# STABLE VECTOR BUNDLES ON ALGEBRAIC SURFACES   II

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This paper is a continuation of "Stable vector bundles on algebraic surfaces" [10].   For simplicity we deal with non-singular projective varieties over the field of complex numbers. Let $W$ be a variety whose fundamental group is solvable, let $H$ be an ample line bundle on $W$, and let $f \colon V \to W$ be an unramified covering.   Then we show in section 1 that if $E$ is an $f^*H$-stable vector bundle on $V$ then $f_*E$ is a direct sum of $H$-stable vector bundles.   In particular $f_*L$ is a direct sum of simple vector bundles if $L$ is a line bundle on $V$.   This result is a corollary of the following: Let $A$ be a finite solvable group of automorphisms of a variety $V$.   Suppose $A$ acts freely on $V$.   Let $W$ be the quotient of $V$ by $A$ and let $f$ be the natural morphism $V \to W$.   Then the direct image of an $f^*H$-stable vector bundle on $V$ by $f$ is a direct sum of $H$-stable vector bundles, and the inverse image of an $H$-stable vector bundle on $W$ by $f$ is a direct sum of $f^*H$-stable vector bundles.   In section 2 we prove the independence of $H$ in the definition of the $H$-stability.   Namely, let $S$ be a relatively minimal surface, and let $E$ be a vector bundle of rank two on $S$ with $c_1^2(E) \geqq 4c_2(E)$.   Then $E$ is $H$-stable if and only if $E$ is $H'$-stable, where $H$ and $H'$ are ample line bundles on $S$.   We have proven this in our previous paper [10] in case $c_1^2(E) > 4c_2(E)$ without the assumption of relative minimality of $S$, and we obtained several results about $H$-stable vector bundles $E$ with $c_1^2(E) = 4c_2(E)$ [10].   For instance, an $H$-stable vector bundle with $c_1^2 = 4c_2$ of rank two on an abelian surface is the direct image of a line bundle under an isogeny of a special type. And an $H$-stable vector bundle with $c_1^2 = 4c_2$ of rank two on a geometrically ruled surface is the vector bundle induced from a stable vector bundle on the base curve tensored with a line bundle on the surface.   In connection with these results, we show in section 4 that on an elliptic bundle the vector bundle induced from a stable vector bundle of rank

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two on the base curve is $H$-stable (in case $c_1^2 = 4c_2$). For its proof we give in section 3 the necessary and sufficiently condition about the ampleness of a line bundle on an elliptic bundle: A line bundle $L$ is ample if and only if $(L^2) > 0$ and $(L, C) > 0$ where $C$ is a fibre. In section 5 we study $H$-stable vector bundles on an elliptic bundle of a special type i.e. a hyperelliptic surface. Let $A(S)$ be the set of all $H$-stable vector bundles of rank two with $c_1^2 = 4c_2$ on a hyperelliptic surface $S$, let $B(S)$ be the set of all indecomposable vector bundles of rank two on $S$ each of which is the direct image of a line bundle under an unramified covering, and let $C(S)$ be the set of all vector bundles on $S$ each of which is the tensor product of a line bundle on $S$ and the vector bundle induced from a stable vector bundle of rank two on the base curve. Then the following holds: 1) $A(S) \supset B(S) \supset C(S)$. 2) If $S$ is basic, then $A(S) \neq C(S)$. 3) If $4K_S = 0$, then $A(S) = B(S)$. 4) If $4K_S \neq 0$, then 4. a) in case $3K_S = 0$ with $S$ basic, we have $B(S) = C(S)$, and 4. b) in case $3K_S \neq 0$ we have $B(S) \neq C(S)$, but $B(S) = C(S)$ under a suitable restriction about vector bundles. Here $K_S$ is the canonical line bundle on $S$.

## 1. The direct image of a line bundle under an unramified covering

All the varieties considered below will be assumed to be over an algebraically closed field $k$, non-singular, projective and irreducible. And all the sheaves will be assumed to be coherent.

DEFINITION. A torsion-free sheaf $F$ of finite rank on a variety $V$ is *quasi locally free* if $\operatorname{depth}_S F \geq 2$ for any closed subvariety $S$ of $V$ with $\operatorname{codim}_V S \geq 2$.

*Remark* 1). Let $F$ and $S$ be as above. We put $U = V - S$. Then the restriction homomorphism $H^0(V, F) \to H^0(U, F)$ is bijective. Hence for any coherent sheaf $G$ on $V$, $H^0(V, \operatorname{Hom}(G, F)) \xrightarrow{\sim} H^0(U, \operatorname{Hom}(G, F))$.

*Remark* 2). Let $f : W \to V$ be an unramified covering and let $F$ be quasi locally free on $V$. Then $f^*F$ is also quasi locally free.

LEMMA (1.1). *Let $F$ be a torsion-free sheaf of finite rank on $V$. Then there exists a quasi locally free sheaf $G$ such that $F \subset G$ and $\operatorname{codim}(\operatorname{Supp}(G/F)) \geq 2$.*

*Proof.* We put $U = \{x \in V \mid F$ is locally free at $x\}$ and $Y = V - U$.

$Y$ is closed in $V$ and $\operatorname{codim}_V Y \geq 2$. Let $i : U \to V$ be the natural morphism. We put $G = i_* i^* F$. It is clear that $F \subset G$ and $\operatorname{codim}(\operatorname{Supp}(G/F)) \geq 2$. Since $G = i_* i^* G$ we have $H^0_Y(G) = H^1_Y(G) \doteq 0$, hence $\operatorname{depth}_Y G \geq 2$. This proves the lemma by the definition of $U$.

*Remark.* If $F$ is a coherent subsheaf of a vector bundle $E$, then above $G$ is a coherent subsheaf of $E$.

Let $H$ be an ample line bundle on a variety $V$. For any non-torsion coherent sheaf $F$, we put $x(F, H) = d(F, H)/r(F)$ where $d(F, H) = (\operatorname{Inv}(F), H^{s-1})$ and $r(F) = \operatorname{rank} F$ $(s = \dim V)$.

DEFINITION. A non-torsion coherent sheaf $F$ on $V$ is *H-stable* (resp. *H-semi-stable*) if every non-torsion coherent subsheaf $G$ of $F$ with $r(G) < r(F)$ we have $x(G, H) < x(F, H)$ (resp. $x(G, H) \leq x(F, H)$).

This is a generalization of the definition of $H$-stability (resp. $H$-semi-stability) given in [10] for vector bundles.

PROPOSITION (1.2). *Let $E$ and $F$ be $H$-stable quasi locally free sheaves of finite rank with $r(E) = r(F)$ and $d(E, H) = d(F, H)$. If $f : E \to F$ is a non-zero homomorphism, then $f$ is an isomorphism.*

*Proof.* Put $G = \operatorname{Image}$ of $f$. By definition we have $x(E, H) \leq x(G, H) \leq x(F, H)$. By assumption we readily have $r(E) = r(G)$, hence $f$ is injective, i.e. $E \xrightarrow{\sim} G$. We put $\operatorname{Supp}(F/E) = S$. Since $F/E$ is torsion and $x(F/E, H) = 0$, we have $\operatorname{codim} S \geq 2$. Since $f$ is an isomorphism on $V - S, f$ is an isomorphism on $V$ by Remark 1).

COROLLARY (1.3). *Let $E$ be an $H$-stable quasi locally free sheaf of finite rank. Then $E$ is simple. i.e. $H^0(V, \operatorname{Hom}(E, E)) = k$.*

*Remark.* Prop. (1.2) and Cor. (1.3) are generalizations of Prop. (1.7) and Cor. (1.8) in [10].

LEMMA (1.4). *Let $E_1$ and $E_2$ be $H$-semi-stable vector bundles on $V$ with $x(E_1, H) = x(E_2, H)$. Then an extension $E$ of $E_2$ by $E_1$ is $H$-semi-stable with $x(E, H) = X(E_i, H)$.*

*Proof.* Suppose $E$ is not $H$-semi-stable. There exists a coherent subsheaf $F$ of $E$ such that $x(F, H) > x(E, H)$. Let $f$ be the natural homomorphism $F \to E \to E_2$. Since $x(F, H) > x(E, H) = x(E_1, H)$, $f$ is non-zero by the $H$-semi-stability of $E_1$. We put $F_1 = \operatorname{kernel}(f)$ and $F_2 =$

image $(f)$.   Case 1) If $x(F_1, H) \geq x(F, H)$, then $x(F_1, H) > x(E_1, H)$ which contradicts the $H$-semi-stability of $E_1$ since $F_1 \subset E_1$.   Case 2) If $x(F_1, H) < x(F, H)$, then $x(F_2, H) > x(F, H) > x(E, H) = x(E_2, H)$ which contradicts the $H$-semi-stability of $E_2$.                                    q.e.d.

Let $A$ be a finite group of automorphisms of $V$.  Suppose $A$ acts freely on $V$.  Let $W$ be the quotient of $V$ by $A$ and let $f$ be the natural morphism $V \to W$ which is an unramified covering by assumption.  These notations remain fixed in this section.  We remark that for any non-torsion coherent sheaf $F$ on $W$ $x(f^*F, f^*H) = \deg f \cdot x(F, H)$.  Hence if $f^*F$ is $f^*H$-semi-stable, then $F$ is $H$-semi-stable.

PROPOSITION (1.5).  *Let $E$ be an $f^*H$-semi-stable vector bundle on $V$ where $H$ is an ample line bundle on $W$.  Then $f_*(E)$ is $H$-semi-stable.*

*Proof.*  Since $f^*f_*(E) = \bigoplus_{a \in A} a^*(E)$, $f^*f_*(E)$ is $f^*H$-semi-stable by Lemma (1.4).  We have the desired result by the above remark.

LEMMA (1.6).  *Assume $A$ is a cyclic group of prime order $l$ which is different from the characteristic of $k$.  Let $E$ be an $f^*H$-stable vector bundle on $V$ for an ample line bundle $H$ on $W$.  If $E$ is not isomorphic to $f^*E_1$ for any vector bundle $E_1$ on $W$, then $f_*(E)$ is $H$-stable.*

*Proof.*  Suppose $f_*E$ is not $H$-stable.  Since $f_*E$ is $H$-semi-stable by Prop. (1.5), there exists a quasi locally free subsheaf $F$ of $f_*E$ such that $x(F, H) = x(f_*E, H)$ with $r(F) < r(f_*E)$.  We may assume $F$ is $H$-stable by taking such $F$ with the smallest $r(F)$.  Since $f^*f_*E = \bigoplus_{a \in A} a^*E$ and $a^*f^*F = f^*F$ for any $a \in A$, we have a non-zero homomorphism $\alpha: f^*F \to E$.  $f^*F$ is $f^*H$-semi-stable quasi locally free because $x(f^*F, f^*H) = x(f^*f_*E, f^*H)$ and $f^*f_*E$ is $f^*H$-semi-stable.  We put $G_0 = f^*F$ and $\alpha_0 = \alpha$.  We fix a generator $a$ of $A$.  Inductively we define $f^*H$-semi-stable coherent subsheaves $G_i$ of $f^*F$ with $r(G_i) = r(F) - ir(E)$ and $x(G_i, f^*H) = x(f^*F, f^*H)$ such that the natural homomorphism $\alpha_i: G_i \to f^*F \to (a^*)^i(E)$ is non-zero and the kernel of $\alpha_i$ is $G_{i+1}$ $(i = 0, 1, 2, \cdots, l-1)$.  Assume $G_0, G_1, \cdots, G_i$ are already defined.  First we show $G_{i+1} \neq 0$.  By Lemma (1.1), there exists a quasi locally free subsheaf $G_i'$ of $f^*F$ such that $G_i \subset G_i'$ and codim $(\mathrm{Supp}\,(G_i'/G_i)) \geq 2$.  It is clear that $G_i'$ is $f^*H$-semi-stable.  If $G_{i+1} = 0$ i.e. $\alpha_i$ is injective, then the natural homomorphism $\alpha_i': G_i' \to f^*F \to (a^*)^i(E)$ is also injective modulo codim $\geq 2$.  Since $x(G_i', f^*H)$

$= x((a^*)^i(E), f^*H)$ and $(a^*)^i(E)$ is $f^*H$-stable, hence $G'_i$ is $f^*H$-stable with $r(G'_i) = r((a^*)^i(E))$, the cokernel of $\alpha'_i$ is torsion and codim Supp (Cokernel of $\alpha'_i$) $\geq 2$. Hence by Remark 1), $\alpha'_i$ is an isomorphism. Thus $f^*F$ contains $(a^*)^i(E)$ as a direct summund. Since $a^*f^*F = f^*F$, $f^*F$ contains $\bigoplus_{i=0}^{l-1} (a^*)^i(E)$ which contradicts $r(f^*F) < l \cdot r(E)$. This proves $G_{i+1} \neq 0$. We remark the following fact before we show $\alpha_{i+1}$ is non-zero. Let $E_0$ and $F_0$ be quasi locally free sheaves of finite rank on a variety $V$ such that $E_0$ is $H$-semi-stable and $F_0$ is $H$-stable for an ample line bundle $H$ on $V$ with $x(E_0, H) = x(F_0, H)$. Then any non-zero homomorphism from $E_0$ to $F_0$ is surjective modulo a closed subvariety of codim $\geq 2$. We have the following diagram

$$0 \longrightarrow G_1 \longrightarrow G_0 = f^*F \overset{\alpha_0}{\longrightarrow} E$$
$$\|$$
$$(a^*)^{i+1} f^*F \longrightarrow (a^*)^{i+1}(E)$$

where the two arrows on the right are surjective modulo codim $\geq 2$. Note that $a^*E \neq E$ by assumption. Hence the natural homomorphism $G_1 \to f^*F \to (a^*)^{i+1}(E)$ is non-zero and the natural homomorphism $G_2 \to G_1 \to f^*F \to (a^*)^{i+1}(E)$ is non-zero since $\alpha_1 : G_1 \to a^*E$ is surjective modulo codim $\geq 2$. Continuing in this fashion, we see that the natural homomorphism $\alpha_{i+1} : G_{i+1} \to G_i \to \cdots \to G_1 \to f^*F \to (a^*)^{i+1}(E)$ is non-zero. Since $G_{i+1} = \ker(\alpha_i : G_i \to (a^*)^i(E))$, $G_i$ is $f^*H$-semi-stable, $(a^*)^i(E)$ is $f^*H$-stable, $x(G_i, f^*H) = x((a^*)^i E, f^*H)$ and $\alpha_i$ is surjective modulo codim $\geq 2$, we have $x(G_{i+1}, f^*H) = x(f^*F, f^*H)$. Hence $G_{i+1}$ is $f^*H$-semi-stable, and $r(G_{i+1}) = r(G_i) - r(E)$. Thus we have $G_0, G_1, \cdots, G_{l-1}$. Since $\alpha_{l-1}$ is non-zero, we have $r(G_{l-1}) \geq r(E)$ for the same reason, hence $r(F) \geq lr(E)$. This contradicts $r(F) < r(f_*E)$.                    q.e.d.

If $E = f^*E_1$ for a vector bundle $E_1$ of $W$ in Lemma (1.6), then $f_*E = \bigoplus_i E_1 \otimes N_i$ since $f_*(\mathcal{O}_V) = \bigoplus_i N_i$ with $N_i$ line bundles on $W$ such that $f^*N_i \simeq \mathcal{O}_V$. It is clear that $E_1$ is $H$-stable since $f^*E_1 = E$.

PROPOSITION (1.7). *Assume $A$ is solvable and the order of $A$ is prime to the characteristic of $k$. If $E$ is an $f^*H$-stable vector bundle on $V$ for an ample line bundle $H$ on $W$, then $f_*E = \oplus E_i$ where $E_i$ is an $H$-stable vector bundle on $W$ with $x(f_*E, H) = x(E_i, H)$.*

*Proof.* It is clear by Lemma (1.6) and the remark above, by successive reduction to the case of prime order.

COROLLARY (1.8). *Assume $A$ is solvable and the order of $A$ is prime to the characteristic. Let $L$ be a line bundle on $V$. If $f_*L$ is indecomposable, then $f_*L$ is $H$-stable for any ample line bundle $H$ on $W$.*

PROPOSITION (1.9). *Assume $A$ is solvable and $A$ has order prime to the characteristic. If $E$ is an $H$-semi-stable vector bundle on $W$ for an ample line bundle $H$ on $W$, then $f^*E$ is $f^*H$-semi-stable.*

*Proof.* We may assume $A$ is cyclic. We have $f_*(\mathscr{O}_V) = \oplus_i N_i$ with $N_i$ line bundles on $W$. Since $f_*f^*E = \oplus_i E \otimes N_i$ and $x(E \otimes N_i, H) = x(E, H)$ by Lemma (1.4), $f_*f^*E$ is $H$-semi-stable. Hence for any coherent subsheaf $F$ of $f^*E$, $x(f_*F, H) \leqq x(f_*f^*E, H)$. We have the cartesian diagram

$$
\begin{array}{ccc}
V \times A & \xrightarrow{\ \sigma\ } & V \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\
V & \xrightarrow{\ f\ } & W
\end{array}
$$

where $\sigma$ is the action of $A$ on $V$ and $p_1$ is the projection. We put $t = $ order of $A$. Since $f$ is flat,

$$(\mathrm{Inv}\,(f_*F), H) = \frac{1}{t}\,(\mathrm{Inv}\,(f^*f_*F), f^*H) = \frac{1}{t}\,(\mathrm{Inv}\,(\sigma_* p_1^* F), f^*H)$$

$$= \frac{1}{t}\left(\mathrm{Inv}\Big(\bigoplus_{a \in A} a^*F\Big), f^*H\right) = (\mathrm{Inv}\,(F), f^*H)\ .$$

It is clear that $d(f_*f^*E, H) = d(f^*E, f^*H)$.                    q.e.d.

PROPOSITION (1.10). *Let $A$ be as above. If $E$ is an $H$-stable vector bundle on $W$ for an ample line bundle $H$ on $W$, then $f^*E = \oplus E_i$ where $E_i$ is an $f^*H$-stable vector bundle on $V$ with $x(f^*E, f^*H) = x(E_i, f^*H)$.*

*Proof.* We may assume $A$ to be a cyclic group of prime order $l$. As above we have $f_*(\mathscr{O}_V) = \oplus N_i$. If $f^*E$ is $f^*H$-stable, we are done. Suppose $f^*E$ is not $f^*H$-stable. By Prop. (1.9) $f^*E$ is $f^*H$-semi-stable, hence there exists a quasi locally free subsheaf $F$ of $f^*E$ such that $x(F, f^*H) = x(f^*E, f^*H)$ and $1 \leqq r(F) < r(E)$. We may assume $F$ to be $f^*H$-stable by choosing such $F$ with the smallest $r(F)$. Since $f_*(F)$ is the subsheaf of the $H$-semi-stable bundle $f_*f^*E$ (cf. Prop. (1.7)) and $x(f_*F, H) = x(f_*f^*E, H)$, we have $f_*(F)$ is $H$-semi-stable. On the other hand $f_*(F) \otimes N_i = f_*(F)$ and $f_*(F) \subset \oplus E \otimes N_i = f_*f^*E$, hence there ex-

ists a non-zero homomorphism $\beta: f_*(F) \to E$. Then $x(E, H) = x(f_* f^* E, H)$ $= x(f_* F, H) \leq x$ (Image of $\beta, H$). By the $H$-stability of $E$ we thus $r(E)$ $= r$ (Image of $\beta$) $\leq r(f_* F)$ i.e. $r(E) \leq l r(F)$. We have $a^* F \neq F$ for any $a \neq 1 \in A$. Indeed if $a^* F = F$ for some $a \neq 1$, hence for all $a \in A$ since $A$ is cyclic of prime order, then $F = f^* F_1$ for some coherent subsheaf $F_1$ of $E$, and $x(F_1, H) = x(E, H)$ which contradicts the $H$-stability of $E$. We fix a generator $a$ of $A$. Inductively we define $f^* H$-semi-stable coherent sheaves $G_i$ with $x(F, H) = x(G_i, f^* H)$ and codim Supp (the torsion part of $G_i) \geq 2$, and inclusions $\alpha_i: F \to a^* G_i$ $(i = 0, 1, \cdots, l - 1)$ such that we have exact sequences

$$0 \longrightarrow F \xrightarrow{\alpha_i} a^* G_i \xrightarrow{\beta_i} G_{i+1} \longrightarrow 0$$

and $\alpha_i = a^*(\beta_{i-1}) a^{*2}(\beta_{i-2}) \cdots a^{*i}(\beta_0) \cdot \alpha_0$. We put $G_0 = f^* E$ and $\alpha_0 = $ the canonical inclusion. Assume $G_0, G_1 \cdots, G_i$ are already defined. Since $x(a^* G_i, f^* H) = x(F, f^* H)$ and $a^* G_i$ is $f^* H$-semi-stable, $G_{i+1}$ is $f^* H$-semi-stable and codim Supp (the torsion part of $G_{i+1}) \geq 2$. It is sufficient to show $\alpha_{i+1}$ is an inclusion. If $\alpha_{i+1} = 0$, then $a^{*2}(\beta_{i-1}) \cdots a^{*i+1}(\beta_0) \cdot \alpha_0 = 0$ because

$$0 \longrightarrow a^* F \longrightarrow a^{*2} G_i \xrightarrow{a^*(\beta_i)} a^* G_{i+1} \longrightarrow 0 \text{ (exact) },$$

$a^{*2}(\beta_{i-1}) \cdots a^{*i+1}(\beta_0) \cdot \alpha_0: F \to a^{*2} G_i$ and $F \neq a^* F$. Continuing in this fashion, we have a contradiction. Thus $\alpha_{i+1}$ is non-zero, hence it is an inclusion because $F$ is $f^* H$-stable and $a^* G_{i+1}$ is $f^* H$-semi-stable. Therefore we have $G_0, G_1, \cdots, G_{l-1}$. Since $r(G_{l-1}) = r(E) - (l - 1) r(F) \leq r(F)$ for the same reason as above, we have codim Supp (Coker of $\alpha_{l-1}) \geqslant 2$. By Remark 1) there exists a homomorphism $\delta: f^* E \to \alpha_{l-1}(F) \cong F$ such that $\delta \cdot \alpha_0 = \alpha_{l-1}$.

$$
\begin{array}{ccc}
0 \longrightarrow F \xrightarrow{\alpha_0} f^* E & & \\
\quad\vert \quad\quad \Vert & & \\
\quad\vert \quad a^{*l-1} f^* E \xrightarrow{a^*(\beta_{l-2}) \cdots a^{*l-2}(\beta_0)} a^* G_{l-1} \longrightarrow 0 \\
\quad\vert \quad\quad\quad\quad\quad \exists \delta \quad\quad \cup \\
\quad L - - - \alpha_{l-1} - - - - - \dashrightarrow \alpha_{l-1}(F) = F
\end{array}
$$

Hence the inclusion $F \to f^* E$ splits, and so $f^* E$ contains $\bigoplus_{i=0}^{l-1} a^{*i} F$ as a direct summund. By comparing the ranks, we have $f^* E = \bigoplus_{i=0}^{l-1} a^{*i} F$.

q.e.d.

COROLLARY (1.11). *Let $W$ be a non-singular projective variety over the complex number field. Assume that the fundamental group of $W$ is solvable. Let $H$ be an ample line bundle on $W$ and let $f : V \to W$ be an unramified covering. Then*

1) *If $E$ is an $H$-stable vector bundle on $W$, then $f^*E = \oplus E_i$ where $E_i$ is an $f^*H$-stable vector bundle on $V$ with $x(E_i, f^*H) = x(f^*E, f^*H)$.*

2) *If $E$ is an $f^*H$-stable vector bundle on $V$, then $f_*E = \oplus E_i$ where $E_i$ is an $H$-stable vector bundle on $W$ with $x(E_i, H) = x(f_*E, H)$.*

*Proof.* There exist a non-singular projective variety $U$, a finite group $A$ of automorphisms of $U$, and a normal subgroup $B$ of $A$ such that $W$ is the quotient of $U$ by $A$ and $V$ is the quotient of $U$ by $B$. By assumption, $A$ and $B$ are solvable. Hence we have Cor. (1.11) by Prop. (1.7) and Prop. (1.10).

*Remark.* If $W$ is an abelian variety, then the above corollary holds in arbitrary characteristic.

Let $f : V \to W$ be an unramified covering such that $f_*(\mathcal{O}_V)$ contains a non-trivial line bundle $J$ as a direct summund. Then $f^*J = \mathcal{O}_V$ and we have $f_*E \simeq f_*E \otimes J$ for a vector bundle $E$ on $V$. Conversely,

LEMMA (1.12). *Let $E$ be a simple vector bundle on $V$. Assume $r(E)$ is prime to the characteristic of $k$. If $E$ is isomorphic to $E \otimes J$ for a non-trivial line bundle $J$ on $V$, then there exist a non-trivial cyclic unramified covering $f : U \to V$ and a simple vector bundle $E_1$ on $U$ such that $f_*(E_1) = E$.*

*Proof.* Obviously we have $J^{\otimes r} = \mathcal{O}_V$ with $r = \text{rank of } E$. Let $d$ be the smallest positive integer such that $J^{\otimes d} = \mathcal{O}_V$. Then a locally free sheaf $B = \oplus_{i=0}^{d-1} J^{\otimes i}$ can be considered as an $\mathcal{O}_V$-algebra by defining the multiplication by $J^{\otimes d} = \mathcal{O}_V$. $f : U = \text{Spec}(B) \to V$ is an unramified covering of degree $d$. $H^0(f^* \text{End}(E)) = H^0(f_*f^* \text{End}(E)) = \oplus_{i=0}^{d-1} H^0(\text{Hom}(E, E \otimes J^{\otimes i}))$. Hence $H^0(\text{End}(f^*E)) = k[X]/(X^d - 1)$ as $k$-algebras, which is a direct sum of $k$ since $k$ is algebraically closed. Therefore $f^*E = \oplus_{i=1}^{d} E_i$. Thus $E^{\oplus d} = f_*f^*E = \oplus_{i=1}^{d} f_*E_i$, hence $E = f_*E_i$ for any $i$. On the other hand, $d = \dim_k H^0(\text{End}(f^*E)) = \sum_{1 \leqslant i, j \leqslant d} \dim_k H^0(\text{Hom}(E_i, E_j))$, hence in particular $E_i$ is simple for any $i$.

COROLLARY (1.13). *Let $E$ be a simple vector bundle on $V$ of prime*

*rank which is different from the characteristic. The following conditions are equivalent to each other*

1)  *There exists a non-trivial line bundle $J$ on $V$ such that $E$ is isomorphic to $E \otimes J$.*

2)  *There exist an unramified covering $f \colon U \to V$ and a line bundle $L$ on $U$ such that $E$ is isomorphic to $f_*(L)$. In this case, $E$ is $H$-stable for any ample line bundle $H$ on $V$.*

*Remark.* Let $E$ be isomorphic to $E \otimes J$ for a line bundle $J$. Then $\operatorname{End}(E)$ contains $J$ as a direct summand. Conversely, let $E$ be a vector bundle such that $\operatorname{End}(E)$ contains $J$ as a direct summand. Moreover, assume $E$ is $H$-stable for an ample line bundle $H$. Then $E$ is isomorphic to $E \otimes J$. Indeed, let $E$ be as above. $H^0(\operatorname{Hom}(E, E \otimes J^{-1})) \neq 0$, so $d(J^{-1}, H) \geqq 0$ by the $H$-stability of $E$. Since $\operatorname{End}(E)$ is self-dual, it also contains $J^{-1}$ as a direct summand, hence similarly $d(J, H) \geqq 0$. Thus $d(J, H) = 0$. Therefore $E \simeq E \otimes J$ by Prop. (1.2).

## 2.   *H*-stability on minimal models

PROPOSITION (2.1).   *Let $S$ be a relatively minimal non-singular projective surface over the complex number field, and let $E$ be a vector bundle of rank two on $S$ with $N(E) = c_1^2(E) - 4c_2(E) \geqq 0$. If $E$ is $H$-stable for some ample line bundle $H$ on $S$, then $E$ is $H'$-stable for any ample line bundle $H'$ on $S$.*

*Remark.* When $N(E)$ is positive, we have shown this in Prop. (2.7) of [10] without the assumption of relative minimality of $S$. We do not know whether this holds in case $N(E) = 0$ without the assumption of relative minimality of $S$.

First we show the following lemma:

LEMMA (2.2).   *Let $S$ be a non-singular projective surface which satisfies one of the following conditions, and $E$ be a vector bundle of rank two on $S$ with $N(E) \geqq 0$. If $E$ is $H$-stable for some ample line bundle $H$ on $S$, then $E$ is $H'$-stable for any ample line bundle $H'$ on $S$.*

(1)  *The Euler-Poincaré characteristic $\chi(\mathcal{O}_S)$ of $\mathcal{O}_S$ is positive*

(2)  *Let $H$ be an ample line bundle on $S$, $K$ the canonical line bundle on $S$ and $L$ a line bundle on $S$. If $(L, K) = 0$, $(L^2) = 0$, $H^1(L^{-1}) \neq 0$ and $(L, H) > 0$, then $(L, H') > 0$ for any ample line bundle $H'$ on $S$.*

*Proof of Lemma* (2.2).   By definition, $E$ is $H$-stable if and only if we have $(L_2 \otimes L_1^{-1}, H) > 0$ for any morphism $f : T \to S$ obtained by successive dilatations and any extension $0 \to f^*L_1 \otimes M \to f^*E \to f^*L_2 \otimes M^{-1} \to 0$ where $L_1$ and $L_2$ are line bundles on $T$ and $M$ is a positive exceptional line bundle on $T$.   Put $L = L_2 \otimes L_1^{-1}$.   Since $N(E) = (L^2) + 4(M^2) \geqq 0$, we have $(L^2) \geqq -4(M^2) \geqq 0$.   Now $H^2(L^{\otimes n}) = 0$ for sufficiently large $n$ because $(L, H) > 0$.   By Riemann-Roch theorem,

$$\chi(L^{\otimes n}) = h^0(L^{\otimes n}) - h^1(L^{\otimes n}) + h^2(L^{\otimes n})$$
$$= \frac{1}{2} n^2 (L^2) - \frac{1}{2} n(L, K) + \chi(\mathcal{O}_S) ,$$

where $h^i(L^{\otimes n}) = \dim_k H^i(S, L^{\otimes n})$.

Case 1.   $(L^2) > 0$.   Then we have the desired result because $H^0(L^{\otimes n}) \neq 0$ for sufficiently large $n$.

Case 2.   $(L^2) = 0$.   Then $(M^2) = 0$ and hence $M = \mathcal{O}_S$.   By tensoring with $L_1^{-1}$, we obtain the extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \otimes L_1^{-1} \longrightarrow L \longrightarrow 0 .$$

a)   $(L, K) < 0$.   Then we have the desired result because $H^0(L^{\otimes n}) \neq 0$ for sufficiently large $n$.

b)   $(L, K) = 0$.   Then $\chi(L^{\otimes n}) = \chi(\mathcal{O}_S)$.   If $\chi(\mathcal{O}_S) > 0$, then $H^0(L^{\otimes n}) \neq 0$ for sufficiently large $n$.   On the other hand the above extension is non-trivial.   Hence $H^1(L^{-1}) \neq 0$ and we have the desired result by assumption.

c)   $(L, K) > 0$.   Then $\chi(L^{\otimes -n}) > 0$ for sufficiently large $n$.   Assume $(L, H') \leqq 0$ for some ample line bundle $H'$ on $S$.   If $(L, H') = 0$, then $L$ is numerically equivalent to $\mathcal{O}_S$ since $(L^2) = 0$.   This contradicts $(L, H) > 0$.   On the other hand, if $(L, H') < 0$, then for the same reason $H^0(L^{\otimes -n}) \neq 0$ for sufficiently large $n$.   This contradicts $(L, H) > 0$.

*Remark.*   If a line bundle $L$ on $S$ with $\chi(\mathcal{O}_S) = 0$ satisfies the condition (2) of Lemma (2.2), then $0 \neq h^2(L^{-1}) = h^0(L \otimes K)$ i.e. $L \otimes K = \mathcal{O}_S(D)$ for some positive divisor $D$.

*Proof of Prop.* (2.1).   By the classification theorems of surfaces and our previous result (Th. 3.7, Prop. 4.1 and Prop. 5.1 of [10]), we may assume $S$ is (a) an elliptic surface, (b) a $K3$ surface or (c) a surface of general type.   (c) If $S$ is of general type, then $c_2(S) > 0$ by Van de Ven's result [11].   On the other hand $c_1^2(S) > 0$ and hence $\chi(\mathcal{O}_S) > 0$.   (b) If $S$

is a $K3$ surface, then $\chi(\mathcal{O}_S) > 0$. In cases (b) and (c), we have the desired result by Lemma (2.2). Therefore we may assume $S$ is an elliptic surface. Then $\chi(\mathcal{O}_S) = (1/12)c_2(S)$. On the other hand, $c_2(S)$ is equal to the topological Euler-Poincaré characteristic $\chi^t(S)$ of $S$. Let $p: S \to \Delta$ be a morphism from $S$ to a non-singular projective curve $\Delta$ such that the inverse image $p^{-1}(u) = C_u$ of any general point $u \in \Delta$ is an elliptic curve. Then we have an equality $\chi^t(S) = \chi^t(\Delta)\chi^t(F) + \Sigma(\chi^t(F_b) - \chi^t(F)) = \Sigma\chi^t(F_b)$ where $\chi^t(*)$ is the topological Euler-Poincaré characteristic and $F$ is any general fibre of $p$ and $F_b$ is a singular fibre. Since $S$ has no exceptional curve of the first kind, $\chi^t(F_b) \geqq \chi^t(F) = 0$ [8] p. 60 and hence $\chi^t(S) \geqq 0$. Hence $\chi(\mathcal{O}_S) \geqq 0$. By Lemma (2.2) we may assume $\chi(\mathcal{O}_S) = 0$ and hence $\chi^t(F_b) = \chi^t(F)$. Thus $F_b = m_b C_b$ where $C_b$ is an elliptic curve [8] p. 60. On the other hand, $K = p^*(K_\Delta \otimes I) \otimes \mathcal{O}_S(\Sigma(m_b - 1)C_b)$ where $I$ is a line bundle on $\Delta$ with $\deg I = 0$ and $K_\Delta$ is the canonical line bundle on $\Delta$ [3]. Assume a line bundle $L$ on $S$ satisfies the condition of Lemma (2.2) (2). Then by the above Remark, $L \otimes K = \mathcal{O}_S(D)$ for some positive divisor $D$. Now $0 = (D, K) = (2g - 2 + \Sigma(1 - 1/m_b))(D, F)$ where $g$ is the genus of $\Delta$. If $D = 0$, then $L = K^{-1}$ and so $(L, H) = -(2g - 2 + \Sigma(1 - 1/m_b))(F, H)$. The sign of $(L, H)$ is thus independent of an ample line bundle $H$. It remains to treat the case $D \neq 0$. If $p_*(D) \neq \Delta$, then $D$ is a linear combination of general fibres and $C_b$. We can thus write $L$ as $p^*(M) \otimes \mathcal{O}_S(\Sigma n_b C_b)$ where $M$ is a line bundle on $\Delta$ with $\deg M = m$. Since $(L, H) = (m + \Sigma(n_b/m_b))(F, H)$, we have the desired result for the same reason as above. Therefore we may assume $p_*(D) = \Delta$, hence $(F, D) \neq 0$. Since $0 = (D, K) = (2g - 2 + \Sigma(1 - 1/m_b))(D, F)$, we have $2g - 2 + \Sigma(1 - 1/m_b) = 0$. This equality holds only in $g = 1$ or $0$. In case $g = 1$, i.e., $\Delta$ is an elliptic curve, $S$ has no singular fibre by this equality and hence $K$ is numerically equivalent to $\mathcal{O}_S$. Thus $L$ is numerically equivalent to the non-zero positive divisor $D$. Therefore we have the desired result. In case $g = 0$, i.e., $\Delta$ is the projective line, we have $\Sigma(1 - 1/m_b) = 2$. Hence by Suwa's result [9], $S$ can be expressed as an elliptic surface $p': S \to \Delta'$ where $\Delta'$ is an elliptic curve, and $S$ has no singular fibre. We have already treated this case above.

### 3. A criterion for ampleness on an elliptic bundle

As a preparation for the next section we study in this section ampleness and cohomologies of a line bundle on an elliptic bundle.

DEFINITION (Suwa [9]).   A non-singular projective surface $S$ is said to be an *elliptic bundle* if there exists a proper, smooth morphism $p$ of $S$ onto a non-singular curve $\Delta$ such that the inverse image $p^{-1}(u) = C_u$ of any point $u \in \Delta$ is an elliptic curve.

Note that all the fibres are isomorphic to the same elliptic curve.

Let $S$ be an elliptic bundle.   Then we have the following exact sequence

$$0 \longrightarrow p^*(\Omega^1_\Delta) \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/\Delta} \longrightarrow 0 \ .$$

On the other hand $\Omega^1_{S/\Delta} = p^*(I)$ for some line bundle $I$ on $\Delta$ since $\Omega^1_{S/\Delta} \otimes \mathcal{O}_{C_u} \cong \Omega^1_{C_u} \cong \mathcal{O}_{C_u}$ for any point $u \in \Delta$.   Hence we have $\chi(\mathcal{O}_S) = 0$ since $c_2(\Omega^1_S) = (p^*\Omega^1_\Delta, \Omega^1_{S/\Delta}) = 0$, and $K_S = p^*(I \otimes K_\Delta)$ where $K_\Delta$ (resp. $K_S$) is the canonical line bundle of $\Delta$ (resp. $S$).   It is well-known that $I = p_*(\Omega^1_{S/\Delta})$ and $R^1 p_*(\mathcal{O}_S)$ are dual to each other.

LEMMA (3.1) (Kodaira).   $I^{\otimes n} = \mathcal{O}_\Delta$ *for some positive integer* $n$.   (*In particular* $\deg I = 0$.)

*Proof.*   By Riemann-Roch theorem, $p_*(ch(\mathcal{O}_S) \cdot T(S)) = ch(p!(\mathcal{O}_S)) \cdot T(\Delta)$. Since $p!(\mathcal{O}_S) = \Sigma(-1)^i R^i p_*(\mathcal{O}_S) = \mathcal{O}_\Delta + I$, the right hand side is $I$.   On the other hand, the left hand side is 0 since $T(S) = 1 + (1/2)p^*(I \otimes K_\Delta)$.   Therefore $I^{\otimes n} = \mathcal{O}_S$ for some $n > 0$.

In case $\Delta$ is an elliptic curve, $K_S^{\otimes n} = \mathcal{O}_S$ for some $n > 0$.   Indeed it is well-known that $n = 12$.

From now on $S$ is an elliptic bundle defined over the complex number field.

PROPOSITION (3.2).   *Let* $S$ *be an elliptic bundle with a fibre* $C$ *and let* $L$ *be a line bundle on* $S$.   *Then* $L$ *is ample if and only if* $(L^2) > 0$ *and* $(L, C) > 0$.

*Remark* (3.3).   Let $D$ be an irreducible curve on $S$ which is not a fibre.   Then $(D^2) \geqq 0$.   Indeed, $2g(D) - 2 \geqq (D, C)(2g(\Delta) - 2)$ by Hurwitz' theorem.   On the other hand $2g(D) - 2 = (D + K, D) = (D^2) + (2g(\Delta) - 2)$ $(D, C)$.

*Remark* (3.4).   Let $D$ be a positive divisor on $S$ with $(D^2) > 0$.   Then $D$ is ample.   Indeed $D$ contains an irreducible curve which is not a fibre. Hence $(D, C) > 0$ and $D$ is ample by Prop. (3.2).

*Proof of Prop.* (3.2). The necessity is clear. By Nakai's criterion for ampleness, it is enough for sufficiency to show that $(L, D)$ is positive for any irreducible curve $D$ different from the fibre on $S$. By Hodge index theorem we have (cf. [2], p. 208)

$$\begin{vmatrix} (L^2) & (L, D) & (L, C) \\ (L, D) & (D^2) & (D, C) \\ (L, C) & (D, C) & (C^2) \end{vmatrix} \geqq 0 .$$

Put $(L^2) = a$, $(L, C) = b$ and $(D, C) = n$. Here $a, b$ and $n$ are positive. Now $2nb(L, D) \geqq b^2(D^2) + an^2 \geqq an^2 > 0$. Therefore $(L, D) > 0$.

We now give results about the cohomology of a line bundle $L$ on $S$. We have the following spectral sequence

$$H^j(\Delta, R^i p_*(L)) \Longrightarrow H^{i+j}(S, L) .$$

Since $R^i p_*(L) = 0$ for $i \geqq 2$,

$$H^0(S, L) = H^0(\Delta, p_*(L)) , \qquad H^2(S, L) = H^1(\Delta, R^1 p_*(L))$$

and $0 \to H^0(\Delta, R^1 p_*(L)) \to H^1(S, L) \to H^1(\Delta, p_*(L)) \to 0$ (exact). Put $h^i(L) = \dim_k H^i(S, L)$.

By Riemann-Roch theorem,

$$h^0(L) - h^1(L) + h^2(L) = \tfrac{1}{2}(L^2) - (g - 1)(L, C)$$

where $g$ is the genus of $\Delta$.

LEMMA (3.5). 1) *If* $(L, C) = n > 0$, *then* $R^1 p_*(L) = 0$ *and* $p_*(L)$ *is a locally free sheaf of rank* $n$. *Hence* $h^2(L) = 0$.

2) *If* $(L, C) = 0$, *then* $(L^2) \leqq 0$. *If* $L$ *is, moreover, not isomorphic to* $p^*(M)$ *for any line bundle* $M$ *on* $\Delta$, *then* $p_*(L) = 0$, $h^0(L) = h^2(L) = 0$ *and* $h^1(L) = -(1/2)(L^2)$.

3) *If* $(L^2) < 0$, *then* $h^0(L) = 0$.

4) *If* $(L^2) > 0$, *then either* $L$ *or* $L^{-1}$ *is ample.*

5) *If* $(L, C) = n > 0$ *and* $(L^2) = 0$, *then* $h^0(L) \leqq n + 1$.

*Proof.* 1) By the base change theorem [5] p. 53, we have $R^1 p_*(L) \otimes k(u) \simeq H^1(C_u, L | C_u) = 0$ for any point $u \in \Delta$, hence $R^1 p_*(L) = 0$. By the base change theorem $p_*(L) \otimes k(u) \simeq H^0(C_u, L | C_u)$ for any point $u \in \Delta$. On the other hand $h^0(L | C_u) = n$ for any point $u \in \Delta$. Therefore $p_*(L)$ is locally free of rank $n$.

2)   Put $Y = \{u \in \Delta : L \,|\, C_u$ is trivial on $C_u\}$ and $U = \Delta - Y$.   Then $Y$
is closed in $\Delta$.   Assume $U$ is not empty.   The proof of 1) shows that
$R^1 p_*(L) = 0$ on $U$, hence $p_*(L) \otimes k(u) \simeq H^0(C_u, L \,|\, C_u) = 0$ for any point
$u \in U$.   Thus $p_*(L) = 0$ on $U$.   On the other hand $p_*(L)$ is torsion free
since $p_*(L) \subset p_*(L \otimes H)$ for a very ample line bundle $H$ on $S$ and
$p_*(L \otimes H)$ is locally free by 1).   Therefore $p_*(L) = 0$ and so $h^0(L) = 0$.
Now $h^2(L) = h^1(R^1 p_*(L)) = 0$ since $\mathrm{Supp}\,(R^1 p_*(L)) = Y$.   If $U$ is empty,
i.e. $\Delta = Y$, then $L = p^*(M)$ for some line bundle $M$ on $\Delta$, hence $(L^2) = 0$.
   3)   It follows from Remark (3.3).
   4)   It follows from 2) and Prop. (3.2).
We omit the proof of 5) because we don't use this result in this paper.
   We can get a little more detailed result for an elliptic bundle of
a special type i.e. a hyperelliptic surface.

DEFINITION (Suwa [8]).   A *hyperelliptic surface* is an elliptic bundle
over an elliptic curve whose first Betti number is equal to 2.

THEOREM (Suwa [8]).   *Any hyperelliptic surface $S$ can be expressed
as the quotient space of an abelian surface $A$ by the group generated
by an automorphism $a$ of $A$.   (Here $a^d = 1$ if and only if $K_S^{\otimes d} = \mathcal{O}_S$.)*

LEMMA (3.6).   *Let $S$ be a hyperelliptic surface and let $L$ be a line
bundle on $S$.*
   1)   *If $L$ is ample, then $h^1(L) = h^2(L) = 0$.*
   2)   *If $L^{-1}$ is ample, then $h^1(L) = h^0(L) = 0$.*
   3)   *If $(L^2) < 0$, then $h^0(L) = h^2(L) = 0$.*
   4)   *If $(L, C) = 0$, then $(L^2) = 0$.   If moreover $L$ is not isomorphic to
$p^*(M)$ for any line bundle $M$ on $\Delta$, then $R^i p_*(L) = 0$ for any $i \geqq 0$.*

   *Proof.*   Let $f : A \to S$ be the natural morphism in the above theorem.
Since $f$ is finite,

$$H^i(A, f^*(L)) = H^i(S, f_* f^*(L)) = \bigoplus_{j=0}^{d-1} H^i(S, L \otimes K^{\otimes j}) \supset H^i(S, L) \,.$$

Hence 1), 2) and 3) are clear by the cohomology theorem about line bundles
on abelian varieties.   Let $H$ be an ample line bundle on $S$.   We now
prove 4).   $L$ is homologous to $H^{\otimes r} \otimes \mathcal{O}_S(sC)$ where $r$ and $s$ are rational
numbers, since the second Betti number is equal to 2.   Hence $(L^2) = r^2(H^2) + 2rs(H, C) = r^2(H^2) \geqq 0$ since $0 = (L, C) = r(H, C)$.   Hence we obtain
$(L^2) = 0$ by Lemma (3.5) 2).   Assume $L$ is not isomorphic to $p^*(M)$ for

any line bundle $M$ on $\varDelta$. It follows from the proof of Lemma (3.5) 2) that $0 = -(1/2)(L^2) = h^1(L) = h^0(R^1 p_*(L))$ is equal to the number of points contained in $Y$ where $Y = \{u \in \varDelta : L \,|\, C_u$ is trivial on $C_u\}$. Hence $Y$ is empty.                                                                  q.e.d.

### 4. Stable vector bundles on an elliptic bundle

In [9] we studied $H$-stable vector bundles of rank two on geometrically ruled surfaces, $P^2$, and abelian surfaces. In this section and the next section, we continue the study for elliptic bundles and especially hyperelliptic surfaces.

Let $p : S \to \varDelta$ be a proper, smooth morphism from a surface $S$ onto a non-singular curve $\varDelta$, and let $E$ be a stable vector bundle of rank two on $\varDelta$. We proved $p^*E$ is $H$-stable for any ample line bundle $H$ when the fibre is the projective line [9]. We now prove this when the fibre is an elliptic curve, i.e. $S$ is an elliptic bundle. We do not know whether this holds when the genus of the fibre is more than 1.

PROPOSITION (4.1). *Let $p : S \to \varDelta$ be an elliptic bundle over the complex number field and $E$ a stable vector bundle on $\varDelta$ of rank two. Then $p^*E$ is $H$-stable for any ample line bundle $H$. (Here $N(p^*E) = 0$.)*

*Proof.* Looking closely at proofs of Prop. (2.1) and Lemma (2.2), we see that it is sufficient to show the following: for any morphism $f : T \to S$ obtained by successive dilatations and any quotient line bundle $f^*L \otimes M^{-1}$ of $f^*p^*E$ (where $L$ is a line bundle on $S$ and $M$ is a positive exceptional line bundle), there exists an ample line bundle $H$ on $S$ such that $(1/2)d(p^*E, H) < d(L, H)$. We may assume $\deg c_1(E) = m > 0$. Restricting the given exact sequence $f^*p^*E \to f^*L \otimes M^{-1} \to 0$ to $C$, we have the exact sequence $\mathcal{O}_C \oplus \mathcal{O}_C \to L \otimes \mathcal{O}_C \to 0$. Hence $(L, C) \geqq 0$. On the other hand, $0 = N(p^*E) = (L^{\otimes 2} \otimes p^*(c_1(E)^{-1}))^2 + 4(M^2)$, hence $(L^2) \geqq m(L, C)$, since $(M^2) \leqq 0$. If $(L, C) > 0$, then $(L^2) > 0$ and hence $L$ is ample by Prop. (3.2). In this case we may assume $H = L$, hence $d(L, L) - (1/2)d(p^*E, L) = (L^2) - (1/2)m(L, C) \geqq (m/2)(L, C) > 0$ and we are done. If $(L, C) = 0$, then $(L^2) \leqq 0$ by Lemma (3.5) and hence $(L^2) = 0$. We thus have $(M^2) = 0$ and so $M = \mathcal{O}_S$. Therefore the above exact sequence is of the following form: $0 \to L_1 \to p^*E \to L \to 0$, where $L_1$ is a line bundle on $S$. If $L \neq p^*(L_2)$ for any line bundle $L_2$ on $\varDelta$, then the same property holds for $L_1$ and moreover $(L_1, C) = 0$. Thus $p_*(L) = p_*(L_1) = 0$ by Lemma

(3.5). Hence $p_* p^* E = E = 0$, a contradiction. So there exist line bundles $L_2$ and $L_3$ on $\Delta$ such that $p^* L_2 = L$ and $p^* L_3 = L_1$. And we have an exact sequence $0 \to p^* L_3 \to p^* E \to p^* L_2 \to 0$. By applying $p_*$ we have an exact sequence $0 \to L_3 \to E \to L_2 \to 0$, hence $(1/2) \deg E < \deg L_2$ by the stability of $E$. Hence $(1/2) d(p^* E, H) < d(p^* L_2, H)$ for any ample line bundle $H$.

q.e.d.

PROPOSITION (4.2). *Let $S$ be an elliptic bundle, and let $E$ be an $H$-stable vector bundle of rank two on $S$ for some ample line bundle $H$. Assume $c_1(E)$ is numerically equivalent to $p^*(L)$ for some line bundle $L$ on $\Delta$. Then $N(E) \leq 0$.*

*Proof.* Suppose there exists $E$ with $N(E) > 0$ which satisfies the above condition. We may assume $\deg L$ is sufficiently large. Since $4\chi(E) = N(E)$ and $d(E^* \otimes K, H) < 0$, we have $H^0(E) \neq 0$ by Lemma (2.1) of [9]. Thus there exist a morphism $f : T \to S$ obtained by successive dilatations and an extension of line bundles:

$$0 \longrightarrow f^*(\mathscr{O}_S(D)) \otimes M^{-1} \longrightarrow f^* E \longrightarrow f^* L_1 \otimes M^{-1} \longrightarrow 0$$

where $M$ is a positive exceptional line bundle on $T$, $D$ is a positive divisor on $S$ and $L_1$ is a line bundle on $S$. Assume $D$ contains an irreducible curve $D_1$ which is not a fibre. Since $N(E) > 0$, $E$ is $p^*(L_2^{\otimes n}) \otimes H$-stable for any $n > 0$ where $L_2$ is a line bundle on $\Delta$ with $\deg L_2 = 1$. Hence we have inequalities

$$0 \leq (D, p^*(L_2^{\otimes n}) \otimes H) < \tfrac{1}{2} d(E, p^*(L_2^{\otimes n}) \otimes H) ,$$

i.e. $0 \leq n(D_1, C) + (D_1, H) < (1/2) d(E, H)$ for any $n > 0$. Hence $(D_1, C) = 0$, i.e. $D_1 = 0$ which contradicts our assumption. Therefore all irreducible components of $D$ are fibres and hence $N(E) = 4(M^2) \leq 0$. This contradicts $N(E) > 0$.

DEFINITION. An elliptic bundle $p : S \to \Delta$ is said to be *basic* if there exists a section $o$ of $p$, i.e. $p \cdot o = 1_\Delta$.

The following Lemma is well-known.

LEMMA (4.3). *Let $S$ be a basic elliptic bundle with a fibre $C$, and let $L$ be a line bundle on $S$. Put $(L, C) = n$. Then there exist a section $s$ of $p$ and a line bundle $M$ on $\Delta$ such that*

$$L \simeq p^*(M) \otimes \mathscr{O}_S(s(\Delta) + (n - 1)o(\Delta)) .$$

*Proof.* Tensoring $\mathcal{O}_S((-n + 1)o(\Delta))$, we may assume $(L, C) = 1$, hence there exists a unique point $P_u$ on $C_u$ such that $L\,|_{C_u} \cong \mathcal{O}_{C_u}(P_u)$ for every $u \in \Delta$. We can take a section $s$ such that $s(u) = P_u$. Since $L\,|_{C_u} \simeq \mathcal{O}_{C_u}(s(\Delta))$ for every point $u \in \Delta$, we have $L \cong \mathcal{O}_S(s(\Delta)) \otimes p^*(M)$ for some line bundle $M$.

PROPOSITION (4.4). *Let $S$ be a basic elliptic bundle with a fibre $C$, and let $E$ be a vector bundle of rank two on $S$ with $(c_1(E), C)$ odd. If $E$ is $H$-stable for some ample line bundle $H$ on $S$, then $N(E) \leqq 4g$, where $g$ is the genus of $\Delta$.*

*Proof.* Note first that $N(E) =$ even. Indeed, $(s(\Delta), s(\Delta)) = 0$ for every section $s$ of p. [3]. Hence for a line bundle $L, (L^2) = 2n \deg (M) + 2(n - 1)(o(\Delta), s(\Delta))$ by Lemma (4.3). Suppose there exists a vector bundle $E$ with $N(E) \geq 4g + 2$ which satisfies the above conditions. Since $(c_1(E^*) \otimes \mathcal{O}_S(no(\Delta)), C) = (c_1(E^*), C) + 2n$, we may assume $(c_1(E^*), C) = 1$. Thus $c_1(E)$ is numerically equivalent to $mC - s(\Delta)$ for some section $s$ of $p$ by Lemma (4.3). Since

$$2g - 2 + \tfrac{1}{2}N(E) - 4g + 3 \geqq 2$$

and

$$c_1(E \otimes \mathcal{O}_S(kC)) = c_1(E) \otimes \mathcal{O}_S(2kC) \, ,$$

we may assume $2g - 2 + (1/2)N(E) > m \geqq 4g - 3$. On the other hand, the Euler-Poincaré characteristic $\chi(E)$ of $E$ is $-(1/2)m + g - 1 + (1/4) N(E) > 0$ and $(c_1(E^* \otimes K), \mathcal{O}_S(s(\Delta)) \otimes \mathcal{O}_S(C)) = 4g - 3 - m \leqq 0$. Thus $H^0(E) \neq 0$ by Lemma (2.1) of [9], since $\mathcal{O}_S(s(\Delta) + C)$ is ample. Hence there exist a morphism $f: T \to S$ obtained by successive dilatations and an extension of line bundles on $T$

$$0 \longrightarrow f^*(\mathcal{O}_S(D)) \otimes M \longrightarrow f^*E \longrightarrow f^*(L) \otimes M^{-1} \longrightarrow 0$$

where $M$ is a positive exceptional line bundle on $T, D$ is a positive divisor on $S$ and $L$ is a line bundle on $S$. By the $H$-stability of $E$, we have inequalities

$$0 \leqq (D, nC + s(\Delta)) < \tfrac{1}{2}(mC - s(\Delta), nC + s(\Delta))$$

i.e. $0 < (1/2)(-n + m)$ for any $n > 0$, a contradiction.

### 5.  Stable vector bundles on a hyperelliptic surface

PROPOSITION (5.1).  *Let $S$ be a non-singular projective surface over the complex number field with $K_S^{\otimes n} = \mathcal{O}_S$ for some positive integer $n$ where $K_S$ is the canonical line bundle on $S$, and let $E$ be a simple vector bundle of rank two on $S$.  Then $N(E) = c_1^2(E) - 4c_2(E) \leqq 0$.*

*Proof.*  By Riemann-Roch theorem, $N(E) + 4\chi(\mathcal{O}_S) = h^0\,(\mathrm{End}\,(E)) - h^1\,(\mathrm{End}\,(E)) + h^2\,(\mathrm{End}\,(E)) \leqq 1 - h^1(\mathcal{O}_S) + h^0\,(\mathrm{Hom}\,(E, E \otimes K_S))$ since the canonical injection $\mathcal{O}_S \to \mathrm{End}\,(E)$ splits.  On the other hand, by the classification theory of surfaces, $S$ is one of the following:

   (1)  Enriques surface $K_S^{\otimes 2} = \mathcal{O}_S$
   (2)  regular surface with $K_S = \mathcal{O}_S$
   (3)  two-dimensional abelian variety
   (4)  hyperelliptic surface.

Hence Prop. (5.1) is clear in cases (2) and (3).  If $E$ is $H$-stable for some ample line bundle $H$ on $S$ in case (1), $h^0\,(\mathrm{Hom}\,(E, E \otimes K_S)) \leqq 1$ by Prop. (1.2).  And if $E$ is not $H$-stable and simple, we may assume there exist a morphism $f: T \to S$ obtained by successive dilatations and an extension of line bundles on $T: 0 \to M \to f^*E \to M^{-1} \otimes f^*L \to 0$ where $L$ is a line bundle with $(L, H) \leq 0$ and $M$ is a positive exceptional line bundle on $T$.  If $H^0\,(\mathrm{Hom}\,(E, E \otimes K_S)) \neq 0$, then there exists a non-zero homomorphism $x: E \to E \otimes K_S$.  Hence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\ y\ } & f^*E & \longrightarrow & M^{-1} \otimes f^*L & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle x \neq 0} & & & & \\
0 & \longrightarrow & M \otimes f^*K_S & \longrightarrow & f^*(E \otimes K_S) & \xrightarrow[z]{} & M^{-1} \otimes f^*(L \otimes K_S) & \longrightarrow & 0\,.
\end{array}$$

If $zxy = 0$, then the natural homomorphism $M^{-1} \otimes f^*L \to M^{-1} \otimes f^*(L \otimes K_S)$ is non-zero and hence $H^0(K_S) \neq 0$ which contradicts our assumption.  Therefore $zxy \neq 0$ and so $0 \neq H^0(M^{-2} \otimes f^*(L \otimes K_S)) \subset H^0(L \otimes K_S)$.  Thus $L \otimes K_S = \mathcal{O}_S$ and hence $M = \mathcal{O}_S$ since $(L \otimes K_S, H) \leqq 0$.  The above extension is thus of the following form: $0 \to \mathcal{O}_S \to E \to K_S^{-1} \to 0$ a contradiction, since $H^1(K_S) = 0$.  If $E$ is $H$-stable for some ample line bundle $H$ on $S$ in case (4), then $h^0\,(\mathrm{Hom}\,(E, E \otimes K_S)) \leqq 1$, hence $N(E) \leqq 0$ because $N(E)$ is even.  Therefore it is sufficient to show the following:

PROPOSITION (5.2).  *Let $S$ be a hyperelliptic surface, and let $E$ be a vector bundle of rank two on $S$ with $N(E) \geqq 0$.  Then $E$ is simple if and*

*only if E is either H-stable for some ample line bundle H or of the form $E_0 \otimes L$, where we have a non-trivial extension $0 \to \mathcal{O}_S \to E_0 \to K_S^{-1} \to 0$.*

*Remark.* Such $E_0$ is unique up to isomorphism since $H^1(K_S) = C$.

*Proof.* $E_0$ is simple by Oda's lemma. Assume $E$ is simple but not $H$-stable for some ample line bundle $H$. Tensoring a suitable line bundle on $S$, we may assume there exist a morphism $f : T \to S$ and an extension of line bundles on $T$: $0 \to M \to f^*E \to f^*L \otimes M^{-1} \to 0$ where $L$ is a line bundle on $S$ with $(L, H) \leqq 0$, $h^0(L) = h^0(L^{-1}) = 0$ and $M$ is a positive exceptional line bundle on $T$ by Prop. (2.9) of [9]. Since $N(E) \geqq 0$, we have $(L^2) \geqq 0$. But $(L^2) = 0$ since $(L, H) \leqq 0$ and $h^0(L^{-1}) = 0$. Hence $M = \mathcal{O}_S$ and the above extension is of the following form: $0 \to \mathcal{O}_S \to E \to L \to 0$. Since $0 = \chi(L^{-1}) = -h^1(L^{-1}) + h^2(L^{-1})$ and $E$ is simple, we have $0 \neq h^1(L^{-1}) = h^2(L^{-1}) = h^0(L \otimes K_S)$. Thus $L \otimes K_S \simeq \mathcal{O}_S$ because $(L \otimes K_S, H) \leqq 0$.

<div align="right">q.e.d.</div>

From now on we assume $S$ is a hyperelliptic surface and $E$ is a vector bundle on $S$ of rank two. On $S$ one of the following holds [9]:

( I )   $K_S^{\otimes 2} = \mathcal{O}_S$   $(K_S \neq \mathcal{O}_S)$      (II)   $K_S^{\otimes 3} = \mathcal{O}_S$   $(K_S \neq \mathcal{O}_S)$

(III)   $K_S^{\otimes 4} = \mathcal{O}_S$   $(K_S^{\otimes 2} \neq \mathcal{O}_S)$     (IV)   $K_S^{\otimes 6} = \mathcal{O}_S$   $(K_S^{\otimes 2}, K_S^{\otimes 3} \neq \mathcal{O}_S)$

PROPOSITION (5.3). *Assume $K_S^{\otimes 4} = \mathcal{O}_S$ (i.e. case (I) or (III)). If E is stable with $N(E) = 0$, then there is a non-trivial line bundle J such that E is isomorphic to $E \otimes J$.*

*Proof.* Put $\operatorname{End}(E) = \mathcal{O}_S \oplus E_1$. By Suwa's theorem which we mentioned in §3, there is an unramified covering $f : A \to S$ of degree $d$, where $d = 2$ in case (I), and $d = 4$ in case (III), and $A$ is an abelian surface. Since $h^0(\operatorname{End}(f^*E)) = \sum_{i=0}^{d-1} h^0(\operatorname{Hom}(E, E \otimes K_S^{\otimes i}))$, we may assume $f^*E$ is simple because $f^*E$ is not simple if and only if $E$ is isomorphic to $E \otimes K_S^{\otimes j}$, where $j = 1$ in case (I) and $j = 2$ in case (III). By Morikawa and Oda [6] there exist an isogeny $g : B \to A$ and a line bundle $L$ on $B$ such that $g_*L = f^*E$. Hence $f^*E_1 = L_1 \oplus L_2 \oplus L_3$ for some line bundles $L_i$ on $A$ ($i = 1, 2, 3$) because $\operatorname{End}(f^*E) = \operatorname{End}(g_*L) = \oplus L'$ where $L'$ runs over all the line bundles $L'$ on $A$ such that $g^*L' = T_a^*L \otimes L^{-1}$ for some point $a$ in $\ker(g)$ [6]. Hence

$$ f_*f^*E_1 = \bigoplus_{i=0}^{d-1} E_1 \otimes K_S^{\otimes i} = f_*L_1 \oplus f_*L_2 \oplus f_*L_3 $$

and the rank of $f_* L_i$ is $d$. Thus $E_1$ is decomposable, i.e. $E_1 = J \oplus E_2$ for some line bundle $J$ on $S$, which is non-trivial since $E$ is simple. Hence $E$ is isomorphic to $E \otimes J$ by the last Remark in section 1.

From now on we assume $S$ is a basic hyperelliptic surface with the typical fibre $C$. Fix a global section $o$ of $p$. For every point $u \in \varDelta \ p^{-1}(u)$ $= C_u$ is an elliptic curve with the zero point which is the intersection point of $C_u$ and $o(\varDelta)$, and $C_u$ is isomorphic to the same curve $C$. Then there exists an isogeny $\varDelta' \to \varDelta$ such that the fibre product $\varDelta' \times_\varDelta S$ is isomorphic to $\varDelta' \times C$ [9], i.e. the diagram is cartesian

$$
\begin{array}{ccc}
\varDelta' \times C & \longrightarrow & S \\
{\scriptstyle pr_1}\downarrow & & \downarrow \\
\varDelta' & \longrightarrow & \varDelta \ .
\end{array}
$$

Let $H^0(\varDelta, S)$ be the set of all sections of $p$. Now we have a homomorphism of abelian groups $f: H^0(\varDelta, S) \to H^0(\varDelta', \varDelta' \times C)$. On the other hand, for any section $s$ different from 0, $s(\varDelta)$ is numerically equivalent to $o(\varDelta)$ since the base number of $S$ is 2, hence the intersection of $s(\varDelta)$ and $o(\varDelta)$ is empty. Thus the image of $\varDelta' \xrightarrow{s'} \varDelta' \times C \xrightarrow{pr_2} C$ does not contain the zero point of $C$, and it defines one point of $C$. Hence $f$ factors: $H^0(\varDelta, S)$ $\to C(k) \subset H^0(\varDelta', \varDelta' \times C)$. By means of Suwa's result [9] and the concrete representation of $S$ by Kodaira [3], we can calculate $H^0(\varDelta, S)$ easily.

LEMMA (5.4).
$$
H^0(\varDelta, S) = \begin{cases} Z/2Z + Z/2Z & \text{in case } \mathrm{I} \\ Z/3Z & \text{in case } \mathrm{II} \\ Z/2Z & \text{in case } \mathrm{III} \\ 0 & \text{in case } \mathrm{IV} \ . \end{cases}
$$

By Lemma (4.3) and Lemma (5.4), we get the following:

PROPOSITION (5.5). *If $S$ is a basic hyperelliptic surface with $K_S^{\otimes 4} \neq \mathcal{O}_S$, then we have a canonical isomorphism* $\mathrm{Pic}\,(\varDelta)_2 \simeq \mathrm{Pic}\,(S)_2$.

Here we define $\mathrm{Pic}\,(V)_2 = \{L \in \mathrm{Pic}\,(V): L^{\otimes 2} = \mathcal{O}_V\}$ for a variety $V$. And we define $\mathrm{Pic}'\,(S/\varDelta) = \{L \in \mathrm{Pic}\,(S): o^*(L) \simeq \mathcal{O}_\varDelta\}$, where $o$ is a global section of $p$. It is well-known that we have an exact sequence

$$
0 \longrightarrow \mathrm{Pic}\,(\varDelta) \longrightarrow \mathrm{Pic}\,(S) \longrightarrow \mathrm{Pic}'\,(S/\varDelta) \longrightarrow 0 \ .
$$

By Lemma (4.3), we can express $\mathrm{Pic}'\,(S/\varDelta)$ by $H^0(\varDelta, S)$ and $I^{-1} = R^1 p_*(\mathcal{O}_S)$

$= \mathcal{O}_{o(\Delta)}(o(\Delta))$. Let $\tilde{\Delta}$ be an unramified covering of $\Delta$. Put $\tilde{S} =$ the fibre product $S \times_{\Delta} \tilde{\Delta}$. If $S$ and $\tilde{S}$ are in the same class, then Pic$'(S/\Delta) \simeq$ Pic$'(\tilde{S}/\tilde{\Delta})$ by Lemma (5.4).

*Remark.* Let $f : T \to S$ be an unramified covering of degree two. Then $T$ and $S$ are in the same class if and only if $J \neq K_S^{\otimes i}$ where $f_*(\mathcal{O}_T) = \mathcal{O}_S \oplus J$.

**PROPOSITION (5.6).** *Let $S$ be a basic hyperelliptic surface with $K_S^{\otimes 4} \neq \mathcal{O}_S$ and $E$ a stable vector bundle of rank two on $S$ with $N(E) = 0$. If $E$ is isomorphic to $E \otimes J$ for some non-trivial line bundle $J$ with $J \neq K_S^{\otimes i}$, then there exist a stable vector bundle $F$ on $\Delta$ and a line bundle $M$ on $S$ such that $E$ is isomorphic to $p^*(F) \otimes M$.*

*Remark.* 1) We have $J^{\otimes 2} = \mathcal{O}_S$ since $r(E) = 2$, hence the above condition $J \neq K_S^{\otimes i}$ always holds in case II and is equivalent to $J \neq K_S^{\otimes 2}$ in case IV. 2) A hyperelliptic surface in case IV is always basic [9].

*Proof.* Since $J$ is contained in Pic $(S)_2$, so $J = p^*(J_1)$ for some $J_1 \in$ Pic $(\Delta)_2$ by Prop. (5.5). Put $\Delta' =$ Spec $(\mathcal{O}_{\Delta} \oplus J_1)$. Then $g : \Delta' \to \Delta$ is an unramified covering of degree two and so is $f : S' =$ Spec $(\mathcal{O}_S \oplus J) \to S$. We have the cartesian diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{f} & S \\
{\scriptstyle p'} \downarrow & & \downarrow {\scriptstyle p} \\
\Delta' & \xrightarrow{g} & \Delta \ .
\end{array}
$$

By Cor. (1.12), there exists a line bundle $L_1$ on $S'$ such that $E = f_*(L_1)$. On the other hand, by our assumption, $S$ and $S'$ are in the same class. Hence we have exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Pic }(\Delta) & \longrightarrow & \text{Pic }(S) & \longrightarrow & \text{Pic}'(S/\Delta) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| \wr & & \\
0 & \longrightarrow & \text{Pic }(\Delta') & \longrightarrow & \text{Pic }(S') & \longrightarrow & \text{Pic}'(S'/\Delta') & \longrightarrow & 0 \ .
\end{array}
$$

Therefore $L_1 = f^*(L_2) \otimes p'^*(L_3)$ for some line bundle $L_2$ on $S$ and $L_3$ on $\Delta$. Hence $E \otimes L_2^{-1} = f_* p'^*(L_3) = p^* g_*(L_3)$, and $g_*(L_3)$ is stable on $\Delta$ by Cor. (1.8). Thus we have the desired result.

**EXAMPLE (5.7).** There is an $H$-stable vector bundle $E$ of rank two

on a basic hyperelliptic surface with $N(E) = 0$ which is not of the form in Prop. (5.6). Let $S$ be a basic hyperelliptic surface with a section $o$ of $p$ and the fibre $C$. Put $H = \mathcal{O}_S(o(\Delta)) \otimes p^*(M)$ and $L = \mathcal{O}_S(o(\Delta)) \otimes p^*(I)$, where $M$ is a line bundle on $\Delta$ with $\deg M = 1$ and $I = R^1 p_*(\mathcal{O}_S)$. We have an exact sequence

$$0 \longrightarrow L^{-1} \longrightarrow p^*(I^{-1}) \longrightarrow \mathcal{O}_{o(\Delta)} \otimes p^*(I^{-1}) = I^{-1} \longrightarrow 0 .$$

Since $(L, C) = 1$ and hence $p_*(L^{-1}) = 0$ by the base change theorem, we have

$$0 \longrightarrow I^{-1} \longrightarrow I^{-1} \longrightarrow R^1 p_*(L^{-1}) \longrightarrow R^1 p_*(p^*(I^{-1})) \longrightarrow 0 .$$

Thus $R^1 p_*(L^{-1}) = \mathcal{O}_\Delta$ since $R^1 p_* p^* I^{-1} = R^1 p_*(\mathcal{O}_S) \otimes I^{-1} = \mathcal{O}_\Delta$. Hence $H^1(L^{-1}) = H^0(R^1 p_* L^{-1}) = H^0(\mathcal{O}_\Delta) = C$ by Leray spectral sequence. And we have a non-trivial extension: $0 \to \mathcal{O}_S \to E \to L \to 0$. Setting $H = \mathcal{O}_S(o(\Delta) + C)$, we see that $E$ is $H$-stable with $N(E) = 0$ since $(c_1(E), H) = 1$ by Lemma (3.8) of [10], and $c_1(E) = L$ is numerically equivalent to $o(\Delta)$. Thus $E$ is not isomorphic to $p^*(F) \otimes N$ where $F$ is stable on $\Delta$ and $N$ is a line bundle on $S$.

Let $S$ be a hyperelliptic surface. Let $A(S)$ be the set of stable vector bundles $E$ of rank two on $S$ with $N(E) = 0$. Let $B(S)$ be the set of simple vector bundles of rank two which can be expressed as $f_*(L)$ where $f : T \to S$ is an unramified covering, $L$ is a line bundle on $T$, and $a^*L$ is numerically equivalent to $L$ but not isomorphic to $L$. Here $a$ is an automorphism of $T$ and $S = T/\langle a \rangle$. Let $C(S)$ be the set of vector bundles on $S$ which can be expressed as $p^*F \otimes M$ where $F$ is a stable vector bundle of rank two on $\Delta$ and $M$ is a line bundle on $S$.

PROPOSITION (5.8).   i)   $A(S) \supset B(S) \supset C(S)$.

  ii)   *If $S$ is basic, then $A(S) \neq C(S)$.*

  iii)   *In cases* I *and* III, *we have* $A(S) = B(S)$.

  iv)   *In case* II *and $S$ is basic, we have* $B(S) = C(S)$.

  v)   *In case* IV, $B(S) \cap \{E : E \neq E \otimes K_S^{\otimes 3}\} = C(S) \cap \{E : E \otimes K_S^{\otimes 3} \neq E\}$ *and* $B(S) \neq C(S)$.

*Proof.* i) The first inclusion follows from Cor. (1.8). The second inclusion follows from either Oda's result [5], or Atiyah's result [1] and Lemma (1.12). ii) follows from Example (5.7). iii) follows from Prop. (5.3) and Cor. (1.13). iv) follows from Prop. (5.6). v) The first state-

ment follows from Prop. (5.6). Put $S' = \mathrm{Spec}\,(\mathcal{O}_S \oplus K_S^{\otimes 3})$. Then we have the cartesian diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{\ f\ } & S \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\
\varDelta' & \longrightarrow & \varDelta\ .
\end{array}
$$

Let $L$ be $\mathcal{O}_{S'}(s'(\varDelta') - o'(\varDelta'))$ such that $o' \neq s' \in H^0(\varDelta', S')$. We see that $f_*L$ is contained in $B(S)$, but is not contained in $C(S)$. Indeed, if $f_*L \in C(S)$, then $f_*L = p^*(E) \otimes L_1$ where $E$ is a stable vector bundle on $\varDelta$ and $L_1$ is a line bundle on $S$. Since $p^*(I_\varDelta) \cong K_S$ and $f_*L \otimes K_S^{\otimes 3} \simeq f_*L$, we have $E \otimes I_\varDelta^{\otimes 3} \simeq E$, hence we see easily that $L \simeq p'^*(L_2) \otimes f^*(L_1)$, a contradiction. It is clear that $f_*L \in B(S)$ since $L$ is numerically equivalent to $0$.

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