm{rank}(F) = r - k$, either $\deg(F) = b$ or $\deg(F) = b - 1$. If $\deg(F) = b$ we have $F \cong G$ and $(b-1)$ splits. Since $\mathrm{Ext}^1(X, G, H) \cong H^1(X, \mathrm{Hom}(G, H)) \oplus H^0(X, \mathrm{Ext}^1(G, H))$ and $h^1(X, \mathrm{Hom}(G, H)) \neq 0$ by Riemann-Roch, this is not the case for a sufficiently general extension (1). Hence we may assume $\deg(F) = b - 1$. Thus G/F is supported by a unique point, P, of X. By construction E has a subsheaf, D, with $D \cong H \oplus F$ and $E/D \cong \boldsymbol{K}_P$. First assume $P \in X_{\mathrm{reg}}$. Hence F is obtained from G making a negative elementary transformation supported by P and E is obtained from D making a positive elementary transformation supported by P. Since $\dim(X_{\mathrm{reg}}) = 1$, and $G|X_{\mathrm{reg}}$ is locally free of rank $r-k$, for fixed G the set of all isomorphic classes of all possible F's depends on at most $r-k$ parameters. Given G, H and F, we have a unique P because $\det(G) \cong \det(F)(P)$ and $g > 0$. Since E is obtained from D making a positive elementary transformation supported by P. Hence for fixed G and H the set of all isomorphic classes of all possible

r has dimension at most $r-k+r-1=2r-k-1$. We claim that for a general such E the sheaf E has at most finitely many subsheaves isomorphic to H. By the differential properties of the Quot-scheme $\mathrm{Quot}(E^*)$ of E^* to check the claim it is sufficient to prove that $h^0(X, G\otimes H^*)=0$. Since every vector bundle on X is a flat limit of stable vector bundles, and both G and H are general, it is sufficient to find two vector bundles G_1 and H_1 with $\mathrm{rank}(G_1)=r-k$, $\deg(G_1)=b$, $\mathrm{rank}(H_1)=k$, $\deg(H_1)=a$ and $h^0(X, G\otimes H^*)=0$. Since $g\geq 2$ and $\mu(G)-\mu(H)<2$, it is sufficient to take as G_1 and H_1 a direct sum of general line bundles, each of them of degree $[a/k]$ or $[(a+k-1)/k]$ or $[b/(r-k)]$ or $[(b+r-k-1)/(r-k)]$. Since $h^1(X, \mathrm{Hom}(G, H))>2r-k-1$ by Riemann-Roch, we obtain a contradiction. Now assume $P\in\mathrm{Sing}(X)$. Since $\mathrm{Sing}(X)$ is finite, in our count of parameters we gain 1. Instead of k, now we have to take $\dim_K(H|\{P\})$ and instead of $r-k$ we have to take $\dim_K(G|\{P\})$; the latter dimension appears twice. Thus we have to compare $\dim_K(H|\{P\})-1+2(\dim_K(G|\{P\})-1)$ with the Riemann-Roch term $\deg(\mathrm{Hom}(G, H))+k(r-k)(g-1)$. Notice that $\dim_K(H|\{P\})-k\leq l(H^\wedge)$ and $\dim_K(G|\{P\})-(r-k)\leq l(G^\wedge)$. Since $g>g''$ and our lower bound for g'' gives a contradiction. Since $\mathrm{Est}^1(X, G, H)\cong$ $\cong H^1(X, \mathrm{Hom}(G, H))\oplus H^0(X, \mathrm{Est}^1(G, H))$ and $(H^\wedge, E^\wedge, G^\wedge)$ is a compatible triple, every class in $H^0(X, \mathrm{Est}^1(G, H))$ may be represented in an exact sequence (1) with E with formal singularity type $E^\wedge$. The first part of the proof shows that we may obtain simultaneously the stability of E. ■

2. – Measures of the order of stability.

In this section we make no restriction on $\mathrm{char}(K)$. Let E be a rank r torsion free sheaf on the integral projective curve X and G a rank k subsheaf of E such that $\deg(G)$ is the maximal degree of a rank k subsheaf of E. In particular G is saturated in E, i.e. E/G has no torsion. Set $s_k(E):=k(\deg(E))-r(\deg(G))$. Each integer $s_k(E)$, $1\leq k\leq$ $\leq r-1$, will be called a Lange invariant of E because these invariants were used in a nice way in [L1]. If $L\in\mathrm{Pic}(X)$ we have $s_k(F)=s_k(F\otimes L)$ because the map $H\to H\otimes L$ between rank k subsheaves of E and rank k subsheaves of $E\otimes L$ has as inverse the map $T\to T\otimes L^*$ and $\deg(E\otimes L)=\deg(E)+r(\deg(L))$, $\deg(H\otimes L)=\deg(H)+$ $+k(\deg(L))$. In particular we have $s_k(E)=s_k(E(-Q))$ for every $Q\in X_{\mathrm{reg}}$.

In this section we will prove refinements of Theorems 1.9 and 1.15 for the first and last Lange invariant of a «generic» torsion free sheaf. We will obtain for free similar results for the following related numerical invariants.

DEFINITION 2.1. – Let X be an integral projective curve and E a rank r torsion free sheaf on X. Assume $r\geq 2$. Fix a rank 1 subsheaf L of E computing $s_1(E)$, i.e. such that $\deg(L)$ is the maximal degree of a rank 1 saturated subsheaf of E. If $r=2$ we stop. Assume $r\geq 3$. Since $\deg(L)$ is maximal, E/L is torsion free and $\mathrm{rank}(E/L)=r-1\geq 2$. Let L_2 be a rank 1 subsheaf of E/L computing $s_1(E/L)$. Equivalently, let M_2 be a rank 2 subsheaf of E containing L and with $\deg(M_2)=\deg(L)+\deg(L_2)$ maximal. Unfortunately, a priori the integer $\deg(L_2)$ may depend on the choice of L. However there is a unique integer, t, such that t is the maximum of the integers $\deg(L_2)$ when L vary among the saturated rank 1 subsheaves of E with maximal degree. If $r=3$ we stop. Assume $r\geq 4$. We fix one such L and one such $L_2=M_2/L$ with $(\deg(L), \deg(M_2/L))$ maximal in the lexicographic order. Then consider a maximal degree rank 1 subsheaf, T, of E/M_2 with maximal degree; we choose (L_1, M_2) such that $\deg(T)$ is maximal. And so on. At the end we obtain $r-1$ integers $(\deg(M_1), \ldots, \deg(M_{r-1}))$ and an increasing filtration $\{M_i\}_{0 \leq i \leq r}$ of E with $M_0 = \{0\}$, $M_r = E$, each M_i, $1 \leq i \leq r-1$, saturated in M_{i+1}, $\mathrm{rank}(M_i) = i$, and such that the ordered set of $r-1$ integers $(\deg(M_1), \ldots, \deg(M_{r-1}))$ is maximal in the lexicographic order among all ordered set of $r-1$ integers obtained from such filtrations. We set $d_i(E) := \deg(M_{i+1}) - \deg(M_i)$, $1 \leq i \leq r-1$.

DEFINITION 2.2. – Let X be an integral projective curve and E a rank r torsion free sheaf on X. For every $P \in \mathrm{Sing}(X)$, fix a filtration, say $\{M_{iP^\wedge}\}_{0 \leq i \leq r}$ of the torsion free $O_{X,P^\wedge}$-module $E_{P^\wedge}$ with $M_{0P^\wedge} = 0$, $M_{rP^\wedge} = E_{P^\wedge}$, $\mathrm{rank}(M_{iP^\wedge}) = i$ for every i and $M_{iP^\wedge}/M_{i-1P^\wedge}$ torsion free for every i. Call $\Psi := \{\{M_{iP^\wedge}\}_{0 \leq i \leq r}\}_{P \in \mathrm{Sing}(X)}$ the data of these filtrations. Now we repeat the construction of Definition 2.1 but restricting to saturated subsheaves, M_i, of E such that for every integer i and every $P \in \mathrm{Sing}(X)$ the completion of M_i at P is $M_{iP^\wedge}$. Set $d_i((E, Y)) := \deg(M_i/M_{i-1})$, $1 \leq i \leq r-1$.

In a similar way we refine the Lange invariants.

DEFINITION 2.3. – Fix integers r, k with $r > k$. Let X be an integral projective curve and E a rank r torsion free sheaf on X. For every $P \in \mathrm{Sing}(X)$, fix a compatible formal singularity data $\Pi := (H^\wedge, E^\wedge, G^\wedge)$ for the integers r, k with $E^\wedge$ formal singularity type of E along $\mathrm{Sing}(X)$. In the definition of Lange invariant given at the beginning of this section take only rank k saturated subsheaves, H, of E with $H^\wedge$ as formal singularity type and such that E/H has formal singularity type $G^\wedge$. Set $s_k((E,)) := k(\deg(E)) - r(\deg(H))$. If E is locally free and Π is the compatible formal singularity data of the trivial modules, we will write $s_k((E, \emptyset))$ instead of $s_k((E, \Pi))$.

We stress that in the definitions 2.2 and 2.3 we impose that the subsheaves are saturated. Following [B1] instead of fixing the formal singularity type of a sheaf or of some of its subsheaves, we may give an «algebraic family» of «admissible» formal singularity types for the corresponding subsheaves. This is often very reasonable because, except in very particular cases, a formal singularity type is not rigid and the set of interesting formal singularity types may depend by continuos parameters.

THEOREM 2.4. – *Fix integers r, k, d with $r > k > 0$. Let X be an integral projective curve with $g := p_a(X) \geq 2$. For a general $E \in U(X; r, d)$ we have $s_k((E, \emptyset)) \geq k(r-k)(g-1)$.*

PROOF. – Fix an exact sequence (1) with $E \in U(X; r, d)$ and H and G locally free. Since H (resp. G) is a flat limit of a family of stable vector bundles ([Re]), we may justify the sentence «H (resp. G) depends on at most $k^2(g-1)+1$ (resp. $(r-k)^2(g-1)+1$) parameters». Since E is simple, we have $h^0(X, \mathrm{Hom}(G, H)) = 0$. Hence by Riemann-Roch we have $\dim(\mathrm{Est}^1(X, G, H)) = k(r-k)(\mu(G) - \mu(H) + g - 1)$. Since proportional extensions give isomorphic bundles and $\dim(U(X; r, d)) = r^2(g-1)+1$ we see that for a general $E \in M(X; r, d)$ we have $\mu(G) - \mu(H) \geq g-1$, as wanted. ∎

THEOREM 2.5. – *Fix an integer $r \geq 2$, integers $\{d_i\}_{1 \leq i \leq r-1}$ and an integral projective curve X. Set $g := p_a(X)$ and assume $2d_1 \leq g-3$, $d_i \leq g-1$ for every i, $d_{i+1} +$*

$+ (i+1)(g-1) < \sum_{1 \leqslant j \leqslant i} d_j$ *for every i with* $1 \leqslant i \leqslant r-2$, $d_i \leqslant d_j + g - 1$ *for all integers* i, j *with* $1 \leqslant i \leqslant i \leqslant r-1$. *Then there exists a rank r vector bundle E on X with* $d_i((E, \emptyset)) = d_i$ *for every i and a filtration* $\{M_j\}_{0 \leqslant j \leqslant r}$ *of E such that* $M_0 = \{0\}$, $M_r = E$, M_i, $1 \leqslant i \leqslant r-1$, *is a rank i saturated subbundle of E,* $\deg(M_1) = 0$, $\deg(M_j) = \sum_{1 \leqslant a < j} d_a$ *for* $j > 1$ *(i.e. every* M_i *computes* $d_i((E, \emptyset))$*),* M_i *is the unique rank i saturated subbundle, A, of E such that* $\deg(A) \geqslant \sum_{1 \leqslant a < i} d_a$ *and the ordered set of r line bundles* $(M_1, M_2/M_1, \ldots, M_r/M_{r-1}) \in \mathrm{Pic}^0(X) \times \mathrm{Pic}^{d_1}(X) \times \ldots \times \mathrm{Pic}^{d_{r-1}}(X)$ *is general in* $\mathrm{Pic}^0(X) \times \mathrm{Pic}^{d_1}(X) \times \ldots \times \mathrm{Pic}^{d_{r-1}}(X)$.

PROOF. – We use induction on r. Assume $r = 2$. Fix a general pair $(A, F) \in \mathrm{Pic}^0(X) \times \mathrm{Pic}^{d_1}(X)$. Consider a general extension

$$0 \to A \to E \to F \to 0 \tag{4}$$

of F by A. By the definition of the invariants $d_i((E, \emptyset))$ to complete the proof for the case $r = 2$ it is sufficient to check that the line subbundle A of E induced by (4) is the unique line subbundle, R, of E with $\deg(R) \geqslant 0$. Fix such a line subbundle R and set $b := \deg(R) \geqslant 0$. Since R is not the line subbundle induced by the exact sequence (4), there is a non-zero map $R \to F$. Thus there is an effective Cartier divisor D with $\deg(D) = d_1 - b$ and such that $R = F(-D)$. In particular $d_1 \geqslant b \geqslant 0$. Furthermore, any such E has a subsheaf, T with $T \cong A \oplus R$ and $E/T \cong \boldsymbol{O}_D$. We count the parameters. The integer b is fixed. Since $F \in \mathrm{Pic}^{d_1}(X)$ is general and $d_1 \geqslant b$, F is not trivial even if $d_1 = 0$, i.e. $h^0(X, F^* \otimes A) = 0$.

Hence by Riemann-Roch we have $\dim(\mathrm{Ext}^1(X, F, A)) = d_1 + g - 1$. Since $d_1 \leqslant g - 1$ and F is general, we have $h^0(X, F \otimes A^*) = 0$. Hence $h^0(X, E) = 0$. Thus any such E fits in a unique extension (4) (up to a multiplicative constant). We claim that the set of all effective Cartier divisors of degree $d_1 - b$ depends on at most $d_1 - b$ parameters even if we do not assume that X has only planar singularities. To check the claim, just use that since a Cartier divisor D is locally principal, D satisfies the locally lifting property for the Quot-scheme $\mathrm{Quot}(\boldsymbol{O}_X)$ and hence by the differential study of the Quot-scheme the scheme $\mathrm{Quot}(\boldsymbol{O}_X)$ is smooth of dimension $\deg(D)$ at the point representing D ([G], Exp. 221, § 5, or [HH], § 4). Since T is locally free of rank 1 and D is a Cartier divisor, a local computation shows that $\dim(\mathrm{Ext}^1(X, \boldsymbol{O}_D, T)) = h^0(X, \mathrm{Ext}^1(\boldsymbol{O}_D, T)) = 2(\deg(D))$. Thus the set of all isomorphism classes of all possible bundles E containing some line subbundle R of non-negative degree has dimension at most $\dim(\mathrm{Pic}^0(X)) + \dim(\mathrm{Pic}^0(X)) + 3d_1 - 1 = 2g - 1 + 3d_1$. We just saw that the set of isomorphism classes of bundles arising as extension (4) has dimension $3g - 2 + d_1$. Hence we obtain a contradiction. Now assume $r > 2$. Take as A a rank $r-1$ vector bundle on X satisfying the thesis of the theorem with respect to a filtration $\{M_j\}_{0 \leqslant j \leqslant r-1}$ of A with $(M_1, M_2/M_1, \ldots, M_{r-1}/M_{r-2})$ general in $\mathrm{Pic}^0(X) \times \mathrm{Pic}^{d_1}(X) \times \ldots \times \mathrm{Pic}^{d_{r-2}}(X)$. Take a general $F \in \mathrm{Pic}^{d_{r-1}}(X)$ and let E be a general bundle fitting as middle term in an extension (4) of F by A. By the definition of the invariants $d_i((E, \emptyset))$ check that E solves our problem with $F \cong M_r/M_{r-1}$ it is sufficient to check that M_1 is the unique saturated line subbundle, R, of E with $\deg(R) \geqslant 0$. Assume that this is not the case and let R be one such saturated subbundle with $b := \deg(R) \geqslant 0$. Since A satisfies the thesis of 2.5 for the integer $r-1$, R is not contained in A and hence the inclusion of R in E induces an inclusion of R in F, i.e. an effective Cartier divisor D with $\deg(D) = d_{r-1} - b$. Hence $d_{r-1} \geq b \geq 0$ and we may repeat the proof of the case $r = 2$ just using d_{r-1} instead of d_1; now $h^0(X, \mathrm{Ext}^1(O_D, T)) = r(\deg(B))$ because T is a rank r vector bundle. First we check that for any fixed E fitting in an exact sequence (4) E has only finitely many subsheaves isomorphic to A. By the differential study of the Quot-scheme $\mathrm{Quot}(E^*)$ of E^* ([G], Exp. 221, §5, or [HH], §4) it is sufficient to have $h^0(X, A^* \otimes F) = 0$. By the generality of the line bundles M_1, M_i/M_{i-1}, $2 \leq i \leq r-1$, and F, to obtain $h^0(X, A^* \otimes F) = 0$ it is sufficient to have $d_{r-1} \leq g-1$ and $d_{r-1} - d_i \leq g-1$ for $1 \leq i \leq r-2$. We have $\dim(\mathrm{Ext}^1(X, F, A)) \geq -\chi(\mathrm{Hom}(F, A)) = (r-1)(\mu(F) - \mu(A) + g - 1) = (r-1)\, d_{r-1} - \sum_{1 \leq i \leq r-2} d_i + (r-1)(g-1)$. Hence for fixed A the set of isomorphic classes of bundles, E, fitting in (4) depends exactly on $g + (r-1)\, d_{r-1} - \sum_{1 \leq i \leq r-2} d_i + (r-1)(g-1) - 1$ parameters, while, for fixed A the set of all possible bundles which may arise as extension of a possible T by a possible Cartier divisor has dimension at most $g + rd_{r-1} - 1$, contradiction. ■

Now we will give partial extensions of Theorem 2.5 to the case of non locally free rank 2 sheaves. We give the following definitions which are related to [B3], Def. 1.4 and 1.5, which considered the case in which N is the trivial module.

DEFINITION 2.6. – Let R be a 1-dimensional local Cohen-Macaulay ring containing $\boldsymbol{K}$ and $\boldsymbol{m}$ its maximal ideal. We assume $R/\boldsymbol{m} \cong \boldsymbol{K}$. We are interested in the case of a locally Cohen-Macaulay curve and $P \in X$; we take as R either its local ring $\boldsymbol{O}_{X,P}$ or the completion $\boldsymbol{O}_{X,P^\wedge}$ of its local ring and call $\boldsymbol{m} := \boldsymbol{m}_P$ or $\boldsymbol{m} := \boldsymbol{m}_{P^\wedge}$ the corresponding maximal ideal. Let M and N be rank 1 torsion free R-moduls. Let k be a non-negative integer. We will say that the pair (N, M) have singularity distance k if k is the minimal integer $t \geq 0$, such that there are R-modules $N' \cong N$, $M' \cong M$ with $N' \subseteq M'$ and $\dim_{\boldsymbol{K}}(M'/N') = t$.

Notice that $N \cong M$ if and only if the pair (N, M) have singularity distance 0.

DEFINITION 2.7. – Let X be an integral projective curve. Set $z := \mathrm{card}(\mathrm{Sing}(X))$ and fix an order $P(1), \ldots, P(z)$ of the points of $\mathrm{Sing}(X)$. Let M and N be rank 1 torsion free sheaves on X. We will say that the pair (N, M) has singularity distance $(k(1), \ldots, k(z))$ if for every i, $1 \leq i \leq z$, the pair of the completions of the stalks $(N_{X, P(i)}, M_{X, P(i)})$ of (N, M) at $P(i)$ has singularity distance $k(i)$ as $\boldsymbol{O}_{X, P(i)}$-module. Let k be a non-negative integer. The integer $k := \sum_{1 \leq i \leq z} k(i)$ will be called the total singularity distance of the pair (N, M). Instead of a pair (N, M) of rank 1 torsion free sheaves we may take a pair $(N^\wedge, M^\wedge)$ of formal singularity types along $\mathrm{Sing}(X)$ for rank 1.

REMARK 2.8. – Let k be the total singularity type of a pair $(N^\wedge, M^\wedge)$ of formal singularity tipes along $\mathrm{Sing}(X)$ for rank 1. Then k is the minimal integer, t, such that there are rank 1 torsion free sheaves A, B on X and an inclusion $j: A \to B$ such that A has formal singularity type $N^\wedge$, B has formal singularity type $M^\wedge$ and $h^0(X, \mathrm{Coker}(j)) = t$. Furthermore, using general positive elementary transformations we see that for every integer $t \geq k$ there is such a triple (A, B, j).

REMARK 2.9. – Let R be a 1-dimensional local Gorenstein Cohen-Macaulay ring containing $\boldsymbol{K}$ and with maximal ideal $\boldsymbol{m}$ such that $R/\boldsymbol{m} \cong \boldsymbol{K}$. We have $\mathrm{Ext}^2_R(\boldsymbol{K}, R) = 0$ ([Ba]), $\mathrm{Ext}^1_R(\boldsymbol{K}, R) = 0$ and $\mathrm{Ext}^1_R(\boldsymbol{K}, R) \cong \boldsymbol{K}$ ([M]).

We will prove simultaneously the next four propositions concerning rank 2 torsion free sheaves.

PROPOSITION 2.10. – *Let X be an integral projective curve. Set $g := p_a(X)$. Fix a formal singularity data $(A^\wedge, E^\wedge, B^\wedge)$ along* $\mathrm{Sing}(X)$ *with respect to the integers* 2 *and* 1. *Let k be the total singularity distance of the pair $(A^\wedge, B^\wedge)$. Fix an integer t with $t < k$. Then there exists an exact sequence* (1) *with H with formal singularity type $A^\wedge$, G with formal singularity type $B^\wedge$, E with formal singularity type $E^\wedge$, $\deg(G) = \deg(H) + t$, with $d_1((E, A^\wedge)) = s_1((E, A^\wedge)) = t$ and such that H is the unique rank* 1 *subsheaf, D, of E (even not a saturated one) with formal singularity type $A^\wedge$ and $\deg(D) \geqslant \deg(H)$. Furthermore, in such exact sequence we may take as H and G any two rank* 1 *torsion free sheaves with the prescribed formal singularity types and with $\deg(G) = \deg(H) + t$.*

PROPOSITION 2.11. – *Let X be an integral projective curve with only planar singularities. Set $g := p_a(X)$. Fix a formal singularity data $(A^\wedge, E^\wedge, B^\wedge)$ along* $\mathrm{Sing}(X)$ *with respect to the integers* 2 *and* 1 *and such that $A^\wedge$ is the formal singularity type of $\boldsymbol{O}_X$. Fix an integer t with $t < (g-1)/2$. Then there exists an exact sequence* (1) *with H with formal singularity type $A^\wedge$, G with formal singularity type $B^\wedge$, E with formal singularity type $E^\wedge$, $\deg(G) = \deg(H) + t$, with $d_1((E, A^\wedge)) = s_1((E, A^\wedge)) = t$ and such that H is the unique rank* 1 *locally free subsheaf, D, of E with formal singularity type $A^\wedge$ and $\deg(D) \geqslant \deg(H)$. Furthermore, in such exact sequence we may take as H and G any two rank* 1 *sheaves with the prescribed formal singularity types and with $\deg(G) = \deg(H) + t$.*

PROPOSITION 2.12. – *Let X be an integral projective curve with only planar singularities. Set $g := p_a(X)$. Fix a formal singularity data $(A^\wedge, E^\wedge, B^\wedge)$ along* $\mathrm{Sing}(X)$ *with respect to the integers* 2 *and* 1 *and such that $B^\wedge$ is the formal singularity type of $\boldsymbol{O}_X$. Fix an integer t with $t < (g-1)/2$. Then there exists an exact sequence* (1) *with H with formal singularity type $A^\wedge$, G with formal singularity type $B^\wedge$, E with formal singularity type $E^\wedge$, $\deg(G) = \deg(H) + t$, with $d_1((E, A^\wedge)) = s_1((E, A^\wedge)) = t$ and such that H is the unique rank* 1 *locally free subsheaf, D, of E with formal singularity type $A^\wedge$ and $\deg(D) \geqslant \deg(H)$. Furthermore, in such exact sequence we may take as H and G any two rank* 1 *sheaves with the prescribed formal singularity types and with $\deg(G) = \deg(H) + t$.*

PROPOSITION 2.13. – *Let X be an integral projective curve with only planar singularities. Fix a formal singularity data $(A^\wedge, E^\wedge, B^\wedge)$ along* $\mathrm{Sing}(X)$ *with respect to the integers* 2 *and* 1. *Set $g := p_a(X)$, $\varepsilon_1 := \varepsilon(A^\wedge) := \sup_{P \in \mathrm{Sing}(X)} \dim_{\boldsymbol{K}}(\mathrm{Ext}^1(\boldsymbol{K}, A_P^\wedge))$, $\varepsilon_2 := \varepsilon(B^\wedge) := \sup_{P \in \mathrm{Sing}(X)} \dim_{\boldsymbol{K}}(\mathrm{Ext}^1(\boldsymbol{K}, B_P^\wedge))$ and $g := \sup_{P \in \mathrm{Sing}(X)} \dim_{\boldsymbol{K}}(G^\wedge \mid \{P\})$. Fix an integer t with $t(\varepsilon_1 + \varepsilon_2 + g) \leqslant g - 2$. Then there exists an exact sequence* (1) *with H with formal singularity type $A^\wedge$, G with formal singularity type $B^\wedge$, E with formal singularity type $E^\wedge$, $\deg(G) = \deg(H) + t$, with $d_1((E, A^\wedge)) = s_1((E, A^\wedge)) = t$ and* *such that H is the unique rank 1 subsheaf, D, of E (even not a saturated one) with formal singularity type $A^{\wedge}$ and* $\deg(D) \geqslant \deg(H)$. *Furthermore, in such exact sequence we may take as H and G any two rank 1 torsion free sheaves with the prescribed formal singularity types and with* $\deg(G) = \deg(H) + t$.

Proofs of 2.10, 2.11, 2.12 and 2.13. Fix any two rank 1 torsion free sheaves H and G with the prescribed formal singularoties types and with $\deg(G) = \deg(H) + t$. Since $\mathrm{Ext}^1(X, G, H) \cong H^1(X, \mathrm{Hom}(G, H)) \oplus H^0(X, \mathrm{Ext}^1(G, H))$, there is an extension of G by H which induces an exact sequence (1) with E having formal singularity type $E^{\wedge}$ and the set of all such extensions is an affine space with $H^1(X, \mathrm{Hom}(G, H))$ as associated vector space. We fix a general extension (1) of G by H with $E^{\wedge}$ as formal singularity type of E. Let D be a rank 1 subsheaf of E (even not saturated) with formal singularity type $A^{\wedge}$ and $\deg(D) \geqslant \deg(H)$. In order to obtain a contradiction we assume $D \neq H$. Hence there is a non-zero map $j: A \to G$ and E contains a subsheaf, T, with $T \cong H \oplus D$. If $t < k$ the existence of j contradicts Remark 2.8. Hence we have proven 2.10. Notice that E/T is a skyscraper sheaf with $h^0(X, E/T) = t$. Now assume H locally free and $3t \leqslant g - 2$. Since H is locally free, E/T is isomorphic to the degree t Weil divisor on X associated to a section of $G \otimes H^*$. Since X has only planar singularities, E/T depends on at most t parameters (see e.g. [Re] or, in characteristic 0, [BGS], Prop. 1.4). Since H is locally free and X is Gorenstein, we have $\deg(\mathrm{Hom}(G, H)) = -t$ (see e.g. [Co], Prop. 3.1.6, part 2). Hence by Riemann-Roch we have $h^1(X, \mathrm{Hom}(G, H)) \geqslant t + g - 1$. Since T is locally free with rank 2, we have $h^0(X, \mathrm{Ext}^1(T, E/T)) = 2t$. Since $3t < t + g - 1$, we obtain 2.11. Since in the statement of 2.12 we assume that X is Gorenstein, 2.12 is obtained from 2.11 taking duals (and vice versa). To obtain 2.13 we repeat the same proof just using that $h^0(X, \mathrm{Ext}^1(T, E/T)) \leqslant t(\varepsilon_1 + \varepsilon_2)$ by the definition of the integers ε_1 and ε_2. Since T is a quotient of G, by definition of γT is a quotient of $\boldsymbol{O}_{X \oplus \gamma}$ supported at least in part on $\mathrm{Sing}(X)$, while the part of T supported on X_{reg} is a Cartier divisor. Hence if $\gamma \geqslant 2$, i.e. G is not locally free, T depends on at most $t\gamma - 1$ parameters. ∎

LEMMA 2.14. – *Make all the assumptions of* 2.11 (*resp.* 2.12, *resp.* 2.13). *Make the further assumption* $t = 2$. *Then the thesis of* 2.11 (*resp.* 2.12, *resp,* 2.13) *is satified by an exact sequence* (1) *with E stable.*

PROOF. – By Proposition 1.6 we know that we may find an exact sequence (1) with E semistable. Assume that for every such exact sequence E is not stable and let L be a rank 1 subsheaf of E with $\deg(L) \geqslant \deg(E)/2$, i.e. $\deg(L) = \deg(H) + 1 = \deg(G) - 1$. Since L is not contained in H we have an inclusion $j: L \to G$ with $\mathrm{Coker}(j) \cong \boldsymbol{K}_P$ for some $P \in X$. First assume $P \in X_{\mathrm{reg}}$. Hence P is a Cartier divisor and $L \cong G(-P)$. For fixed H and G, the set of the isomorphism classes of the possible sheaves L depends only on one parameter, because $\dim(X) = 1$. Since E is obtained from $H \oplus L$ making a positive elementary transformation supported by P, for fixed H and G the set of the isomorphism classes of the possible sheaves E depends on at most two parameters, contradicting the computation of $\dim(\mathrm{Ext}^1(X, G, H))$ and the fact that each such E contains a unique saturated subsheaf isomorphic to H. Now assume $P \in \mathrm{Sing}(X)$. Since P is not a Cartier divisor, the pair (L, G) has total singularity distance 1 and L and G are formally isomorphic at every point of $X \backslash \{P\}$. L is the kernel of a surjection $G \to \boldsymbol{K}_P$. Since $\mathrm{Sing}(X)$ is finite, the set of the isomorphism classes of the possible sheaves L depends on at most $\dim_K(G|\{P\})$ parameters. Under the assumptions of 2.12 (i.e. G locally free) we have $\dim_K(G|\{P\})=1$ and $\dim_K(\mathrm{Ext}^1(X, \boldsymbol{K}_P, L))=2$. Thus the set of the isomorphism classes of all such E depends on at most $1+\dim_K(\mathrm{Ext}^1(X, \boldsymbol{K}_P, H))$ parameters. However, we have to count only the sheaves, E, whose formal singularity type is the product of $H^\wedge$ and a trivial factor. Since P is not a Cartier divisor this means that we take only extensions, $\boldsymbol{e}=(e_1, e_2)$, whose component e_1 in the decomposition $\mathrm{Ext}^1(X, \boldsymbol{K}_P, H\oplus L)\cong \mathrm{Ext}^1(X, \boldsymbol{K}_P, H)\oplus \mathrm{Ext}^1(X, \boldsymbol{K}_P, L)$ vanishes. Hence we obtain a contradiction in the set-up of 2.12. By duality we obtain the stability of E under the assumptions of 2.11. Now we consider 2.13. We just use the definition of $\varepsilon(A^\wedge)$, $\varepsilon(B^\wedge)$, γ, the previous proof and the proof of 2.13. ∎

THEOREM 2.15. – *Make all the assumptions of* 2.11 (*resp.* 2.12, *resp.* 2.13). *Make the further assumption* $t>0$. *Then the thesis of* 2.11 (*resp.* 2.12, *resp.* 2.13) *is satified by an exact sequence* (1) *with E stable.*

PROOF. – If $t=1$, this is a very particular case of Proposition 1.7. If $t=2$ the result was proved in Lemma 2.16. For $t\geq 3$ we use inductively the result for the integer $t' := t-2$, Lemma 1.2 and Remark 1.3. ∎

REMARK 2.16. – Make all the assumptions of 2.10 (resp. 2.11, resp. 2.12, resp. 2.13). Make the further assumption $t=0$. Then the thesis of 2.10 (resp. 2.11, resp, 2.12, resp. 2.13) is satified by an exact sequence (1) with E semistable (Proposition 2.6).

REMARK 2.17. – As in [B3] it is obvious from the proof of Theorem 2.15 that if we make all the assumptions of 2.11 or 2.12, except any assumption on t, for every integer $t>0$ we may find an exact sequence (1) with (H, E, G) with the given degrees and formal singularity types and with E stable; for 2.13 we need to assume $t>0$ and $2(\varepsilon_1+\varepsilon_2+\gamma)\leq g-2$ to handle the case $t=2$.

3. – Stable curves.

In this section we consider a related problem for stable curves. Let X be a reducible, connected curve with $g:=p_a(X)\geq 2$ and with only ordinary nodes as singularities. Let D_i, $1\leq i\leq x$, be the irreducible components of X. Let E be a torsion free sheaf on X. E is said to have multirank $(r_1, \ldots, r_x)$ if its restriction to D_i has generic rank r_i, $1\leq i\leq x$. A polarization, H, on X is given by an order set $(h_1, \ldots, h_x)$ of strictly positive real numbers. Fix a polarization, H, on X and two ordered sets of integers $(r_1, \ldots, r_x)$ and $(d_1, \ldots, d_x)$ with $r_i\geq 0$ for every i and $r_i\neq 0$ for at least one integer i, $1\leq i\leq x$. Let $M(X;(r_1, \ldots, r_x),(d_1, \ldots, d_x), H)$ be the scheme of H-slope stable torsion free sheaves on X with multirank $(r_1, \ldots, r_x)$ and multidegree $(d_1, \ldots, d_x)$ (see e.g. [P] or [OS]). Let E be a vector bundle on $\boldsymbol{P}^1$ with splitting type $a_1\geq \ldots \geq a_r$; E is rigid (i.e. every small deformation of E is isomorphic to E) if and only if $a_1-a_r\geq 1$.

THEOREM 3.1. – *Assume* $\mathrm{char}(\boldsymbol{K})=0$. *Let X be a stable curve of genus* $g\geq 3$. *Fix integers r, k with* $r>k>0$. *For every irreducible component D of X fix two integers $a(D)$ and $b(D)$ such that* $a(D)/k\leq b(D)/(r-k)$ *for every D. If* $p_a(D)\geq 2$, *assume also*

$a(D)/k < b(D)/(r-k)$. Then there exists an exact sequence

$$0 \to H \to E \to G \to 0 \tag{5}$$

of vector bundles on X with the following properties:

a) $\operatorname{rank}(E) = r$, $\operatorname{rank}(H) = k$, $\operatorname{rank}(G) = r - k$ *and for every irreducible component, D, of X we have* $\deg(H|D) = a(D)$ *and* $\deg(G|D) = b(D)$ (*hence* $\deg(E|D) = a(D) + b(D)$);

b) for every irreducible component D of X with $p_a(D) \geq 3$ the vector bundles $H|D$, $E|D$ and $G|D$ are stable;

c) for every irreducible component D of X which is smooth and of genus 2 the vector bundles $H|D$, $E|D$ and $G|D$ are semistable;

d) for every irreducible component D of X with $p_a(D) = 1$ the vector bundles $H|D$, $E|D$ and $G|D$ are semistable;

e) for every irreducible component D of X with $p_a(D) = 2$ the vector bundles $H|D$ and $G|D$ are stable and the vector bundle $E|D$ is semistable and simple;

f) for every irreducible component D of X which is smooth and rational the vector bundles $H|D$, $E|D$ and $G|D$ are rigid.

Furthermore, the vector bundles H and G are simple. The vector bundle E is simple except perhaps if every irreducible component of X has arithmetic genus ≤ 1. For any polarization, J, on X the vector bundles H and G are J-stable (*or J-semistable*) *if there are J-stable* (*or J-semistable*) *vector bundles with the numerical invariants of H and G.*

PROOF. – We divide the proof into three parts.

Part (i): We fix an irreducible component D of the stable curve X. Set $S := \operatorname{Sing}(X) \cap D$, $S' := \operatorname{Sing}(D)$ and $S'' := S \setminus S'$. Call Y the closure in X of $X \setminus D$. Hence $S'' = D \cap Y$. Call $\pi: D' \to D$ the normalization map. Let

$$0 \to A \to F \to B \to 0 \tag{6}$$

be an exact sequence of vector bundles on D with $\operatorname{rank}(A) = k$, $\deg(A) = a(D)$, $\operatorname{rank}(B) = r - k$, $\deg(B) = b(D)$ and with A, F and B stable. Assume given any exact sequence

$$0 \to A' \to F' \to B' \to 0 \tag{7}$$

of vector bundles on Y with $\operatorname{rank}(A') = k$ and $\operatorname{rank}(B') = r - k$. Here we will show how to construct vector bundles A'', F'', B'' on X with $A''|Y = A'$, $A''|D = A$, $F''|Y = F'$, $F''|D = F$, $B''|Y = B'$, $B''|D = B$ and fitting in an exact sequence

$$0 \to A'' \to F'' \to B'' \to 0 \tag{8}$$

of vector bundles on X. For every $P \in D \cap Y$ call $i(P)$ (resp. $q(P)$) the injective map (resp. surjective map) obtained restricting the exact sequence (6) to the fibers over $\{P\}$ and $i(P)'$, $q(P)'$ the corresponding linear maps obtained from the exact sequence (7). Since D and Y are smooth at every point of $Y \cap D$, a vector bundle on X is uniquely determined by its restriction to D, its restriction to Y and its gluing data of the corresponding fibers over every point of $Y \cap D$. Fix $P \in D \cap Y$. We fix any gluing data for $A|\{P\}$ and $A'|\{P\}$ and for $B|\{P\}$ and $B'|\{P\}\}$ and we give the following gluing data for the fibers $F|\{P\}$ and $F'|\{P\}$. We glue the subspaces $i(P)(A|\{P\})$ and $i(P)'(A'|\{P\})$. We fix complementary subspaces T of $i(P)(A|\{P\})$ in $F|\{P\}$ and T' of $i(P)(A'|\{P\})$ in $F'|\{P\}$. We glue together T and T' in the unique way compatible with the isomorphisms $q(P)|T: T \to B|\{P\}$, $q(P)'|T': T' \to B'|\{P\}$ and the chosen gluing of $B|\{P\}\}$ with $B'|\{P\}$. We obtain a unique gluing of $F|\{P\}\}$ with $F'|\{P\}$. Hence we chose as A'' and B'' any pair of vector bundles with $A''|Y = A'$, $A''|D = A$, $B''|Y = B'$, $B''|D = B$, while we gave a strong restriction for F''.

Part (ii): In this part of the proof we distinguish 7 cases.

Case 1). Here we assume D smooth and of genus $q \geqslant 2$. By assumption we have an exact sequence (6) with numerical datum $(k, a(D), r-k, b(D))$. By [RT] we may find such exact sequence with A, F and B stable.

Case 2). Here we assume D smooth and of genus 1. By [BR], § 1, there exists an exact sequence (6) on D with numerical datum $(k, a(D), r-k, b(D))$ and with A, F and B semistable. Since $p_a(X) > 1$ we have $S'' := D \cap Y \neq \emptyset$.

Case 3). Here we assume D singular and $q := p_a(D) \geqslant 3$. By [B2] we have an exact sequence (6) on D with numerical datum $(k, a(D), r-k, b(D))$ and with A, F and B stable. Hence we conclude as in Case 1).

Case 4). Here we assume D singular $p_a(D) = 2$ and $p_a(D') = 1$. Hence D has an ordinary node as unique singularity. Set $\{P, Q\} := \pi^{-1}(\mathrm{Sing}(\mathrm{D}))$. We take an exact sequence (6) on D' (not on D) with numerical type $(k, a(D), r-k, b(D))$. By [BR], § 1, we may find such exact sequence with A, F and B semistable. We glue the fibers of the vector bundles A, F and B over the point P and over the point Q in the following way. We fix any gluing data for $A|\{P\}$ and $A|\{Q\}\}$ and for $B|\{P\}\}$ and $B|\{Q\}\}$ and we give the following gluing data for the fibers $F|\{P\}\}$and $F|\{Q\}$. We glue the subspaces $i(P)(A|\{P\})$ and $i(Q)(A|\{Q\})$. We fix complementary subspaces T of $i(P)(A|\{P\})$ in $F|\{P\}$ and T' of $i(Q)(A|\{Q\})$ in $F|\{Q\}$. We glue together T and T' in the unique way compatible with the isomorphisms $q(P)|T: T \to B|\{P\}$, $q(Q)|T': T' \to B|\{Q\}$ and the chosen gluing of $B|\{P\}$ with $B|\{Q\}$. We obtain a unique gluing of $F|\{P\}$ with $F|\{Q\}$. In this way we obtain 3 vector bundles on D fitting in an exact sequence of type (6). Given an exact sequence of type (7) on Y we apply the gluing recipe described in Part (i) of the proof to obtain an exact sequence (8) on X.

Case 5). Here we assume D smooth and rational. It is well-known the existence of an exact sequence of type $(a-1)$ with any numerical type $(k, a(D), r-k, b(D))$ in which A, F and B are rigid, i.e. if $a_1 \geqslant \ldots \geqslant a_k$ (resp. $f_1 \geqslant \ldots \geqslant f_r$, resp. $b_1 \geqslant \ldots \geqslant b_{r-k}$) is the splitting type of A (resp. F, resp. B) we have $a_k \geqslant a_1 - 1$ (resp. $f_r \geqslant f_1 - 1$, resp. $b_{r-k} \geqslant b - 1$).

Case 6). Here we assume D singular and rational. Hence D has q ordinary nodes. We take an exact sequence of type $(a-1)$ on D' with A, F and B rigid and we apply q times the gluing procedure given in Case 4.

Case 7). Here we assume D smooth and of genus 2. We may take as $H|D$ and

$G|D$ general stable bundles with their numerical invariants. The proof of [B2] gives that we may take $E|D$ semistable and simple.

Part (iii): We may do the construction of Part (ii) in such a way that for every irreducible component D of X with $q := p_a(D) \geq 2$ the sheaves A and B are general respectively in $U(D; k, a(D))$ and in $U(D, r-k, b(D))$. Since the gluing data for H and G are arbitrary, we may obtain the J-stability of H and G just from the existence of J-stable bundles on X with their numerical invariants. Let D be an irreducible component of X with either arithmetic genus at least 2 or smooth and with genus 2; by Part (ii) the bundles $H|D$, $E|D$ and $G|D$ are simple. Let R be an irreducible component of X with $p_a(R) = 1$ and $\mathrm{Sing}(R) \neq \emptyset$; by Part (ii) $H|D$, $E|D$ and $G|D$ are semistable; fix $P \in R$ with P contained in another irreducible component of X; by Part (ii) we have $h^0(R, \mathrm{End}(H|R)(-P)) = h^0(R, \mathrm{End}(E|R)(-P)) = h^0(R, \mathrm{End}(G|R)(-P)) = 0$. Let T be an irreducible component of X which is smooth and elliptic; fix $P \in T$ with P contained in another irreducible component of X; by Part (ii) we have $h^0(T, \mathrm{End}(H|T)(-P)) = h^0(T, \mathrm{End}(E|T)(-P)) = h^0(T, \mathrm{End}(G|T)(-P)) = 0$. Let C be an irreducible component of X which is smooth and rational; fix two points P, P' of T contained in other irreducible components of X; by Part (ii) we have

$$h^0(C, \mathrm{End}(H|C)(-P-P')) =$$

$$= h^0(C, \mathrm{End}(E|C)(-P-P')) = h^0(C, \mathrm{End}(G|C)(-P-P')) = 0 .$$

Hence it is easy to check that every endomorphism of H (resp. E, resp. G) whose restriction to a point is an homotethy is an homotethy. Furthermore, every endomorphism of H (resp. E, resp. G) is an homotethy except perhaps if every irreducible component of X has arithmetic genus at most 1. The bundles H and G are stable because we may take as «good» polarization the canonical polarization ω_X (see for instance [P]) and every J-stable sheaf is simple.

REMARK 3.2. – As «good» polarization such that H and G are J-stable we may always take the canonical polarization ω_X (see e.g. [P]).

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