STABLE SMALL-AMPLITUDE SOLUTIONS IN REACTION-DIFFUSION SYSTEMS*

By

G. BARD ERMENTROUT

National Institutes of Health, Bethesda

Abstract. Bifurcation and perturbation techniques are used to construct small-amplitude periodic wave-trains for general systems of reaction and diffusion. All solutions are characterized by the amplitude a and the wavenumber k. For scalar diffusion, $k \sim a$, while for certain types of nonscalar diffusion, k is bounded away from zero as $a \searrow 0$. For certain ranges of a and k, linear stability of waves is demonstrated.

1. Introduction. Recently, there has been a great deal of interest in the existence and stability of waves for reaction-diffusion equations. These nonlinear parabolic systems are believed to play a role in biological and chemical pattern formation. A variety of wave forms have been demonstrated for these systems including wavefronts, pulse waves, and periodic wavetrains (see for example [3] and [7]). In this paper we are concerned only with small-amplitude, periodic wavetrains: those occurring as a zero rest state loses stability. There are numerous rigorous and formal results concerning the existence of small-amplitude wavetrains, but very little is known of their stability; most results have been negative [2, 5, 7]. In [4, 7] it was conjectured that large-amplitude waves "near" a stable homogeneous limit cycle were stable. Rinzel [8] demonstrated neutral stability for large-amplitude waves occurring in a simple class of reaction-diffusion equations modeling nerve-axons.

The most complete stability results are for a very simple class of two-component systems introduced by Kopell and Howard [5]. These so-called λ-ω-systems exhibit stable waves which can be explicitly constructed. Many general reaction-diffusion equations can be reduced to λ-ω-systems when the kinetics are near a Hopf bifurcation (see [1] and Sec. 3 of this paper). Cohen *et al.* [2] recently proved the existence of bifurcating waves for a fairly general class of equations, but these waves were shown to be unstable. Based on these results, it was suggested that all small-amplitude bifurcating waves are unstable.

In this paper we show that there are many small-amplitude wavetrains which are, in some sense, stable. The principal difference between our techniques and those of [2, 5, 7] is the introduction of a small bifurcation parameter into the reaction kinetics. In Secs. 2 and 3, stable long waves are constructed near a small-amplitude bifurcating limit cycle. By long waves, we mean that the wave number is of the same order as the amplitude;

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thus, as the amplitude vanishes, so does the wave number. In Secs. 4 and 5 we examine the existence and stability of truly bifurcating wavetrains, that is, wavetrains which are not "near" a homogeneous limit cycle. For this second type of wave, instability arises because of differences in the diffusion constants and *not* because of the kinetics; in the absence of diffusion the uniform state is asymptotically stable. Other investigators have demonstrated diffusion-induced instability which leads to *stationary* spatially periodic structures. In our case this diffusion-induced instability leads to traveling waves or standing spatially periodic temporal oscillations. At the end of Sec. 4 we discuss an example of a system with this unusual instability.

2. A simple λ-ω system. Kopell and Howard [5] introduced a simple class of nonlinear reacting dynamics of the form:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda(r) & -\omega(r) \\ \omega(r) & \lambda(r) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}; \qquad r = (u^2 + v^2)^{1/2}. \tag{2.1}$$

If we let $u = r\cos\theta$ and $v = r\sin\theta$ be the polar coordinate representations of (u, v), (2.1) becomes

$$r_t = r\lambda(r), \qquad \theta_t = \omega(r). \tag{2.2}$$

If $\lambda(0) > 0$, $\lambda(\bar{r}) = 0$ for some $\bar{r} > 0$, $\lambda'(\bar{r}) < 0$, and $\omega(\bar{r}) \neq 0$, then (2.2), and equivalently (2.1), admit a stable limit cycle:

$$u = \bar{r}\cos\omega(\bar{r})t, \qquad v = \bar{r}\sin\omega(\bar{r})t.$$

Suppose we put $\omega(r) \equiv 1$ and $\lambda(r) = \gamma - r^2$, where γ is a parameter. Then, for $\gamma < 0$, the rest state, $u = v = 0$ is asymptotically stable, while for $\gamma > 0$ it is unstable. It is readily shown that the eigenvalues at the instability are complex and thus (2.1) undergoes a Hopf bifurcation to a stable limit cycle at $\gamma = 0$. The amplitude of these cycles is $\sqrt{\gamma}$.

Consider now the system (2.1) with scalar diffusion terms:

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \gamma - (u^2 + v^2) & -1 \\ 1 & \gamma - (u^2 + v^2) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \frac{\partial^2}{\partial x^2}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.3}$$

The change to polar coordinates yields:

$$r_t = r(\gamma - r^2) + r_{xx} - r\theta_x^2, \qquad \theta_t = 1 + \frac{1}{r^2}(r^2\theta_x)_x \tag{2.4}$$

(2.4) admits solutions of the form:

$$\begin{matrix} r = r_0, & \theta = \sigma t - \alpha x, \\ \sigma = 1, & \alpha^2 = \gamma - r_0^2; \end{matrix} \qquad r_0 < \sqrt{\gamma}, \tag{2.5}$$

i.e.,

$$u = r_0\cos(t - \sqrt{\gamma - r_0^2}\,x), \qquad v = r_0\sin(t - \sqrt{\gamma - r_0^2}\,x).$$

These are wavetrains with amplitude r_0 and wavelength $2\pi/(\gamma - r_0^2)^{1/2}$. In Fig. 1 we have sketched one such wave.

We now examine the stability of this wave. There are a number of different concepts of stability—in this paper we are concerned only with linear stability. That is, suppose $u^*(x, t)$ is a solution to:

$$u_t = F(u).$$

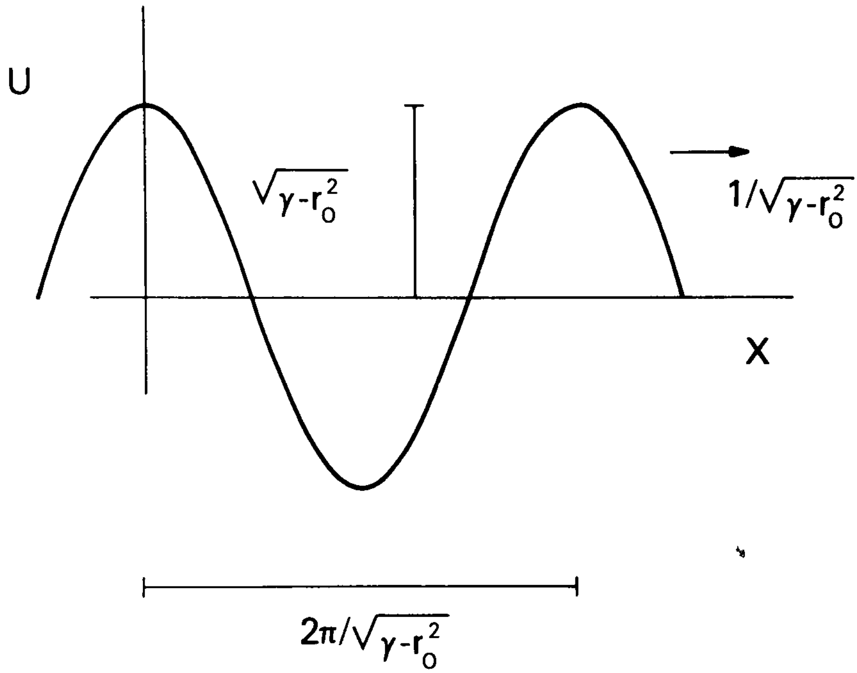

FIG. 1. Typical profile of a bifurcating wave. The wavelength is $2\pi/(\gamma - r_0^2)^{1/2}$, the velocity $1/(\gamma - r_0^2)^{1/2}$, the amplitude $(\gamma - r_0^2)^{1/2}$, and the frequency 1.

Then the linear stability problem is:

$$v_t = DF(u^*(x, t))v. \tag{2.5}$$

If this linear problem admits solutions which are spatially bounded but grow without bound as t increases, we say that $u^*(x, t)$ is linearly unstable. Ideally, we would like to be able to say that if we start near the solution u^*, we remain close to u^* (up to translations and rotations in x or t) for all $t > 0$. In certain cases, it has been shown that linear stability implies stability in this latter sense (see e.g. [9]). To determine stability of the solutions (2.5), we apply a theorem due to Kopell and Howard [5]:

THEOREM 2.6 [5, p. 317]. The wave solution with amplitude r_0 and wavenumber α is linearly stable if and only if:

$$4\alpha^2(1 + (\omega'(r_0)/\lambda'(r_0))^2) + r_0\lambda'(r_0) \le 0.$$

For our system, we find this implies:

$$\sqrt{\gamma} \ge r_0 \ge (\tfrac{2}{3}\gamma)^{1/2}. \tag{2.7}$$

Thus, we can always find stable waves with arbitrarily small amplitude which "bifurcate" from a uniform rest state. We note that these are *long* waves with a wavelength proportional to $1/\sqrt{\gamma}$. There is an entire range for which these waves are stable, including the infinite-wavelength "wave" corresponding to the bulk kinetic oscillation, $r_0 = 1$. In Fig. 2, we illustrate the bifurcation diagram for this system, showing the domain of stability.

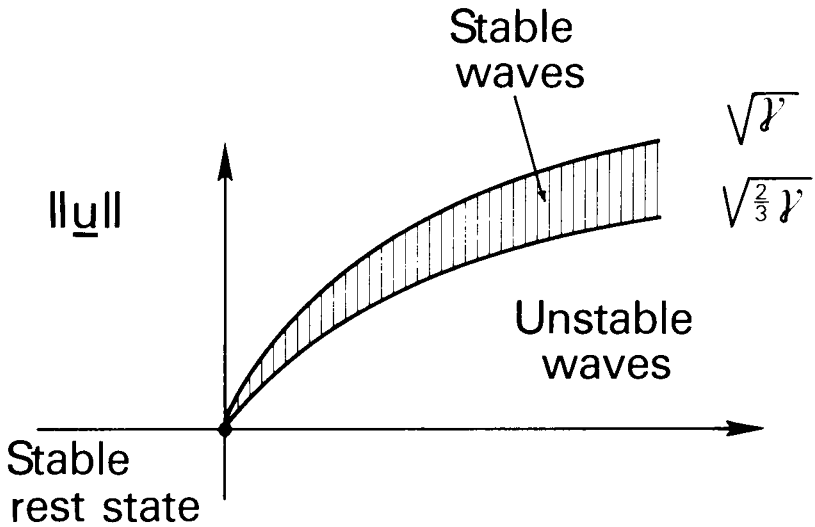

FIG. 2. "Bifurcation diagram" for waves arising as the rest state loses stability.

This simple example shows that we cannot eliminate the possibility of stable small-amplitude wavetrains.

3. Long waves in general systems. In Sec. 2, we examined an extremely simple system constructed in such a way as to demonstrate stable long waves explicitly. The key point in the analysis was that (2.3) displayed a stable small-amplitude bifurcating limit cycle. The presence of diffusion enables spatial variations in phase to develop and traveling waves to arise. If the bifurcation parameter γ is fixed and of order ε^2, $\varepsilon \ll 1$, then the amplitude and wave number are both of order ε.

Using a multi-scaling perturbation approach, we derive a λ-ω-type system starting with a general reaction-diffusion system as it passes through a bulk Hopf bifurcation. To simplify the present computations, we write the general system as:

$$\frac{\partial u}{\partial t} = Au + \gamma Bu + D\frac{\partial^2 u}{\partial x^2} + F(u) \tag{3.1}$$

where $u \in \mathbb{R}^n$ and $F(u) = O(|u|^2)$. Define $H(\gamma, k^2) = A + \gamma B - k^2 D$ and let $\mu_1(\gamma, k^2) = s(\gamma, k^2) + im(\gamma, k^2)$ be an eigenvalue with maximal real part s. We make the following assumption:

Assumption A (Fife [3]). Assume that for (γ, k) in some rectangle R with positive area, μ_1 is algebraically simple and hence differentiable in (γ, k). Let μ_1 be unique up to complex conjugate, and assume for some (γ_0, k_0) in the interior of R that:

(i) $s(\gamma_0, k_0^2) = 0$,

(ii) $s(\gamma_0, k^2) < 0$ for all $|k| \neq |k_0|$,

(iii) $s_\gamma(\gamma_0, k_0^2) \equiv \frac{\partial s}{\partial \gamma}(\gamma, k_0^2)\Big|_{\gamma=\gamma_0} > 0$,

(iv) $m(\gamma_0, k_0^2) \equiv m_0 \neq 0$.

There are two cases which can occur. If D is nearly scalar then $k_0 = 0$, while in other cases, we may have $k_0 \neq 0$. In this section the case $k_0 = 0$ is treated, leaving the remaining case for the next two sections.

With no loss in generality, we assume $\gamma_0 = 0$ and define Φ_0 and Ψ_0 by:

$$A\Phi_0 = im_0\Phi_0, \qquad A^T\Psi_0 = -im_0\Psi_0, \qquad (\Phi_0, \Psi_0) = 1, \qquad (\Phi_0, \Phi_0) = 1, \quad (3.2)$$

where A^T is the transpose of A and $(\cdot, \cdot)$ is the inner product on $\mathbb{C}^n$. We introduce the fast time scale, $t^* = m_0 t$. Clearly the operator $L_0 \equiv m_0(\partial/\partial t^*) + A$ has eigenfunctions:

$$\phi_1(t^*) = \Phi_0 e^{it^*}, \qquad \phi_2(t^*) = \bar{\Phi}_0 e^{-it^*} \quad (3.3a)$$

and the adjoint, $L_0^* \equiv -m_0(\partial/\partial t^*) + A^T$, has eigenfunctions:

$$\psi_1(t^*) = \Psi_0 e^{it^*}, \qquad \psi_2(t^*) = \bar{\Psi}_0 e^{-it^*}. \quad (3.3b)$$

Here "$-$" denotes complex conjugate. Certainly from (3.2) we must have

130 $$\langle \phi_i, \phi_j \rangle \equiv \frac{1}{\pi}\int_0^{2\pi} (\phi_i(t^*), \phi_j(t^*))\, dt^* = \langle \phi_i, \psi_j \rangle = \delta_{ij}.$$

We seek solutions to (3.1) which are real, bounded, periodic in the fast variable t^* and slowly varying:

$$u(x, t) = \varepsilon\{Z(\xi, \tau; \varepsilon)\phi_1(t^*) + w(\xi, \tau, t^*; \varepsilon)\} \sim \quad (3.5a)$$

$$\gamma = \varepsilon^2\hat{\gamma}(\varepsilon), \qquad \xi = \varepsilon x,\ \tau = \varepsilon^2 t, \quad (3.5b)$$

where

$$\langle \phi_j, w \rangle = 0; \qquad j = 1, 2. \quad (3.6)$$

The symbol "$\sim$" means "plus the complex conjugate of the preceding quantity" and is necessary here since real solutions are desired. $Z(\xi, \tau; \varepsilon) = R(\xi, \tau; \varepsilon)e^{i\Theta(\xi, \tau; \varepsilon)}$ incorporates the slowly varying amplitude, $R(\xi, \tau; \varepsilon)$ and the slowly varying phase, $\Theta(\xi; \tau; \varepsilon)$. These depend only on ε and the slow variables ξ and τ. At the onset of this section, we justified the scaling for the slow spatial variation ξ and the amplitude ε. Our choice for τ is made based on the second-order frequency corrections required for the Hopf bifurcation. We expect w to be $O(\varepsilon)$ since all components which are outside of the null space decay exponentially to zero at $\gamma = 0$. To derive the equations governing Z, we write $F(u) = B^{(2)}(u, u) + B^{(3)}(u, u, u) + \cdots$, where $B^{(k)}$ is a homogeneous multinomial of degree k. We expand w and $\bar{w}$ in a power series in ε:

$$w(\xi, \tau, t^*; \varepsilon)\sim\ = \varepsilon w_1(\xi, \tau, t^*)\sim\ + \varepsilon^2 w_2(\xi, \tau, t^*)\sim\ + \cdots \quad (3.7)$$

and substitute this and (3.5) into (3.1) to obtain:

$$\begin{aligned} L_0(w_1 + \bar{w}_1 + \varepsilon(w_2 + \bar{w}_2)) &= B^{(2)}(Z\phi_1 + \bar{Z}\bar{\phi}_1, Z\phi_1 + \bar{Z}\bar{\phi}_1) \\ &+ \varepsilon\{B^{(2)}(Z\phi_1 + \bar{Z}\bar{\phi}_1, w_1 + \bar{w}_1) + B^{(3)}(Z\phi_1 + \bar{Z}\bar{\phi}_1, Z\phi_1 + \bar{Z}\bar{\phi}_1, Z\phi_1 + \bar{Z}\bar{\phi}_1) \\ &+ \gamma(\varepsilon)ZB\phi_1 + \gamma(\varepsilon)\bar{Z}B\bar{\phi}_1 + D[Z_{\xi\xi}\phi_1 + \bar{Z}_{\xi\xi}\bar{\phi}_1] - Z_\tau\phi_1 - \bar{Z}_\tau\bar{\phi}_1\} + O(\varepsilon^2). \end{aligned} \quad (3.8)$$

This is an equation of the form $L_0 w = f$; since L_0 has a two-dimensional null-space spanned by ϕ_1 and $\phi_2 = \bar{\phi}_1$, (3.8) can be solved in the space of continuous 2π-periodic functions C_0 if and only if f is orthogonal to the adjoint eigenfunctions ψ_1 and ψ_2. Clearly the first term on the right-hand side of (3.8) is orthogonal to ψ_j so we may solve for w_1:

$$L_0 w_1 = Z^2 B^{(2)}(\phi_1, \phi_1) + Z\bar{Z}B^{(2)}(\phi_1, \bar{\phi}_1). \tag{3.9}$$

Seeking solutions in C_0, we find:

$$w_1 = Z^2 \Phi^{(2)} e^{2it^*} + Z\bar{Z}\Phi^{(0)} + C_1\phi_1 + C_2\phi_2 \tag{3.10}$$

where $\Phi^{(2)}$ and $\Phi^{(0)}$ satisfy the algebraic equations:

$$-A\Phi^{(0)} = B^{(2)}(\Phi_0, \bar{\Phi}_0), \qquad (2im_0 - A)\Phi^{(2)} = B^{(2)}(\Phi_0, \Phi_0) \tag{3.11}$$

and C_1, C_2 are arbitrary constants. From (3.6) it is seen that $C_1 = C_2 = 0$, so that the complete lowest-order solution is:

$$w_1 + \bar{w}_1 = Z^2\Phi^{(2)}e^{2it^*} + \bar{Z}^2\bar{\Phi}^{(2)}e^{-2it^*} + Z\bar{Z}\{\Phi^{(0)} + \bar{\Phi}^{(0)}\}. \tag{3.12}$$

To find w_2, we apply the orthogonality principle to the order-ε equations in (3.8), obtaining:

$$Z_\tau = dZ_{\xi\xi} + Z(\hat{\gamma}(\varepsilon)b_1 - b_3 Z\bar{Z}) + O(\varepsilon) \tag{3.13}$$

with

$$d \equiv \mathscr{D}_1 + i\mathscr{D}_2 = \langle\psi_1, D\phi_1\rangle, \qquad b_1 \equiv b_{1,1} + ib_{2,1} = \langle\psi_1, B\phi_1\rangle, \qquad b_3 \equiv b_{1,3} + ib_{2,3}$$
$$= -\langle\psi_1, B^{(2)}(\Phi^{(2)}, \bar{\Phi}_0) + B^{(2)}(\Phi^{(0)} + \bar{\Phi}^{(0)}, \Phi_0) + 3B^{(3)}(\Phi_0, \Phi_0, \Phi_0)\rangle. \tag{3.14}$$

The following lemma characterizes b_1 and d.

LEMMA 3.15. Let $q = k^2$; then from Assumption A:

(a) $b_1 = s_\gamma + im_\gamma$; $m_\gamma = \dfrac{\partial m}{\partial\gamma}(\gamma, 0)\Big|_{\gamma=0}$;

(b) $d \equiv \mathscr{D}_1 + i\mathscr{D}_2 = -s_q - im_q$; $s_q = \lim\limits_{q\to 0^+} \dfrac{\partial s}{\partial q}(0, q) > 0, \qquad m_q = \lim\limits_{q\to 0^+} \dfrac{\partial m}{\partial q}(0, q)$.

Proof. We consider the eigenfunction equation:

$$(A + \gamma B - qD)\Phi(\gamma, q) = \mu_1(\gamma, q)\Phi(\gamma, q) \tag{3.16}$$

where $\Phi(0, 0) = \Phi_0$ and $\mu_1(0, 0) = im_0$. Differentiating this with respect to γ at $\gamma = 0$ and taking the inner product with Ψ_0 yields (3.15a).

From Assumption A(ii), for q small and positive:

$$\mu_1(0, q) = qs_q + im_0 + iqm_q + O(q^2).$$

Since $\mu_1(0, q)$ has a *maximal* real part at $k = \sqrt{q} = 0$, $s_q < 0$. Note that the case $s_q = 0$ is excluded since it is nongeneric. Expanding (3.16) in powers of q, we observe that to order q:

$$A\Phi_q - D\Phi_0 = s_q\Phi_0 + im_q\Phi_0 + im_0\Phi_q.$$

Taking the inner product of this expression with Ψ_0 gives (3.15b).

We remark that if D is a scalar matrix, say $D \equiv 1$, then $d = 1$. Furthermore, note that from Assumption A(iii) that $b_{1,1} > 0$. To derive the zero-order modulation equations we expand Z and $\hat{\gamma}$ in a series in ε:

$$Z(\xi, \tau; \varepsilon) = Z_0(\xi, \tau) + \varepsilon Z_1(\xi, \tau) + \cdots, \qquad \hat{\gamma}(\varepsilon) = \gamma_0 + \varepsilon\gamma_1 + \cdots. \tag{3.17}$$

Substituting (3.17) into (3.13) and collecting the zero-order terms leads to:

$$Z_{0\tau} = dZ_{0\xi\xi} + Z_0(\gamma_0 b_1 - b_3 Z_0 \bar{Z}_0). \tag{3.18}$$

Before continuing the analysis of (3.18), γ_0 must be determined. To do this, we return to the problem (3.1). In the *absence of diffusion* this system admits a small-amplitude periodic orbit, $u(t)$, for $\gamma > 0$, i.e. a Hopf bifurcation. As a further normalization requirement, we demand that $\|u(t)\| = \langle u(t), u(t)\rangle^{1/2} = \varepsilon$, whence we find $Z_0(\tau)\overline{Z_0}(\tau) = 1$ and $\gamma_0 = b_{1,3}/b_{1,1}$. One could pick any arbitrary amplitude normalization in order to uniquely specify γ_0; we have chosen the usual one (see Sattinger [9]). With this value of γ_0, the spatially uniform solution to (3.18) is:

$$Z_0(\tau) = e^{i(\gamma_0 b_{2,1} - b_{2,3})\tau}.$$

In the event that $b_{1,3} = 0$, then $\gamma(\varepsilon) = \varepsilon^4\hat{\gamma}(\varepsilon)$ and we must rescale the space and time variables ξ and τ. Rather than a nonlinearity of the form $Z^2\bar{Z}$, we would find $Z^3\bar{Z}^2$. We shall not consider this latter case, and hence assume $b_{1,3} \neq 0$. The spatially uniform solution $u(\tau)$ is stable if and only if $b_{1,3} > 0$, as can be readily shown.

To see the similarity between (3.18) and the simple system studied in Sec. 2, we note that by setting $W = re^{i\theta}$, (2.4) may be written as:

$$W_\tau = W_{\xi\xi} + W(\gamma + i - W\bar{W}).$$

Clearly, (3.18) is a generalization of (2.4) with complex diffusion and a complex nonlinearity.

With γ_0 as defined above, we look for space-dependent solutions to (3.18). The solutions analogous to the long waves in Sec. 2 are:

$$Z_0(\xi, \tau) = r_0 e^{i[\sigma\tau + \alpha\xi]}; \qquad 0 \leq r_0 \leq 1 \tag{3.19}$$

where

$$\mathscr{D}_1\alpha^2 = b_{1,3}(1 - r_0^2) \equiv \lambda(r_0), \qquad \sigma = -\mathscr{D}_2\alpha^2 + \gamma_0 b_{2,1} - b_{2,3}r_0^2$$
$$\equiv -\mathscr{D}_2\alpha^2 + \omega(r_0).$$

These solutions correspond to periodic wavetrains in the original system, (3.1), with amplitude εr_0, frequency, $m_0 + \varepsilon^2\sigma$, and wave number $\varepsilon\alpha$. We now examine their stability. Rather than determining stability of (3.19) and (3.5) as a solution to (3.1), we instead determine the stability of (3.19) as a solution to (3.18). While stability of the modulating solutions does not necessarily imply stability of the solution, it is not an unreasonable possibility. From Theorem 2.6, with λ and ω as defined in (3.19), we can determine stability when $\mathscr{D}_2 = 0$:

COROLLARY 3.20. Suppose $\mathscr{D}_2 = 0$. Then the solution with amplitude r_0 is linearly stable if and only if:

(a) $b_{1,3} > 0$ and
(b) $r_0^2 \geq 2/[2 + 1/(1+\delta)]$; $\quad \delta = (b_{2,3}/b_{1,3})^2$.

(3.20a) simply states that the spatially homogeneous Hopf bifurcation must be supercritical ($\gamma_0 > 0$) and hence stable. We can parametrize these waves by either the amplitude or the wave-number using (3.19). As in the case of the λ-ω-systems, there is an entire interval of stable periodic wavetrain solutions to (3.1) with amplitude εr_0. For $b_{2,3} = 0$ the frequency does not change to lowest order and the bounds on r_0 are identically those obtained for the system studied in Sec. 2. Thus, general reaction-diffusion equations with scalar diffusion can always be reduced to λ-ω-systems when their kinetics are near a Hopf bifurcation.

In the event that $\mathscr{D}_2 \neq 0$, the above result is not valid and we state a somewhat weaker version of the linear stability theorem, (2.6):

THEOREM 3.21. Consider the problem:

(a) $Z_\tau = dZ_{\xi\xi} + Z(\lambda(r) + i\omega(r))$; $\qquad r^2 = Z\bar{Z}$.

The traveling wave solutions

$$Z_0(\tau, \xi) = r_0 e^{i(\sigma\tau + \alpha\xi)},$$

where $i\sigma + d\alpha^2 = \lambda(r_0) + i\omega(r_0)$, are stable to infinitesimal perturbations of sufficiently small spatial frequency κ, if:

(b) $\lambda'(r_0) < 0$
(c) $r_0\lambda'(r_0) + 4\mathscr{D}_1\alpha^2[1 + (\omega'(r_0)/\lambda'(r_0))^2] + r_0\omega'(r_0)\mathscr{D}_2/\mathscr{D}_1 < 0$.

Here $d = \mathscr{D}_1 + i\mathscr{D}_2$.

Proof. Linearizing (a) about $Z_0(\tau, \xi) = r_0 e^{i(\sigma_0\tau + \alpha\xi)}$ leads to the problem:

$$\begin{aligned} J_0 W \equiv & -W_\tau + dW_{\xi\xi} + (\lambda(r_0) + i\omega(r_0))W \\ & + r_0(\lambda'(r_0) + i\omega'(r_0))[W + \bar{W}e^{2i(\sigma\tau + \alpha\xi)}] = 0. \end{aligned} \tag{3.22}$$

We are interested in solutions bounded in space which grow exponentially in time; in this case the waves are said to be linearly unstable. We look for solutions of the form $W(\tau, \xi) = V(\tau, \xi)e^{i(\sigma\tau + \alpha\xi)}$. Substituting this into (3.22) and taking real and imaginary parts of V, $V = x + iy$, we see that x and y satisfy:

$$\begin{aligned} x_\tau &= \mathscr{D}_1 x_{\xi\xi} - \mathscr{D}_2 y_{\xi\xi} - 2\alpha\mathscr{D}_2 x_\xi - 2\alpha\mathscr{D}_1 y_\xi + r_0\lambda'(r_0), \\ y_\tau &= \mathscr{D}_1 y_{\xi\xi} + \mathscr{D}_2 x_{\xi\xi} + 2\alpha\mathscr{D}_1 x_\xi - 2\alpha\mathscr{D}_2 y_\xi + \omega'(r_0)r_0. \end{aligned} \tag{3.23}$$

(3.23) is linear with constant coefficients, so it suffices to examine perturbations of the form $\binom{f_1}{f_2}e^{\rho\tau + i\kappa\xi}$, $\binom{f_1}{f_2}$ a constant vector, and $\kappa \in \mathbb{R}$. If there are values of κ for which $\rho = \rho(\kappa)$ has a positive real part, then the waves are unstable; otherwise, they are said to be linearly stable. Substituting this exponential form into (3.23) leads to the algebraic system:

$$\begin{bmatrix} -\rho - \mathscr{D}_1\kappa^2 - 2\mathscr{D}_2\alpha i\kappa + r_0\lambda'(r_0) & -2\alpha i\kappa\mathscr{D}_1 + \mathscr{D}_2\kappa^2 \\ 2\alpha i\kappa\mathscr{D}_1 - \mathscr{D}_2\kappa^2 + \omega'(r_0)r_0 & -\rho - \mathscr{D}_1\kappa^2 - 2\alpha i\kappa\mathscr{D}_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = 0. \tag{3.24}$$

This has nontrivial solutions if and only if

$$\rho = \rho^{\pm}(\kappa) = -\mathscr{D}_1\kappa^2 + \frac{r_0\lambda'(r_0)}{2} - 2\alpha i\kappa\mathscr{D}_2$$

$$\pm\left\{\left[\frac{r_0\lambda'(r_0)}{2}\right]^2 + 4\alpha^2\mathscr{D}_1^2\kappa^2 + \mathscr{D}_2\kappa^2[-\mathscr{D}_2^2 + r_0\omega'(r_0)]\right.$$

$$\left. + i[+4\mathscr{D}_1\mathscr{D}_2\kappa^3 - 2\alpha\mathscr{D}_1 r_0\omega'(r_0)\kappa]\right\}^{1/2}.$$

When $\kappa = 0$, the roots are $r_0\lambda'(r_0)/2 \pm |r_0\lambda'(r_0)|/2$.

If $r_0\lambda'(r_0) > 0$, then there is a positive root and the wave is unstable, thus we require $\lambda'(r_0) < 0$. In this case, when $\kappa = 0$, $\rho^+(0) = 0$ and we must examine higher-order κ terms. Expanding Re $\rho^+(\kappa)$ in a power series in κ, we show to order κ^2:

$$\operatorname{Re}\rho^+(\kappa) = -\kappa^2\left[\mathscr{D}_1 + \frac{4\mathscr{D}_1^2\alpha^2}{r_0\lambda'(r_0)} + \mathscr{D}_2\frac{\omega'(r_0)}{\lambda'(r_0)} + \frac{4\alpha^2\mathscr{D}_1^2\omega'(r_0)^2}{r_0\lambda'(r_0)^3}\right].$$

For stability, the quantity in the brackets must be positive. Since $r_0\lambda'(r_0)/\mathscr{D}_1 < 0$, multiplying through by this gives us (3.21c). Unfortunately, because of $\mathscr{D}_2$, we are unable to prove the stronger result that stability to long wave perturbations implies stability to all wavelength perturbations, as was achieved in [5]. By setting $\mathscr{D}_2 = 0$ and $\mathscr{D}_1 = 1$, we recover Theorem 2.6. As a corollary to Theorem 3.21, we obtain:

COROLLARY 3.25. The solution (3.19 to the modulation equations (3.18) is stable to infinitesimal perturbations of sufficiently long wavelength if and only if:

(1) $b_{1,3} > 0$ and

(b) $2(1 + \delta^2)/[(1 + \delta c) + 2(1 + \delta^2)] \le r_0^2$; $\quad \delta = \dfrac{b_{2,3}}{b_{1,3}}$, $\quad c = \dfrac{\mathscr{D}_2}{\mathscr{D}_1}$.

For general systems, the presence of the term $\mathscr{D}_2$ prevents us from obtaining a global linear stability result. Nevertheless, in the subclass of systems with near diagonal, scalar diffusion, there is global linearized stability and we conclude there are in fact formally stable small-amplitude periodic wavetrains bifurcating from the initially uniform rest state. These long waves have the form:

$$u(x, t) = \varepsilon\{r_0\phi_1([\omega_0 + \sigma\varepsilon^2]t + \alpha\varepsilon x) + O(\varepsilon)\}.$$

4. Existence of short waves and a Poincaré-Lindstedt series. In the previous two sections, we constructed long waves using bifurcation and perturbation methods. Here, we construct periodic wave trains with a wave number k bounded away from zero. We call these short or slow waves, since their wavelength and velocity are both $O(1)$ with respect to the amplitude a, and not $O(1/a)$ as the waves in Secs. 2 and 3. The construction in this section is motivated by an existence theorem of Fife [3] and depends on two independent parameters: the amplitude a and the change in wavenumber q. In this respect, it differs from the small-amplitude waves constructed and demonstrated by Kopell and Howard [5] and Ortoleva and Ross [7] where the change in wavenumber is the only parameter. Ortoleva and Ross [7] briefly discuss the case considered here in the appendix of their paper.

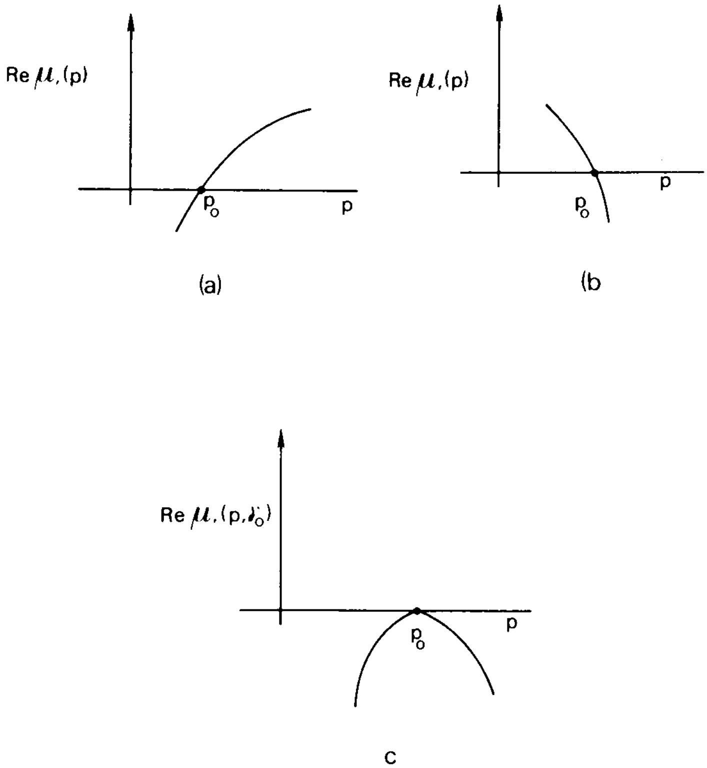

FIG. 3. Real part of the maximal eigenvalue plotted as a function of wavenumber k^2. (a, b) stability lost when there is no additional parameter; (c) dependence on a parameter γ.

In [3], Fife discusses why the above-mentioned one-parameter expansions are expected to generate only unstable wavetrains. We briefly sketch his argument. In [2, 5, 7] there are no "external" bifurcation parameters; rather, the wavenumber, $k^2 = p$, plays the role of the parameter. Let $\mu_1(p)$ denote the eigenvalue of maximal real part for the matrix $A - pD$ (this corresponds to the linearized version of (3.1) when $\gamma = 0$). We suppose that for $p = p_0$, $\operatorname{Re} \mu_1(p_0) = 0$ and $\operatorname{Im} \mu_1(p_0) \neq 0$. As long as $(d/dp)\operatorname{Re} \mu_1(p_0) \neq 0$, we expect a Hopf bifurcation to periodic wave solutions. Under this latter assumption, the graph of $\mu_1(p)$ versus p must have the form shown in Figs. 3a and 3*b*. From these figures, it is clear that there will always be eigenvalues of the linear equation with large positive real parts and instability is consequently always expected. We contrast this to the situation depicted in Fig. 3c, where, the additional bifurcation parameter γ has been introduced. For $\gamma = 0$, all wavenumbers but p_0 lead to eigenvalues with negative real parts. As γ increases beyond 0, an interval of values of p, centered about p_0, have *small* positive real parts. Below, we show that this can sometimes be compensated for by the nonlinear terms and for each *fixed* value of γ, a one-parameter family of stable waves results. These waves arise in a manner quite different from those discussed in [1, 2, 5] and Sec. 3 of this paper. The loss of stability of the rest state is due to unequal diffusion of the reactants—in fact, in Lemma 4.6c we show that scalar diffusion is precluded for the genesis of this phenomenon. It is known that spatial pattern formation arises from differences in the diffusion coefficients; here we demonstrate similar structures which, additionally, are time-dependent. It is important to emphasize that these Turing-like structures arise as the rest state becomes unstable to spatially *inhomogeneous* perturbations while remaining *stable* to homogeneous perturbations.

We turn now to a statement of Fife's existence theorem. Using this theorem as a starting point, we construct the lowest-order terms in a two-parameter expansion of the wave solutions. In Sec. 5, we determine a restricted linear stability of these waves. Recalling Assumption A (Sec. 3), we also assume that $k_0 \neq 0$, $m_0 \neq 0$, and that D is nonsingular. Let Φ_0 be the complex eigenfunction of $H(\gamma_0, k^2)$ with eigenvalue im_0 such that $(\Phi_0, \Phi_0) = 1$.

THEOREM 4.1 [3, p. 711]. Under Assumption A with s and m as before and the additional assumptions that $m_0 \neq 0$, $k_0 \neq 0$, and that D be nonsingular, there exists a two-parameter family $[U(z; a, q), \gamma(a, q), \bar{\omega}(a, q)]$ of solutions to (3.1) defined and continuous in (a, q) for a and q sufficiently small, such that

(i) U is 2π-periodic and continuous in $z = \bar{\omega}(a, q)t - (k_0^2 + q)^{1/2}x$
(ii) $\langle \phi_1, U \rangle = a$, $\langle \phi_2, U \rangle = 0$, where $\phi_1 = \operatorname{Re} \Phi_0 e^{iz}$ and $\phi_2 = -\phi_1'$
(iii) $\gamma(0, 0) = \gamma_0 = 0$, $\bar{\omega}(0, 0) = m_0 \neq 0$.

To lowest order, it will be shown that:

$$U(z; a, q) = a[\phi_1(z) + qw_{0,1}(z) + aw_{1,0}(z) + O(a^2, q^2, aq)], \tag{4.2a}$$

$$\bar{\omega}(a, q) = m_0 + \zeta_{0,1} q + \zeta_{0,2} q^2 + \zeta_{2,0} a^2 + O(a^2 q, q^3, a^3, aq^2), \tag{4.2b}$$

$$\gamma(a, q) = \gamma_{0,2} q^2 + \gamma_{2,0} a^2 + O(a^2 q, q^3, a^3, aq^2). \tag{4.2c}$$

Thus, the lowest-order expansion for the parameter γ depends only on a^2 and q^2 and there are no terms of the form aq. Before continuing we define the following quantities:

$$L_0 v \equiv m_0 \frac{dv}{dz} - k_0^2 D \frac{d^2 v}{dz^2} - Av, \tag{4.3a}$$

$$L_0^* v \equiv -m_0 \frac{dv}{dz} - k_0^2 D^T \frac{d^2 v}{dz^2} - A^T v, \tag{4.3b}$$

$$H(0, k_0^2)\Phi_0 \equiv H_0 \Phi_0 = im_0 \Phi_0, \qquad (\Phi_0, \Phi_0) = 1, \tag{4.3c}$$

$$H_0^T \Psi_0 = -im_0 \Psi_0, \qquad (\Phi_0, \Psi_0) = 1. \tag{4.3d}$$

We assume $k_0 \neq 0$ so that from Assumption A(ii), Fig. 3c obtains. Let $p = k^2$, $p_0 = k_0^2$, and $q = p - p_0$. Then $s(\gamma, p)$ is a function of q and because s vanishes at $q = 0$ and has a maximum at $q = 0$ ($\gamma = 0$), we must have:

$$s(0, k_0^2 + q) = s_{qq} q^2 + O(q^3) \tag{4.4}$$

with $s_{qq} \leq 0$. We contrast this with the situation in Sec. 3, where s depended linearly on q for q small and positive. That $k_0 \neq 0$ and Fig. 3c obtains precludes this possibility from the present case. The case $s_{qq} = 0$ and Fig. 3c imply that s must have fourth-order degeneracy. This is "non-generic" and thus we do not consider this case. Define:

$$\begin{aligned} \phi_1 &= \operatorname{Re} \Phi_0 e^{iz}, \qquad \phi_2 = \operatorname{Im} \Phi_0 e^{iz} = -\phi_1', \\ \psi_1 &= \operatorname{Re} \Psi_0 e^{iz}, \qquad \psi_2 = \operatorname{Im} \Psi_0 e^{iz} = -\psi_1', \end{aligned} \tag{4.5}$$

where $'$ denotes d/dz. Note that:

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}, \qquad \langle \psi_i, \phi_j \rangle = \delta_{ij}.$$

We have the following lemma, which corresponds to Lemma 3.15:

LEMMA 4.6.

(a) $\langle \psi_1, B\phi_1 \rangle = s_\gamma(0, k_0^2) \equiv s_\gamma > 0$,
(b) $\langle \psi_2, B\phi_1 \rangle = m_\gamma(0, k_0^2) \equiv m_\gamma$,
(c) $\langle \psi_1, D\phi_1 \rangle = \langle \psi_2, D\phi_2 \rangle = 0$,
(d) $\langle \psi_2, D\phi_1 \rangle = -\langle \psi_1, D\phi_2 \rangle = m_q(0, k_0^2) \equiv m_q$.

The proof of this lemma is similar to that of (3.15) and has been omitted. Letting C_0 denote the space of continuous 2π-periodic functions, we see that ϕ_1 and ϕ_2 are the independent eigenfunctions generating the two-dimensional null-space of L_0 in C_0. Similarly ψ_1, ψ_2 generate the null-space of the adjoint, L_0^*. The key point in this section is (4.6c). This follows from the assumption that k_0^2 is the only wavenumber leading to a nonnegative eigenvalue when $\gamma = 0$.

Here we see the importance of nonscalar diffusion for this type of instability. For if D were, say, dI, where I is the identity, the inner products, $\langle \psi_j, D\phi_j \rangle = d\langle \psi_j, \phi_j \rangle = d$, could vanish if and only if $d \equiv 0$. Thus in order for (4.6c) to hold it is necessary that D be nonscalar.

Motivated by the preceding existence theorem we write:

$$U(z; a, q) = a(\phi_1(z) + w_{a,q}(z)), \qquad \langle \phi_j, w_{a,q} \rangle = 0, j = 1, 2, \tag{4.7}$$

and expand $w_{a,q}$, $\gamma(a, q)$, and $\bar{\omega}(z, q)$ in a power series in a and q:

$$\begin{aligned} w_{a,q} &= aw_{1,0} + qw_{0,1} + a^2 w_{2,0} + qaw_{1,1} + q^2 w_{0,2} + \cdots, \\ \gamma(a, q) &= a\gamma_{1,0} + q\gamma_{0,1} + a^2\gamma_{2,0} + aq\gamma_{1,1} + q^2\gamma_{0,2} + \cdots, \\ \bar{\omega}(a, q) &= m_0 + a\zeta_{1,0} + q\zeta_{0,1} + a^2\zeta_{2,0} + aq\zeta_{1,1} + q^2\zeta_{0,2} + \cdots. \end{aligned} \tag{4.8}$$

As in Sec. 3, we assume that the nonlinear terms are sufficiently differentiable and can be expanded into homogeneous multinomials:

$$F(u) = B^{(2)}(u, u) + B^{(3)}(u, u, u) + \cdots.$$

Substituting (4.8) into (3.1) leads to the following linearly inhomogeneous equations:

$$L_0 w_{1,0} = -\zeta_{1,0}\phi_1' + B^{(2)}(\phi_1, \phi_1) + \gamma_{1,0}B\phi_1, \tag{4.9a}$$

$$L_0 w_{0,1} = -\zeta_{0,1}\phi_1' + D\phi_1'' + \gamma_{0,1}B\phi_1, \tag{4.9b}$$

$$\begin{aligned} L_0 w_{1,1} = &-\zeta_{1,1}\phi_1' - \zeta_{1,0}w_{0,1}' - \zeta_{0,1}w_{1,0}' + 2B^{(2)}(\phi_1, w_{0,1}) \\ &+ Dw_{1,0}'' + \gamma_{1,1}B\phi_1 + \gamma_{1,0}Bw_{0,1} + \gamma_{0,1}Bw_{1,0}, \end{aligned} \tag{4.9c}$$

$$\begin{aligned} L_0 w_{2,0} = &-\zeta_{1,0}w_{1,0}' - \zeta_{2,0}\phi_1' + 2B^{(2)}(\phi_1, w_{1,0}) \\ &+ B^{(3)}(\phi_1, \phi_1, \phi_1) + \gamma_{1,0}Bw_{1,0} + \gamma_{2,0}B\phi_1, \end{aligned} \tag{4.9d}$$

$$L_0 w_{0,2} = -\zeta_{0,1}w_{0,1}' - \zeta_{0,2}\phi_1' + Dw_{0,1}'' + \gamma_{0,1}Bw_{0,1} + \gamma_{0,2}B\phi_1. \tag{4.9e}$$

Each of the equations, (4.9 a–e), is of the form $L_0 w = f$. This has a solution in C_0 if and only if f is orthogonal to the two adjoint eigenfunctions, ψ_1 and ψ_2. At each stage in our expansion f takes the form of

$$\gamma_{n,m} B\phi_1 - \zeta_{n,m}\phi_1' + g_{nm}$$

where g_{nm} depends on lower-order terms. The orthogonality conditions require

$$\begin{pmatrix} 0 & s_\gamma \\ 1 & m_\gamma \end{pmatrix}\begin{pmatrix} \gamma_{n,m} \\ \zeta_{n,m} \end{pmatrix} = \begin{pmatrix} \langle \psi_1, g_{nm} \rangle \\ \langle \psi_2, g_{nm} \rangle \end{pmatrix}.$$

Since $s_\gamma \neq 0$, $\gamma_{n,m}$ and $\zeta_{n,m}$ can always be uniquely determined. Applying these conditions to (4.9a), we find $\gamma_{1,0} = \zeta_{1,0} = 0$ and that $w_{1,0}$ satisfies:

$$L_0 w_{1,0} = B^{(2)}(\phi_1, \phi_1).$$

Using the normalization (4.7), we see that this has a solution in C_0:

$$w_{1,0}(z) = \text{Re}\{\Phi^{(0)} + \Phi^{(2)} e^{2iz}\} \tag{4.10}$$

where $\Phi^{(0)}$ and $\Phi^{(2)}$ satisfy:

$$\begin{aligned} A\Phi^{(0)} &= -\tfrac{1}{2}B^{(2)}(\Phi_0, \bar{\Phi}_0) \\ (2i\omega_0 - A + 4k_0^2 D)\Phi^{(2)} &= \tfrac{1}{2}B^{(2)}(\Phi_0, \Phi_0). \end{aligned} \tag{4.11}$$

Applying the orthogonality principle to (4.7b) and using Lemma 4.5 shows $\gamma_{0,1} = 0$ and $\zeta_{0,1} = m_q$. To find $w_{0,1}$, we examine the algebraic eigenvalue problem:

$$H(0, q)\Phi(q) = \psi_1(0, q)\Phi(q); \qquad H(0, q) = H_0 - qD. \tag{4.12}$$

Letting $\Phi(q) = \Phi_0 + q\Phi_q$ and substituting this into (4.12) yields

$$(H_0 - im_0)\Phi_q = D\Phi_0 + m_q\Phi_0$$

Multiplying both sides by e^{iz}, taking real parts, and comparing to (4.9b) shows

$$w_{0,1} = \text{Re}\ \Phi_q e^{iz}, \qquad \Phi_q = \left.\frac{d\Phi(q)}{dq}\right|_{q=0} \tag{4.13}$$

where, again, we have used the normalization requirement (4.7). Applying the orthogonality condition to (4.9c) demonstrates that $\zeta_{1,1}$ and $\gamma_{1,1}$ are both zero. Since we are only interested to determining $\gamma_{2,0}$, $\gamma_{0,2}$, $\zeta_{2,0}$, and $\zeta_{0,2}$, we do not need to solve (4.9c) for $w_{1,1}$. The orthogonality conditions applied to (4.9d) imply:

$$\begin{aligned} \gamma_{2,0} &= -\langle \psi_1, 2B^{(2)}(\phi_1, w_{1,0}) + B^{(3)}(\phi_1, \phi_1, \phi_1)\rangle / s_\gamma \\ &\equiv b_{1,3}/s_\gamma, \\ \zeta_{2,0} &= -\langle \psi_2, 2B^{(2)}(\phi_1, w_{1,0}) + B^{(3)}(\phi_1, \phi_1, \phi_1)\rangle - \gamma_{2,0} m_\gamma \\ &\equiv b_{2,3} - \gamma_{2,0} m_\gamma. \end{aligned} \tag{4.14}$$

In the event that $b_{1,3} = 0$ we must continue to higher orders than $w_{1,1}$, $w_{0,2}$, and $w_{2,0}$, so we assume that $b_{1,3} \neq 0$.

Finally, we turn to (4.9e); applying the orthogonality condition to this equation yields:

$$\gamma_{0,2} s_\gamma = \langle \psi_1, \zeta_{0,1} w'_{0,1} - D w''_{0,1} \rangle,$$
$$\zeta_{0,2} = -\gamma_{0,2} m_\gamma + \langle \psi_2, m_q w'_{0,1} - D w''_{0,1} \rangle. \tag{4.15}$$

In order to determine $\gamma_{0,2}$, we expand (4.12) to order q^2:

$$H_0 \Phi_{qq} + 2D\Phi_q = im_0 \Phi_{qq} + 2im_q \Phi_q + \mu_{1qq} \Phi_0. \tag{4.16}$$

Applying Ψ_0 to both sides and using the definitions of $w_{0,1}$, ϕ_j, and ψ_j, we find $\gamma_{0,2} = -\frac{1}{2}s_{qq}/s_\gamma$ and $\zeta_{0,2} = \frac{1}{2}m_{qq} - \gamma_{0,2} m_\gamma$. From (4.4), $\gamma_{0,2} > 0$ and we have obtained the solution to desired order. Recapitulating, we tabulate the above computations:

$$\begin{aligned}
\gamma_{1,0} &= \gamma_{0,1} = \gamma_{1,1} = \zeta_{1,1} = \zeta_{1,0} = 0, \\
\gamma_{0,2} &= -\tfrac{1}{2}s_{qq}/s_\gamma > 0, \\
\gamma_{2,0} &= -\{\langle \psi_1, 2B^{(2)}(\phi_1, w_{1,0}) + B^{(3)}(\phi_1, \phi_1, \phi_1)\rangle\}/s_\gamma \equiv b_{1,3}/s_\gamma; \\
\zeta_{0,1} &= m_q, \qquad \zeta_{0,2} = \tfrac{1}{2}m_{qq} - \gamma_{0,2} m_\gamma, \\
\zeta_{2,0} &= -\{\langle \gamma_{2,0} m_\gamma + \langle \psi_2, 2B^{(2)}(\phi_1, w_{1,0}) + B^{(3)}(\phi_1, \phi_1, \phi_1)\rangle\rangle\}, \\
&\equiv -\gamma_{2,0} m_\gamma + b_{2,3}, \\
w_{0,1} &= \mathrm{Re}\, \Phi_q e^{iz}, \qquad w_{1,0} = \mathrm{Re}\{\Phi^{(0)} + \Phi^{(2)} e^{2iz}\}, \\
\phi_1 &= \mathrm{Re}\, \Phi_0 e^{iz}, \qquad \psi_1 = \mathrm{Re}\, \Psi_0 e^{iz}, \qquad \psi_2 = -\psi'_1.
\end{aligned} \tag{4.17}$$

We have constructed the leading terms in an asymptotic expansion for the wave solutions of a system of reaction and diffusion equations. Unlike other expansions, there are two independent parameters, a, the amplitude, and q, the deviation from the critical wavenumber, k_0^2. For each fixed γ, these two parameters are related by the equation:

$$\gamma = a^2 \gamma_{2,0} + q^2 \gamma_{0,2} + \text{higher-order terms.} \tag{4.18}$$

Thus, when all external parameters are fixed, there is a family of periodic wavetrains parametrized by the amplitude, i.e.

$$\begin{aligned}
k^2 &= k_0^2 \pm \hat{q}(a) \equiv k_0^2 \pm ((\gamma - a^2\gamma_{2,0})/\gamma_{0,2})^{1/2} + \text{higher-order terms,} \\
\bar{\omega} &= m_0 \pm \zeta_{0,1} \hat{q}(a) + \zeta_{0,2} \hat{q}(a)^2 + \zeta_{2,0} a^2 + \text{higher-order terms.}
\end{aligned} \tag{4.19}$$

(4.19) is the dispersion relation for the small-amplitude waves and in Sec. 5, conditions for stability are given. It should be noted that (4.18) is very similar to the expression relating the wavenumber α^2 to the amplitude r_0 in (2.5), for (4.18) can be written as:

$$q^2 = (\gamma - \gamma_{2,0} a^2)/\gamma_{0,2}. \tag{4.20}$$

At the beginning of this section we made some very specific assumptions about the reaction-diffusion system. In particular, we required that the real part of the maximal eigenvalue $s(\gamma, p)$ have a maximum at $p = p_0 \neq 0$ and that $m(\gamma, p_0) \neq 0$. It is not easy to construct such a system—we show, here, that a minimum of three reactants are needed. The second example is not general but the method of construction can be generalized for any systems.

We first show that Assumption A(iii) is easily satisfied by setting $B \equiv I$. For then, from Lemma 3.15a and Lemma 4.6a, $s_\gamma = 1 > 0$. Thus, we need only verify Assumption A(i, ii, iv), and disregard γ. Clearly one cannot expect oscillations in a one-component first-order system, so we consider the general second-order system:

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}u_1\\u_2\end{pmatrix} &= \begin{pmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix} + \begin{pmatrix}d_{11} & d_{12}\\d_{21} & d_{22}\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}u_1\\u_2\end{pmatrix}\\ &\equiv A\begin{pmatrix}u_1\\u_2\end{pmatrix} + D\frac{\partial^2}{\partial x^2}\begin{pmatrix}u_1\\u_2\end{pmatrix}.\end{aligned} \tag{4.21}$$

In this case, the matrix $H(0, k^2)$ is given by:

$$H(0, p) = \begin{pmatrix}a_{11} - d_{11}p & a_{12} - d_{12}p\\a_{21} - d_{21}p & a_{22} - d_{22}p\end{pmatrix}; \qquad p = k^2. \tag{4.22}$$

For a 2×2 system, a necessary condition for imaginary eigenvalues is that the trace, $\mathrm{Tr}(p) \equiv a_{11} + a_{22} - (d_{11} + d_{22})p$, of $H(0, p)$ vanish while the determinant remain positive. Since the trace is a monotonically decreasing function of p, it has a maximum value at $p = 0$, thus if $\mathrm{Tr}(p_0) = 0$, $\mathrm{Tr}(0) > 0$. This violates Assumption A(ii) since, a positive trace implies an eigenvalue occurs with positive real part. Hence, we see that Assumption A can hold only if $p_0 = 0$ and then the instability is due *not* to diffusion but to the chemical kinetics.

This, of course, is not true for higher-order systems, as is illustrated by the following third-order system:

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} &= \begin{pmatrix}0.1452 & 1 & 0\\-6.2501 & -0.3452 & 1\\0 & 3.1000 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 4\end{pmatrix}\frac{\partial^2}{\partial x^2}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}\\ &\equiv A\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} + D\frac{\partial^2}{\partial x^2}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}.\end{aligned} \tag{4.23}$$

The solutions to (4.23) are of the form

$$\Phi(p)\exp(\lambda(p)t + ikx); \qquad p = k^2,$$

where $\Phi(p)$ and $\lambda(p)$ are respectively the eigenvectors and eigenvalues of the matrix

$$H(p) = A - Dp.$$

We find that λ satisfies the characteristic equation

$$\chi(\lambda) \equiv \lambda^3 + T_2(p)\lambda^2 + T_1(p)\lambda + T_0(p) = 0,$$

$$T_2(p) = 6p + 0.2, \qquad T_1(p) = 9p^2 + p + 3.1, \qquad T_0(p) = 4p^3 + 0.8p^2 + 21.7p + 0.45.$$

The Routh-Hurwitz criteria demand that each of the coefficients $T_j(p)$ be positive for stability. This is clearly the case since $p \geq 0$. The last requirement for stability is that the penultimate Routh-Hurwitz determinant:

$$D(p) = T_2(p)T_1(p) - T_0(p)$$

be non-negative. If for some value of p, $D(p)$ vanishes while $T_j(p) > 0$, the eigenvalue equation $\chi(\lambda)$ has a pair of conjugate imaginary roots, $\pm im_0$, and a real negative root, $-r^2$. For our example we find:

$$D(p) = (p - p_0)^2(50p + 17); \qquad p_0 = 0.1.$$

Clearly, for all $p \geq 0$, $p \neq p_0$, $D(p) > 0$ and $T_j(p) > 0$, so that (4.23) is asymptotically stable to perturbations with $k \neq k_0 \pm \sqrt{p_0}$. For $p = p_0$, $D(p) = 0$ and $T_j(p_0) > 0$, so that there is an imaginary pair of eigenvalues, $\pm im_0$, with $m_0 \neq 0$. In fact, for $p = p_0$, we find

$$m_0 = 1.8125, \qquad -r^2 = -0.8.$$

One could actually compute the eigenvalues of $\chi(\lambda)$ exactly and verify Assumption A, but this is unnecessary; the above calculations are sufficient verification of the hypothesis. To the author's knowledge, this is the first example of a diffusion-induced instability at imaginary eigenvalues for a reaction-diffusion system. The use of the Routh-Hurwitz criteria presents a simple way to verify Assumption A since it only requires examining four polynomials in $p = k^2$ and, particularly, the form of $D(p)$.

Summarizing, we have presented an example of a reaction-diffusion scheme for which diffusion plays a major role in inducing pattern formation. In this 3×3-system, the appearance of nonscalar diffusion leads to a loss of stability of the rest state through a *complex* eigenvalue at a *nonzero* wave number, $k_0 = \sqrt{p_0}$. This is unlike previous examples (see e.g. [2, 5]) where stability is initially lost at the zero wave number $k_0 = 0$.

5. Stability of small-amplitude short waves. In the last section, we used a two-parameter series to construct small-amplitude short waves near a bifurcation point. Here we compute the linearized stability of these waves following the method proposed in [5] and explicitly carried out in [4].

To formulate the stability problem, we linearize (3.1) about the solution $U(z; a, q)$ given in (4.1a), obtaining:

$$\frac{\partial V}{\partial t} - N(U(z; a, q))V - D\frac{\partial^2 V}{\partial x^2} = 0, \tag{5.1a}$$

$$N(w)V \equiv (A + 2B^{(2)}(w, V) + 3B^{(3)}(w, w, V) + \cdots. \tag{5.1b}$$

Changing variables to the moving coordinate z leads to:

$$\frac{\partial V}{\partial t} + \bar{\omega}\frac{\partial V}{\partial z} - N(U(z; a, q))V - (k_0^2 + q)D\frac{\partial^2 V}{\partial z^2} = 0. \tag{5.2}$$

Since the coefficients of (5.2) are independent of t, we look for solutions of the form

$$V(z, t) = e^{\rho t}\hat{V}(z),$$

where $\hat{V}(z)$ is bounded. From the results of Floquet theory (see e.g. [5]), $\hat{V}(z)$ is of the form:

$$\hat{V}(z) = e^{i\kappa z}P(z)$$

where κ is real and $P(z)$ is 2π-periodic. Thus, we seek solutions to (5.2) of the form

$$V(z, t) = e^{\rho t}e^{i\kappa z}P(z). \tag{5.3}$$

If for some real value of κ, there is a ρ such that Re $\rho > 0$, then we say the solution $U(z; a, q)$ is linearly unstable; otherwise, the solution is said to be linearized stable. A necessary condition for linearized stability is that U be stable to perturbations with κ small. In the following we determine the stability condition for small κ. Substituting (5.3) into (5.2), we find that the relevant problem is:

$$\bar{\omega}\left(\frac{d}{dz} + i\kappa\right)P - N(U(z; a, q))P - (k_0^2 + q)D\left(\frac{d}{dz} + i\kappa\right)^2 P + \rho P = 0. \tag{5.4}$$

At $\kappa = 0$, (5.4) becomes:

$$\bar{\omega}\frac{dP}{dz} - N(U(z; a, q))P - (k_0^2 + q)D\frac{d^2P}{dz^2} + \rho P \equiv \mathscr{L}_0 P + \rho P = 0. \tag{5.5}$$

(5.5) has a solution $\rho = 0$ and $P(z) = U_z(z; a, q)$. Clearly, for stability we require that all of the remaining eigenvalues of L_0 remain in the left-half complex plane. In addition to the zero eigenvalue corresponding to neutral stability of the phase, there is another eigenvalue with a real part near zero (see [9] for a complete discussion of this point for periodic problems). We can compute this small eigenvalue.

Lemma 5.6. The problem (5.5) has a zero eigenvalue $\rho = 0$ and, for (a, q) sufficiently small, an eigenvalue:

$$\rho(a, q) \equiv \tilde{\rho}(a) = -2\gamma_{2,0}a^2 + O(a^3, qa^2), \tag{5.6}$$

in particular, this eigenvalue is q-independent to lowest order.

The proof of this lemma is in Appendix A. From (5.6), $\gamma_{2,0}$ must be nonnegative; this condition is equivalent to the requirement (3.25a) that $b_{1,3}$ be nonnegative. If (5.6) is to hold to order a^2, we demand that $\gamma_{2,0}$, and therefore $b_{1,3}$, are positive. The case $\gamma_{2,0} = 0$ will not be considered since it is then necessary to continue to higher orders. For a problem with quadratic or cubic terms, it is rare that $\gamma_{2,0} \equiv 0$. If we set $q = 0$ in the expansion for the bifurcation parameter γ, this condition implies supercritical bifurcation. As is typical in most bifurcation problems, supercritical bifurcation is necessary for stability (see [9] for a discussion of this point). We remark that the condition $\gamma_{0,2} > 0$ is implicit in Assumption A. As was discussed in the opening paragraphs of Sec. 4, the case $\gamma_{0,2} < 0$, leads to unstable waves, since there are always solutions to the linear equation $L_0 w = 0$ with positive real parts. It is now clear that the lowest-order expression for γ, (4.1c), describes an ellipse, $\gamma_{0,2}q^2 + \gamma_{2,0}a^2 = \gamma$, and so when γ is small, so are both a and q.

From the above lemma, we know that there is a zero eigenvalue for $\kappa = 0$; thus we examine how this changes with κ small. To determine this, we expand ρ and P in a series in κ:

$$\rho = 0 + \kappa\rho_1 + \kappa^2\rho_2, \qquad P = U_z(z; a, q) + \kappa P_1 + \kappa^2 P_2 + \cdots. \tag{5.7a, b}$$

Rather than explicitly performing the rather tedious computations, we rely on the stability condition calculated for general plane waves in Howard and Kopel [4]. We assume γ is a fixed parameter, thus determining the relationship between the amplitude a and the wavenumber q:

$$\gamma = \gamma_{2,0}a^2 + \gamma_{0,2}q^2. \tag{5.8}$$

Using (5.8), we can parametrize $U(z; a, q)$ and $\bar{\omega}(a, q)$ by the wave number q. If we let $\dot{U}$ denote the derivative of U with respect to q, we find

$$\frac{d\bar{\omega}}{dq}(a, q) \equiv \bar{\omega}' = m_q + 2a\dot{a}\zeta_{2,0} + 2q\zeta_{0,2} + \cdots,$$

$$\dot{U}(z; a, q) = \dot{a}\phi_1 + 2a\dot{a}w_{1,0} + aw_{0,1} + \dot{a}qw_{1,0} + \cdots, \tag{5.9}$$

where, by differentiating (5.8), the expression

$$2a\dot{a}\gamma_{2,0} + 2q\gamma_{0,2} = 0 \tag{5.10}$$

determines $\dot{a}$. Let $\mathscr{L}_0^*$ denote the adjoint of the operator $\mathscr{L}_0$:

$$\mathscr{L}_0^* V \equiv -\bar{\omega}\frac{dV}{dz} - N^T(U(z; a, q))V - (k_0^2 + q)D^T\frac{d^2V}{dz^2}. \tag{5.11}$$

We assume that $\mathscr{L}_0^*$ has a unique eigenfunction η^* with a zero eigenvalue such that:

$$\langle \eta^*, U_z(z; a, q)\rangle = a^2. \tag{5.12}$$

(5.12) is a convenient normalization and uniquely determines η^*. We may compute η^* as a function of a and q for small (a, q) using a simple perturbation scheme. The result of this is:

LEMMA 5.13.
(a) $\eta^* = a\{-\psi_2(z) + (f + ac_{1,0} + qc_{0,1})\psi_1(z) + a\eta_{1,0}(z) + q\eta_{0,1}(z) + \cdots\}$

where

(b) $\psi_1(z) = \operatorname{Re}\Psi_0 e^{iz}$, $\psi_2(z) = -\psi_1'(z)$,
(c) $\eta_{0,1}(z) = f\operatorname{Re}\Psi_q e^{iz} - Im\Psi_q e^{iz}$,
(d) $\eta_{1,0}(z) = f\operatorname{Re}\{\Psi^{(0)} + \Psi^{(2)}e^{2iz}\} - Im\{\Psi^{(0)} + \Psi^{(2)}e^{2iz}\}$,
(e) $f = b_{2,3}/b_{1,3}$,
(f) $c_{1,0}$ and $c_{0,1}$ arbitrary at this order.

(Note that Ψ_q, $\Psi^{(0)}$, $\Psi^{(2)}$, are defined in Appendix C.) We shall see that the terms $c_{1,0}$ and $c_{0,1}$ are unnecessary for the lowest-order stability results. This lemma is proved in Appendix C. With these preliminaries, we state the stability result of Howard and Kopell.

LEMMA 5.14 ([4], Eqs. A8–A10). The unique eigenvalue $\rho(\kappa)$ of (5.4) such that $\rho(0) = 0$, has the following form for sufficiently small κ:

$$\rho = 0 + i[-\bar{\omega} + 2(k_0^2 + q)\bar{\omega}']\kappa + k_0^2 T_1\kappa^2 + \cdots,$$

$$T_1 = -\langle \eta^*, DU_z(z; a, q)\rangle + 4(k_0^2 + q)\bar{\omega}\langle \eta^*, \dot{U}(z; a, q)\rangle$$
$$- 4(k_0^2 + q)\langle \eta^*, D\dot{U}_z(z; a, q)\rangle.$$

Thus for stability of the waves to low-wavenumber perturbations (κ small), $T_1 \le 0$. Substituting (5.9) and (5.13) into (5.14) and using the relationship from (5.10) that $a\dot{a}q = -q^2(\gamma_{0,2}/\gamma_{2,0})$, we obtain to order a^2, q^2, aq:

$$-\gamma_{2,0}a^2\left\{1 + \frac{b_{2,3}}{b_{1,3}}\frac{d_2}{d_1}\right\} + 2q^2\gamma_{0,2}\left\{1 + \left(\frac{b_{2,3}}{b_{1,3}}\right)^2 - \frac{b_{2,3}}{b_{1,3}}\frac{\zeta_{2,0}}{b_{1,3}}\right\} \le 0 \tag{5.15}$$

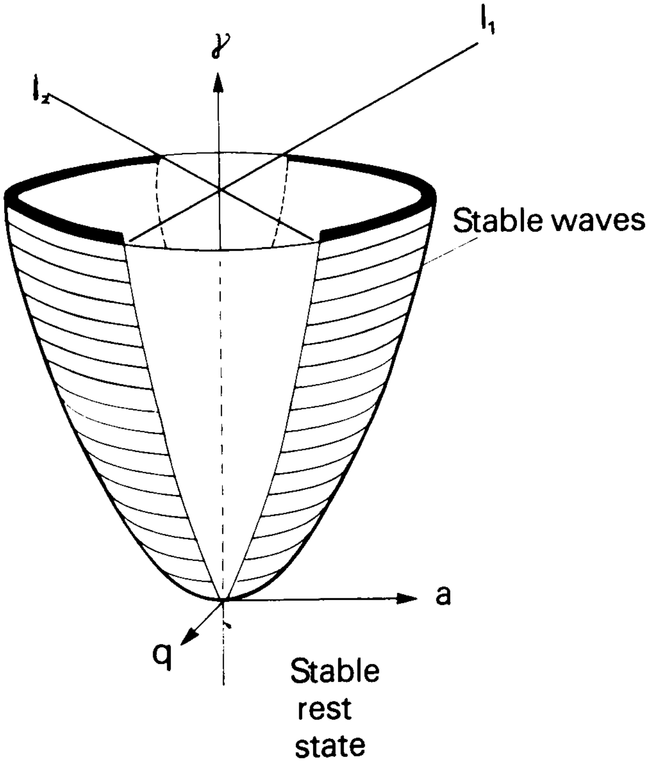

FIG. 4. Complete "bifurcation" picture for traveling waves. For each value of γ, there is an ellipse of values of q and a corresponding to traveling waves. Only a band of these is stable (shaded).

with $d_1 = -s_{qq}$ and $d_2 = -m_{qq} - m_q/8k_0^2$. This expression is quite similar to (3.21b) when one makes the identifications of $\mathscr{L}_j$ with d_j and the nonlinear terms $\omega'(r_0)$ and $\lambda'(r_0)$ with $b_{2,3}$ and $b_{1,3}$ respectively. If we combine (5.15) with the restriction (5.8) we observe the stability diagram shown in Fig. 4. The lines l_1 and l_2 represent solutions to $F(a, q) = 0$ where F is the left-hand side of (5.15). Projecting this figure onto the a-γ plane leads to Fig. 2, showing that the basic stability diagram of this very general system is qualitatively the same as that of the simplest λ-ω-system. Once again there is a restriction on both the amplitude and the wavenumber for stability.

The expansion of $\rho(\kappa)$ as given in Lemma 5.14 only holds for sufficiently small κ. Thus a natural question to ask is: how small? $\rho(\kappa)$ can be written in the form

$$\rho(\kappa) = C_1(a, q)\kappa + C_2(a, q)\kappa^2 + \cdots + C_j(a, q)\kappa^j + \cdots$$

where $C_j(a, q)$ are complex coefficients which depend on a and q. A trivial computation shows that the odd coefficients are imaginary and the even coefficients are real. The real part of $\rho(\kappa)$ is: $C_2(a, q)\kappa^2 + C_4(a, q)\kappa^4 + \cdots$. If the coefficient $C_4(a, q)$ is, for example, $O(1)$ in a and q (i.e. $C_4(0, 0) \neq 0$), then for (a, q) sufficiently small the fourth-order terms are important since $C_2(0, 0) = 0$. Thus the size of κ must somehow be restricted so as to prevent these problems near $(a, q) = (0, 0)$. To get bounds on the allowable range of κ, we introduce the natural scaling:

$$a = \alpha\sqrt{\gamma}, \qquad q = \beta\sqrt{\gamma}, \qquad \kappa = \delta\gamma^{\nu},$$

where α, β, δ are $O(1)$ in γ, and ν remains to be determined. Note that α, β are restricted by (5.8):

$$1 = \gamma_{2,0}\alpha^2 + \gamma_{0,2}\beta^2 + O(\gamma^{1/2}).$$

When these scales are substituted into the expression for $\rho(\kappa)$, we obtain:

$$\text{Re } \rho(\kappa) = \text{Re } \rho^*(\gamma) = \delta^2 C_2^*(\alpha, \beta)\gamma^{2\nu+1} + \delta^4 C_4^*(\gamma, \alpha, \beta)\gamma^{4\nu} + \cdots .$$

where C_2^* is $O(1)$ in γ and C_4^* is some polynomial in γ. By dividing through by $\delta^2\gamma^{2\nu+1}$, we obtain:

$$\delta^{-2}\gamma^{-2\nu-1} \text{ Re } \rho^*(\gamma) = C_2^*(\alpha, \beta) + O(\gamma^{2\nu-1}).$$

Thus, for validity we must have $\nu > \frac{1}{2}$ and the allowable range of perturbations in κ becomes vanishingly small as $\gamma \searrow 0$, i.e. $\kappa = \delta\gamma^{\nu}$ with $\nu > \frac{1}{2}$.

We have shown that there exist formally stable small-amplitude wavetrains which bifurcate from the rest state at a non-vanishing critical wavenumber k_0. Our results bear a close resemblance to those of Newell and Whitehead [6] in which two-parameter families of stationary periodic solutions were constructed. More recently, Fife [3] has proven an existence theorem for a two-parameter family of periodic stationary solutions, under Assumption A, where $m_0 = 0$. This corresponds to a bifurcation at a nonzero wavenumber through a real eigenvalue. While this case was not studied in the present paper, the results of Secs. 4 and 5 can be readily adapted. Indeed, since the maximal eigenvalue is assumed real for small $|k^2 - k_0^2| = |q|$, $m(\gamma, k^2) \equiv 0$. In this case $\zeta_{0,2}$, $\zeta_{2,0}$, $b_{2,3}$, and d_2 vanish, the solutions are:

$$U(x) = a[\Phi_0 \cos((k_0^2 + q)^{1/2}x) + \cdots], \qquad \gamma = a^2\gamma_{2,0} + q^2\gamma_{0,2} + \cdots ,$$

and these are stable as long as:

$$2\gamma_{0,2}q^2 - \gamma_{2,0}a^2 \leq 0.$$

This represents a one-parameter family of stable stationary periodic structures for each fixed γ. Thus, the case of stationary periodic patterns can be analyzed in the same manner as the traveling periodic patterns.

In computing the stability of these waves, we have made a serious symmetry restriction: we have excluded the possibility of standing oscillatory patterns. These are of the form:

$$a \text{ Re}\{\Phi_0 e^{i(\omega t - kx)} + \Phi_0 e^{i(\omega t + kx)}\} + \cdots$$

In order to select between the traveling waves and the standing waves, a complete unrestricted bifurcation analysis is necessary. By restricting our analysis to traveling waves we often are led to incorrect stability results with regard to the full bifurcation problem.

We sketch the arguments for the stability of the wave with maximal amplitude, $q = 0$, $a = (\gamma/\gamma_{2,0})^{1/2}$. Assuming solutions of the form $u \sim \sum_{i=1}^{4} z_i\phi_i$, where z_i are complex numbers:

$$\phi_1 = \tfrac{1}{2}\Phi_0 e^{i(m_0 t + k_0 x)}, \qquad \phi_2 = \tfrac{1}{2}\Phi_0 e^{-i(m_0 t - k_0 x)}, \qquad \phi_3 = \bar{\phi}_1, \phi_4 = \bar{\phi}_2$$

and $z_i = \bar{z}_j$ if $\phi_i = \bar{\phi}_j$, the bifurcation equations for this system can be computed.

In Appendix D, we verify that these equations have the form:

$$\begin{aligned} r_1(d\hat{\gamma} + i\hat{\omega} + Q_1 r_1^2 + Q_2 r_2^2) &= 0, \\ r_2(d\hat{\gamma} + i\hat{\omega} + Q_1 r_2^2 + Q_2 r_1^2) &= 0 \end{aligned} \qquad z_j = r_j e^{i\theta_j},\ \theta_j \text{ arbitrary}, \tag{5.16}$$

where

$$d = s_\gamma + im_\gamma, \qquad Q_1 = -b_{1,3} - ib_{2,3}, \qquad Q_2 = -C_{1,3} - iC_{2,3}. \tag{5.17}$$

The expressions for $C_{1,3}$ and $C_{2,3}$ are in the appendix, while those for d and $b_{j,3}$ are given in Sec. 4. The solutions analogous to those in Sec. 4 with $q = 0$ are $r_1 = 1$, $r_2 = 0$, whence we find

$$\hat{\gamma} = b_{1,3}/s_\gamma, \qquad \hat{\omega} = b_{2,3} - m_\gamma \hat{\gamma}.$$

These agree with the lowest-order terms for $\hat{\gamma}$ and $\hat{\omega}$ in (4.14) when we set $q = 0$. In addition to $r_1 = 1$, $r_2 = 0$, there are also solutions of the form $r_1 = r_2 = \frac{1}{2}$ which correspond to standing patterns. Only one of these distinct possibilities is generally possible. Thus, we examine the stability of the traveling wave against both standing- and traveling-wave perturbations. To do this, we linearize (5.16) about the solution $r_1 = 1$, $r_2 = 0$. This two-by-two matrix has eigenvalues $2Q_1$ and $Q_2 - Q_1$; thus for stability we require:

$$b_{1,3} > 0 \quad \text{and} \quad b_{1,3} < C_{1,3}.$$

The condition on the relative sizes of $C_{1,3}$ and $b_{1,3}$ was ignored in the stability analysis of Sec. 4 because only perturbations of the form $e^{i\kappa\zeta}$ were examined. As a particular example, if there are no quadratic terms in the system, then from the appendix $b_{1,3} = \frac{1}{2}C_{1,3}$ and (5.18) always holds. More generally, if (5.18) is violated, we do not expect always to observe stable traveling waves. Thus, we cannot ignore the possibility of standing waves near an instability with $k_0 \neq 0$ and $m_0 \neq 0$.

6. Discussion and conclusions. The results of this paper indicate that there are in fact stable small-amplitude solutions to reaction-diffusion systems which "bifurcate" from a homogeneous rest state. The actual computation of these waves can be done using a two-parameter series expansion or a formal multi-scaling technique. When there is a Hopf bifurcation of the reaction kinetics, we expect a one-parameter family of stable long waves for each fixed small value of the bifurcation parameter. This is qualitatively similar to the situation occurring near a large-amplitude limit cycle. In both [5] and [7] existence of such long waves near limit cycles was established. In simple λ-ω-systems, we can explicitly demonstrate these long waves and establish their stability. For more general systems with scalar diffusion, the multi-scaling procedure enables us to reduce the problem to that of finding stable solutions to a λ-ω-system.

For systems with nonscalar diffusion, the possibility exists for a Hopf bifurcation at a nonzero wavenumber k_0. Such a bifurcation can lead to a stable (at least to long-wave perturbations) family of plane waves for each fixed value of γ as long as certain additional requirements are satisfied. The first of these is shown in Fig. 3c and corresponds to an assumption about the *maximal* eigenvalue of the linearized equations. The second assumption requires supercritical bifurcation with respect to the amplitude. The major point is that the lowest order expansion of the bifurcation parameter must be a positive definite form in the two variables a and q. This resembles the requirement in λ-ω-systems of positive diffusion and that $\lambda'(r_0) < 0$. Dispersive effects (those involving $\bar{\omega}(a, q)$) play an important role in determining the stability of solutions with amplitude a and wave number $(k_0^2 + q)^{1/2}$. In absence of these dispersive effects, the stability requirement depends only on the relative sizes of $\gamma_{2,0}$ and $\gamma_{0,2}$. In this case, the stability results are identical to those obtained in section 2 for a simple λ-ω-system with no dispersion $(\tilde{\omega}(r) \equiv \omega_0)$.

While we studied only a simple class of reaction-diffusion models, we expect these results to hold for much more general systems. To this end, we have derived a system of modulation equations for a nonlinear, homogeneous, isotropic, stationary medium and under certain restrictions the equations reduce to the λ-ω-system described in Sec. 3. It is hoped that these results will answer some of the many questions on the stability of waves in biological, chemical, and physical systems.

Appendix A. Eigenvalues of $\mathscr{L}_0$. We wish to study the eigenvalue problem:

$$\mathscr{L}_0 V = -\rho V \tag{A1}$$

where $\mathscr{L}_0$ is as in (5.5). V and ρ are typically functions of a and q, so it is natural to expand in powers of a and q. Much of this computation is simplified by observing that when $a \equiv 0$, $U(z; 0, q)$ is an eigenfuction of $\mathscr{L}_0$ with *zero* eigenvalue. Thus $\rho \equiv \rho(a, q)$ vanishes when $a = 0$. The periodic symmetry of the eigenfunctions ϕ_1, ϕ_2, imply, as in Sec. 4, that all terms of the form aq^n vanish as well. Thus $\rho(a, q)$ is of the form $\rho_0 a^2$ + terms in a^3, a^2q^2, and so on. To lowest order we can with no loss in generality study the q-independent problem. We follow Sattinger [8] and look for solutions of the form:

$$\begin{aligned} &V = a\{c(a)[\phi_1' + aw_{1,0}' + \cdots] + \chi\}, \\ &\rho = a^2\{\rho_0 + a\rho_1 + \cdots\}, \qquad c = c_0 + ac_1 + \cdots, \\ &\chi = \phi_1 + a\hat{\chi}, \langle \psi_1, \hat{\chi}\rangle = \langle \psi_2, \hat{\chi}\rangle = 0, \qquad \hat{\chi} = \chi_1 + a\chi_2 + \cdots . \end{aligned} \tag{A2}$$

Substituting this into (A1) and setting $q = 0$ leads to the following equations:

$$L_0\chi_1 = 2B^{(2)}(\phi_1, \phi_1), \tag{A3a}$$

$$\begin{aligned} L_0\chi_2 = {} & 3B^{(3)}(\phi_1, \phi_0, \phi_1) + 2B^{(2)}(w_{1,0}, \phi_1) + 2B^{(2)}(\phi_1, \chi_1) \\ & + \gamma_{2,0}B\phi_1 - \zeta_{2,0}\phi_1' - \rho_0 c_0 \phi_1' - \rho_0\phi_1 \end{aligned} \tag{A3b}$$

(A3a) has a solution since the right-hand side is always orthogonal to the two eigenfunctions of L_0, ψ_1 and ψ_2. We find:

$$\chi_1 = 2w_{1,0}. \tag{A4}$$

Applying the orthogonality principle to (A3b) we find:

$$-\rho_0 + \langle \psi_1, 3B^{(3)}(\phi_1, \phi_1, \phi_1) + 6B^{(2)}(w_{1,0}, \phi_1) + \gamma_{2,0}B\phi_1\rangle = 0, \tag{A5a}$$

$$\rho_0 c_0 + \zeta_{2,0} + \langle \psi_2, 3B^{(3)}(\phi_1, \phi_1, \phi_1) + 2B^{(2)}(w_{1,0}, \phi) + \gamma_{2,0}B\phi_1\rangle. \tag{A5b}$$

Using the fact that $\langle \psi_1, B^{(3)}(\phi_1, \phi_1, \phi_1) + 2B^{(2)}(w_{1,0}, \phi_1)\rangle = -\gamma_{2,0}$, we obtain:

$$\rho_0 = -2\gamma_{2,0}S_{;}. \tag{A6}$$

Since $\rho_0 \neq 0$, c_0 may be similarly obtained. We have shown that the small eigenvalue of $\mathscr{L}_0^*$ is:

$$\rho(a, q) = 0 - 2\gamma_{2,0} s_\gamma a^2 + a^2 O(a, q); \tag{A7}$$

thus stability occurs if and only if $\gamma_{2,0} > 0$.

Appendix B. Computation of the adjoint eigenfunction η^*. We must solve the following equation:

$$\mathscr{L}_0^* \eta^* \equiv -\bar{\omega}\eta_z^* - N^T(U(z; a, q), \eta^*) - (k_0^2 + q)D^T \eta_{zz}^* = 0;$$
$$N^T(u, \eta^*) = (A^T + \gamma B^T)\eta^* + 2B^{(2)T}(U(z; a, q), \eta^*) \tag{B1}$$
$$+ 3B^{(3)T}(U(z; a, q), U(z; a, q), \eta^*)$$

subject to the normalization condition:

$$\langle \eta^*, U' \rangle = a^2; \qquad ' = d/dz. \tag{B2}$$

Expanding η^* in a series in a and q:

$$\eta^* = a\{\eta_0 + a\eta_{1,0} + q\eta_{0,1} + a^2\eta_{2,0} + q^2\eta_{0,2} + aq\eta_{1,1} + \cdots\} \tag{B3}$$

and substituting this into (B1–B2) lead to the following equations:

$$L_0^*\eta_0 \equiv -\omega_0\eta_0' - A^T - D^T\eta_0'' = 0; \qquad \langle \eta_0, \phi_1' \rangle = 0 \tag{B4a}$$

$$L_0^*\eta_{1,0} = 2B^{(2)T}(\phi_1, \eta_0); \qquad \langle \eta_0, w_{1,0}' \rangle + \langle \eta_{1,0}, \phi_1' \rangle = 0 \tag{B4b}$$

$$L_0^*\eta_{0,1} = \omega_q\eta_0' + D^T\eta_0''; \qquad \langle \eta_0, w_{0,1}' \rangle + \langle \eta_{0,1}, \phi_1' \rangle = 0 \tag{B4c}$$

$$L_0^*\eta_{2,0} = 3B^{(3)T}(\phi_1, \phi_1, \eta_0) + \gamma_{2,0}B^T\eta_0 + 2B^{(2)T}(\phi_1, \eta_{1,0})$$
$$+ 2B^{(2)T}(w_{1,0}, \eta_0); \qquad \langle \eta_0, w_{0,2}' \rangle + \langle \eta_{1,0} w_{1,0}' \rangle + \cdots = 0 \tag{B4d}$$

Each of these is of the form $L_0^* w = g$ and thus has a solution if and only if g is orthogonal to the nullspace of $(L_0^*)^* = L_0$. So we must have $\langle \phi_1, g \rangle = \langle \phi_2, g \rangle = 0$. The solution to (B4a) is:

$$\eta_0 = -\psi_2 + f\psi_1 \tag{B5}$$

where f is an arbitrary constant determined in higher orders. Turning to (B4b), we must solve:

$$L_0^*\eta_{1,0} = 2fB^{(2)T}(\phi_1, \psi_1) - 2B^{(2)T}(\phi_1, \psi_2).$$

The right-hand side is orthogonal to ϕ_1 and ϕ_2 so this has a solution:

$$\eta_{1,0} = -2Im[\Psi^{(0)} + \Psi^{(2)}e^{2iz}] + f2\,\mathrm{Re}[\Psi^{(0)} + \Psi^{(2)}e^{2iz}] + c_{1,0}\psi_1, \tag{B6a}$$

$$-A\Psi^{(0)} = \tfrac{1}{2}B^{(2)T}(\Phi_0, \bar{\Psi}_0), \tag{B6b}$$

$$H^{(2)}\Psi^{(2)} \equiv (-2i\omega_0 - A^T - 4k_0^2 D^T)\Psi^{(2)} = \tfrac{1}{2}B^{(2)T}(\Phi_0, \Psi_0). \tag{B6c}$$

Again, $c_{1,0}$ is an arbitrary constant determined at some higher order. The right-hand side of (B4c) is orthogonal to ϕ_1 and ϕ_2 and $\eta_{0,1}$ is readily determined:

$$\eta_{0,1} = -Im\Psi_q e^{iz} + f\,\mathrm{Re}\,\Psi_q e^{iz} + c_{0,1}\psi_1 \tag{B7}$$

where Ψ_q is a solution to the algebraic equation:

$$(H_0^T + i\omega_0)\Psi_q = D^T\Psi_0 - i\omega_0\Psi_0$$

and $c_{0,1}$ is an arbitrary constant. f is found by applying the orthogonality condition to (B4d):

$$\langle \phi_j, 3B^{(3)T}(\phi_1, \phi_1, -\psi_2 + f\psi_1) + \gamma_{2,0}B^T(-\psi_2 + f\psi_1) + 2B^{(2)T}(\phi_1, \eta_{1,0}) + 2B^{(2)T}(w_{1,0}, -\psi_2 + f\psi_1)\rangle = 0, \qquad j = 1, 2. \tag{B8}$$

To evaluate the expressions in (B8) we note, for example, that:

$$\langle \psi_1, B^{(3)T}(\phi_1, \phi_1, \psi_2)\rangle = \langle B^{(3)}(\phi_1, \phi_1, \phi_1), \psi_2\rangle,$$
$$\langle \phi_2, B^{(3)T}(\phi_1, \phi_1, \psi_2)\rangle = \langle B^{(3)}(\phi_1, \phi_1, \phi_2)\psi_2\rangle = \tfrac{1}{3}\langle \psi_1, B^{(3)}(\phi_1, \phi_1, \phi_1)\rangle. \tag{B9}$$

The quadratic terms are slightly more difficult. The terms of the form $\langle \phi_k, B^{(2)T}(w_{1,0}, \psi_j)\rangle$ can be computed as in (B9). This leaves us with terms like

$$\langle \phi_1, B^{(2)T}(\phi_1, \eta_1^*)\rangle,$$

where $\eta_1^* \equiv \mathrm{Re}[\Psi^{(0)} + \Psi^{(2)}e^{2iz}]$. η_1^* and $w_{1,0}$ satisfy respectively:

$$L_0^*\eta_1^* = B^{(2)T}(\phi_1, \psi_1), \qquad L_0 w_{1,0} = B^{(2)}(\phi_1, \phi_1) \tag{B10a, b}$$

We apply $w_{1,0}$ to both sides of (B10a), η_1^* to both sides of (B10b) and use the definition of the adjoint to show:

$$\langle \phi_1, B^{(2)T}(\phi_1, \eta_1^*)\rangle = \langle \psi_1, B^{(2)}(w_{1,0}, \phi_1)\rangle.$$

Using this and similar relationships, we obtain:

$$f = b_{2,3}/b_{1,3}. \tag{B11}$$

It is unnecessary to continue with further terms since $c_{1,0}$ and $c_{0,1}$ do not appear in the lowest-order stability expression.

Summarizing, we have shown:

$$\eta^* = a\{-\psi_2 + (f + c_{1,0}a + c_{0,1}q)\psi_1 + q\eta_{0,1} + a\eta_{1,0} + \cdots\}, \tag{B12a}$$

$$f = b_{2,3}/b_{1,3}, \tag{B12b}$$

$$\eta_{0,1} = f\eta^1 - \eta^2; \qquad L_0^*\eta^j = L_1^*\psi_j \equiv \omega_q\psi_j' + D^T\psi_j''. \tag{B12c}$$

Appendix C. Computation of the full bifurcation equations. As is the usual case in Hopf bifurcation problems, we expect the frequency to change as the amplitude a increases. Thus, in addition to the parameter $\gamma(a)$, we introduce the frequency $\omega(a)$, and let $\tau = \omega(a)t$, $y = k_0 x$. We seek solutions to (3.1) which are 2π-periodic in τ and y. Substituting the new time and space variables into (3.1) yields:

$$\left(\omega(a)\frac{\partial}{\partial \tau} - k_0^2 D\frac{\partial^2}{\partial y^2} - A\right)u = \gamma(a)Bu + B^{(2)}(u, u) + B^{(3)}(u, u, u) + \cdots \tag{C1}$$

We expand u, ω, and γ in a power series in the amplitude a:

$$u = aU_0 + a^2U_1 + \cdots, \qquad \omega(a) = m_0 + a^2\hat{\omega} + \cdots, \qquad \gamma(a) = a^2\hat{\gamma} + \cdots. \tag{C2}$$

Solutions which are 2π-periodic in y and τ are sought with the amplitude normalization:

$$\|u(y, \tau)\|_2^2 \equiv \frac{1}{2\pi^2}\int_0^{2\pi} d\tau \int_0^{2\pi} dy(u(y, \tau), u(y, \tau)) = a^2.$$

Again (,) is the inner product on $\mathbb{C}^n$. Substituting (C2) into (C1) produces a sequence of linear inhomogeneous equations:

$$L_0 U_0 = 0, \qquad L_0 U_1 = B^{(2)}(U_0, U_0) \tag{C4a, b}$$

$$L_0 U_2 = -\hat{\omega}\frac{\partial U_0}{\partial \tau} + \hat{\gamma} B U_0 + 2B^{(2)}(U_0, U_1) + B^{(3)}(U_0, U_0, U_0) \tag{C4c}$$

The solution to (C4a) is:

$$U_0 - Z_1\eta_1 + Z_2\eta_2 + \bar{Z}_1\bar{\eta}_1 + \bar{Z}_2\bar{\eta}_2 \tag{C5}$$

where

$$\eta_1 = \tfrac{1}{2}\Phi_0 e^{i(\tau - y)}, \qquad \eta_2 = \tfrac{1}{2}\Phi_0 e^{i(\tau + y)}$$

and Z_1, Z_2 are arbitrary complex numbers determined by (C3) and the remaining equations (C4b, c). L_0, Φ_0 are the same as in Sec. 4. Each of the remaining equations (C4b, c ...) is of the form $L_0 u = f$. Since L_0 has a nontrivial nullspace, a continuous solution 2π-periodic in τ and y exists if and only if f is orthogonal to the adjoint eigenfunctions of L_0; η_1^*, η_2^*, $\bar{\eta}_1^*$, $\bar{\eta}_2^*$. Here the η_j^* satisfy:

$$L_0^*\eta_j^* = 0$$

and are given by:

$$\eta_1^*(y, \tau) = \Psi_0 e^{i(\tau - y)}, \qquad \eta_2^*(y, \tau) = \Psi_0 e^{i(\tau + y)}.$$

Note that $\langle \eta_j, \eta_k^* \rangle = \delta_{ij}$.

For (C4b), this orthogonality requirement automatically holds and the solution to (C4b) is:

$$\begin{aligned} U_1(y, \tau) = {} & \Phi_{(0,0)}(Z_1\bar{Z}_1 + Z_2\bar{Z}_2) + \Phi_{(0,2)}e^{2iy}Z_2\bar{Z}_1 \\ & + \Phi_{(0,2)}e^{-2iy}Z_1\bar{Z}_2 + 2\Phi_{(2,0)}e^{2i\tau}Z_1 Z_2 \\ & + \Phi_{(2,2)}[e^{i(2\tau - 2y)}Z_1 Z_1 + e^{i(2\tau + 2y)}Z_2 Z_2] + \text{complex conjugates.} \end{aligned} \tag{C6}$$

The $\Phi_{(k,j)}$ satisfy:

$$\begin{aligned} & -A\Phi_{(0,0)} = \tfrac{1}{4}B^{(2)}(\Phi_0, \bar{\Phi}_0), \qquad (2im_0 - 4k_0^2 D - A)\Phi_{(2,2)} = \tfrac{1}{4}B^{(2)}(\Phi_0, \Phi_0), \\ & (-4k_0^2 D - A)\Phi_{(0,2)} = \tfrac{1}{4}B^{(2)}(\Phi_0, \bar{\Phi}_0), \qquad (2im_0 - A)\Phi_{(2,0)} = \tfrac{1}{4}B^{(2)}(\Phi_0, \Phi_0). \end{aligned} \tag{C7}$$

The amplitude normalization (C3) implies there are no additional terms of the form $c_j\eta_j$ in (C6).

Applying the orthogonality requirement to (C4c) and using the expression in (C6) for U_1 yields the following equations:

$$\begin{aligned} & (i\hat{\omega} + \hat{\gamma}(\Psi_0, B\Phi_0) + Z_1\bar{Z}_1 Q_1 + Z_2\bar{Z}_2 Q_2)Z_1 = 0, \\ & (i\hat{\omega} + \hat{\gamma}(\Psi_0, B\Phi_0) + Z_2\bar{Z}_2 Q_1 + Z_1\bar{Z}_1 Q_2)Z_2 = 0 \end{aligned} \tag{C8}$$

and their complex conjugates, where

$$Q_1 = (4B^{(2)}(\Phi_0, \Phi_{(0,0)} + \tfrac{1}{2}\Phi_{(2,2)} + \tfrac{3}{4}B^{(3)}(\Phi_0, \Phi_0, \bar{\Phi}_0), \bar{\Psi}_0),$$
$$Q_2 = (4B^{(2)}(\Phi_0, \Phi_{(0,0)} + \Phi_{(0,2)} + \Phi_{(2,0)} + \tfrac{3}{2}B^{(3)}(\Phi_0, \Phi_0, \bar{\Phi}_0), \bar{\Psi}_0). \tag{C9}$$

(C9) is a system of four equations for the unknowns $\bar{Z}_j$, Z_j. We can considerably simplify them by writing:

$$Z_j = r_j e^{i\theta_j}, \qquad r_j \geq 0, \qquad \theta_j \in (0, 2\pi).$$

Substituting this into (C8) yields:

$$r_1(d\hat{\gamma} + i\hat{\omega} + Q_1 r_1^2 + Q_2 r_2^2) = 0, \qquad r_2(d\hat{\gamma} + i\hat{\omega} + Q_1 r_2^2 + Q_2 r_1^2) = 0. \tag{C10}$$

Comparing the expression for Q_1 with that in (4.17), we see that $Q_1 = -(b_{1,3} + ib_{2,3})$. Furthermore, in absence of quadratic terms, note that $Q_2 = 2Q_1$.

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