STANDARD MODELS FOR SOME COMMUTING MULTIOPERATORS

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ABSTRACT. An analogue to the Sz.-Nagy-Foiaş dilation theory is presented for several commuting operators on a Hilbert space.

The dilation theory of Hilbert space contractions, a reference of which is the excellent monograph [9], has not yet found a complete counterpart valid for several (commuting) Hilbert space operators. Apart from the results of this type already contained in [9] and other results of general character (i.e., Arveson's or Stinespring's extension theorem), there are not too many contributions specialized to the multiparameter dilation theory. Among these contributions, we quote the works [7, 3–5, 10], which are closer to our topics.

The aim of this paper is to analyse some positivity conditions for commuting multioperators, which ensure the unitary equivalence of these objects to some standard models consisting of backwards multishifts. We emphasize, in particular, a situation that seems to satisfy the requirements of an appropriate extension for the dilation theory of Hilbert space contractions and that exploits the results already obtained in [4, 10].

Our finer methods also imply better statements of some assertions from [10] and provide an answer to Question 4.5 from [5].

Let $n \geq 1$ be an integer, and let $\mathbf{Z}_+^n$ be the set of all n-tuples of nonnegative integers (i.e., the multi-indices of length n). If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n$, we set, as usual, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. For $\alpha, \beta \in \mathbf{Z}_+^n$ we write $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, and $\alpha \leq \beta$ whenever $\alpha_i \leq \beta_i$ $(i = 1, \ldots, n)$.

Let H be a Hilbert space, and let $\mathscr{L}(H)$ be the algebra of all bounded linear operators on H. An element $T = (T_1, \ldots, T_n) \in \mathscr{L}(H)^n$ such that $T_1, \ldots, T_n$ mutually commute will be designated as a commuting multioperator (briefly, a c.m.).

Let $T \in \mathscr{L}(H)^n$ be a c.m. We define the operator $M_T : \mathscr{L}(H) \to \mathscr{L}(H)$ by the formula

$$(1) \qquad M_T(X) = \sum_{i=1}^n T_i^* X T_i, \qquad X \in \mathscr{L}(H).$$

Note that

$$(2) \qquad M_T^s(X) = \sum_{\substack{\alpha \in \mathbf{Z}_+^n \\ |\alpha|=s}} \frac{s!}{\alpha!} T^{*\alpha} X T^\alpha, \qquad s = 0, 1, 2, \dots.$$

Here, as usual, $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ and $T^* = (T_1^*, \dots, T_n^*)$. It is clear that $M_T^s(X) \geq 0$ whenever $X \geq 0$.

We also define

$$(3) \qquad \Delta_T^{(m)} = (1 - M_T)^m(1), \qquad m = 0, 1, 2, \dots.$$

We have

$$\Delta_T^{(m)} = \sum_{j=0}^m (-1)^j \binom{m}{j} M_T^j(1)$$

and, by (2),

$$(4) \qquad \Delta_T^{(m)} = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \frac{m!}{\alpha!(m-|\alpha|)!} T^{*\alpha} T^\alpha.$$

Lemma 1. *Let* $T \in \mathscr{L}(H)^n$ *be a commuting multioperator. Then* $\Delta_T^{(1)} \geq 0$ *if and only if* $\|M_T\| \leq 1$.

Proof. If $\Delta_T^{(1)} \geq 0$ (i.e., $T_1^* T_1 + \cdots + T_n^* T_n \leq 1$), a direct estimate shows that

$$|\langle M_T(X)x, y \rangle| = \left| \sum_{i=1}^n \langle X T_i x, T_i y \rangle \right| \leq \|X\| \sum_{i=1}^n \|T_i x\| \|T_i y\|$$

$$\leq \|X\| \left(\sum_{i=1}^n \|T_i x\|^2 \right)^{1/2} \left(\sum_{i=1}^n \|T_i y\|^2 \right)^{1/2} \leq \|X\| \|x\| \|y\|$$

for all $X \in \mathscr{L}(H)$ and $x, y \in H$. Hence $\|M_T\| \leq 1$.

Conversely, if $\|M_T\| \leq 1$, then $\|M_T(1)\| = \|T_1^* T_1 + \cdots + T_n^* T_n\| \leq 1$, whence $T_1^* T_1 + \cdots + T_n^* T_n \leq 1$.

Lemma 2. *Let* $T \in \mathscr{L}(H)^n$ *be a c.m., and let* $m \geq 1$ *be an integer. The following conditions are equivalent*:

    (a) $\sup_s \|M_T^s(1)\| < \infty$ *and* $\Delta_T^{(m)} \geq 0$.
    (b) $\Delta_T^{(1)} \geq 0$ *and* $\Delta_T^{(m)} \geq 0$.
    (c) $\Delta_T^{(k)} \geq 0$ *for* $k = 1, \dots, m$.

Proof. The implication (c) $\Rightarrow$ (b) is clear and the implication (b) $\Rightarrow$ (a) follows from Lemma 1. We prove (a) $\Rightarrow$ (c).

Suppose condition (a) is satisfied. We may assume $m \geq 2$. Let $x \in H$. For $s = 0, 1, 2, \dots$ let $a_s = \langle M_T^s(\Delta_T^{(m-1)})x, x \rangle$. We have

$$a_s - a_{s+1} = \langle M_T^s(1 - M_T)(\Delta_T^{(m-1)})x, x \rangle = \langle M_T^s(\Delta^{(m)})x, x \rangle \geq 0$$

as $\Delta_T^{(m)} \geq 0$. Thus a_s is a nonincreasing sequence of real numbers. Further

$$\left| \sum_{s=0}^{N} a_s \right| = \left| \sum_{s=0}^{N} \langle (1 - M_T) M_T^s (\Delta_T^{(m-2)}) x , x \rangle \right|$$

$$= |\langle \Delta_T^{(m-2)} x , x \rangle - \langle M_T^{N+1} (\Delta_T^{(m-2)}) x , x \rangle|$$

$$\leq \left(1 + \sup_s \| M_T^s(1) \| \right) \| (1 - M_T)^{m-2} \| \, \| x \|^2 < \infty.$$

We conclude that $a_s \geq 0$ for every s. In particular, $a_0 = \langle \Delta_T^{(m-1)} x , x \rangle \geq 0$, hence $\Delta_T^{(m-1)} \geq 0$.

The assertion (c) now follows by a simple induction.

Corollary 3. *If* $T \in \mathscr{L}(H)^n$ *is a c.m. such that* $\Delta_T^{(1)} \geq 0$ *and* $\Delta_T^{(m)} \geq 0$ *for some integer* $m \geq 1$, *then* $0 \leq \Delta_T^{(m)} \leq \Delta_T^{(m-1)} \leq \cdots \leq \Delta_T^{(1)} \leq 1$.

Proof. For $k = 1, 2, \ldots, m - 1$, we have $\Delta_T^{(k)} - \Delta_T^{(k+1)} = (1 - M_T)^k(1) - (1 - M_T)^{k+1}(1) = M_T(1 - M_T)^k(1) \geq 0$ as $(1 - M_T)^k(1) = \Delta_T^{(k)} \geq 0$ by Lemma 2.

Lemma 4. *Let* $T \in \mathscr{L}(H)^n$ *be a c.m., and let* $m \geq 1$, $N \geq 1$ *be integers. Then*

$$\sum_{s=0}^{N} \binom{s + m - 1}{m - 1} M_T^s \Delta_T^{(m)} \Delta_T^{(m)} = 1 - \sum_{j=0}^{m-1} \binom{N + j}{j} M_T^{N+1} \Delta_T^{(j)}.$$

Proof. We prove the assertion by induction with respect to m. If $m = 1$, we have

$$\sum_{s=0}^{N} M_T^s \Delta_T^{(1)} = \sum_{s=0}^{N} M_T^s (1 - M_T)(1) = 1 - M_T^{N+1}(1).$$

Now suppose that the assertion holds for $m - 1$. Then

$$\sum_{s=0}^{N} \binom{s + m - 1}{m - 1} M_T^s \Delta_T^{(m)}$$

$$= \sum_{s=0}^{N} \binom{s + m - 1}{m - 1} (1 - M_T) M_T^s \Delta_T^{(m-1)}$$

$$= \Delta_T^{(m-1)} + \sum_{s=1}^{N} \left[\binom{s + m - 1}{m - 1} - \binom{s + m - 2}{m - 1} \right] M_T^s \Delta_T^{(m-1)}$$

$$- \binom{N + m - 1}{m - 1} M_T^{N+1} \Delta_T^{(m-1)}$$

$$= \sum_{s=0}^{N} \binom{s + m - 2}{m - 2} M_T^s \Delta_T^{(m-1)} - \binom{N + m - 1}{m - 1} M_T^{N+1} \Delta_T^{(m-1)}$$

$$= 1 - \sum_{j=0}^{m-1} \binom{N + j}{j} M_T^{N+1} \Delta_T^{(j)},$$

by the induction hypothesis.

Lemma 5. *Let $T \in \mathscr{L}(H)^n$ be a c.m. such that $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$ for some integer $m \geq 1$. Let also $x \in H$.*

(a) *The limit $\lim_{N \to \infty} N^k \langle M_T^N(\Delta_T^{(k)})x, x \rangle$ exists for all $k = 0, \ldots, m-1$ and is null for $1 \leq k \leq \max\{1, m-1\}$.*

(b) *We have $\sum_{s=0}^{\infty} s^{k-1} \langle M_T^s(\Delta_T^{(k)})x, x \rangle < \infty$, $k = 1, \ldots, m$.*

Proof. The assertions will be proved simultaneously by induction with respect to k. Since $\Delta_T^{(1)} \geq 0$ and $M_T^N \Delta_T^{(1)} \geq 0$ for each integer $N \geq 1$, we have

$$\langle M_T^N(1)x, x \rangle - \langle M_T^{N+1}(1)x, x \rangle = \langle M_T^N(\Delta_T^{(1)})x, x \rangle \geq 0.$$

Thus $\{\langle M_T^N(1)x, x \rangle\}_N$ is a nonincreasing sequence of nonnegative numbers, and so the limit

$$(5) \qquad\qquad \lim_{N \to \infty} \langle M_T^N(1)x, x \rangle$$

exists. Hence (a) holds for $k = 0$.

Note also that

$$\sum_{s=0}^{\infty} \langle M_T^s(\Delta_T^{(1)})x, x \rangle = \lim_{N \to \infty} \sum_{s=0}^{N} \langle (1 - M_T)M_T^s(1)x, x \rangle$$

$$= \lim_{N \to \infty} \langle (1 - M_T^{N+1})(1)x, x \rangle = \|x\|^2 - \lim_{N \to \infty} \langle M_T^{N+1}(1)x, x \rangle < \infty,$$

where the limit exists by (5). Thus (b) holds for $k = 1$.

Let us prove now that if (b) holds for some k, $1 \leq k \leq \max\{1, m-1\}$, and (a) holds for $k - 1$, then (a) also holds for k. Indeed, if we set $a_s = \langle M_T^s(\Delta_T^{(k)})x, x \rangle$, we have $a_s \geq 0$ and $a_s - a_{s+1} = \langle M_T(\Delta_T^{(k+1)})x, x \rangle \geq 0$ for all s, since $\Delta_T^{(k)} \geq 0$ and $\Delta_T^{(k+1)} \geq 0$ by Lemma 2. Thus $\{a_s\}_s$ is a nonincreasing sequence of nonnegative numbers satisfying $\sum_{s=0}^{\infty} s^{k-1} a_s < \infty$ in virtue of the induction assumption. By [8, Lemma 3.8] it follows that $\lim_{s \to \infty} s^k a_s = 0$, which shows that (a) holds for k.

Conversely, if (a) holds for some k, $1 \leq k \leq \max\{1, m-1\}$, and (b) holds for k, then (b) also holds for $k + 1$. Indeed,

$$\sum_{s=0}^{\infty} s^k \langle M_T^s(\Delta_T^{(k+1)})x, x \rangle$$

$$(6) \qquad = \lim_{N \to \infty} \sum_{s=0}^{N} s^k \langle (1 - M_T)M_T^s(\Delta_T^{(k)})x, x \rangle$$

$$= \lim_{N \to \infty} \left[\sum_{s=1}^{N} (s^k - (s-1)^k) \langle M_T(\Delta_T^{(k)})x, x \rangle - N^k \langle M_T^{N+1}(\Delta_T^{(k)})x, x \rangle \right]$$

$$+ \langle (1 - M_T)(\Delta_T^{(k)})x, x \rangle.$$

As $\lim_{N \to \infty} N^k \langle M_T^{N+1}(\Delta_T^{(k)})x, x \rangle = 0$ by the induction hypothesis and $s^k - (s-1)^k \leq k s^{k-1}$, we infer from (6) that

$$\sum_{s=0}^{\infty} s^k \langle M_T^s(\Delta_T^{(k+1)})x, x \rangle < \infty$$

via (b) for k, which finishes the proof of Lemma 5.

Let m be a positive integer. We define the function $\rho_m\colon \mathbf{Z}_+^n \to \mathbf{Z}_+$ by the formula

$$\rho_m(\alpha) = \frac{(m + |\alpha| - 1)!}{\alpha!(m - 1)!}, \qquad \alpha \in \mathbf{Z}_+^n.$$

Lemma 6. *The numbers $\rho_m(\alpha)$ satisfy the equation*

$$(7) \qquad \sum_{\substack{|\beta| \leq m \\ \beta \leq \alpha}} (-1)^{|\beta|} \frac{m!}{\beta!(m - |\beta|)!} \rho_m(\alpha - \beta) = 0, \qquad \alpha \neq 0.$$

Proof. If λ is a complex number such that $|\lambda| < 1$, then we have the following well-known series expansion

$$(8) \qquad (1 - \lambda)^{-m} = \sum_{k=0}^{\infty} \binom{m + k - 1}{k} \lambda^k.$$

Therefore, if $z = (z_1, \ldots, z_n) \in C^n$ satisfies $\sum_{j=1}^{n} |z_j| < 1$, we can write

$$(9) \qquad \left(1 - \sum_{j=1}^{n} z_j\right)^{-m} = \sum_{k=0}^{\infty} \binom{m + k - 1}{k} \sum_{|\gamma| \leq k} \frac{k!}{\gamma!} z^\gamma = \sum_{\gamma \geq 0} \rho_m(\gamma) z^\gamma.$$

Note also that

$$(10) \qquad \left(1 - \sum_{j=1}^{n} z_j\right)^{m} = \sum_{|\beta| \leq m} (-1)^{|\beta|} \frac{m!}{\beta!(m - |\beta|)!} z^\beta.$$

By multiplying (9) and (10) we obtain the identity

$$\sum_{\alpha \geq 0} \sum_{\substack{|\beta| \leq m \\ \beta \leq \alpha}} (-1)^{|\beta|} \frac{m!}{\beta!(m - |\beta|)!} \rho_m(\alpha - \beta) z^\alpha = 1,$$

whence we easily derive (7).

Consider the Hilbert space $\mathscr{H} = l^2(\mathbf{Z}_+^n, H)$ consisting of all functions $f\colon \mathbf{Z}_+^n \to H$ such that

$$(11) \qquad \|f\|^2 = \sum_{\alpha \in \mathbf{Z}_+^n} |f(\alpha)|^2 < \infty.$$

Clearly, $\mathscr{H}$ has a Hilbert space structure with the norm given by (11). We define on the space $\mathscr{H}$ the operators $S_j^{(m)}$ by the formula

$$(12) \qquad (S_j^{(m)} f)(\alpha) = [\rho_m(\alpha + e_j)^{-1} \rho_m(\alpha)]^{1/2} f(\alpha + e_j), \qquad j = 1, \ldots, n,$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{Z}_+^n$ with 1 on the jth position and $m \geq 1$ is a fixed integer. Then $S^{(m)} = (S_1^{(m)}, \ldots, S_n^{(m)}) \in \mathscr{L}(H)^n$ is a c.m., which is called the *backwards multishift* of type (n, m).

It is easy to see that $S_j^{(m)*}$ is given by

$$(13) \qquad (S_j^{(m)*} f)(\alpha) = \begin{cases} 0 & \text{if } \alpha_j = 0, \\ [\rho_m(\alpha)^{-1} \rho_m(\alpha - e_j)]^{1/2} f(\alpha - e_j) & \text{if } \alpha_j \geq 1. \end{cases}$$

Lemma 7. *The backwards multishift* $S^{(m)}$ *has the following properties*:

(a) $\Delta^{(1)}_{S^m} \geq 0$.

(b) $\Delta^{(m)}_{S^{(m)}} \geq 0$.

(c) $\lim_{s \to \infty} M^s_{S^{(m)}}(1) = 0$ *in the strong operator topology*.

Proof. For every $h \in H$ and $\alpha \in \mathbf{Z}^n_+$ we denote by $h_\alpha \in \mathscr{H}$ the function $h_\alpha \colon \mathbf{Z}^n_+ \to H$ defined by $h_\alpha(\alpha) = h$, $h_\alpha(\beta) = 0$ if $\beta \neq \alpha$. The finite sums of elements from the set $\{h_\alpha, \ \alpha \in \mathbf{Z}^n_+, \ h \in H\}$ form a linear subspace dense in $\mathscr{H}$.

Note that for all $h \in H$ and $\alpha \in \mathbf{Z}^n_+$ we have

$$S^{(m)\beta} h_\alpha = \begin{cases} [\rho_m(\alpha)^{-1}\rho_m(\alpha - \beta)]^{1/2} h_{\alpha - \beta} & \text{if } \beta \leq \alpha, \\ 0 & \text{if } \beta \not\leq \alpha \end{cases}$$

and, by (13),

$$(14) \qquad S^{(m)*\beta} S^{(m)\beta} h_\alpha = \begin{cases} \rho_m(\alpha)^{-1}\rho_m(\alpha - \beta) h_\alpha & \text{if } \beta \leq \alpha, \\ 0 & \text{if } \beta \not\leq \alpha. \end{cases}$$

Therefore, using (14) for $\beta = e_j$ $(j = 1, \ldots, n)$,

$$\Delta^{(1)}_{S^{(m)}} h_\alpha = h_\alpha - \sum_{\substack{1 \leq j \leq n \\ \alpha_j \neq 0}} \rho_m(\alpha)^{-1} \rho_m(\alpha - e_j) h_\alpha$$

$$= \left(1 - \sum_{j=1}^n \frac{\alpha_j}{|\alpha| + m - 1}\right) h_\alpha = \frac{m-1}{|\alpha| + m - 1} h_\alpha,$$

whence we infer $\Delta^{(1)}_{S^{(m)}} \geq 0$, and so (a) holds.

To prove (b) we observe that if $h \in H$ and $\alpha \in \mathbf{Z}^n_+$, $\alpha \neq 0$, then

$$\Delta^{(m)}_{S^{(m)}} h_\alpha = \sum_{|\beta| \leq m} (-1)^{|\beta|} \frac{m!}{\beta!(m - |\beta|)!} S^{(m)*\beta} S^{(m)\beta} h_\alpha$$

$$= \sum_{\substack{|\beta| \leq m \\ \beta \leq \alpha}} (-1)^{|\beta|} \frac{m!}{\beta!(m - |\beta|)!} \frac{\rho_m(\alpha - \beta)}{\rho_m(\alpha)} h_\alpha = 0,$$

by (14) and Lemma 6.

If $\alpha = 0$, then $\Delta^{(m)}_{S^{(m)}} h_\alpha = h_\alpha$. Therefore $\Delta^{(m)}_{S^{(m)}} \geq 0$, which establishes (b).

Let again $h \in H$ and $\alpha \in \mathbf{Z}^n_+$. If $s > |\alpha|$ and $\beta \in \mathbf{Z}^n_+$, $|\beta| = s$, then $\beta_j > \alpha_j$ for some $j \in \{1, \ldots, n\}$. Thus $S^{(m)*\beta} S^{(m)\beta} h_\alpha = 0$ by (14) and hence $M^s_{S^{(m)}}(1) h_\alpha = 0$ by (2). Therefore $\lim_{s \to \infty} M^s_{S^{(m)}}(1) f = 0$ for every linear combination of elements of the form h_α, which is a dense subspace of $\mathscr{H}$. As $\|M^s_{S^{(m)}}\| \leq 1$ for all s by Lemma 1, it follows that (c) also holds.

Let $T \in \mathscr{L}(H)^n$ be a c.m. such that $\Delta^{(1)}_T \geq 0$ and $\Delta^{(m)}_T \geq 0$ for some integer $m \geq 1$. We define an operator $V \colon H \to \mathscr{H}$ by the relation

$$(15) \qquad (Vh)(\alpha) = (\rho_m(\alpha)\Delta^{(m)}_T)^{1/2} T^\alpha h, \qquad h \in H, \ \alpha \in \mathbf{Z}^n_+.$$

Notice that for all $h \in H$ and $\alpha \in \mathbf{Z}^n_+$ we have

$$(VT_j h)(\alpha) = (\rho_m(\alpha)\Delta^{(m)}_T)^{1/2} T^{\alpha + e_j} h$$

and
$$(S_j^{(m)} V h)(\alpha) = [\rho_m(\alpha + e_j)^{-1} \rho_m(\alpha)]^{1/2} (\rho_m(\alpha + e_j) \Delta_T^{(m)})^{1/2} T^{\alpha + e_j} h .$$

Hence

(16) $$V T_j = S_j^{(m)} V , \qquad j = 1 , \dots , n .$$

Lemma 8. *With the previous notation one has*
$$\|V h\|^2 = \|h\|^2 - \lim_{s \to \infty} \langle M_T^s(1) h , h \rangle$$

for every $h \in H$.

Proof. We have
$$\|V h\|^2 = \sum_{\alpha \in \mathbf{Z}_+^n} \rho_m(\alpha) \langle \Delta_T^{(m)} T^\alpha h , T^\alpha h \rangle$$

$$= \sum_{k=0}^\infty \frac{(k + m - 1)!}{k!(m-1)!} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \langle \Delta_T^{(m)} T^\alpha h , T^\alpha h \rangle$$

$$= \sum_{k=0}^\infty \binom{k + m - 1}{m - 1} \langle M_T^k(\Delta_T^{(m)}) h , h \rangle$$

$$= \lim_{N \to \infty} \sum_{k=0}^N \binom{k + m - 1}{m - 1} \langle M_T^k(\Delta_T^{(m)}) h , h \rangle$$

$$= \|h\|^2 - \lim_{N \to \infty} \sum_{j=0}^{m-1} \binom{N + j}{j} \langle M_T^{N+1}(\Delta_T^{(j)}) h , h \rangle$$

$$= \|h\|^2 - \lim_{N \to \infty} \langle M_T^{N+1}(1) h , h \rangle$$

by Lemmas 4 and 5.

Theorem 9. *Let $T \in \mathscr{L}(H)^n$ be a c.m., and let $m \geq 1$ be an integer. The following conditions are equivalent:*

(a) $\Delta_T^{(1)} \geq 0$, $\Delta_T^{(m)} \geq 0$, *and* $\lim_{s \to \infty} M_T^s(1) = 0$ *in the strong operator topology.*

(b) T *is unitarily equivalent to the restriction of $S^{(m)}$ to an invariant subspace.*

Proof. (a) $\Rightarrow$ (b) Consider the operator $V : H \to \mathscr{H}$ defined by (15). It follows from Lemma 8 that V is an isometry. Moreover, the space $V H$ is invariant under $S^{(m)}$ by (16), and so (b) holds.

(b) $\Rightarrow$ (a) Let $K \subset \mathscr{H}$ be a (closed, linear) subspace invariant under $S^{(m)}$. If $T_j = S_j^{(m)} | K$ and $x \in K$, it is easily checked that
$$\langle \Delta_T^{(k)} x , x \rangle = \langle \Delta_{S^{(m)}}^{(k)} x , x \rangle , \qquad k = 1 , 2 , \dots .$$

In particular, $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$ by Lemma 7.

Moreover, if P_K is the orthogonal projection of $\mathscr{H}$ onto K, as $T^{*\alpha} = P_K S^{(m)*\alpha}$ for all α, we have
$$\lim_{s \to \infty} \|M_T^s(1) x\| = \lim_{s \to \infty} \|P_K M_{S^{(m)}}^s(1) x\| \leq \lim_{s \to \infty} \|M_{S^{(m)}}^s(1) x\| = 0 ,$$

again by Lemma 7. This establishes (a).

We recall that a c.m. $T \in \mathscr{L}(H)^n$ is said to be spherical isometry if $\Delta_T^{(1)} = 0$, i.e., $T_1^* T_1 + \cdots + T_n^* T_n = 1$ (see [4]).

Lemma 10. *Let* $T \in \mathscr{L}(H)^n$ *be a c.m. satisfying* $\Delta_T^{(1)} \geq 0$. *Then there exists a Hilbert space* $\mathscr{K}$, *a spherical isometry* $W \in \mathscr{L}(\mathscr{K})^n$, *and an operator* $V_0 \colon H \to \mathscr{K}$ *such that* $V_0 T_j = W_j V_0$ $(j = 1, \ldots, n)$ *and*

$$\|V_0 x\|^2 = \lim_{s \to \infty} \langle M_T^s(1)x, x \rangle$$

for every $x \in H$.

Proof. We have $1 \geq M_T(1) \geq M_T^2(1) \geq \cdots \geq 0$, so that $\lim_{s \to \infty} \langle M_T^s(1)x, x \rangle$ exists for every $x \in H$.

Define on H the seminorm

(17) $$\|x\|_0^2 = \lim_{s \to \infty} \langle M_T^s(1)x, x \rangle.$$

Let $N = \{x \in H, \|x\|_0 = 0\}$, and let $\mathscr{K}$ be the completion of the quotient H/N with respect to the norm induced by (17), which is clearly a Hilbert space. Let also $V_0 \colon H \to \mathscr{K}$ be given by $V_0 h = h + N$ $(h \in H)$. Then $\|V_0 x\|^2 = \|x\|_0^2 = \lim_{s \to \infty} \langle M_T^s(1)x, x \rangle$.

Let $W_1, \ldots, W_n$ be the linear operators defined by

(18) $$W_j(h + N) = T_j h + N, \qquad h \in H, \ j = 1, \ldots, n.$$

Since, by (2),

$$\|T_j h + N\|_0^2 = \lim_{s \to \infty} \langle M_T^s(1)T_j h, T_j h \rangle$$

$$\leq \lim_{s \to \infty} \sum_{k=1}^n \langle M_T^s(1)T_k h, T_k h \rangle = \lim_{s \to \infty} \langle M_T^{s+1}(1)h, h \rangle = \|h\|_0^2.$$

It follows that each W_j is correctly defined and it can be continuously extended to the whole space $\mathscr{K}$; this extension will also be denoted by W_j. Note that $V_0 T_j = W_j V_0$, which follows from (18).

It remains to prove that $W = (W_1, \ldots, W_n)$ is a spherical isometry. Indeed, for every $h \in H$ we have

$$\sum_{j=1}^n \|W_j(h + N)\|_0^2 = \sum_{j=1}^n \lim_{s \to \infty} \langle M_T^s(1)T_j h, T_j h \rangle$$

$$= \sum_{j=1}^n \langle T_j^* M_T^s(1)T_j h, h \rangle = \langle M_T^{s+1}(1)h, h \rangle,$$

whence

$$\sum_{j=1}^n \|W_j(h + N)\|_0^2 = \|h\|_0^2.$$

Theorem 11. *Let* $T \in \mathscr{L}(H)^n$ *be a c.m., and let* $m \geq 1$ *be an integer. The following statements are equivalent:*

(a) $\Delta_T^{(1)} \geq 0$ *and* $\Delta_T^{(m)} \geq 0$.

(b) T *is unitarily equivalent to the restriction of* $S^{(m)} \oplus W$ *to an invariant subspace, where* W *is a spherical isometry on some Hilbert space* $\mathscr{K}$.

(c) T *is unitarily equivalent to the restriction of* $S^{(m)} \oplus N$ *to an invariant subspace, where* $N = (N_1, \ldots, N_n)$ *and* $N_1, \ldots, N_n$ *are commuting normal operators on a Hilbert space* $\mathscr{K}'$ *satisfying* $N_1^* N_1 + \cdots + N_n^* N_n = 1$.

Proof. (a) $\Rightarrow$ (b) Let $V: H \rightarrow \mathscr{H}$ be the operator defined by (15) and let $\mathscr{K}$, $V_0: H \rightarrow \mathscr{K}$ and $W \in \mathscr{L}(\mathscr{K})^n$ be given by Lemma 10. If we set $V_1 h = Vh \oplus V_0 h$ for each $h \in H$, then $V_1: K \rightarrow \mathscr{H} \oplus \mathscr{K}$ is an isometry, which satisfies $V_1 T^\alpha h = (S^{(m)\alpha} \oplus W^\alpha) V_1 h$ for all $\alpha \in Z_+^n$ and $h \in H$ by (18) and Lemmas 8 and 10.

(b) $\Rightarrow$ (c) By a result of Athavale [4], the spherical isometry $W \in \mathscr{L}(K)^n$ has an extension $N = (N_1, \cdots, N_n) \in \mathscr{L}(\mathscr{K}')^n$, where $\mathscr{K}'$ is a Hilbert space containing $\mathscr{K}$, $N_1, \ldots, N_n$ are commuting normal operators on $\mathscr{K}'$ satisfying $N_1^* N_1 + \cdots + N_n^* N_n = 1$, and $N_j | H = W_j$ $(j = 1, \ldots, n)$. This gives (c).

(c) $\Rightarrow$ (a) We have $\Delta_{S^{(m)}}^{(1)} \geq 0$ and $\Delta_{S^{(m)}}^{(m)} \geq 0$ by Lemma 8. Further $\Delta_N^{(1)} = 0$, and so $\Delta_N^{(m)} = (1 - M_N)^{m-1} \Delta_N^{(1)} = 0$. Thus $\Delta_{S^{(m)} \oplus N}^{(1)} \geq 0$ and $\Delta_{S^{(m)} \oplus N}^{(m)} \geq 0$. As these inequalities remain true for any restriction of $S^{(m)} \oplus N$ to an invariant subspace, (a) is established.

Remarks. $1°$. The case $m = n = 1$. This is the classical Sz.-Nagy-Foiaş dilation theory [9]. In this case, $\Delta_T^{(1)} = 1 - T_1^* T_1$ is the square of the defect operator and $\Delta_T^{(1)} \geq 0$ holds if and only if $\|T_1\| \leq 1$. The condition $\lim_{s \to \infty} M_T^s(1)x = 0$ for every $x \in H$ reduces in this case to the C_0 condition $\lim_{s \to \infty} T_1^s x = 0$ $(x \in H)$.

$2°$. The case $n = 1$, $m \geq 2$. This is the case of m-hypercontractions of Agler [1] (cf. also [8]).

$3°$. The case $n \geq 2$, $m = 1$. This case has been studied by Drury [7] (see also [5]).

$4°$. The case $m = n$. This case has been studied in [10], (see also [4]), and is particularly interesting since one has the additional property $\Delta_{S^{(n)*}}^{(1)} = 0$, i.e., $S^{(n)*}$ is a spherical isometry. Indeed, if $h \in H$ and $\alpha \in Z_+^n$, then, with the notation of Lemma 8,

$$\sum_{j=1}^n S_j^{(n)} S_j^{(n)*} h_\alpha = \sum_{j=1}^n \frac{\rho_n(\alpha)}{\rho_n(\alpha + e_j)} h_\alpha = \left(\sum_{j=1}^n \frac{\alpha_j + 1}{|\alpha| + n} \right) h_\alpha = h_\alpha,$$

and so the same equality holds for all $f \in \mathscr{H}$. By the above-mentioned result of Athavale [4], it follows from Theorem 11 that every c.m. $T \in \mathscr{L}(H)^n$ satisfying $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(n)} \geq 0$ has a spherical dilation, i.e., there exist a Hilbert space $K \supset H$ and a c.m. $N = (N_1, \ldots, N_n) \in \mathscr{L}(K)^n$ consisting of normal operators such that $N_1^* N_1 + \cdots + N_n^* N_n = 1$ on K and $T^\alpha = P_H N^\alpha | H$ for all $\alpha \in Z_+^n$, where P_H is the orthogonal projection of K onto H. So in this case a complete analogy with the Sz.-Nagy-Foiaş theory might be developed (see [10]).

In particular, for a c.m. $T \in \mathscr{L}(H)^n$ such that $\Delta_T^{(1)} \geq 0$, $\Delta_T^{(n)} \geq 0$, and $\lim_{s \to \infty} M_T^s(1)x = 0$, $x \in H$, we derive the existence of a contractive unital algebra homomorphism $\Phi_T: H^\infty(B) \rightarrow \mathscr{L}(H)$ as in Theorem 4.20 from [10].

$5°$. If $m = n + 1$, then the multishift operator $S_j^{(m)*}$ $(j = 1, \ldots, n)$ is unitarily equivalent to the multiplication by the variables z_j $(j = 1, \ldots, n)$ on the Bergman space. The Bergman space consists of all analytic functions in the open unit ball B of $\mathbf{C}^n$, which lie in $L^2(B, d\nu)$, where ν is the normalized Lebesgue measure.

Analogously, for $m > n$, $S_j^{(m)*}$ $(j = 1, \ldots, n)$ is unitarily equivalent to the multiplication by z_j in the weighted Bergman space, which consists of all analytic functions in B that lie in $L^2(B, d\mu)$, where $d\mu = c_m(1-|z|^2)^{m-n-1}d\nu(z)$ and c_m is a normalization constant chosen such that $\mu(B) = 1$ (see, e.g., [2, 5]).

6°. Condition $\Delta_T^{(1)} \geq 0$ is necessary but not sufficient to ensure the existence of a spherical dilation for a c.m. $T \in \mathscr{L}(H)^n$ if $n \geq 2$ (cf. also [7]). Indeed, with K and N as in 4°, we have

$$\sum_{j=1}^n \langle T_j^* T_j x, x \rangle = \sum_{j=1}^n \langle P_H N_j x, P_H N_j x \rangle \leq \sum_{j=0}^n \langle N_j x, N_j x \rangle = \|x\|^2$$

for each $x \in H$, i.e., $\Delta_T^{(1)} \geq 0$. Similarly, $\Delta_{T^*}^{(1)} \geq 0$. On the other hand, if $1 \leq m < n$, with $S^{(m)}$, h, and h_α as in Lemma 7, one deduces from (12) and (13) that

$$\langle \Delta_{S^{(m)*}}^{(1)} h_\alpha, h_\alpha \rangle = \left(1 - \sum_{j=1}^n \frac{\rho_m(\alpha)}{\rho_m(\alpha + e_j)}\right)\|h\|^2$$

$$= \left(1 - \sum_{j=1}^n \frac{\alpha_j + 1}{m + |\alpha|}\right)\|h\|^2 = \frac{m - n}{|\alpha| + m}\|h\|^2 < 0$$

if $h \neq 0$. Hence $S^{(m)}$ cannot have a spherical dilation.

7°. If $\Delta_T^{(1)} \geq 0$, then the joint (Taylor) spectrum $\sigma(T)$ is contained in the closed unit ball of $\overline{B}$ of $\mathbf{C}^n$. To prove this, denote by

$$\sigma_\pi(T) = \left\{\lambda \in \mathbf{C}^n, \inf\left\{\sum_{i=1}^n \|(T_i - \lambda_i)x\|, \ x \in H, \ \|x\| = 1\right\} = 0\right\}$$

the joint approximate point spectrum of T.

If $\lambda \in \sigma_\pi(T)$, then there exists a sequence $\{x_k\}_k \subset H$, $\|x_k\| = 1$ with $\lim_{k\to\infty}(T_i - \lambda_i)x_k = 0$ $(i = 1, \ldots, n)$ and $1 = \|x_k\| \geq \sum_{i=1}^n \|T_i x_k\|^2 \to \sum_{i=1}^n |\lambda_i|^2$. So $\sigma_\pi(T) \subset \overline{B}$ and, by [6], $\sigma(T) \subset \overline{B}$ (this is true for any reasonable joint spectrum).

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