*Communication*

# Starobinsky–Bel–Robinson Gravity

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**Abstract:** A novel string-inspired gravitational theory in four spacetime dimensions is proposed as a sum of the modified $(R + \alpha R^2)$ gravity motivated by the Starobinsky inflation and the leading Bel–Robinson-tensor-squared correction to the gravitational effective action of superstrings/M-theory compactified down to four dimensions. The possible origin of the theory from higher dimensions is revealed. The proposed Starobinsky–Bel–Robinson action has only two free parameters, which makes it suitable for verifiable physical applications in black hole physics, cosmological inflation and Hawking radiation.

## 1. Introduction

General relativity theory with the Einstein–Hilbert (EH) action for gravity in $D = 4$ spacetime dimensions is well confirmed by precision measurements inside the solar system. However, the EH action has to be modified in the UV regime (for high energies and curvatures, in the early universe), in the IR regime (for cosmological distances), and (beyond any doubt) in quantum gravity. When preserving the general coordinate invariance and locality, the EH gravity action can only be modified (besides a cosmological constant) by extra terms of the higher order in spacetime curvature. These higher-derivative terms in the gravitational effective action are supposed to describe quantum gravity corrections to the EH action, while they are necessary for physical applications in the UV regime because the EH gravity is non-renormalizable. This issue is well known in the literature about gravity but the main problem is a well-motivated derivation of the gravitational effective action from quantum gravity or a reasonable selection of the higher-order terms because there are infinitely many of them.

Solving this problem requires a practical framework for quantum gravity in order to perform calculations, which is another problem. Superstring theory is a mathematically consistent framework for quantum gravity, but its applications to observed physics are limited by the necessity of compactification of extra $(D - 4)$ spacetime dimensions and related huge uncertainties in verifiable predictions. Moreover, superstring theory in $D = 10$ dimensions is defined as a quantum perturbation theory and is not background-independent. In fact, string theory can only be formulated on Ricci-flat backgrounds and does not allow de Sitter vacua.

Nevertheless, it makes sense to use insights from superstrings/M-theory together with other insights into quantum gravity, coming from early universe cosmology, black hole physics and particle physics beyond the standard model, in order to motivate the *leading* quantum gravity corrections to the EH action of gravity.

In this manuscript, we formulate a new proposal for those corrections needed in the high-curvature regime, motivated by the compactified $D = 10$, type-II superstrings or M-theory in $D = 11$ dimensions, together with the simplest viable model of cosmological

inflation in four spacetime dimensions, known as the Starobinsky model. The resulting gravity action is called the Starobinsky–Bel–Robinson (SBR) gravity.

Our paper is organized as follows. In Section 2, we recall the Starobinsky model of inflation in four spacetime dimensions and its possible origin from higher dimensions. In Section 3, we recall the low-energy effective action of M-theory in eleven dimensions and its compactification to four dimensions. In Section 4, we formulate the novel SBR gravity in four dimensions and give our conclusions.

## 2. Starobinsky Gravity and Extra Dimensions

The Starobinsky model of inflation is defined by the modified gravity action [1]

$$S_{\text{Star.}} = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left( R + \frac{1}{6m^2} R^2 \right),$$ (1)

where we have introduced the scalar curvature $R$, reduced Planck mass $M_{\text{Pl}} = 1/\sqrt{8\pi G_{\text{N}}} \approx 2.4 \times 10^{18}$ GeV and the mass parameter $m$. We use the spacetime signature $(-, +, +, +, )$.

In the low-curvature spacetime, the $R^2$ term can be ignored and the action (1) reduces to the EH action. During inflation in the early universe (with strong spacetime curvature), the EH term can be ignored and the action (1) reduces to the no-scale $R^2$ gravity with the dimensionless coupling constant in front of the action. The $R^2$ term with a positive coupling constant is the *only* term in a generic Lagrangian, quadratic in the curvature tensor, that does not lead to ghosts, i.e., it is well theoretically motivated. The quantized gravity theory (1) is, however, non-renormalizable, with the UV cutoff being given by $M_{\text{Pl}}$; see, e.g., Ref. [2] for more details.

The action (1) is also well phenomenologically motivated due to its excellent agreement with WMAP/Planck/BICEP/KECK precision measurements of the cosmic microwave background radiation [3]. Then, the parameter $m$ is the inflaton mass that can be fixed by the COBE/WMAP normalization as

$$m \approx 3 \times 10^{13} \text{ GeV} \quad \text{or} \quad \frac{m}{M_{\text{Pl}}} \approx 1.3 \times 10^{-5}.$$ (2)

The action (1) can be deduced from higher ($D$) spacetime dimensions, while preserving the hierarchy of physical scales, $H_{\text{inf.}} \ll M_{\text{KK}} \ll M_{\text{Pl}}$, where $H_{\text{inf.}}$ is the Hubble scale of inflation, and $M_{\text{KK}}$ is the Kaluza–Klein scale. The relevant field theory in $D > 4$ dimensions should have the modified gravity Lagrangian of the form $(R + \alpha R^n)$ in terms of the $D$-dimensional scalar curvature $R$, coupled to a $(p-1)$-form gauge field $A$ and a having cosmological constant. Then, the warped compactification from $D$ dimensions on a sphere $S^{D-4}$ with a non-vanishing flux of the gauge field strength $F = d \wedge A$ down to four spacetime dimensions leads to the Starobinsky action (1). As was demonstrated in Refs. [4,5], it is only possible when $n = D/2$ and $p = n$. In particular, when $D = 8$, the modulus (radius) of the hidden four-sphere $S^4$ of extra dimensions can be stabilized, while the modified $D = 8$ gravity can be embedded into the modified $D = 8$ (Salam–Sezgin) gauged supergravity [6]. In turn, as was argued in Ref. [5], the modified $D = 8$ supergravity may be derived by compactifying the modified 11-dimensional supergravity on a three-sphere $S^3$. As a result, the Starobinsky action (1) may be derivable from the 11-dimensional supergravity modified by the term quartic in the scalar curvature and compactified on the product $S^4 \times S^3$ of extra dimensions. Unfortunately, a supersymmetric completion of such action in $D = 11$ was never found. The significance of eleven dimensions is due to the fact that the supergravity theory in $D = 11$ is *unique* [7], while its extension to quantum gravity, known under the name of M-theory, is presumably also unique, with local supersymmetry in $D = 11$ playing the essential role. [1]

### 3. Bel–Robinson-Tensor-Squared Term

Having recognized the importance of eleven spacetime dimensions, it is natural to ask what could be the next term in the effective gravity Lagrangian beyond the $R^2$ term in $D = 4$, when starting from M-theory beyond the supergravity action in $D = 11$, after its (flux) compactification to four spacetime dimensions.

The bosonic terms (when all fermionic fields are ignored) of M-theory in the leading order beyond the $D = 11$ supergravity action read [8,9]

$$S_{\text{M}} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left[ R - \frac{1}{2 \cdot 4!} F^2 - \frac{1}{6 \cdot 3! \cdot (4!)^2} \varepsilon_{11} CFF \right] \tag{3}$$

$$- \frac{T_2}{(2\pi)^4 \cdot 3^2 \cdot 2^{13}} \int d^{11}x \sqrt{-g} \left( J_{11} - \frac{1}{2} E_8 \right) + T_2 \int C \wedge X_8,$$

where $\kappa_{11}$ is the 11-dimensional gravitational constant, $T_2$ is the M2-brane tension,

$$T_2 = \left( \frac{2\pi^2}{\kappa_{11}^2} \right)^{1/3}, \tag{4}$$

$C$ is the three-form gauge field of the eleven-dimensional supergravity [7], $F = d \wedge C$ is the four-form gauge field strength, $R$ is the gravitational scalar curvature in $D = 11$, $\varepsilon_{11}$ stands for the Levi–Civita symbol in eleven dimensions, while $(J, E_8, X_8)$ are the *quartic* polynomials with respect to the eleven-dimensional Riemann curvature. In particular, the $J_{11}$ is given by

$$J_{11} = 3 \cdot 2^8 \left( R^{mijn} R_{pijq} R_m{}^{rsp} R^q{}_{rsn} + \frac{1}{2} R^{mnij} R_{pqij} R_m{}^{rsp} R^q{}_{rsn} \right), \tag{5}$$

the $E_8$ can be written in terms of the Euler density in eight dimensions,

$$E_8 = \frac{1}{3!} \varepsilon^{abcm_1 n_1 \ldots m_4 n_4} \varepsilon_{abcm'_1 n'_1 \ldots m'_4 n'_4} R^{m'_1 n'_1}{}_{m_1 n_1} \cdots R^{m'_4 n'_4}{}_{m_4 n_4} \tag{6}$$

and the $X_8$ is given by the gravitational eight-form

$$X_8 = \frac{1}{192 \cdot (2\pi^2)^4} \left[ \text{tr} \hat{R}^4 - \frac{1}{4} (\text{tr} \hat{R}^2)^2 \right], \tag{7}$$

where $\hat{R}$ stands for the spacetime curvature two-form in eleven dimensions, and the traces are taken with respect to (implicit) Lorentz indices in $D = 11$ dimensions. All (Latin) vector indices take values $i, j, k, \ldots = 0, 1, 2, \ldots, 10$, while they are all suppressed in Equation (3) for simplicity. See Ref. [10] for the notation of the exterior differential forms describing gravitational quantities in any dimension.

The $D = 11$ gravity theory (3) can be compactified with the warp factors on the product $S^3 \times S^4$ of extra dimensions down to four spacetime dimensions, in the presence of fluxes needed for moduli stabilization [11]. Being interested in the gravitational sector of the effective field theory in $D = 4$ dimensions, we can apply a simple dimensional reduction to the action (3) and ignore details of its compactification together with all moduli. Then, only one term survives [12],

$$S_4 = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R + \kappa^6 \beta J_4 \right) \tag{8}$$

where all quantities are now in $D = 4$ with $\kappa = 1/M_{\text{Pl}}$, and $\beta$ is the new dimensionless coupling constant whose value is supposed to be determined from the compactification. In particular, the $J_4$ is dictated by its $D = 11$ origin from $J_{11}$ of Equation (5) and has the same structure in $D = 4$,

$$J_4 = R^{mijn} R_{pijq} R_m{}^{rsp} R^q{}_{rsn} + \frac{1}{2} R^{mnij} R_{pqij} R_m{}^{rsp} R^q{}_{rsn}. \tag{9}$$

with $i, j, k, \ldots = 0, 1, 2, 3$.

The peculiar structure and physical meaning of the quartic curvature terms in Equation (9) is revealed via their connection to the four-dimensional *Bel–Robinson* (BR) tensor that is well known in general relativity [13–15]. The BR tensor is defined as

$$T^{iklm} \equiv R^{ipql}R^{k}{}_{pq}{}^{m} + {}^{*}R^{ipql*}R^{k}{}_{pq}{}^{m} = R^{ipql}R^{k}{}_{pq}{}^{m} + R^{ipqm}R^{k}{}_{pq}{}^{l} - \frac{1}{2}g^{ik}R^{pqrl}R_{pqr}{}^{m} \quad (10)$$

by analogy with the energy–momentum tensor of the Maxwell theory of electromagnetism,

$$T_{ij}^{\text{Maxwell}} = F_{ik}F_{j}{}^{k} + {}^{*}F_{ik}{}^{*}F_{j}{}^{k}, \quad F_{ij} = \partial_{i}A_{j} - \partial_{j}A_{i}, \quad (11)$$

where the superscript $(*)$ means the dual tensor in $D = 4$. As regards the curvature tensor, we have

$$^{*}R_{iklm} = \frac{1}{2}E_{ikpq}R^{pq}{}_{lm}, \quad (12)$$

where $E_{iklm} = \sqrt{-g}\,\varepsilon_{iklm}$ is the Levi–Civita tensor.

Then, one finds [12,15]

$$T_{ijkl}^{2} = 8J_{4} = -\frac{1}{4}({}^{*}R_{ijkl}^{2})^{2} + \frac{1}{4}({}^{*}R_{ijkl}R^{ijkl})^{2} = \frac{1}{4}(P_{4}^{2} - E_{4}^{2}) = \frac{1}{4}(P_{4} + E_{4})(P_{4} - E_{4}), \quad (13)$$

where we have also introduced the Euler and Pontryagin topological densities in $D = 4$,

$$E_{4} = \frac{1}{4}\varepsilon_{ijkl}\varepsilon^{mnpq}R^{ij}{}_{mn}R^{kl}{}_{pq} = {}^{*}R_{ijkl}{}^{*}R^{ijkl} \quad (14)$$

and

$$P_{4} = {}^{*}R_{ijkl}R^{ijkl}, \quad (15)$$

respectively. We use the book-keeping notation $A_{ijlk}^{2} \equiv A^{ijlk}A_{ijlk}$ for any rank-4 tensor $A$.

On-shell (by using the equations of motion in the EH gravity), one can show [13,14] that the BR tensor is fully symmetric and traceless,

$$T_{ijkl} = T_{(ijkl)}, \qquad T^{i}{}_{ikl} = 0, \quad (16)$$

is covariantly conserved,

$$\nabla^{i}T_{ijkl} = 0, \quad (17)$$

and has a *positive* "energy' density",

$$T_{0000} > 0. \quad (18)$$

The BR tensor is also related to the symmetric gravitational Landau–Lifshitz (LL) energy–momentum *pseudo*-tensor [16]

$$(t_{LL})^{ij} = -\eta^{ip}\eta^{jq}\Gamma^{k}_{pm}\Gamma^{m}_{qk} + \Gamma^{i}_{mn}\Gamma^{j}_{pq}\eta^{mp}\eta^{nq} - \left(\Gamma^{m}_{np}\Gamma^{j}_{mq}\eta^{in}\eta^{pq} + \Gamma^{m}_{np}\Gamma^{i}_{mq}\eta^{jn}\eta^{pq}\right) + h^{ij}\Gamma^{m}_{np}\Gamma^{n}_{mq}\eta^{pq} \quad (19)$$

and the non-symmetric Einstein (E) gravitational energy–momentum pseudo-tensor [17]

$$(t^{E})^{i}_{j} = \left(-2\Gamma^{i}_{mp}\Gamma^{m}_{jq} + \delta^{i}_{j}\Gamma^{n}_{pm}\Gamma^{m}_{qn}\right)\eta^{pq} \quad (20)$$

in Riemann normal coordinates as follows [15]:

$$T_{ijkl} = \partial_{k}\partial_{l}\left(t^{LL}_{ij} + \frac{1}{2}t^{E}_{ij}\right). \quad (21)$$

## 4. Conclusions

Having motivated the presence of the scalar curvature squared term from inflation and the Bel–Robinson–tensor-squared term from superstrings/M-theory in the $D = 4$ gravitational effective action, we propose the SBR gravity as a sum of them, with the following action:

$$
\begin{aligned}
S_{\text{SBR}}[g_{ij}] &= \frac{M_{\text{Pl}}^2}{2} \int \text{d}^4 x \sqrt{-g} \left[ R + \frac{1}{6m^2} R^2 + \frac{\beta}{M_{\text{Pl}}^6} J_4 \right], \\
&= \frac{M_{\text{Pl}}^2}{2} \int \text{d}^4 x \sqrt{-g} \left[ R + \frac{1}{6m^2} R^2 + \frac{\beta}{8M_{\text{Pl}}^6} T^2 \right], \\
&= \frac{M_{\text{Pl}}^2}{2} \int \text{d}^4 x \sqrt{-g} \left[ R + \frac{1}{6m^2} R^2 + \frac{\beta}{32M_{\text{Pl}}^6} (P_4^2 - E_4^2) \right],
\end{aligned}
\tag{22}
$$

where we have used Equation (13).

Equation (22) is our proposal for the $D = 4$ gravitational effective action in the high-curvature regime and our main result in this manuscript. The high-curvature regime is defined by a situation where the values of the second and third terms in Equation (22) are comparable to or larger than the value of the first one. Of course, the action (22) is still an approximation that is not valid at the scales close to the Planck scale.

More comments are in order.

The proposed action (22) is gravitational (no matter added) and geometric; it does not have arbitrary functions and extra scalars (beyond the spacetime metric and Starobinsky's scalaron that is the physical excitation of the higher-derivative $(R + \alpha R^2)$ gravity).

The SBR action has only two parameters $(m, \beta)$, which implies its predictive power for a wide range of physical applications, such as black hole entropy [18], inflation [19] and Hawking radiation [20]. These parameters should be subject to renormalization, i.e., they should depend upon a physical scale. Being subject to the renormalization of the parameters, the action (22) may be applied at any scale.

The SBR action is string-inspired in the sense that the BR term quartic in the curvature comes from superstrings/M-theory as the theory of quantum gravity that is not only renormalizable but is also unitary and ghost-free by construction [21]. In other words, the popular Lovelock-like or Horndeski-like criteria demanding the absence of the higher derivatives (beyond the second order) in the equations of motion do not apply.

The Ricci-tensor-dependent terms in the gravitational effective action of strings/M-theory cannot be determined from the known S-matrix in superstring theory, but they are not forbidden either [22]. From the viewpoint of string theory, the coupling constants $(m, \beta)$ are supposed to be given by the vacuum expectation values of moduli fields (including dilaton) as a result of compactification. However, all those moduli and dilaton have to be stabilized, which is a difficult problem [11,23].

The SBR action includes two topological densities (or four-forms) $E_4$ and $P_4$ that can be locally represented as the wedge derivatives of the corresponding Chern–Simons three-forms. However, they do contribute to the equations of motion in the SBR gravity because they enter the action (22) as the squared terms. Since the Euler density $E_4$ is the same as the Gauss–Bonnet density $G = R^2 - 4R_{ij}^2 + R^{ijkl} R_{ijkl}$, when ignoring the $P_4^2$ term in Equation (22), the SBR action reduces to the particular modified gravity action of the type $F(R, G)$ with the quadratic terms in $R$ and $G$ only (cf. Ref. [24]).

Some specific applications of the proposed SBR theory will be studied elsewhere.

## Notes

[1] Another possibility is to start from F-theory in $D = 12$ dimensions and compactify it on a product of two Kummer surfaces $K3 \times K3$.

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