STAR OPERATIONS ON OVERRINGS AND SEMISTAR OPERATIONS

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Abstract. The purpose of this paper is to deepen the study of the relation between semistar operations on an integral domain D and the (semi)star operations (that is, the semistar operations that restricted to the set of the fractional ideals are star operations) on the overrings of D. First, we define the composition of two semistar operations and we study when this composition is a semistar operation. Then we show that there is a bijection between the set of all semistar operations on a domain D and the set of all (semi)star operations on the overrings of D. To do this, we prove that semistar operations on D have a canonical decomposition as the composition of a semistar operation given by the extension to an overring and a (semi)star operation on this overring. Moreover, we study which properties of semistar operations are preserved by this bijection. Finally, we give some applications to the study of semistar operations on valuation and Prüfer domains and we give, by using the techniques introduced in this paper, a characterization of generalized Dedekind domains in terms of the H-domains introduced by Glaz and Vasconcelos

1. Introduction and background results

In 1994, Okabe and Matsuda [26] introduced the concept of semistar operation to extend the notion of classical star operations as described in [14, Section 32]. Star operations have been proven to be an essential tool in multiplicative ideal theory, allowing one to study different classes of integral domains. Semistar operations, thanks to a higher flexibility, permit a finer study and classification of integral domains.

After the introduction of semistar operations, several authors have worked on this topic, extending to semistar operations results and notions previously studied in the context of star operations, see for instance [7], [8], [11], [25], [22] and, for similar results in the language of monoids and module systems, [17] (see in particular [17, Remark 3.4] for the connection between module systems and semistar operations). Already in the paper of Okabe and Matsuda [26], it was shown how a semistar operation $\star$ on a domain D induces on the overring $D^\star$ of D a (semi)star operation, that is, a semistar operation that restricted to fractional ideals is a star operation.

The purpose of this paper is to deepen the study of this relation between semistar operations and the (semi)star operations induced on the overrings. This should permit one to have a clearer idea of which of the results already obtained for star operations can be transferred to the context of semistar operations.

After recalling the main results needed in this paper, in Section 2 we introduce, in a natural way (by using the fact that semistar operations are maps of a particular

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type), the composition of two semistar operations. We show that it is not in general a semistar operation and we give some conditions for the composition of two semistar operations to be a semistar operation. In Section 3, we show that there is a bijection between the set of semistar operations on a domain D and the set of all (semi)star operations on the overrings of D. To do this, we show that semistar operations on D have a canonical decomposition as the composition of a semistar operation given by the extension to an overring and a (semi)star operation on this overring. Moreover, we study which properties of semistar operations are preserved by this bijection. More precisely, we show that the finite character and some relevant cancellation properties are preserved by this bijection and, then, that there is a bijection between the set of all semistar operations on D with these properties and the set of all (semi)star operation with the same properties on the overrings of D. By contrast, we show that properties like stability and the spectral property do not behave so well with respect to this map, but that they need some additional flatness-like property on the overring to be preserved.

Finally, we give some examples of how the techniques developed in Section 3 can be applied to the study of semistar operations. We prove in a different way some results (some of them already proven in [22] and [25] in the finite dimensional case) about valuation domains and about Prüfer domains in which all semistar operations are of finite type. Then, we characterize local domains in which each semistar operation is stable and finally we give a characterization of generalized Dedekind domains in terms of the H-domains introduced by Glaz and Vasconcelos, [15]. (For a generalization of H-domains in the semistar setting, see [13].)

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Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D–submodules of K and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e. $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D–submodules of K. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

A *semistar operation* on D is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D), E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

cf. for instance [8]. Recall that [8, Theorem 1.2 and p. 174], for all $E, F \in \overline{\boldsymbol{F}}(D)$, we have :

$$\begin{array}{c}(EF)^\star = (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star \ ; \\ (E+F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star \ ; \\ (E:F)^\star \subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star \ ; \\ (E \cap F)^\star \subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star \ , \quad \text{if } E \cap F \neq (0)\,;\end{array}$$

A *(semi)star operation* is a semistar operation that, restricted to $\boldsymbol{F}(D)$, is a star operation (in the sense of [14, Section 32]). It is easy to see that a semistar operation $\star$ on D is a (semi)star operation if and only if $D^\star = D$.

A quasi–$\star$–ideal I of D is a nonzero ideal such that $I = I^\star \cap D$. A quasi–$\star$–prime is a quasi–$\star$–ideal that is also a prime ideal. A quasi–$\star$–maximal ideal is an ideal that is a maximal element in the set of quasi–$\star$–prime ideals.

If $\star_1$ and $\star_2$ are two semistar operations on D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$.

Example 1.1. **(1)** The easiest semistar operation on D is the *identity semistar operation*, denoted by d_D (or simply d), defined by $E \mapsto E^d := E$, for each $E \in \overline{\boldsymbol{F}}(D)$. Another trivial semistar operation on D is the e operation, given by $E \mapsto E^e := K$, for each $E \in \overline{\boldsymbol{F}}(D)$. It is clear that $d \leq \star \leq e$, for each semistar operation $\star$ on D.

(2) A star operation $\star$ on D induces canonically a (semi)star operation $\star_e$ on D (the *trivial extension* of $\star$) defined by $E^{\star_e} := E^\star$, if $E \in \boldsymbol{F}(D)$, and $E^{\star_e} := K$ otherwise. (For a discussion of the trivial extension of the identity star operation, see Remark 3.10.)

(3) Denote by v_D (or, simply, v) the *v–(semi)star operation* on D defined by $E^v := (E^{-1})^{-1}$, for each $E \in \overline{\boldsymbol{F}}(D)$, with $E^{-1} := (D :_K E) := \{z \in K \mid \ zE \subseteq D\}$. It is easy to see that this semistar operation coincides with the trivial extension of the classical v-operation ([14, Section 34]). We note that if $\star$ is a (semi)star operation on D, then $\star \leq v_D$ (see [14, Theorem 34.1(4)]).

(4) If T is an overring of D, the map $\star_{\{T\}}$, given by $E \mapsto E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\boldsymbol{F}}(D)$ is a semistar operation (called *extension to the overring* T). More generally, if $\{R_\alpha\}_{\alpha \in A}$ is a set of overrings of D, and, for each $\alpha \in A$, $\star_\alpha$ is a semistar operation on R_α, the map $E \mapsto \bigcap\{(ER_\alpha)^{\star_\alpha} \mid \alpha \in A\}$, for each $E \in \overline{\boldsymbol{F}}(D)$, is a semistar operation.

(5) Among the semistar operations in the class defined in **(4)**, particularly interesting are the semistar operations induced by overrings that are localizations of D at prime ideals. More precisely, if $\Delta \subseteq \mathrm{Spec}(D)$, we denote by $\star_\Delta$ the semistar operation defined by $E \mapsto E^{\star_\Delta} := \bigcap\{ED_P \mid P \in \mathrm{Spec}(D)\}$. We refer to these semistar operations as *spectral semistar operations*. If $\Delta = \{P\}$, where $P \in \mathrm{Spec}(D)$, we denote $\star_\Delta$ simply by $\star_{\{P\}}$.

(6) If, in the construction given in **(4)**, we let all $R_\alpha = D$, the semistar operation $E \mapsto \bigcap\{E^{\star_\alpha} \mid \alpha \in A\}$ is denoted by $\wedge\star_\alpha$ and is the largest semistar operation $\star$ on D such that $\star \leq \star_\alpha$ for each α. Moreover, if we have a family $\{\star_\beta\}_{\beta \in B}$, we define a new semistar operation as $\vee\star_\beta := \wedge\{\star \mid \star_\beta \leq \star,\ \beta \in B\}$. This is the smallest semistar operation $\star$ on D such that $\star_\beta \leq \star$ for each $\beta \in B$. In the case of star operations, these constructions are investigated in [1].

(7) If $\star$ is a semistar operation on D, then we can consider a map $\star_{_f} : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined for each $E \in \overline{\boldsymbol{F}}(D)$ as follows: $E^{\star_{_f}} := \bigcup\{F^\star \mid F \in \boldsymbol{f}(D) \text{ and } F \subseteq E\}$. It is easy to see that $\star_{_f}$ is a semistar operation on D, called *the semistar operation of finite type associated to* $\star$. Note that, for each $F \in \boldsymbol{f}(D)$, $F^\star = F^{\star_{_f}}$. A semistar operation $\star$ is called a *semistar operation of finite type* if $\star = \star_{_f}$. It is easy to see that $(\star_{_f})_{_f} = \star_{_f}$ (that is, $\star_{_f}$ is of finite type).

(8) The semistar operation of finite type $(v_D)_f$ (or, simply, $v_{_f}$) associated to v_D is denoted by t_D (or, simply, t) and it is called the *t–(semi)star operation* on D. We note that, if $\star$ is a (semi)star operation of finite type on D, then $\star \leq t$ (by **(3)** and the fact that the passage to the finite type semistar operation associated is order preserving).

Note also that, for each overring T of D, the semistar operation $\star_{\{T\}}$ on D is a semistar operation of finite type.

A semistar operation $\star$ is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$, for each $E, F \in \overline{\boldsymbol{F}}(D)$. Spectral semistar operations are stable, [8, Lemma 4.1(3)]. For the semistar operations given by the extension to an overring we have the following Proposition.

Proposition 1.2. *Let D be an integral domain and T an overring of D. The following are equivalent:*

(i) *T is flat over D.*
(ii) *The semistar operation $\star_{\{T\}}$ on D is stable.*
(iii) *The semistar operation $\star_{\{T\}}$ on D is spectral.*

Proof. (i)$\Leftrightarrow$ (ii) It follows from [23, Theorem 7.4(i)] and [28, Proposition 1.7].
(i)$\Rightarrow$(iii) It is easy to see that $\star_{\{T\}} = \star_{\Delta(T)}$, where $\Delta(T) := \{M \cap D \,|\, M$ maximal ideal of $T\}$.
(iii) $\Rightarrow$ (ii) It is clear, since spectral semistar operations are stable, [8, Lemma 4.1(3)]. □

We recall [9, Chapter V] that a localizing system of D is a family $\mathcal{F}$ of ideals of D such that:

(LS1) If $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$.
(LS2) If $I \in \mathcal{F}$ and J is an ideal of D such that $(J :_D iD) \in \mathcal{F}$, for each $i \in I$, then $J \in \mathcal{F}$.

A localizing system $\mathcal{F}$ is *finitely generated* if, for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$.

The relation between semistar operations (in particular, stable semistar operations) and localizing systems has been deeply investigated by M. Fontana and J. Huckaba in [8] and by F. Halter-Koch in the context of module systems [17]. We summarize some results about it in the following Proposition (see [8, Proposition 2.8, Proposition 3.2, Proposition 2.4, Corollary 2.11, Theorem 2.10(B)] for the proofs).

Proposition 1.3. *Let D be an integral domain.*

(1) *If $\star$ is a semistar operation on D, then $\mathcal{F}^\star := \{I$ ideal of $D \,|\, I^\star = D^\star\}$ is a localizing system (called* the localizing system associated to $\star$*).*
(2) *If $\star$ is a semistar operation of finite type, then $\mathcal{F}^\star$ is a finitely generated localizing system.*
(3) *If $\mathcal{F}$ is a localizing system, the map $E \mapsto E^{\star_\mathcal{F}} := \bigcup\{(E : J) \,|\, J \in \mathcal{F}\}$, for each $E \in \overline{\boldsymbol{F}}(D)$, is a stable semistar operation.*
(4) *If $\mathcal{F}$ is a finitely generated localizing system, then $\star_\mathcal{F}$ is a finite type semistar operation.*
(5) *If $\mathcal{F}'$ and $\mathcal{F}''$ are two localizing systems of D, then $\mathcal{F}' \subseteq \mathcal{F}''$ if and only if $\star_{\mathcal{F}'} \leq \star_{\mathcal{F}''}$.*
(6) *$\star_{\mathcal{F}^\star} = \star$ if and only if $\star$ is stable.* □

If $\star$ is a semistar operation, the semistar operation $\tilde{\star} := \star_{\mathcal{F}^{\star_f}}$ associated to the localizing system $\mathcal{F}^{\star_f}$ is of particular interest. It coincides with the spectral semistar operation defined by the (nonempty) set $\mathcal{M}(\star_f)$ of the quasi–$\star_f$–maximal ideals. We note that if v is the v-semistar operation, $\tilde{v} = w$, where w is the (semi)star operation studied by Wang Fanggui and R.L. McCasland in [29].

Proposition 1.4. [8, Corollary 3.9, Proposition 4.23] *Let D be an integral domain, $\star$ a semistar operation on D. Then:*

(1) $\star_{\mathcal{F}^\star} = \tilde{\star}$ *if and only if* $\mathcal{F}^\star$ *is finitely generated if and only if* $\star_{\mathcal{F}^\star}$ *is a semistar operation of finite type.*
(2) $\star = \tilde{\star}$ *if and only if* $\star$ *is stable of finite type if and only if* $\star$ *is spectral of finite type.* □

Proposition 1.5. *Let D be an integral domain and $\star$ a stable semistar operation on D. Then, $P^\star$ is a prime ideal of $D^\star$, for each prime ideal P of D such that $P^\star \neq D^\star$. Moreover, $D^\star_{P^\star} = D_P$.*

Proof. It follows from Proposition 1.3(6) and [2, Theorem 1.1]. □

We say that a semistar operation $\star$ is *cancellative* (or that D has the $\star$-*cancellation law*) if, for each $E, F, G \in \overline{\boldsymbol{F}}(D)$, $(EF)^\star = (EG)^\star$ implies $F^\star = G^\star$. We say that $\star$ is *a.b.* if the same holds for each $E \in \boldsymbol{f}(D)$, $F, G \in \overline{\boldsymbol{F}}(D)$ and that $\star$ is *e.a.b.* if the same holds for each $E, F, G \in \boldsymbol{f}(D)$. Clearly, a cancellative semistar operation is *a.b.* and an *a.b.* semistar operation is *e.a.b.*.

Let T be an overring of D. Let $\star$ be a semistar operation on D and $\star'$ a semistar operation on T. Then, we say that T is $(\star, \star')$*–linked to* D, if

$$F^\star = D^\star \;\Rightarrow\; (FT)^{\star'} = T^{\star'},$$

for each nonzero finitely generated ideal F of D. Finally, recall that we say that T is $(\star, \star')$*–flat over* D if it is $(\star, \star')$–linked to D and, in addition, $D_{Q \cap D} = T_Q$, for each quasi–$\star'_f$ –prime ideal Q of T. More details on these notions can be found in [6] and in [18].

Let D be an integral domain and T an overring of D. In the following, we denote by $\textbf{SStar}(D)$ the set of all semistar operations on D, by $\textbf{SStar}_f(D)$ the set of all semistar operation of finite type on D, by $\textbf{SStar}(D,T)$ (resp., $\textbf{SStar}_f(D,T)$) the set of semistar operations (resp., semistar operations of finite type) $\star$ such that $D^\star = T$. For the sake of simplicity we denote by $\textbf{(S)Star}(D)$ the set $\textbf{SStar}(D,D)$ of all (semi)star operations on D.

Proposition 1.6. ([26, Lemma 45] and [12, Example 1.2]) *Let D be an integral domain, T an overring of T, $\iota : D \to T$ the canonical embedding of D in T, $\star$ a semistar operation on D and $*$ a semistar operation on T. Then:*

(1) *The map $\star_\iota : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$, $E \mapsto E^{\star_\iota} := E^\star$ is a semistar operation on T.*
(2) *The map $*^\iota : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^{*^\iota} := (ET)^*$ is a semistar operation on D.* □

We will denote by $(-)_\iota$ the map $\textbf{SStar}(D) \to \textbf{SStar}(T)$, $\star \mapsto \star_\iota$, and by $(-)^\iota$ the map $\textbf{SStar}(T) \to \textbf{SStar}(D)$, $* \mapsto *^\iota$. We will refer to this map respectively as the "ascent" and the "descent" maps, as in [26]. In next Lemma we observe that these two maps are order preserving.

Lemma 1.7. *Let D be an integral domain, T an overring of D, $\iota : D \to T$ the canonical embedding of D in T, $\star_1, \star_2$ semistar operations on D and $*_1, *_2$ semistar operations on T. Then:*

(1) $\star_1 \leq \star_2$ *implies* $(\star_1)_\iota \leq (\star_2)_\iota$.
(2) $*_1 \leq *_2$ *implies* $(*_1)^\iota \leq (*_2)^\iota$.

Proof. (1) $E^{(\star_1)_\iota} = E^{\star_1} \subseteq E^{\star_2} = E^{(\star_2)_\iota}$.
(2) $E^{(*_1)^\iota} = (ET)^{*_1} \subseteq (ET)^{*_2} = E^{(*_2)^\iota}$. □

Example 1.8. **(1)** Let the notation be as in Proposition 1.6. If $\star_{\{T\}}$ is the semistar operation given by the extension to the overring T and d_T is the identity (semi)star operation on T, it is easy to see that $(\star_{\{T\}})_\iota = d_T$ and $(d_T)^\iota = \star_{\{T\}}$.

(2) Consider a nonzero ideal I of D and let $T := (I : I)$. Let $\star$ be the (semi)star operation on T defined by $J^\star := (I : (I : J))$, for each $J \in \overline{\boldsymbol{F}}(T)$ (it is easy to see that this semistar operation is the trivial extension of the star operation considered in [20, Proposition 3.2] for introducing the m(ultiplicative)-canonical ideals). Then, $v(I) := \star^\iota$ defined by $E^{v(I)} := (I : (I : ET)) = (I : (I : E))$, for each $E \in \overline{\boldsymbol{F}}(T)$ is a semistar operation on D.
More generally, it can be proven that, if $F \in \overline{\boldsymbol{F}}(D)$, the map given by $E^{v(F)} \mapsto (F : (F : E))$ is a semistar operation. The proof of this fact is exactly the same as the proof for ideals (cf. [20, Proposition 3.2]).

(3) Let D be an integral domain and let $\{V_\alpha\}_{\alpha \in A}$ be the set of the valuation overrings of D. It is well-known that, if D is integrally closed, the map defined by $I \mapsto \bigcap \{IV_\alpha \mid \alpha \in A\}$, $I \in \overline{\boldsymbol{F}}(D)$ is a (semi)star operation, that, restricted to $\boldsymbol{F}(D)$, coincides with the "classical" b-star operation, as defined in [14, page 398]. If D is not integrally closed, this map is a semistar operation (the b–semistar operation, [10, Example 2.5(6)]) and it is exactly the "descent" of the $b_{D'}$–(semi)star operation of the integral closure D' of D, *i.e.* $b_D = (b_{D'})^\iota$, where ι is the canonical embedding of D in D'.

2. Composition of semistar operations

Semistar operations are, in fact, maps. Then, it is possible to define, in a "natural" way, a composition of semistar operations. Let $\star_1$ be a semistar operation on an integral domain D and let $\star_2$ be a semistar operation on an integral domain T, with $D \subseteq T \subseteq D^{\star_1}$. It follows that $\overline{\boldsymbol{F}}(D^{\star_1}) \subseteq \overline{\boldsymbol{F}}(T) \subseteq \overline{\boldsymbol{F}}(D)$. We can define the map $\star_1\star_2 : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, by $E \mapsto (E^{\star_1})^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$. This map is well defined, since $E^{\star_1} \in \overline{\boldsymbol{F}}(D^{\star_1}) \subseteq \overline{\boldsymbol{F}}(T)$.

The map $\star_1\star_2$ can be, but in general is not, a semistar operation on D. The properties $(\star_1)$ and $(\star_2)$ of the definition of semistar operation are easily checked, while $(\star_3)$ is not always satisfied.

Example 2.1. **(1)** Let D be an integral domain with quotient field K and $R(\neq K)$ an overring of D, such that $(D : R) = 0$. Let $\star_1 = v$, the v-operation of D, and let $\star_2 = \star_{\{R\}}$, the semistar operation on D given by the extension to R (that is, $E^{\star_2} = ER$, for each $E \in \overline{\boldsymbol{F}}(D)$). Let $\star := \star_1\star_2$. This is a map, defined on the set $\overline{\boldsymbol{F}}(D)$. We prove that it is not a semistar operation, by showing that, in general, if $I \in \overline{\boldsymbol{F}}(D)$, $I^\star \neq (I^\star)^\star$. Then, let $I \in \overline{\boldsymbol{F}}(D)$. We have $I^\star = I^vR$ and $(I^\star)^\star = (I^vR)^vR$. We notice that $(I^vR)^v = (D : (D : I^vR)) = (D : ((D : R) : I^v)) = (D : (0 : I^v)) = (D : 0) = K$. Then, if, for instance, I is a principal ideal (or, more generally, a divisorial ideal) of D, we have $I^\star = IR \subsetneq (I^\star)^\star = K$.

(2) Let D be an integral domain, T an overring of D, let ι be the canonical embedding of D in T and $*$ a semistar operation on T. Then the semistar operation $*^\iota$ on D defined in Proposition 1.6(2)is exactly the composition of $\star_{\{T\}}$ and $*$.

(3) It is well-known ([8, Proposition 1.6]) that, if $\star_1, \star_2$ are semistar operations on an integral domain D, such that $\star_1 \leq \star_2$, then $\star_1\star_2 = \star_2$ (in particular, $\star_1\star_2$ is a semistar operation).

(4) Let D be an integral domain and A and B two overrings of D. Then, the map $\star := \star_{\{A\}}\star_{\{B\}}$ is a semistar operation. It is sufficient to prove property $(\star_3)$, thus, if $E \in \overline{\boldsymbol{F}}(D)$, we have $E^{\star\star} = EABAB = EAABB = EAB = E^{\star}$. More precisely, $\star = \star_{\{R\}}$, where $R = AB$ is the overring of D given by the product of A and B. We notice that, if A and B are not comparable, then $\star$ is different from both $\star_{\{A\}}$ and $\star_{\{B\}}$.

It is natural to ask when the composition of two semistar operations is a semistar operation, that is, under what conditions $(\star_3)$ holds.

Lemma 2.2. *Let D be an integral domain, $\star_1$ a semistar operation on D, T an overring of D, $T \subseteq D^{\star_1}$, $\iota : D \hookrightarrow T$ the canonical embedding of D in T. Let $\star_2$ be a semistar operation on T and $\star := \star_1\star_2$. Then:*

1. *$E^{\star_1} \subseteq E^{\star}$, for each $E \in \overline{\boldsymbol{F}}(D)$. (Then, when $\star$ is a semistar operation, $\star_1 \leq \star$.)*
2. *$\star = \star_1$ if and only if $\star_2 \leq (\star_1)_\iota$. In this case, $\star$ is a semistar operation.*
3. *$E^{(\star_2)^\iota} \subseteq E^{\star}$, for each $E \in \overline{\boldsymbol{F}}(D)$. (Then, when $\star$ is a semistar operation, $(\star_2)^\iota \leq \star$.)*
4. *$\star = (\star_2)^\iota$ if and only if $(\star_1)_\iota \leq \star_2$. In this case, $\star$ is a semistar operation.*

Proof. (1) and (3) are straightforward.
(2) Suppose $\star = \star_1$. Let $E \in \overline{\boldsymbol{F}}(T) \subseteq \overline{\boldsymbol{F}}(D)$. We have $(E^{(\star_1)_\iota})^{\star_2} = (E^{\star_1})^{\star_2} = E^{\star_1} = E^{(\star_1)_\iota}$ and then $\star_2 \leq (\star_1)_\iota$. Conversely, let $\star_2 \leq (\star_1)_\iota$. Then, $(E^{\star_1})^{\star_2} \subseteq ((ET)^{\star_1})^{\star_2} = ((ET)^{(\star_1)_\iota})^{\star_2} = (ET)^{(\star_1)_\iota} \subseteq (ED^{\star_1})^{\star_1} = E^{\star_1}$.
(4) Suppose $\star = (\star_2)^\iota$. Let $E \in \overline{\boldsymbol{F}}(T)$. Then, $E^{(\star_1)_\iota} \subseteq \left(E^{(\star_1)_\iota}\right)^{\star_2} = (E^{\star_1})^{\star_2} = E^{(\star_2)^\iota} = (ET)^{\star_2} = E^{\star_2}$. Hence, $(\star_1)_\iota \leq \star_2$. Conversely, let $(\star_1)_\iota \leq \star_2$. Then, $E^{(\star_2)^\iota} = (ET)^{\star_2} \subseteq (E^{\star_1})^{\star_2} \subseteq ((ET)^{\star_{1\iota}})^{\star_2} = (ET)^{\star_2} = E^{\star_2{}^\iota}$. □

Example 2.3. Let D be an integral domain, $\star_1$ a semistar operation on D and ι the canonical embedding of D in $D^{\star_1}$. Let $\star_2 := v_{D^{\star_1}}$ be the v-operation on $D^{\star_1}$. Consider the composition $\star := \star_1\star_2$. This map, defined by $E^{\star} = (D^{\star_1} : (D^{\star_1} : E^{\star_1}))$, for each $E \in \overline{\boldsymbol{F}}(D)$, is a semistar operation and it coincides with the semistar operation $(v_{D^{\star_1}})^\iota$, by Lemma 2.2(4), since $(\star_1)_\iota$ is a (semi)star operation on $D^{\star_1}$ and so $(\star_1)_\iota \leq \star_2$. As in Example 1.8(2), we will denote this semistar operation on D by $v(D^{\star_1})$. We note that in general, if T is an overring of D, the semistar operation $v(T)$ defined in Example 1.8(2) coincides with the descent to D of the v-operation of T(i.e. $v(T) = (v_T)^\iota$, where $\iota : D \hookrightarrow T$ is the canonical embedding). We will denote by $t(T)$ the descent of the t-operation of T (i.e. $t(T) = (t_T)^\iota$), that coincides with the semistar operation of finite type $(v(T))_f$ associated to $v(T)$ (cf.Proposition 3.2(1)).

Remark 2.4. Let D be an integral domain and let T be an overring of D. Let $\star_1$ be a semistar operation on D and $\star_2$ be a semistar operation on T. We have shown in Lemma 2.2 that, if $(\star_1)_\iota$ and $\star_2$ are comparable in T, then $\star_1\star_2$ is a semistar operation. We notice that this is not a necessary condition. For instance, take A and B two not comparable overrings of D and let $T := A \cap B$. Let $\star_1 := \star_{\{A\}}$ be the semistar operation on D given by the extension to A and let $\star_2$ be the semistar operation defined on T given by the extension to B. It is easy to see that $\star_1\star_2$ is a semistar operation (with an argument similar to the one used in Example 2.1(4)), but it is clear that $\star_1$ and $\star_2$ are not comparable on T, since A and B are not comparable.
This is not a necessary condition even if $T = D^{\star_1}$. For example, let D be an integral domain, that is not conducive and that is not a Prüfer domain (for example, the

domain $K[X,Y]$, where K is a field and X,Y two indeterminates on K, is clearly not a Prüfer domain and it is not conducive by [5, Corollary 2.7]). Let $\star_1 = d_e$ be the trivial extension of the identity star operation on D and let $\star_2 = b$, the b semistar operation of D as defined in Example 1.8. In this case, $T = D = D^{\star_1}$. We have $b \not\leq d_e$ since there exists a nonzero ideal I of D such that $I^b \neq I$ (otherwise D would be a Prüfer domain, [14, Theorem 24.7]). On the other hand, $d_e \not\leq b$. Indeed, since D is not conducive, there exists a valuation overring V of D that is not a fractional ideal, [5, Lemma 2.0(ii)]. Then, $V^{d_e} = K$, by the definition of d_e, but clearly $V^b = V$. Thus, $d_e \not\leq b$. So, these two semistar operations are not comparable on $D = D^{\star_1}$, but it is easy to see that the composition $\star_1\star_2$ is a semistar operation. More precisely, $\star_1\star_2 = b_e$, the trivial extension of the b–semistar operation of D.

Proposition 2.5. *Let $\star_1$ be a semistar operation on an integral domain D and let $\star_2$ be a semistar operation on an integral domain T, with $D \subseteq T \subseteq D^{\star_1}$ and ι the canonical embedding of D in T. Let $\star := \star_1\star_2$. The following are equivalent:*

(i) *$\star$ is a semistar operation.*
(ii) *$((E^{\star_1})^{\star_2})^{\star_1} = (E^{\star_1})^{\star_2}$, for each E in $\overline{\boldsymbol{F}}(D)$.*
(iii) *$(F^{\star_2})^{\star_1} = F^{\star_2}$, for each $F \in \overline{\boldsymbol{F}}^{\star_1}(D)(:= \{E^{\star_1} \mid E \in \overline{\boldsymbol{F}}(D)\})$.*
(iv) *$(E^{\star_2})^{\star_1} \subseteq (E^{\star_1})^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(T)$.*
(v) *$\star_1\star_2 = \star_1 \vee (\star_2)^{\iota}$ (Example 1.1(6)).*

Proof. (ii) $\Leftrightarrow$ (iii) It is straightforward.
(i)$\Rightarrow$ (ii) It is clear, since if $(E^{\star_1})^{\star_2} \subsetneq ((E^{\star_1})^{\star_2})^{\star_1}$, then $E^\star \subsetneq (E^\star)^\star$, and $\star$ is not a semistar operation.
(ii) $\Rightarrow$ (i) We have only to prove $(\star_3)$, that is $(((E^{\star_1})^{\star_2})^{\star_1})^{\star_2} = (E^{\star_1})^{\star_2}$. But this follows immediately from the hypothesis, since we have $(((E^{\star_1})^{\star_2})^{\star_1})^{\star_2} = ((E^{\star_1})^{\star_2})^{\star_2} = (E^{\star_1})^{\star_2}$.
(ii) $\Rightarrow$ (iv) Let $E \in \overline{\boldsymbol{F}}(T)$. Then $(E^{\star_2})^{\star_1} \subseteq ((E^{\star_1})^{\star_2})^{\star_1} = (E^{\star_1})^{\star_2}$.
(iv) $\Rightarrow$ (ii) Let $E \in \overline{\boldsymbol{F}}(D)$. Since $E^{\star_1} \in \overline{\boldsymbol{F}}(D^{\star_1}) \subseteq \overline{\boldsymbol{F}}(T)$, by the hypothesis we have $((E^{\star_1})^{\star_2})^{\star_1} \subseteq ((E^{\star_1})^{\star_1})^{\star_2} = (E^{\star_1})^{\star_2} \subseteq ((E^{\star_1})^{\star_2})^{\star_1}$. Hence, $(E^{\star_1})^{\star_2} = ((E^{\star_1})^{\star_2})^{\star_1}$.
(i) $\Rightarrow$ (v) It is enough to show that, if $*$ is a semistar operation on D such that $\star_1 \leq *$ and ${\star_2}^{\iota} \leq *$ then $E^{\star_1\star_2} \subseteq E^*$ for each $E \in \overline{\boldsymbol{F}}(D)$. So, let $E \in \overline{\boldsymbol{F}}(D)$. Note that $E^* \in \overline{\boldsymbol{F}}(D^*) \subseteq \overline{\boldsymbol{F}}(T)$, since $T \subseteq D^{\star_1} \subseteq D^*$. Then $(E^*)^{\star_2}$ is defined and $(E^*)^{\star_2} = E^{*(\star_2)^{\iota}} = E^*$, by [8, Proposition 1.6], since $(\star_2)^{\iota} \leq *$. So, $\star_1 \leq *$ implies $E^{\star_1} \subseteq E^*$ and then $E^{\star_1\star_2} \subseteq E^{**_2} = E^*$.
(v)$\Rightarrow$ (i) It is obvious, since $\star_1 \vee (\star_2)^{\iota}$ is a semistar operation. $\square$

Example 2.6. Let the notation be like in Example 2.1(1). In this case, $\star_1$ and $\star_2$ are both defined on D. We show again that $\star_1\star_2$ is not a semistar operation by exhibiting an ideal E of D that does not satisfy condition (iv) of Proposition 2.5. Indeed, let $E := xD$, for some $x \in D$. Then $(xD)^{\star_2\star_1} = x(D^{\star_{\{R\}}{}^{v}}) = x(D : (D : R)) = x(D : 0) = K$ that clearly is not contained in $(xD)^{\star_1\star_2} = xD^{v\star_{\{R\}}} = xR$.

Proposition 2.7. *Let D be an integral domain and $\star_1$ a semistar operation on D. Let T be an overring of D, with $T \subseteq D^{\star_1}$, and $\star_2$ a semistar operation on T. Suppose that $\star_1\star_2$ is a semistar operation on D.*

(1) *If $\star_1, \star_2$ are of finite type, then $\star_1\star_2$ is of finite type.*
(2) *If $\star_1, \star_2$ are stable, then $\star_1\star_2$ is stable.*
(3) *If $\star_1 = \widetilde{\star_1}$ and $\star_2 = \widetilde{\star_2}$ then $\star_1\star_2 = \widetilde{\star_1\star_2}$.*

(4) *If* $\star_1$ *and* $\star_2$ *are spectral and of finite type, then* $\star_1\star_2$ *is spectral of finite type.*

Proof. (1) Let $E \in \overline{\boldsymbol{F}}(D)$, $x \in (E^{\star_1})^{\star_2}$. Since $\star_2$ is of finite type, there exists $F \in \boldsymbol{f}(D^{\star_1})$, $F \subseteq E^{\star_1}$, such that $x \in F^{\star_2}$. Let $F = x_1D^{\star_1} + \ldots + x_nD^{\star_1}$. Since $F \subseteq E^{\star_1}$ and $\star_1$ is of finite type, there exists $G_1, \ldots, G_n \subseteq E$, $G_i \in f(D)$, $i = 1, \ldots, n$, such that $x_1 \in {G_1}^{\star_1}, \ldots, x_n \in {G_n}^{\star_1}$. It follows that $F \subseteq {G_1}^{\star_1} + \ldots + {G_n}^{\star_1} \subseteq ({G_1}^{\star_1} + \ldots + {G_n}^{\star_1})^{\star_1} = (G_1 + \ldots + G_n)^{\star_1}$. Let $G = G_1 + \ldots + G_n$. Then, $F \subseteq G^{\star_1}$ implies $F^{\star_2} \subseteq (G^{\star_1})^{\star_2}$. It follows that $x \in (G^{\star_1})^{\star_2}$ with $G \in \boldsymbol{f}(D)$, $G \subseteq E$. Hence, $\star_1\star_2$ is of finite type.
(2) It is straightforward.
(3) It follows from (1) and (2), since a semistar operation $\star$ coincides with $\tilde{\star}$ if and only if $\star$ is stable and of finite type (Proposition 1.4(2)).
(4) It follows immediately from (3), since a semistar operation $\star$ is spectral and of finite type if and only if $\star = \tilde{\star}$ (Proposition 1.4(2)). □

Remark 2.8. (1) The converse of Proposition 2.7 is not true in general. That is, for $\star_1\star_2$ to be of finite type (resp. stable) it is not necessary that $\star_1, \star_2$ are of finite type (stable). For example, if $\star_1$ and $\star_2$ are both defined on D and $\star_1 \leq \star_2$, if $\star_2$ is of finite type (stable) then $\star_1\star_2$ and $\star_2\star_1$ are of finite type (stable) without further conditions on $\star_1$ (since $\star_1\star_2 = \star_2\star_1 = \star_2$).
(2) In the proof of Proposition 2.7, we do not use the fact that $\star_1\star_2$ is a semistar operation. So, even if $\star_1\star_2$ is not a semistar operation, we have that $E^{\star_1\star_2} = \bigcup\{F^{\star_1\star_2} | F \in \boldsymbol{f}(D)\}$, for each $E \in \overline{\boldsymbol{F}}(D)$, when $\star_1$ and $\star_2$ are of finite type, and $(E \cap F)^{\star_1\star_2} = E^{\star_1\star_2} \cap F^{\star_1\star_2}$, for each $E, F \in \overline{\boldsymbol{F}}(D)$, when $\star_1$ and $\star_2$ are stable.

Example 2.9. Let D be an integral domain, P and Q incomparable prime ideals of D. Let $\star_1 := \star_{\{P\}}$ and $\star_2 := \star_{\{Q\}}$. Consider $\star := \star_{\{P\}}\star_{\{Q\}}$. From Example 2.1, it follows that $\star$ is a semistar operation. Since both $\star_1$ and $\star_2$ are spectral and of finite type, $\star$ must be spectral and of finite type (Proposition 2.7(4)). Indeed, it is easy to check that $D_PD_Q = D_S$, the localization of D at the multiplicative set $S := \{ab \,|\, a \in D \setminus P,\ b \in D \setminus Q\}$. Then, $\star = \star_{\{D_S\}}$, that is a semistar operation of finite type. Moreover, since D_S is flat over D, it follows by Proposition 1.2 that $\star$ is spectral defined by the set of the primes P of D such that $P \cap S = \emptyset$.

3. Star operations on overrings and semistar operations

In the following, we recall some properties and prove new ones of the semistar operations defined in Proposition 1.6

Proposition 3.1. *Let D be an integral domain, T an overring of T, $\iota : D \to T$ the canonical embedding of D in T, $\star$ a semistar operation on D and $\star_\iota$ the semistar operation on T defined as in Proposition 1.6.*

(1) *If $\star$ is of finite type, then $\star_\iota$ is of finite type.*
(2) *If $\star$ is stable then $\star_\iota$ is stable.*
(3) *If $\star$ is cancellative on D, then $\star_\iota$ is cancellative on T.*
(4) *If $\star$ is a.b. then $\star_\iota$ is a.b.*
(5) *Assume $T = D^\star$ or $T \in \boldsymbol{f}(D)$. If $\star$ is e.a.b. then $\star_\iota$ is e.a.b.*
(6) *Assume $T = D^\star$. If $\star$ is spectral, then $\star_\iota$ is spectral.*
(7) *If $T = D^\star$ then $\star_\iota$ is a (semi)star operation on T.*

Proof. (1) and (7) are in [10, Proposition 2.8]
(2) is straightforward.
(3) is straightforward, since, on T-modules, $\star$ and $\star_\iota$ coincide.
(4) Let $E \in \boldsymbol{f}(T)$ and $F, G \in \overline{\boldsymbol{F}}(T)$ with $(EF)^{\star_\iota} = (EG)^{\star_\iota}$. There exists $E_0 \in \boldsymbol{f}(D)$ such that $E_0T = E$. Then, $(E_0TF)^\star = (E_0TG)^\star$ and, since $\star$ is *a.b.* and $E_0 \in \boldsymbol{f}(D)$, $F^{\star_\iota} = F^\star = (FT)^\star = (GT)^\star = G^\star = G^{\star_\iota}$. Hence, $\star_\iota$ is an *a.b.* semistar operation.
(5) Let $E, F, G \in \boldsymbol{f}(T)$ such that $(EF)^{\star_\iota} = (EG)^{\star_\iota}$. Then, there exists $E_0, F_0, G_0 \in \boldsymbol{f}(D)$ with $E = E_0T$,$F = F_0T$ and $G = G_0T$. It follows $(E_0F_0T)^{\star_\iota} = (E_0G_0T)^{\star_\iota}$, that is, $(E_0F_0T)^\star = (E_0G_0T)^\star$. If $T \in \boldsymbol{f}(D)$, then F_0T, $G_0T \in \boldsymbol{f}(D)$, thus, since $\star$ is *e.a.b.*, we obtain $(F_0T)^\star = (G_0T)^\star$. Hence, $F^{\star_\iota} = G^{\star_\iota}$. On the other hand, if $T = D^\star$, then $(E_0F_0D^\star)^\star = (E_0G_0D^\star)^\star$ implies $(E_0F_0)^\star = (E_0G_0)^\star$ and thus ${F_0}^\star = {G_0}^\star$. So, $F^{\star_\iota} = (F_0D^\star)^\star = {F_0}^\star = {G_0}^\star = (G_0D^\star)^\star = G^{\star_\iota}$. Hence, in any case, $\star_\iota$ is an *e.a.b.* semistar operation.
(6) Let $\star = \star_\Delta$, for some $\Delta \subseteq \mathrm{Spec}(D)$. We want to prove that $\star_\iota = \star_{\Delta^\star}$, where $\Delta^\star := \{P^\star : P \in \Delta\}$. It is clear that, for $P \in \Delta$, $P^\star \neq D^\star$, so, from Proposition 1.5, it follows that $P^\star$ is prime and that $(D^\star)_{P^\star} = D_P$. Then, if $E \in \overline{\boldsymbol{F}}(D^\star)$, we have $E^{\star_\iota} = \bigcap\{ED_P \mid P \in \Delta\} = \bigcap\{ED^\star_{P^\star} \mid P \in \Delta\} = E^{\star_{\Delta^\star}}$. Hence, $\star_\iota$ is spectral. □

We study now the semistar operation $\star^\iota$ defined in Proposition 1.6(2), when $\star$ is a semistar operation on an overring T of an integral domain D. We recall here (Example 2.1(2)) that this semistar operation (defined on D) is exactly the composition of two semistar operations, the extension $\star_{\{T\}}$ to T and the semistar operation $\star$.

Proposition 3.2. *Let D be an integral domain, T an overring of T, $\iota : D \to T$ the canonical embedding of D in T, $*$ a semistar operation on T and $*^\iota$ the semistar operation on D defined as in Proposition 1.6. Then:*

(1) $(*^\iota)_f = (*_f)^\iota$ *(in particular, if $*$ is of finite type, then $*^\iota$ is of finite type).*
(2) *If $*$ is cancellative, then $*^\iota$ is cancellative.*
(3) *If $*$ is e.a.b. (resp., a.b.) then $*^\iota$ is e.a.b. (resp., a.b.).*

Proof. (1) See [7, Lemma 3.1].
(2) Let $E, F, G \in \overline{\boldsymbol{F}}(D)$ such that $(EF)^{*^\iota} = (EG)^{*^\iota}$. Then, $(ETFT)^* = (ETGT)^*$, and, since $*$ is cancellative, we obtain $(FT)^* = (GT)^*$, that is, $F^{*^\iota} = G^{*^\iota}$.
(3) See [10, Proposition 2.9(2)] □

Remark 3.3. (1) Note that the fact that, with the notation of Proposition 3.2, if $\star$ is a semistar operation of finite type, then $\star^\iota$ is a semistar operation of finite type can be proven also by using Proposition 2.7, Example 2.1(2), and the fact that the extension $\star_{\{T\}}$ is a semistar operation of finite type (Example 1.1(7)).
(2) In Proposition 3.1 we have shown that the map $(-)_\iota :$ $\mathbf{SStar}(D) \to$ $\mathbf{SStar}(T)$ preserves all the main properties of a semistar operation, that is, the finite character, the stability, the property of being spectral, *a.b.* and, under some conditions, *e.a.b.*. The map $(-)^\iota :$ $\mathbf{SStar}(T) \to$ $\mathbf{SStar}(D)$ does not behave so well: in fact, while the finite character and the properties of being *a.b.* and *e.a.b.* are preserved (Proposition 3.2), the properties of being stable or spectral are not preserved. For instance, with the notation of Proposition 3.2, take T not flat over D, and $* = d_T$, the identity semistar operation of T. Clearly $*$ is spectral (defined by the set of all maximal ideals of T) and then it is stable, but $*^\iota = \star_{\{T\}}$ is not stable (and then not spectral), by Proposition 1.2.

For the next Proposition, see for example [26, Lemma 45] or [7, Example 2.1 (e)].

Proposition 3.4. *Let D be an integral domain, T an overring of D, $\iota : D \to T$ the canonical embedding of D in T. Then:*

1. *For each semistar operation $*$ on T, $(*^{\iota})_{\iota} = *$ (that is, $(-)_{\iota} \circ (-)^{\iota} = id_{\mathbf{SStar}(T)}$).*
2. *For each semistar operation $\star$ on D, $\star \leq (\star_{\iota})^{\iota}$.* □

It follows that the map $(-)_{\iota} : \mathbf{SStar}(D) \to \mathbf{SStar}(T)$ is surjective (that is, each semistar operation on T is an "ascent" of a semistar operation on D) and the map $(-)^{\iota} : \mathbf{SStar}(T) \to \mathbf{SStar}(D)$ is injective.

Consider $\mathbf{SStar}(D,T)$ the set of all semistar operations $\star$ on D such that $D^{\star} = T$. From Proposition 3.1(7) it follows that $\{\star_{\iota} \,|\, \star \in \mathbf{SStar}(D,T)\} \subseteq \mathbf{(S)Star}(T)$. We will denote by $(-)^{T}_{\iota}$ the map $(-)_{\iota}$ restricted to $\mathbf{SStar}(D,T)$, i.e. $(-)^{T}_{\iota} : \mathbf{SStar}(D,T) \to \mathbf{(S)Star}(T)$. Analogously, we denote by $(-)^{\iota}_{T}$ the map $(-)^{\iota}$ restricted to $\mathbf{(S)Star}(T)$, i.e. $(-)^{\iota}_{T} : \mathbf{(S)Star}(T) \to \mathbf{SStar}(D,T)$. We prove that these maps are one the inverse of the other.

Proposition 3.5. *Let D be an integral domain, T an overring of T, $\iota : D \to T$ the canonical embedding of D in T. Let $(-)^{T}_{\iota}$ and $(-)^{\iota}_{T}$ be defined as above. Then:*

1. *For each $\star \in \mathbf{SStar}(D,T)$, $(\star_{\iota})^{\iota} = \star$ (i.e. $(-)^{\iota}_{T} \circ (-)^{T}_{\iota} = id_{\mathbf{SStar}(D,T)}$).*
2. *For each $* \in \mathbf{(S)Star}(T)$, $(*^{\iota})_{\iota} = *$ (i.e. $(-)^{T}_{\iota} \circ (-)^{\iota}_{T} = id_{\mathbf{(S)Star}(T)}$).*
3. *$(-)^{T}_{\iota}$ and $(-)^{\iota}_{T}$ are bijective.*

Proof. (1) Suppose $\star \in \mathbf{SStar}(D,T)$ and let $E \in \overline{\boldsymbol{F}}(D)$. We have $E^{(\star_{\iota})^{\iota}} = (ED^{\star})^{\star_{\iota}} = (ED^{\star})^{\star} = (ED)^{\star} = E^{\star}$, that is, $(\star_{\iota})^{\iota} = \star$.
(2) It is immediate by Proposition 3.4.
(3) Straightforward from (1) and (2). □

It follows that each semistar operation on a domain D can be decomposed, in a canonical way, as the composition of two semistar operations, more precisely, the first semistar operation is $\star_{\{T\}}$ for some overring T of D and the second one is a (semi)star operation on T.

Corollary 3.6. *Let D be an integral domain.*

1. *Let $\star$ be a semistar operation on D, let $T = D^{\star}$ and ι the canonical embedding of D in T. Then $\star$ is the composition of the semistar operation $\star_{\{T\}}$ and of a (semi)star operation $*$ on T, i.e. $\star = \star_{\{T\}}*$ (equivalently, $\star = *^{\iota}$, for some (semi)star operation $*$ on T).*
2. *Let T be an overring of D and ι the canonical embedding of D in T. Then $\mathbf{SStar}(D,T) = \{\star^{\iota} \,|\, \star \in \mathbf{(S)Star}(T)\}$.*

Proof. (1) Take $* := \star_{\iota}$ and apply Proposition 3.5(1).
(2) It follows immediately from Proposition 3.5(3). □

Since by Proposition 3.1(1) and Proposition 3.2(1), the finite type property is preserved by the maps $(-)_{\iota}$ and $(-)^{\iota}$, we obtain easily a similar result for finite type semistar operations.

Corollary 3.7. *Let D be an integral domain, T an overring of D.*

(1) *Let $\star$ be a semistar operation of finite type on D, let $T = D^\star$ and ι the canonical embedding of D in T. Then $\star$ is the composition of the semistar operation $\star_{\{T\}}$ and of a (semi)star operation of finite type $*$ on T, i.e. $\star = \star_{\{T\}}*$ (equivalently, $\star = *^\iota$, for some (semi)star operation of finite type $*$ on T).*

(2) *Let T be an overring of D and ι the canonical embedding of D in T. Then $\mathbf{SStar}_f(D,T) = \left\{ \star^\iota \,|\, \star \in \mathbf{(S)Star}_f(T) \right\}$.* □

Corollary 3.8. *Let D be an integral domain, T an overring of D. Let $\star$ be a semistar operation on D, such that $D^\star = T$ (that is, $\star \in \mathbf{SStar}(D,T)$). Then:*

(1) *$\star_{\{T\}} \leq \star \leq v(T)$.*

(2) *$\star_{\{T\}} \leq \star_f \leq t(T)$.*

Proof. (1) By Corollary 3.6(1), there exists a (semi)star operation $*$ on T such that $\star = *^\iota$. Since $*$ is a (semi)star operation on T, $d_T \leq * \leq v_T$ (Example 1.1(3)). It follows (Lemma 1.7) that $\star_{\{T\}} = (d_T)^\iota \leq *^\iota = \star \leq (v_T)^\iota = v(T)$.
(2) Use the same argument of (1), applying Corollary 3.7(1) and Example 1.1(7). □

In the following, we will denote by $\mathcal{O}(D)$ the set of all overrings of an integral domain D. Since it is clear that $\mathbf{SStar}(D) = \bigcup \{ \mathbf{SStar}(D,T) \,|\, T \in \mathcal{O}(D)\}$ and $\mathbf{SStar}_f(D) = \bigcup \{ \mathbf{SStar}_f(D,T) \,|\, T \in \mathcal{O}(D)\}$, and these unions are disjoint, we have the following Theorem.

Theorem 3.9. *Let D be an integral domain. For each $T \in \mathcal{O}(D)$, let ι_T be the canonical embedding of D in T.*

(1) *The map $\star \mapsto \star_{\iota_{D^\star}}$ establishes a bijection between the set $\mathbf{SStar}(D)$ and the set $\bigcup \{ \mathbf{(S)Star}(T) : T \in \mathcal{O}(D)\}$.*

(2) $\mathbf{SStar}(D) = \bigcup \{\star^{\iota_T} \,|\, \star \in \mathbf{(S)Star}(T),\, T \in \mathcal{O}(D)\}$.

(3) *The restriction of the map in (1) establishes a bijection between the set $\mathbf{SStar}_f(D)$ and the set $\bigcup \{ \mathbf{(S)Star}_f(T) : T \in \mathcal{O}(D)\}$.*

(4) $\mathbf{SStar}_f(D) = \bigcup \left\{\star^{\iota_T} \,|\, \star \in \mathbf{(S)Star}_f(T),\, T \in \mathcal{O}(D)\right\}$. □

Remark 3.10. Note that the bijection defined in Theorem 3.9 holds between the set of semistar operations on D and the set of (semi)star operations on the overrings of D (which, in general, contains properly the set of star operations on the overrings under the canonical embedding $\mathbf{Star}(T) \hookrightarrow \mathbf{(S)Star}(T)$, $\star \mapsto \star_e$ (see Example 1.1(2)). Note that, in general, a star operation can be extended to a semistar operation (clearly a (semi)star operation) in different ways. For example, if $\overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D) \neq \{K\}$ (where K is the quotient field of D, the identity star operation can be extended to the identity semistar operation or to the trivial extension defined in Example 1.1(2). The property that the trivial extension of the identity star operation coincides with the identity semistar operation characterizes the conducive domains, that is, the domains such that $\overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D) = \{K\}$ (see [5]).

The following Proposition shows two cases in which the extension is, in some sense, unique.

Proposition 3.11. *Let D be an integral domain.*

(1) *The map* $\star \mapsto \star_{|\boldsymbol{F}(D)}$ *(where* $\star_{|\boldsymbol{F}(D)}$ *is the star operation given by the restriction of* $\star$ *to* $\boldsymbol{F}(D)$*) establishes a bijection between the set* **(S)Star**$_f(D)$ *of all (semi)star operations of finite type on* D *and the set* **Star**$_f(D)$ *of all star operations of finite type on* D*.*

(2) *If* D *is conducive, the map* $\star \mapsto \star_{|\boldsymbol{F}(D)}$ *establishes a bijection between the set* **(S)Star**(D) *of all (semi)star operations on* D *and the set* **Star**(D) *of all star operations on* D*.*

Proof. (1) It follows immediately from the fact that a semistar operation of finite type is completely determined by the image of the finitely generated ideals.
(2) It follows immediately from the fact that $\overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D) = \{K\}$, where K is the quotient field of D, and that $K^\star = K$ for each semistar operation $\star$. □

Corollary 3.12. *Let* D *be an integral domain. For each* $T \in \mathcal{O}(D)$*, let* ι_T *be the canonical embedding of* D *in* T*.*

(1) *The map* $\star \mapsto (\star_{\iota_{D^\star}})_{|\boldsymbol{F}(D)}$ *establishes a bijection between the set* **SStar**$_f(D)$ *and the set* $\bigcup\{$ **Star**$_f(T) : T \in \mathcal{O}(D)\}$*.*

(2) *If* D *is conducive, the map* $\star \mapsto (\star_{\iota_{D^\star}})_{|\boldsymbol{F}(D)}$ *establishes a bijection between the set* **SStar**(D) *and the set* $\bigcup\{$ **Star**$(T) : T \in \mathcal{O}(D)\}$*.*

Proof. (1) Apply Theorem 3.9(3) and Proposition 3.11(1).
(2) Apply Theorem 3.9(1) and Proposition 3.11(2). □

As a direct consequence of Theorem 3.9 and Proposition 3.1, we have the following Corollary concerning the properties preserved by the map $(-)_\iota$.

Corollary 3.13. *Let* D *be an integral domain. Let* $\star$ *be a stable (resp. cancellative,* a.b.*,* e.a.b.*, spectral) semistar operation on* D*, let* $T = D^\star$ *and* ι *the canonical embedding of* D *in* T*. Then* $\star$ *is the composition of the semistar operation* $\star_{\{T\}}$ *and of a stable (resp., cancellative;* a.b.*;* e.a.b.*; spectral) (semi)star operation* $*$ *on* T*, i.e.* $\star = \star_{\{T\}}*$ *[equivalently,* $\star = *^\iota$*, for some stable (resp., cancellative;* a.b.*;* e.a.b.*; spectral) (semi)star operation* $*$ *on* T*].* □

From the previous corollary, we deduce only that, for each semistar operation $\star$ on an integral domain D with a certain property, there exists a particular (semi)star operation $*$ induced by $\star$ on an overring T of D(more precisely $* = \star_\iota$, where ι is the canonical embedding of D in T), with the same property, such that $\star$ is the composition of the extension to T and this (semi)star operation. We can't deduce that, taking a (semi)star operation $*$ on an overring T of D with a certain property, the composition of the extension to T and $*$ has the same property. This is true for the properties preserved by the map $(-)^\iota$ (see Proposition 3.2). In these cases, we have a bijection between the semistar operations on D with a certain property and the (semi)star operations on the overrings of D with the same property. This is the content of the following theorem.

Theorem 3.14. *There is a canonical bijection between the set of all cancellative (resp. a.b., e.a.b.) semistar operations on* D *and the set of all cancellative (resp. a.b., e.a.b.) (semi)star operations on the overrings of* D*.*

Proof. It follows from Proposition 3.1 (3),(4),(5) and Proposition 3.2(2),(3), Proposition 3.5. □

Then, the study of semistar operations (of finite type, cancellative, *a.b.*, *e.a.b.*) on an integral domain D is equivalent to the study of all (semi)star operations (of finite type, cancellative, *a.b.*, *e.a.b.*) on D and on all overrings of D.

We have shown that the properties for which there is a bijection between the semistar operations with that property and the (semi)star operations with that properties on overrings are exactly the properties preserved by the map $(-)^{\iota}$. For example, we have already shown (Remark 3.3(2)) that if $\star$ is a semistar operation on an integral domain D, if T is an overring of D, and ι the canonical embedding of D in T, then, it is not true, in general, that $\star^{\iota}$ is stable (resp. spectral), for a stable (resp. spectral) semistar operation $\star$ on D. We want to study when a canonical bijection also holds for stable and for spectral semistar operations.

Proposition 3.15. *Let D be an integral domain, T an overring of D and ι the canonical embedding of D in T. Let $*$ be a semistar operation on T.*

1. *If $*$ is stable and T is a flat overring of D, then $*^{\iota}$ is stable.*
2. *If T is a flat overring of D, then the map $(-)^T_{\iota}$ induces a bijection between the set of all stable semistar operations $\star$ on D such that $T = D^{\star}$ and the set of all stable (semi)star operations on T.*

Proof. (1) Let $E, F \in \overline{\boldsymbol{F}}(D)$. We notice that $(E \cap F)T = ET \cap FT$, since T is flat over D, [23, Theorem 7.4(i)]. Then, $(E \cap F)^{*^{\iota}} = ((E \cap F)T)^* = (ET \cap FT)^* = (ET)^* \cap (FT)^* = E^{*^{\iota}} \cap F^{*^{\iota}}$, by using the stability of $*$.
(2) It follows from (1) and Proposition 3.1(2). □

Remark 3.16. We have shown in Proposition 3.15(1) that if T is an overring of D, and $\star$ is a stable semistar operation on T, the flatness of T over D is a sufficient condition for $\star^{\iota}$ to be stable. This condition is not necessary: in fact, let $\mathcal{F}$ be a localizing system on D, such that $D_{\mathcal{F}}$ is not flat over D. Let ι be the canonical embedding of D in $D_{\mathcal{F}}$. Take the semistar operation $\star_{\mathcal{F}}$ on D (as defined in Proposition 1.3(3)) and consider the (semi)star operation $* := (\star_{\mathcal{F}})_{\iota}$ on $D_{\mathcal{F}}(= D^{\star_{\mathcal{F}}})$. By Proposition 3.1, $*$ is stable, and, by Proposition 3.5(1), $*^{\iota} = \star_{\mathcal{F}}$. Then, $*^{\iota}$ is stable, but $D_{\mathcal{F}}$, by the choice of $\mathcal{F}$, is not flat over D.

Proposition 3.17. *Let D be an integral domain, T an overring of D and ι the canonical embedding of D in T. Let $\star = \star_{\Delta}$ be a spectral semistar operation on T, defined by $\Delta \subseteq \operatorname{Spec}(T)$.*

1. *If $T_P = D_{P\cap D}$ (in particular if T is flat over D,* [14, Section 40 Exercise 7]*), for each $P \in \Delta$, then $\star^{\iota}$ is spectral.*
2. *If T is $(\star^{\iota}, \star)$-flat over D, then $\star^{\iota}$ is spectral.*
3. *If one of the conditions in (1) or (2) holds, then the map $(-)^T_{\iota}$ induces a bijection between the set of all spectral semistar operations on D such that $T = D^{\star}$ and the set of all spectral (semi)star operations on T.*

Proof. (1) Let $E \in \overline{\boldsymbol{F}}(D)$. Then, $E^{\star^{\iota}} = (ET)^{\star} = \bigcap\{(ET)T_P(= ET_P) \mid P \in \Delta\} = \bigcap\{ED_{P\cap D} \mid P \in \Delta\}$, that is, $\star^{\iota} = \star_{\Delta'}$, where $\Delta' = \{P \cap D \mid P \in \Delta\}$.
(2) For each $P \in \Delta$, P is a quasi–$\star$–prime ideal [8, Lemma 4.1(4)]. So, by definition of $(\star^{\iota}, \star)$-flatness, $T_P = D_{P\cap D}$. Then, apply (1).
(3) It is clear, because in these cases, the map $(-)^{\iota}$ preserves the spectral property. □

Corollary 3.18. *Let D be an integral domain. For each $T \in \mathcal{O}(D)$, let ι_T be the canonical embedding of D in T. Then the following are equivalent:*

(i) *D is a Prüfer domain.*
(ii) *The map $\star \mapsto \star_{\iota_{D^\star}}$ establishes a bijection between the set of all stable semistar operations on D and the set of all stable (semi)star operations on the overrings of D.*
(iii) *The map $\star \mapsto \star_{\iota_{D^\star}}$ establishes a bijection between the set of all spectral semistar operations on D and the set of all spectral (semi)star operations on the overrings of D.*

Proof. (i)⇒(ii),(iii) It follows from the fact that each overring of a Prüfer domain is flat, Proposition 3.15(2) and Proposition 3.17(3).
(ii),(iii) ⇒ (i) In both cases, the fact that the bijection holds implies that the semistar operation given by the extension to an overring of D is stable for each overring of D, since it is the descent of the identity semistar operation, that is obviously spectral and stable. Therefore each overring of D is flat, by Proposition 1.2. Then, D is a Prüfer domain by [9, Theorem 1.1.1]. □

4. Some applications

As a first application of the results proved in Section 3 we study semistar operations on valuation domains and on Prüfer domains. Some of the results we obtain have already been proven, but only for finite dimensional domains, in [22], [21], [25] and [24]. We generalize several statements without restrictions on the dimension, as corollaries of the results proven in the previous section.

First we recall a result about star operations [1, Proposition 12] (see also [16, Theorem 15.3] for the same result in the context of ideal systems):

Proposition 4.1. *Let V be a valuation domain, with maximal ideal M.*
(1) *If $M^2 \neq M$, then each ideal of V is divisorial, that is, $\mathbf{Star}(V) = \{d\}$.*
(2) *If $M^2 = M$, then $\mathbf{Star}(V) = \{d, v\}$.* □

Since a valuation domain is conducive [5, Proposition 2.1], by applying this result and Corollary 3.12, we have the following Proposition:

Proposition 4.2. *Let P be a prime ideal of a valuation domain V.*
(1) *If $P \neq P^2$, then $\mathbf{SStar}(V, V_P) = \{\star_{\{P\}}\}$.*
(2) *If $P = P^2$, then $\mathbf{SStar}(V, V_P) = \{\star_{\{P\}},\, v(V_P)\}$ (where $v(V_P)$ is defined as in Example 2.3).*
(3) $\mathbf{SStar}(V) = \bigcup\{\star_{\{P\}} \mid P \in \mathrm{Spec}(V)\} \cup \bigcup\{v(V_Q) \mid Q \in \mathrm{Spec}(V),\, Q^2 = Q\}$.

Proof. (1) Since $(PV_P) \neq (PV_P)^2$, we have $\mathbf{Star}(V_P) = \{d_{V_P}\}$ by Proposition 4.1(1). Then, by Corollary 3.6, $\mathbf{SStar}(V, V_P) = \{\star_{\{P\}}\}$.
(2) Apply the same argument, using Proposition 4.1(2).
(3) It is immediate, since $\mathbf{SStar}(D) = \bigcup\{\,\mathbf{SStar}(D, T) \mid T \in \mathcal{O}(D)\}$, as we have already observed. □

Remark 4.3. We notice that each semistar operation $\star$ on a valuation domain V is of the type introduced in Example 1.8(2), that is, there exists an ideal I of V such that $\star = v(I)$. Indeed, if $\star = \star_{\{P\}}$ for some prime P of V, then it is easy to see that $\star = v(P)$. So, let $\star = (v_{V_P})^\iota$ $(= v(V_P)$, with the notation of Proposition 4.2(2)). Then $V^\star = V_P$ is a fractional ideal of V (since V is conducive), that is, there exist an ideal I of V and $x \in V$ such that $V_P = x^{-1}I$. It is easy to see that $\star = v(I)$.

Now we prove a result that characterizes semistar operations of finite character on Prüfer domains.

Lemma 4.4. *Let D be a Prüfer domain, $\star$ a semistar operation of finite type on D. Then, $\star = \star_{\{D^\star\}}$.*

Proof. Let $\star$ be a semistar operation of finite type on D and let ι be the canonical embedding of D in $D^\star$. Then, $\star_\iota$ is a finite type (semi)star operation on the Prüfer domain $D^\star$, that is, $\star_\iota = d_{D^\star}$, the identity semistar operation (by [14, Proposition 34.12] and Example 1.1(7)). By Proposition 3.5(1) and Example 1.8(1), we have that $\star = \star_{\{D^\star\}}$. □

Remark 4.5. The result of Lemma 4.4 can be also proven directly using the semistar analogue of [14, Lemma 32.17], that is:

Let $\star$ be a semistar operation on an integral domain D, let I be an invertible ideal of D and $E \in \overline{\boldsymbol{F}}(D)$. Then $(IE)^\star = IE^\star$.

The proof of this result is exactly the same as the proof in the case of star operations. From this, it follows that, if $\star$ is a finite type semistar operation on a Prüfer domain D and I is a finitely generated (then an invertible ideal) of D, we have $I^\star = (ID)^\star = ID^\star$. So, since $\star$ coincides with the extension to $D^\star$ on the set of finitely generated ideals of D, it is clear that, if $\star$ is of finite type, it is the extension to $D^\star$.

We can use these results to characterize Prüfer domains such that each semistar operation is of finite type (cf. [24]). To do this, we need another Lemma:

Lemma 4.6. *Let D be a conducive Prüfer domain such that each nonzero prime ideal is contained in only one maximal ideal. Then, D is a valuation domain.*

Proof. Since D is a Prüfer, conducive domain, Spec(D) is pinched, by [5, Corollary 3.4]. That is, there exists a nonzero prime ideal P comparable under inclusion to each prime of D. Suppose that D has two distinct maximal ideal M and N. Then, both must contain P, a contradiction. Hence, D is a local Prüfer domain, that is, a valuation domain. □

We recall that an integral domain D is *divisorial* if each nonzero fractional ideal of D is divisorial, that is, if $I^v = I$, for each $I \in \boldsymbol{F}(D)$. The domain D is *totally divisorial* if each overring of D is divisorial. (For results on divisorial domains see for example [19], [4]; for totally divisorial domains see [3], [27].)

Theorem 4.7. *Let D be a Prüfer domain. Then, the following are equivalent:*

(i) *Each semistar operation on D is of finite type.*
(ii) *Each semistar operation on all overrings of D is of finite type.*
(iii) *Each semistar operation on D is an extension to an overring of D.*
(iv) *D is conducive and totally divisorial.*
(v) *D is a strongly discrete valuation domain (that is, $P \neq P^2$, for all primes P).*

Proof. (i) $\Leftrightarrow$ (ii) It is a consequence of Proposition 3.4(1) and Proposition 3.1(1).
(i) $\Leftrightarrow$ (iii) It follows immediately by Lemma 4.4 and the fact that the extensions to overrings are semistar operations of finite type (Example 1.1(7)).
(iii) $\Rightarrow$ (iv) Let d_e be the trivial extension of the identity star operation. From the hypothesis, it is the extension to D, that is, the identity semistar operation.

Then, D is conducive (Remark 3.10). To see that D is totally divisorial, take an overring T of D and let ι be the canonical embedding of D in T. Consider v_T the v-(semi)star operation on T. Note that by the hypothesis, $\mathbf{SStar}(D,T) = \{\star_{\{T\}}\}$. Thus, $(v_T)^\iota = \star_{\{T\}}$, since $(v_T)^\iota \in \mathbf{SStar}(D,T)$. So, by Proposition 3.4(1), $v_T = ((v_T)^\iota)_\iota = (\star_{\{T\}})_\iota = d_T$, and T is a divisorial domain. Hence, D is totally divisorial.
(iv) $\Rightarrow$ (v) Since D is divisorial, each nonzero prime of D is contained only in one maximal ideal ([19, Theorem 2.4]). Then, by Lemma 4.6, D is a valuation domain. Let $P \in \mathrm{Spec}(D)$. By the hypothesis, $\mathbf{Star}(D_P) = \{d\}$. Then, by Corollary 3.6(2), $\mathbf{SStar}(D, D_P) = \{\star_{\{P\}}\}$ and $P \neq P^2$, by Proposition 4.2. Hence, D is strongly discrete.
(v)$\Rightarrow$(i) It is a straightforward consequence of Proposition 4.2. □

In particular, for finite dimensional valuation domains,we reobtain the following results (cf. [21, Theorem 4] and [22, Theorem 4]), after recalling that for a finite dimensional valuation domain the notions of "discrete" and "strongly discrete" coincide.

Corollary 4.8. *Let (V,M) be an n-dimensional valuation domain. Then*

(1) $\mathrm{Card}\,(\mathbf{SStar}(V)) = n + 1 + \mathrm{Card}\left(\{P \in \mathrm{Spec}(V) \mid P^2 \neq P\}\right)$.
(2) *V is discrete if and only if* $\mathrm{Card}\,(\mathbf{SStar}(V)) = n + 1$. □

Some of the equivalent conditions of Theorem 4.7 hold also without the assumption that D is a Prüfer domain (cf. [24]).

Proposition 4.9. *Let D be an integral domain. The following are equivalent:*

(i) *Each semistar operation on D is an extension to an overring.*
(ii) *D is conducive and totally divisorial.*

Proof. (i) $\Rightarrow$(ii) The same proof as Theorem 4.7(iii)$\Rightarrow$(iv).
(ii) $\Rightarrow$(i) Let $\star$ be a semistar operation on D and ι the canonical embedding of D in $D^\star$. Then, $\star_\iota$ is a (semi)star operation on the divisorial conducive domain $D^\star$. It follows that $\star_\iota = d_{D^\star}$. Hence, $\star = \star_{\{D^\star\}}$, by Proposition 3.5(1) and Example 1.8(1). □

Next question is: when are all semistar operations on an integral domain spectral? We give a complete characterization of such domains in the local case.

To begin, we give an easy consequence of Proposition 1.2.

Corollary 4.10. *Let D be an integral domain such that each semistar operation on D is stable. Then:*

(1) *D is a Prüfer domain.*
(2) *Each semistar operation on each overring of D is stable.*

Proof. (1) If each semistar operation is stable, in particular each extension to an overring is stable. Then, by Proposition 1.2, each overring of D is flat. It follows that D is a Prüfer domain [9, Theorem 1.1.1].
(2) By (1), it follows that each overring of D is flat over D. Now, the result is a consequence of Proposition 3.15(1), Proposition 3.1(2)and Proposition 3.4. □

A similar result holds for spectral semistar operations.

Proposition 4.11. *Let D be an integral domain such that each semistar operation on D is spectral. Then:*

(1) *D is a Prüfer domain.*
(2) *Each semistar operation on each overring of D is spectral.*

Proof. (1) Note that spectral semistar operations are stable and apply Corollary 4.10(1).
(2) By (1), it follows that each overring of D is flat over D. Now, the result is a consequence of Proposition 3.17(1), Proposition 3.1(6)and Proposition 3.4. □

So, we can restrict to Prüfer domain. We start from the local case, that is, from a valuation domain V. First, we notice that, clearly, if P is a branched prime, $\star_{\{P\}}$ is the unique spectral semistar operation in $\mathbf{SStar}(V, V_P)$.
In the following Proposition, we see what happens when P is unbranched.

Proposition 4.12. *Let V be a valuation domain, $\star$ a semistar operation on V and $P \in \operatorname{Spec} V$ such that $V^\star = V_P$. If P is unbranched, then $\mathbf{SStar}(V, V_P) = \left\{\star_{\{P\}}, \star_\Delta\right\}$, where $\Delta = \{Q \in \operatorname{Spec}(V) \text{ such that } Q \subsetneq P\}$ (in this case, $v(V_P) = \star_\Delta$).*

Proof. We have only to prove that $V^{\star_\Delta} = V_P$ and that $\star_\Delta \neq \star_{\{P\}}$. That $V^{\star_\Delta} = V_P$ follows from the fact that P is unbranched. That $\star_\Delta \neq \star_{\{P\}}$ is straightforward since P is not contained in any $Q \in \Delta$ and then $P^{\star_\Delta} = \bigcap_{Q\in\Delta} PV_Q = \bigcap_{Q\in\Delta} V_Q = V_P \neq P = P^{\star_{\{P\}}}$. □

Thus, we have the following characterization of local domains such that each semistar operation is spectral.

Corollary 4.13. *Let D be a local domain. The following are equivalent:*
(i) *Every semistar operation on D is spectral.*
(ii) *D is a discrete valuation domain (that is, a valuation domain with all idempotent prime ideals unbranched).*

Proof. (i) ⇒ (ii) By Proposition 4.11(1), D is a valuation domain. Let $P \in \operatorname{Spec}(D)$. If $P^2 \neq P$, the only semistar operation in $\mathbf{SStar}(D, D_P)$ is $\star_{\{P\}}$, which is spectral. If $P^2 = P$, with the notations of Proposition 4.2, $\mathbf{SStar}(D, D_P) = \left\{\star_{\{P\}},\, v(V_P)\right\}$. We have already noticed that, if P is branched, only $\star_{\{P\}}$ is spectral. Thus, $v(V_P)$ is not spectral, a contradiction, since each semistar operation on D is spectral. Thus, D has not branched idempotent prime ideals and D is a discrete valuation domain.
(ii)⇒ (i) It is an immediate consequence of Proposition 4.12. □

As a consequence, we have the following:

Proposition 4.14. *Let D be an integral domain, such that each semistar operation on D is spectral. Then D is a Prüfer domain, such that D_P is a discrete valuation domain for each $P \in \operatorname{Spec}(D)$.* □

We recall that a Prüfer domain D is a generalized Dedekind domain (GDD for short) if and only if each localizing system of D is of finite type, [9, Theorem 5.2.1.]. Equivalently, by Proposition 1.3 if and only if each stable semistar operation is of finite type. A domain D is an H-domain if, for each ideal I such that $I^{-1} = D$, there exists a finitely generated ideal $J \subseteq I$ such that $J^{-1} = D$ [15]. To finish, we show another example on how the techniques developed in Section 3 can be used, giving a characterization of generalized Dedekind domains in terms of H-domains. First, we need a characterization of H-domains in terms of semistar operations.

Proposition 4.15. *Let D be an integral domain. The following are equivalent:*

(i) *D is an H-domain.*
(ii) *The localizing system $\mathcal{F}^v$ (associated to the v-operation of D) is finitely generated.*
(iii) $\star_{\mathcal{F}^v} = \tilde{v}(= w)$.

Moreover, if D is a Prüfer domain, these conditions are equivalent to:

(iv) $\star_{\mathcal{F}^v} = d$.

Proof. (i) $\Leftrightarrow$ (ii) It is straightforward, since, for each ideal I of D, we have $I^v = D$ (that is, $I \in \mathcal{F}^v$) if and only if $I^{-1} = D$.
(ii) $\Leftrightarrow$ (iii) It is a consequence of Proposition 1.4(1) and Proposition 1.3(2).
(iii)$\Leftrightarrow$(iv) It follows from the fact that, in a Prüfer domain, $w = d$ (since it is a finite type (semi)star operation). □

Now, we conclude with the following:

Theorem 4.16. *Let D be a Prüfer domain. The following are equivalent:*

(i) *D is a GDD.*
(ii) *D is an H-domain and each overring of D is an H-domain.*
(iii) *Every stable semistar operation on D is an extension to an overring of D.*

Proof. (i) $\Rightarrow$ (ii) The localizing system $\mathcal{F}^v$, where v is the v-(semi)star operation of D is finitely generated by the hypothesis and so D is an H-domain. So, we have proved that a GDD is an H-domain. Since each overring of a GDD is a GDD ([9, Theorem 5.4.1]), then each overring of D is an H-domain.
(ii) $\Rightarrow$ (iii) Let $\star$ be a stable semistar operation on D. Let ι be the canonical embedding of D in $D^\star$. Consider the semistar operation $\star_\iota$. This is a stable (semi)star operation, by Proposition 3.1(2). Then, $\star_\iota = d_{D^\star}$, by Proposition 4.15, since $D^\star$ is a Prüfer domain. Then, $\star = \star_{\{D^\star\}}$, by Example 1.8(1).
(iii) $\Rightarrow$ (i) Since the semistar operation given by the extension to an overring is of finite type (1.1(7)), we have that every stable semistar operation is of finite type. Hence, D is a GDD. □

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