Statistica Sinica **20** (2010), 1637-1653

STARS AND REGULAR FRACTIONAL FACTORIAL DESIGNS WITH RANDOMIZATION RESTRICTIONS

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Abstract: Factorial and fractional factorial designs are widely used for assessing the impact of several factors on a process. Frequently, restrictions are placed on the randomization of the experimental trials. The randomization structure of such a factorial design can be characterized by its set of *randomization defining contrast subspaces.* It turns out that in many practical situations, these subspaces will overlap, thereby making it impossible to assess the significance of some of the factorial effects. In this article, we propose new designs that minimize the number of effects that have to be sacrificed. We also propose new designs, called stars, that are easy to construct and allow the assessment of a large number of factorial effects under an appropriately chosen overlapping strategy.

Key words and phrases: Block design, finite projective geometry, minimal $(t-1)$-cover, split-lot design, split-plot design, $(t-1)$-spread.

1. Introduction

In the initial stages of experimentation, factorial designs with p independent factors, each at q levels (usually $q = 2$), are commonly used to help assess the impact of several factors on a process. Ideally, one performs the experimental trials in a completely random order. Complete randomization of trials is often infeasible, and randomization restrictions are imposed. Indeed, in many situations different factors must be held fixed at each stage of the experimental process (e.g., see Mee and Bates (1998); Vivacqua and Bisgaard (2004); Bingham et al. (2008)). In the analysis of the such experiments, there are variance components associated with each stage of randomization. Preferably the experiment can be designed so that the variance components have as little impact as possible on the variance of the effect estimators. Designs aimed at minimizing the impact of randomization restrictions on the analysis of multistage factorial experiments are the primary focus of this work.

Bingham et al. (2008) proposed using *randomization defining contrast subgroups* (RDCSGs) to describe the randomization structure of multistage factorial designs (e.g., blocked designs, split-plot designs, strip-plot designs, split-lot designs). This representation can be viewed as a generalization of a block defining contrast subgroup (see, e.g., Sun, Wu and Chen (1997)). Recently, Ranjan, Bingham and Dean (2009) developed a finite projective geometric formulation of the RDCSGs, called *randomization defining contrast subspaces* (RDCSSs), that helps establish the existence and construction of such designs in practical settings.

The RDCSSs indicate which effects are impacted by the variance from each stage of randomization. It is important to note that each point in a subspace is associated with a factorial effect. So, for non-overlapping subspaces, each effect appearing in a RDCSS has a variance that is a linear combination of the replication error variance and the variance component associated with that stage of randomization. On the other hand, when RDCSSs are not disjoint, the effects in the overlap will have variance that is a linear combination of all of the variances associated with the overlapping subspaces. As a result, the distribution of an effect estimator depends on its presence in different RDCSSs.

The selection of a design is usually based on properties from a data analysis viewpoint. A common strategy for assessing the significance of factorial effects in unreplicated factorial designs is to use half-normal plots (Daniel (1959)) with the restriction that the effects appearing on the same plot must have the same error variance. In the current setup, the RDCSSs indicate which effects have the same variance and thus can appear on the same half-normal plot.

In this article, we focus on unreplicated experiments. However, if it is possible to replicate an experiment, one has a few choices. First, the experiment could be replicated and the usual variance components analysis can be conducted. Unless the number of replicates is large, one would likely opt to use a half-normal plot analysis since the error degrees of freedom would be small. Instead of replicating the design, one has the option of performing a larger fractional factorial design, perhaps with higher resolution. In this case, one would also use the visual method to identify the significant effects.

A desirable feature for the randomization structure of an unreplicated factorial design is to have disjoint RDCSSs that are large enough to construct useful half-normal plots. However, it is not always feasible to construct a desired number of disjoint RDCSSs that satisfy the size requirement (Ranjan, Bingham and Dean (2009)). Here we focus on regular (fractional) factorial designs with randomization restrictions when an overlap among the distinct RDCSSs cannot be avoided. Two RDCSSs S_i and S_j are said to be distinct if $(S_i \cup S_j)\backslash(S_i \cap S_j)$ is nonempty. We propose two new classes of factorial designs with randomization restrictions: (a) when the overlap among the RDCSSs is minimized, and (b) when overlap among the distinct RDCSSs is used as an advantage for constructing designs that allow for the assessment of a larger number of the factorial effects.

The paper is organized as follows. We introduce the notation necessary to establish the relationship between randomization restrictions and RDCSSs in Section 2. In Section 3, we propose adapting results from a projective geometric structure called a minimal $(t-1)$-cover of $PG(p-1,q)$ to construct (regular fractional) factorial designs with randomization restrictions. An overlapping strategy is proposed in Section 4 that leads to a new geometric structure we call a *star* in $PG(p-1,q)$. The factorial designs constructed from stars allow for the assessment of a relatively large number of effects. The existence and construction of factorial designs based on stars are developed in Section 5. In Section 6, we establish the relationship between stars and the minimal covers of $PG(p-1,q)$. We conclude the paper with a brief discussion in Section 7.

2. Background Review and Notation

Throughout the article, q is a prime or prime power. Let b be a p-dimensional pencil over the Galois field $GF(q)$ (e.g., Dey and Mukerjee (1999)). For non-zero $\alpha \in GF(q)$, b and αb represent the same $q-1$ degrees of freedom pencil. A pencil b represents an r-factor interaction if b has exactly r nonzero elements. Denote the $(p-1)$-dimensional finite projective geometry, given by the set of all p-dimensional pencils (or points) over $GF(q)$, as $PG(p-1,q)$. In this sense, we often refer to $\mathcal{P} = PG(p-1,q)$ as the effect space. For $q=2$, a pencil b with r nonzero elements uniquely corresponds to an r-factor interaction in a 2^p factorial design with a single degree of freedom. Thus, the set of all effects (excluding the grand mean) of a two-level factorial design with p independent factors is equivalent to the set of all points in $PG(p-1,2)$.

The restrictions on the randomization of experimental runs are equivalent to grouping experimental units into sets of trials. We follow the usual approach of forming these sets for factorial experiments by using independent pencils to define the groupings. Blocked factorial designs, for example, use q^t $(t<p)$ combinations of t independent pencils to divide q^p treatment combinations into q^t blocks. These factorial effects are then completely confounded with the block effects and represent t randomization restriction *factors*. The set S of all non-null pencils formed from these t randomization restriction factors in $\mathcal{P}$ forms a $(t-1)$-dimensional projective subspace of $\mathcal{P}$. We call such a subspace a RDCSS.

For a q-level factorial design with p independent factors and m stages of randomization, the m RDCSSs can be denoted by the projective subspaces $S_1,\ldots,S_m$ contained in $\mathcal{P}$. For each i, let S_i be generated from t_i independent pencils $(0<t_i<p)$, so that $|S_i| = (q^{t_i}-1)/(q-1)$. It turns out that the existence of fractional factorial designs with randomization restrictions is equivalent to the existence of distinct projective subspaces S_i's in $\mathcal{P}$ that accommodate the

desired randomization structure. It is not easy to establish the existence of such designs that can be analyzed effectively (Ranjan, Bingham and Dean (2009)).

For a factorial design with m stages of randomization and RDCSSs denoted by S_i, $i = 1, \ldots, m$, the error vector, ε, in the linear regression model is a sum of $m+1$ error terms, $\varepsilon = \varepsilon_0 + \varepsilon_1 + \cdots \varepsilon_m$. Here, ε_0 denotes the observational error vector, and ε_i $(1 \leq i \leq m)$ is the error vector associated with the randomization restriction characterized by S_i. Consequently, if a pencil belongs to more than one RDCSSs (say, in S_i and S_j), then the distribution, and hence the variance of the estimator of any contrast representing the pencil, depend on a linear combination of the variance associated with ε_0, ε_i and ε_j (Ranjan, Bingham and Dean (2009)). This necessitates separate analyses for pencils in $S_i \cap S_j$ and those in $S_i \backslash (S_i \cap S_j)$. From this perspective, factorial designs with randomization restrictions ideally have disjoint RDCSSs. Ranjan, Bingham and Dean (2009) show that the existence of disjoint RDCSSs of equal size is equivalent to that of full or partial $(t-1)$-spreads.

A (full) $(t-1)$-*spread* of $\mathcal{P}$ is a collection of $(t-1)$-dimensional subspaces of $\mathcal{P}$ which partition $\mathcal{P}$, whereas a *partial* $(t-1)$-*spread* of $\mathcal{P}$ is a collection of $(t-1)$-dimensional subspaces of $\mathcal{P}$ that are pairwise disjoint. The existence of full and partial $(t-1)$-spreads have been studied in André (1954) and Eisfeld and Storme (2000). For practical use, however, mere existence is not enough, one needs to find the designs. To this end, Ranjan, Bingham and Dean (2009) proposed a methodology for constructing designs with disjoint RDCSSs, not necessarily of the same size. We focus here on designs where the projective subspaces corresponding to the RDCSSs are such that overlap among them is unavoidable.

3. Minimal Overlap and $(t-1)$-covers

In this section, we adapt results used to study a geometric structure, called a $(t-1)$-cover of $PG(p-1, q)$, to construct designs that maximize the number of distinct subspaces and minimize the overlap among the intersecting subspaces for constructing distinct RDCSSs. The resulting subspaces are used to set the levels of each factor at each stage of randomization, and also to identify which pencils are estimated with the same error variance. Our aim is to construct designs that are easy to analyze and allow the significance assessment of lower order effects.

Definition 1. A $(t-1)$-cover, $\mathcal{C}$, of $\mathcal{P} = PG(p-1, q)$ is a set of distinct $(t-1)$-dimensional subspaces of $\mathcal{P}$ which cover all the points of $\mathcal{P}$.

A $(t-1)$-cover is called *minimal* if no other $(t-1)$-cover contains a smaller number of subspaces. Although the subspaces forming a minimal cover may overlap, the size of the overlap is often small.

Since effects appearing in the overlap have a different error variance than effects not in the overlap itself, it is often preferable to minimize the degree of intersection among the subspaces (we argue later that there are sometimes advantages to not doing so). This makes minimal covers attractive for designs with randomization restrictions where overlap among the RDCSSs is unavoidable.

Remarks. (i) For a half-normal plot analysis, the S_i's have to be reasonably large (e.g., see Bingham et al. (2008)). For instance, in 2-level factorial designs, the size of each S_i should be at least 2^3-1 (i.e., $t_i \geq 3$). (ii) Factorial designs with randomization restrictions are often larger than completely randomized designs. Since, at each stage of randomization, multiple experimental units are processed simultaneously. For example, Jones and Goos (2009) used a 128-run design to analyze a cheese-making experiment, and Mee and Bates (1998) proposed 64-wafer and 81-wafer designs for an integrated circuit experiment.

Example 1 presents a scenario where the overlap among the RDCSSs cannot be avoided, and a minimal $(t-1)$-cover is used to construct a good design.

Example 1. Following Bingham et al. (2008), consider a 2^5 factorial experiment performed in three stages to identify the factors suspected to have a significant impact on a specific plutonium alloy. The three stages of randomization were characterized by $S_1 \supset \{A, B\}$, $S_2 \supset \{C\}$, and $S_3 \supset \{D, E\}$, where A, B represented the casting mechanism for creating a type of plutonium alloy, and C, D, E were the heat treatments applied in the manufacturing process.

For a half-normal plot analysis we need $|S_i| \geq 7$ for all i. Using a result from Ranjan, Bingham and Dean (2009), we find that any two distinct S_i share at least one effect. Bingham et al. (2008) also reached this conclusion after an exhaustive search.

The design proposed by Bingham et al. (2008), is characterized by $S_1 = \langle A, B, ABCDE \rangle$, $S_2 = \langle C, AD, ABCDE \rangle$ and $S_3 = \langle D, E, ABCDE \rangle$ where, for instance, $\langle A, B, ABCDE \rangle$ is the subspace spanned by the pencils representing the factorial effects A, B and $ABCDE$.

Note that S_1, S_2, S_3, together with $S_4 = \langle AC, AE, ABCDE \rangle$ and $S_5 = \langle BC, BD, ABCDE \rangle$, form a minimal 2-cover of $\mathcal{P} = \langle A, B, C, D, E \rangle$. That is, the design proposed in Example 1 can be constructed without using an exhaustive computer search. The following result specifies the size of such a minimal cover.

Lemma 1 (Eisfeld and Storme (2000)). *A minimal* $(t-1)$*-cover of* $\mathcal{P} = PG(p-1, q)$ *contains* $q^s\,[(q^{kt}-1)/q^t-1]+1$ *distinct* $(t-1)$*-dimensional subspaces of* $\mathcal{P}$*, where* $p = kt+s$, $0 < s < t < p$, *and* $k \geq 1$.

It turns out that for any $t < p$, there always exists a minimal $(t-1)$-cover with $q^s\,[(q^{kt}-1)/(q^t-1)]+1$ distinct $(t-1)$-dimensional subspaces of $\mathcal{P}$. This is indeed very useful for design construction. Next, we outline a recursive technique for constructing a minimal $(t-1)$-cover, $\mathcal{C}$, of $\mathcal{P} = PG(p-1,q)$. The proposed method shares features with the construction of a maximal partial $(t-1)$-spread (see Eisfeld and Storme (2000) and Ranjan, Bingham and Dean (2009) for details).

Construction: A minimal $(t-1)$-cover of $\mathcal{P}$ consists of $q^s\,[(q^{kt}-1)/(q^t-1)] - q^s$ disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ if $k > 1$, and q^s+1 distinct $(t-1)$-dimensional subspaces that overlap on a common $(t-s-1)$-dimensional subspace. The construction of $q^s\,[(q^{kt}-1)/(q^t-1)] - q^s$ disjoint elements of the minimal cover $\mathcal{C}$ begins by defining a sequence of indices $w_i = it+s$ for $i = 1,\ldots,k-1$, and setting $\mathcal{P}'_k = \mathcal{P}$. The recursive algorithm starts from $i = k-1$ and goes down to $i = 1$.

1. Construct a projective space $\mathcal{P}_i = PG(2w_i-1,q)$ that contains $\mathcal{P}'_{i+1}$.
2. Construct a (w_i-1)-spread $\mathcal{S}'_i$ of $\mathcal{P}_i$ that contains an (w_i-1)-dimensional subspace, U_i, of $\mathcal{P}'_{i+1}$.
   (a) Construct a (w_i-1)-spread $\mathcal{S}''_i$ of $\mathcal{P}_i$ as in Ranjan, Bingham and Dean (2009).
   (b) Transform the spread $\mathcal{S}''_i$ to $\mathcal{S}'_i$ by finding an appropriate collineation (see Batten (1997) and Ranjan, Bingham and Dean (2009)) such that $U_i \in \mathcal{S}'_i$.
3. Construct $\mathcal{S}_i = \{S \cap \mathcal{P} : S \in \mathcal{S}'_i \backslash \{U_i\}\}$.
4. Define $\mathcal{P}'_i = U_i$ and then set $i = i-1$. If $i > 0$ go to Step 1.

For every $i \in \{1,\ldots,k-1\}$, $\mathcal{S}_i$ is a set of $(t-1)$-dimensional subspaces in $\mathcal{P}$, and $\mathcal{S}_i \cap \mathcal{S}_j = \phi$ for $i \neq j$. Finally, $\mathcal{S} = \cup_{i=1}^{k-1}\mathcal{S}_i$ contains $q^s\,[(q^{kt}-1)/(q^t-1)] - q^s$ disjoint $(t-1)$-dimensional elements of $\mathcal{C}$. The construction of the remaining q^s+1 elements is shown in a more general setup (Section 5), where we also show that the set of such overlapping elements of $\mathcal{C}$ form a new geometric structure called a *star*.

The above technique facilitates the construction of minimal $(t-1)$-covers and hence factorial designs with efficient assessment of many factorial effects except for a few higher order interactions. Although constructing a minimal $(t-1)$-cover does not require an exhaustive computer search, the pencils (or effects) in the subspaces constituting the minimal cover may have to be relabeled to get the desired design. Next, we revisit Example 1 and construct the design proposed in this example using a minimal $(t-1)$-cover approach.

Example 1 (contd.) From Lemma 1, a minimal 2-cover $\mathcal{C}$ of $\mathcal{P} = PG(4,2)$ contains 5 (since $t = 3, k = 1$ and $s = 2$) distinct subspaces. Also note that, any

Table 1. The ANOVA table for the plutonium alloy experiment.

Effects Appearing on the Same Half-normal Plot	Variance	Degrees of Freedom
$S_1\backslash\{ABCDE\}$	$\frac{2^2}{2^5}\sigma_1^2+\frac{1}{2^5}\sigma^2$	6
$S_2\backslash\{ABCDE\}$	$\frac{2^2}{2^5}\sigma_2^2+\frac{1}{2^5}\sigma^2$	6
$S_3\backslash\{ABCDE\}$	$\frac{2^2}{2^5}\sigma_3^2+\frac{1}{2^5}\sigma^2$	6
$\{ABCDE\}$	$\frac{2^2}{2^5}(\sigma_1^2+\sigma_2^2+\sigma_3^2)+\frac{1}{2^5}\sigma^2$	1
$\mathcal{P}\backslash(S_1\cup S_2\cup S_3)$	$\frac{1}{2^5}\sigma^2$	12

pair of 2-dimensional subspaces of $PG(4,2)$ shares at least one effect. That is, $|S_i\cap S_j|\geq 1$ for all $i\neq j$ and $S_i,S_j\in\mathcal{C}$. This also meets the size constraint $(5([2^3-1]-[2^1-1])+[2^1-1]=2^5-1)$ on the subspace structure of $\mathcal{C}$. The construction of a minimal 2-cover (same as a star in this case) of $PG(4,2)$, as outlined in Section 5.2, results in $\mathcal{C}=\{S_1,\ldots,S_5\}$, where $S_1=\langle D,BC,ABCDE\rangle$, $S_2=\langle C,AB,ABCDE\rangle$, $S_3=\langle B,ACD,ABCDE\rangle$, $S_4=\langle A,BD,ABCDE\rangle$, and $S_5=\langle CD,AC,ABCDE\rangle$. Since these subspaces do not satisfy the desired requirements on the RDCSSs, by using the relabeling

$$\begin{array}{lllllllll} A & \longrightarrow & CD & B & \longrightarrow & D & C & \longrightarrow & BD \\ D & \longrightarrow & A & E & \longrightarrow & E & & & \end{array}$$

we get $S_1=\langle A,B,ABCDE\rangle$, $S_2=\langle BD,C,ABCDE\rangle$, $S_3=\langle D,ABC,ABCDE\rangle$, $S_4=\langle CD,AD,ABCDE\rangle$, and $S_5=\langle ABD,ACD,ABCDE\rangle$; these meet the restrictions on the three stages of randomization. Bingham et al. (2008) found the same design via a computer search. Since S_1,S_2, and S_3 intersect in $ABCDE$, the error variance of $ABCDE$ effect estimator is a linear combination of all the components. Constructing a half-normal plot with one point is not informative, and hence $ABCDE$ could not be assessed (see Table 1). Sacrificing the assessment of $ABCDE$ was not an issue here because the impact of the five-factor interaction $ABCDE$ was assumed to be negligible.

There are a few issues worth noting. First, sometimes lower order interactions in the common overlap are unavoidable. For instance, consider a 2^5 regular fractional factorial design setup with two stages of randomization, where $S_1\supset\{A,B,C,D\}$ and $S_2\supset\{E\}$. Since $|S_2|\geq 7$ is required for significance assessment of effects in S_2, if $S_1=\langle A,B,C,D\rangle$ and $S_2\supset\{E\}$ with $|S_2|=2^3-1$, then $|S_1\cap S_2|\geq 3$. Moreover, since $S_1\cap S_2$ is a subspace of S_1, at least one 2-factor interaction is contained in $S_1\cap S_2$. This results in sacrificing the assessment of the three pencils in $S_1\cap S_2$ and four pencils including one main effect in $S_2\backslash(S_1\cap S_2)$. That is, the assessment of seven factorial effects including one main effect and at least one two-factor interaction have to be sacrificed, which is certainly undesirable. Second, if a regular fractional factorial design has to be constructed, sacrificing the assessment of even higher order interactions of the *base factorial design* (constructed from the basic factors only) is not always desirable; many good regular fractional factorial designs (e.g., minimum aberration designs) tend to choose higher order interactions for the fractional generators. As a result, minimizing the size of the overlap among the RDCSSs is also not desirable. In the next section, we propose a strategy for choosing RDCSSs with larger overlaps to allow the assessment of all effects.

4. A New Overlapping Strategy

The key idea in this section is that when an overlap among the RDCSSs is unavoidable, the size of the overlap itself can be made large enough to allow analysis of all the factorial effects. That is, we can use the overlap to our advantage rather than being forced to sacrifice the assessment of the pencils therein.

In Example 1, since the design was a full factorial, sacrificing the assessment of a 5-factor interaction was possible. If instead, a fractional factorial was to be performed, one might construct a design by assigning the *added factors* to the higher order interactions of the *basic factors*. Example 2 presents a scenario where a larger overlap leads to a better design.

Example 2. Consider the plutonium alloy example in Example 1 (Section 3). Suppose the experimenter wishes to introduce two additional factors (F, G) at the second stage of randomization without increasing the run size (i.e., a 2^{7-2} fractional factorial experiment). If we consider the randomization structure of Example 1 for the *base factorial design*, the minimum aberration design has resolution IV with fractional generators $F = ABDE$ and $G = ACE$. This leads to sacrificing the assessment of $FC = ABCDE$, as it is common to all S_i. This is certainly undesirable, as two-factor interactions are of utmost priority.

Instead of minimizing the overlap, we suggest finding a design with large enough overlap to construct a separate half-normal plot for the pencils in the overlap. For instance, the desired 2^{7-2} regular fractional factorial split-lot design with 3 stages of randomization can be constructed by defining $S_1 = \langle A, AB, DE, ACD\rangle$, $S_2 = \langle C, AB, DE, ACD\rangle$ and $S_3 = \langle D, AB, DE, ACD\rangle$ with the same fractional generators $F = ABDE$ and $G = ACE$. The resulting design has minimum aberration and allows the assessment of all the factorial effects using four separate half-normal plots (Table 2).

A key feature of a good overlapping strategy is that all non-disjoint subspaces should have a common overlap. This keeps the number of half-normal plots small,

Table 2. The sets of effects having equal variance in the 2^5 split-lot design.

Effects	Variance	Degrees of Freedom
$S_1 \backslash \langle AB, DE, ACD \rangle$	$\frac{2^1}{2^5}\sigma_1^2 + \frac{1}{2^5}\sigma^2$	8
$S_2 \backslash \langle AB, DE, ACD \rangle$	$\frac{2^1}{2^5}\sigma_2^2 + \frac{1}{2^5}\sigma^2$	8
$S_3 \backslash \langle AB, DE, ACD \rangle$	$\frac{2^1}{2^5}\sigma_3^2 + \frac{1}{2^5}\sigma^2$	8
$\langle AB, DE, ACD \rangle$	$\frac{2^1}{2^5}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{2^5}\sigma^2$	7

Table 3. The ANOVA table for the battery cell experiment.

Effects	Variance	Degrees of Freedom
$S_1 \cap S_2$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{2^3}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	1
$S_1 \backslash (S_1 \cap S_2)$	$\frac{2^2}{2^6}\sigma_1^2 + \frac{1}{2^6}\sigma^2$	14
$S_2 \backslash (S_1 \cap S_2)$	$\frac{2^3}{2^6}\sigma_2^2 + \frac{1}{2^6}\sigma^2$	6
$\mathcal{P} \backslash (S_1 \cup S_2)$	$\frac{1}{2^6}\sigma^2$	42

i.e., with m stages of randomization, at most $m+2$ half-normal plots are needed to assess the significance of all the pencils.

Given the number of independent basic and added factors and the randomization restrictions, the existence and construction of designs with large enough overlap is not straightforward. The RDCSSs are also often stipulated by the experimenters and are likely to be of different sizes (see Example 3), thereby complicating matters.

Example 3. Consider the battery cell experiment in Vivacqua and Bisgaard (2004). In this setup, a company manufacturing electric batteries was interested in identifying the factors that could have significant impact on the open circuit voltage of batteries. A 2^6 factorial experiment was performed, where the manufacturing took place in a two-stage process: (a) assembly - characterized by $S_1 \supset \{A, B, C, D\}$, and (b) curing - characterized by $S_2 \supset \{E, F\}$. The original design used $S_1 = \langle A, B, C, D \rangle$ and $S_2 = \langle E, F \rangle$. Since $|S_2| = 3$ and the half-normal plots require more than six effects per plot, the effects in S_2 could not be assessed. One could instead use a design with $S_2 = \langle E, F, ABCD \rangle$ to allow the assessment of all factorial effects except $ABCD$ (see Table 3).

Since the minimal $(t-1)$-cover approach focusses on projective subspaces of equal size only, it is not possible to appeal to related results. A new geometric structure, a star, is proposed in the next section. This is quite general and accommodates unequal sized RDCSSs. We revisit Example 3 in Section 5.2, where stars are used to construct similar designs with unequal sized RDCSSs.

5. Stars and RDCSSs

The geometric structure called a star was introduced (Shaw and Maks (2003)) for a set of 1-dimensional projective subspaces with a common overlap on a point in $PG(p-1,2)$. Here, we first propose a generalization of the star to $(t-1)$-dimensional subspaces of $\mathcal{P} = PG(p-1,q)$, for arbitrary $1 < t < p$ and prime or prime power q, and then to subspaces of unequal sizes. Next, we develop results for the existence and construction of stars in $\mathcal{P}$. The designs constructed allow large overlaps that facilitate the assessment of all factorial effects.

A star with equal sized subspaces consists of two components: (i) a set of $(t-1)$-dimensional subspaces (π_t's) in $\mathcal{P}$ that we call *rays* of the star, and (ii) the common overlap on a (t_0-1)-dimensional subspace (π_{t_0}), called the *nucleus* of the star, where $t_0 < t < p$. Such a star is also a $(t-1)$-cover of $\mathcal{P}$ if its rays cover the effect space $\mathcal{P}$.

Definition 2. A star $St(\mu, \pi_t, \pi_{t_0})$ is a set of μ rays consisting of $(t-1)$-dimensional subspaces (π_t's) of $\mathcal{P} = PG(p-1,q)$, and a nucleus π_{t_0}, where π_{t_0} ($t_0 < t$) is a (t_0-1)-dimensional subspace of $\mathcal{P}$ contained in each of the μ rays.

If a star $\Omega = St(\mu, \pi_t, \pi_{t_0})$ exists in $\mathcal{P} = PG(p-1,q)$, the maximum number of rays in Ω is given by $(q^p - q^{t_0})/(q^t - q^{t_0})$. For a star with the dimension of rays being fixed, the smaller the nucleus, the fewer the number of rays (μ).

Stars can be further generalized for a set of subspaces of unequal sizes with a common overlap. Suppose a star consists of exactly k distinct-sized rays. Let f_i be the number of rays with dimension $(t_i - 1)$, for $i = 1, \ldots, k$, with the common overlap for every pair of rays a (t_0-1)-dimensional subspace of $\mathcal{P}$. Such a star can be denoted by $St(f_1, \ldots, f_k; \pi_{t_1}, \ldots, \pi_{t_k}; \pi_{t_0})$, where the total number of rays is $\mu = \sum_{i=1}^{k} f_i$. Hereafter, without loss of generality, let $0 < t_0 < t_1 < t_2 < \cdots < t_k < p$. A star is called *balanced* if all its rays are of the same size (i.e., $k = 1$), otherwise it is called *unbalanced* and $k \geq 2$. Next, we establish the existence of both balanced and unbalanced stars.

5.1. Existence of stars

If there exists a star that covers the entire effect space $\mathcal{P} = PG(u-1,q)$, for positive integer $u > 1$, one can select an appropriate subset of rays to construct the desired set of RDCSSs. Thus, our results focus on the existence of stars that cover $\mathcal{P}$. It turns out that stars and spreads are very closely related in terms of their geometric structure.

Definition 3. A $(h_1-1, \ldots, h_\mu-1)$-spread $\mathcal{S}$ of $\mathcal{P} = PG(u-1,q)$ is a collection of μ pairwise disjoint subspaces $S_i, i = 1, \ldots, \mu$, such that $|S_i| = (q^{h_i}-1)/(q-1)$ and $\mathcal{P} = \cup_{i=1}^{\mu} S_i$.

Rains, Sloane and Stufken (2002) refer to such a $(h_1-1,\ldots,h_\mu-1)$-spread as a mixed spread of strength 2. If $h_1=\cdots=h_\mu$ ($=t$, say) then a $(h_1-1,\ldots,h_\mu-1)$-spread reduces to a $(t-1)$-spread; otherwise, we call it a mixed spread of $PG(u-1,q)$. Though the existence of a $(t-1)$-spread of $\mathcal{P}$ is trivial and well established (André (1954)), determining the existence of a mixed spread is nontrivial.

Lemma 2. *For the existence of a* $(h_1-1,\ldots,h_\mu-1)$*-spread* $\mathcal{S}$ *of* $PG(u-1,q)$*, the following conditions are necessary:*

(i) $q^u-1=\sum_{i=1}^{\mu}(q^{h_i}-1)$,

(ii) $h_i+h_j\le u$ *for every* $i\ne j$ $(i,j=1,\ldots,\mu)$.

Proof of Lemma 2(i) follows trivially from the definition of a spread, and Lemma 2(ii) comes from Ranjan, Bingham and Dean (2009, Thm. 6). The conditions in Lemma 2 are not sufficient. For example, let $u=5, q=2$, and $\mu=11$, where $h_1=\cdots=h_{10}=2$ and $h_{11}=1$. Then both Lemma 2(i) and (ii) hold. If such a $(h_1-1,\ldots,h_\mu-1)$-spread exists, then following Wu, Zhang, and Wang (1992), we would get an orthogonal array $L_{32}(4^{10}\times 2^1)$ of strength two and hence $L_{32}(4^{10})$ of strength two, which does not exist due to the Bose and Bush (1952) bound. To our knowledge, no necessary and sufficient conditions are known for the existence of a mixed spread in $PG(u-1,q)$ for arbitrary positive integer u and prime or prime power q. Nevertheless, see in Section 5.2, the cases that are of interest in statistical considerations can be completely settled. Next, we show the equivalence between a star and a spread.

Lemma 3. *The existence of a star* $\Omega=St(f_1,\ldots,f_k;\pi_{t_1},\ldots,\pi_{t_k};\pi_{t_0})$ *in* $PG(p-1,q)$*, that is also a cover of* $PG(p-1,q)$*, is equivalent to the existence of a* $(h_1-1,\ldots,h_\mu-1)$*-spread* $\mathcal{S}$ *of* $PG(u-1,q)$*, where* $u=p-t_0$*, and for each* i*,* f_i *is the number of* h_j*'s that are equal to* t_i-t_0.

Proof. For any $0<t_0<p$, there exist two disjoint subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$ in $PG(p-1,q)$ such that $|\mathcal{U}_1|=(q^{t_0}-1)/(q-1)$ and $|\mathcal{U}_2|=(q^{p-t_0}-1)/(q-1)$. If there exists a $(h_1-1,\ldots,h_\mu-1)$-spread $\mathcal{S}$ of $\mathcal{U}_2$, a star Ω can be constructed with nucleus $\mathcal{U}_1$ and the set of rays defined by $\{R_i=\langle\mathcal{U}_1,S_i\rangle\mid S_i\in\mathcal{S},\,1\le i\le\mu\}$.

Now suppose, there exists a star $\Omega=St(f_1,\ldots,f_k;\pi_{t_1},\ldots,\pi_{t_k};\pi_{t_0})$ that covers $PG(p-1,q)$. Without loss of generality, let $\mathcal{U}_1=\pi_{t_0}=\langle F_1,\ldots,F_{t_0}\rangle$ be the nucleus and $\mathcal{U}_2=\langle F_{t_0+1},\ldots,F_p\rangle$, where the F_i, $i=1,\ldots,p$ form a basis for $PG(p-1,q)$. Then the set of $\mu=\sum_{i=1}^{k}f_i$ rays of Ω can be used to construct a $(h_1-1,\ldots,h_\mu-1)$-spread $\mathcal{S}=\{R_i\cap\mathcal{U}_2\mid R_i$ is a ray of $\Omega,\,1\le i\le\mu\}$ of $\mathcal{U}_2$. Thus, the existence of the star Ω and the spread $\mathcal{S}$ are equivalent.

Combining Lemma 2 and Lemma 3, we obtain necessary conditions for the existence of a possibly unbalanced star that is also a cover of $PG(p-1,q)$.

Lemma 4. *For the existence of a star* $\Omega = St(f_1, \ldots, f_k; \pi_{t_1}, \ldots, \pi_{t_k}; \pi_{t_0})$ *in* $\mathcal{P} = PG(p-1, q)$ *that is also a cover of* $\mathcal{P}$*, the following conditions are necessary:*

(i) $q^{p-t_0} - 1 = \sum_{i=1}^{k} f_i(q^{t_i-t_0} - 1)$,

(ii) $t_i + t_j - t_0 \leq p$ *for every* $i \neq j$ $(i, j = 1, \ldots, k)$,

(iii) $2t_i - t_0 \leq p$ *for every* i *such that* $f_i \geq 2$.

The conditions in Lemma 4are not sufficient. In the special case of balanced stars, however, Lemma 3 suggests that a balanced star $St(\mu, \pi_t, \pi_{t_0})$ covering $\mathcal{P}$ exists if and only if there exists a $(t - t_0 - 1)$-spread of $PG(p - t_0 - 1, q)$. In conjunction with a result in André (1954), this leads to the following lemma.

Lemma 5. *There exists a balanced star* $St(\mu, \pi_t, \pi_{t_0})$ *in* $\mathcal{P} = PG(p - 1, q)$ *that covers* $\mathcal{P}$*, if and only if* $(t - t_0)$ *divides* $(p - t_0)$*. Furthermore, if* $(t - t_0)$ *divides* $(p - t_0)$*, the number of rays is* $\mu = (q^{p-t_0} - 1)/(q^{t-t_0} - 1)$.

Corollary 1. *For every* t $(2 \leq t < p)$ *and* $t_0 = t - 1$*, there exists a balanced star* $St(\mu, \pi_t, \pi_{t_0})$ *in* $\mathcal{P} = PG(p - 1, q)$ *that covers* $\mathcal{P}$*, where* $\mu = (q^{p-t+1} - 1)/(q - 1)$.

Although most of the results developed in this section focus on the general scenario (i.e., the existence of $St(f_1, \ldots, f_k; \pi_{t_1}, \ldots, \pi_{t_k}; \pi_{t_0})$ for $k \geq 2$), balanced stars are more useful for designs with relatively smaller run size. Unbalanced stars that are useful from statistical perspective tend to have large run sizes. For instance, for a 2^p factorial design, unbalanced stars that lead to informative half-normal plots must contain at least 64 experimental units, since S_i's of unequal sizes that overlap on at least 7 effects force $t_1 \geq 4$ and $t_2 \geq 5$ (as $t_2 > t_1$). This further implies that $p \geq 6$. While this may appear to apply for large designs, as we have previously noted, multistage experiments are frequently larger than completely randomized designs.

5.2. Construction

We first consider balanced stars covering $\mathcal{P}$. By Lemma 5, such a star $\Omega = St(\mu, \pi_t, \pi_{t_0})$ exists if and only if $(t - t_0)$ divides $(p - t_0)$. If this holds, then the construction is precisely as in the first paragraph of the proof of Lemma 3 via consideration of disjoint subspaces $\mathcal{U}_1$ and $\mathcal{U}_2$. Cyclic projectivities (Hirschfeld (1998)) can be used to construct a $(t - t_0 - 1)$-spread of $\mathcal{U}_2$. For instance, in Example 1 (contd.), $\mathcal{U}_1 = \{ABCDE\}$ and $\mathcal{U}_2 = \langle A, B, C, D \rangle$. The 1-spread of $\mathcal{U}_2$ obtained by using the primitive polynomial $w^4 + w + 1$ is shown in Table 4 (see Ranjan, Bingham and Dean (2009, Sec. 5.1) for details).

In Example 2 also, the design proposed is a star $St(3, \pi_4, \pi_3)$ with $\mathcal{U}_1 = \langle AB, DE, ACD \rangle$ and $\mathcal{U}_2 = \langle A, C \rangle$. The 0-spread of $\mathcal{U}_2$, given by $\{\{A\}, \{C\}, \{AC\}\}$, was used to construct the three rays (or the three subspaces) S_1, S_2, and S_3.

Table 4. A 1-spread of $PG(3, 2)$.

S_1	S_2	S_3	S_4	S_5
D	C	B	A	CD
BC	AB	ACD	BD	AC
BCD	ABC	$ABCD$	ABD	AD

In this approach the experimenter has control over the choice of pencils in the nucleus, but the spread construction limits the composition of the rays to some extent. If necessary, as in Example 1 (contd.), one can find an appropriate *collineation* (e.g., Batten (1997); Ranjan, Bingham and Dean (2009)) to transform the spread (i.e., rays of Ω) to meet the experimenter's requirement for the RDCSSs.

Turning to unbalanced stars, recall that the conditions in Lemma 4 are not sufficient. While this precludes the development of a general construction, the cases that are of practical interest from statistical considerations can easily be constructed. We discuss the existence and construction of two-level factorial designs that are obtained from stars covering $PG(p-1, 2)$, where $p \le 7$. In terms of a star $St(f_1, \ldots, f_k; \pi_{t_1}, \ldots, \pi_{t_k}; \pi_{t_0})$, the cases of interest are as follows:

(a) $p = 4, 5, 6, 7$; $t_0 = 1$, $t_i \ge 3$ $(i = 1, \ldots, k)$,

(b) $p = 5, 6, 7$; $t_0 = 2$, $t_i \ge 4$ $(i = 1, \ldots, k)$,

(c) $p = 5, 6, 7$; $t_0 = 3$, $t_i \ge 4$ $(i = 1, \ldots, k)$,

(d) $p = 6, 7$; $t_0 = 4$, $t_i \ge 5$ $(i = 1, \ldots, k)$,

(e) $p = 7$; $t_0 = 5$, $t_i \ge 6$ $(i = 1, \ldots, k)$.

The cases listed do not exhaust all feasible configurations of the parameters, but the remaining cases are either trivial or do not lead to good designs. For instance, consider the scenario with $t_0 = 2, t_i = 3$ for each i and $p = 4$. From Corollary 1, there exists a star $St(3, \pi_3, \pi_2)$ that covers $PG(3, 2)$. Denoting the rays of this star by R_1, R_2, R_3, note that $|R_i \cap R_j| = 3$ and $|R_i \backslash (R_i \cap R_j)| = 4$ for all $i \ne j$. The resulting design is not useful because none of the half-normal plots has a sufficient number of effects. In general, designs with $t_0 = 2$ lead to sacrificing the assessment of at least three factorial effects that are assigned to the nucleus of the star. We consider the interesting cases one-by-one.

(a1) $p = 4, t_0 = 1, t_i \ge 3$ $(i = 1, \ldots, k)$. Since $t_i < p$, the only possibility is $t_i = 3$ for all i. Nonexistence follows from Lemma 5.

(a2) $p = 5, t_0 = 1, t_i \ge 3$ $(i = 1, \ldots, k)$. Then, t_i is either 3 or 4. Lemma 4(i) yields $15 = 3f_1 + 7f_2$, with the only solution for (f_1, f_2) as $(5, 0)$. This corresponds to $t_i = 3$ for each i, and existence follows from Lemma 5.

(a3) $p = 6, t_0 = 1, t_i \geq 3$ $(i = 1, \ldots, k)$. In this case, t_i can be either 3, 4 or 5. Lemma 4(i) yields $31 = 3f_1 + 7f_2 + 15f_3$, and the only solution for (f_1, f_2, f_3), meeting Lemma 4(ii) and (iii) as well, is $(8, 1, 0)$. The existence and construction for $(f_1, f_2, f_3) = (8, 1, 0)$ follows from Example (iii) after Lemma 1 in Rains, Sloane and Stufken (2002).

(a4) $p = 7$, $t_0 = 1$, $t_i \geq 3$ $(i = 1, \ldots, k)$. The choices for t_i are 3, 4, 5 and 6. From Lemma 4(i), $63 = 3f_1 + 7f_2 + 15f_3 + 31f_4$, and the only solutions for (f_1, f_2, f_3, f_4), meeting Lemma 4(ii) and (iii) as well, are $(16, 0, 1, 0)$, $(0, 9, 0, 0)$, $(7, 6, 0, 0), (14, 3, 0, 0)$, and $(21, 0, 0, 0)$. The existence and construction for the first four cases follow from Rains, Sloane and Stufken (2002, Theorem 13), while the last case corresponds to a balanced star, and its existence and construction follow from Lemma 5.

(b1) $p = 5, t_0 = 2, t_i = 4$ $(i = 1, \ldots, k)$. Nonexistence follows from Lemma 5.

(b2) $p = 6, t_0 = 2, t_i \geq 4$ $(i = 1, \ldots, k)$. The choices for t_i are 4 and 5. From Lemma 4(i), $15 = 3f_1 + 7f_2$. This is the same scenario as in (a2).

(b3) $p = 7, t_0 = 2, t_i \geq 4$ $(i = 1, \ldots, k)$. In this case, $t_i = 4, 5$ or 6. Lemma 4(i) yields $31 = 3f_1 + 7f_2 + 15f_3$, leading to the same scenario as in (a3).

(c1) $p = 5, t_0 = 3, t_i = 4$ $(i = 1, \ldots, k)$. Existence follows from Corollary 1.

(c2) $p = 6, t_0 = 3, t_i \geq 4$ $(i = 1, \ldots, k)$. The options for t_i are 4 and 5. The necessary condition in Lemma 4(i) yields $7 = f_1 + 3f_2$, and the only solutions for (f_1, f_2) that meet Lemma 4(ii) and (iii) as well are $(7, 0)$ and $(4, 1)$. For each of the two cases, the existence and construction follow in a straightforward manner; see e.g., Wu, Zhang, and Wang (1992).

(c3) $p = 7, t_0 = 3, t_i \geq 4$ $(i = 1, \ldots, k)$. Then $t_i = 4, 5$ or 6. Lemma 4(i) yields $15 = f_1 + 3f_2 + 7f_3$, and the only solutions for (f_1, f_2, f_3) meeting Lemma 4(ii) and (iii) as well are $(8, 0, 1)$ and $(15 - 3j, j, 0)$, $0 \leq j \leq 5$. For each of these, existence and construction follow from Wu, Zhang, and Wang (1992).

(d1) $p = 6, t_0 = 4, t_i = 5$ $(i = 1, \ldots, k)$. Existence follows from Corollary 1.

(d2) $p = 7, t_0 = 4, t_i \geq 5$ $(i = 1, \ldots, k)$. Thus $t_i = 5$ or 6. Lemma 4(i) yields $7 = f_1 + 3f_2$, leading to the same scenario as in (c2).

(e) $p = 7, t_0 = 5, t_i = 6$ $(i = 1, \ldots, k)$. Existence follows from Corollary 1.

We now revisit Example 3 and illustrate how the use of an appropriately chosen star can entail a better experimental plan in the sense of making all half-normal plots informative, thus allowing inference on all factorial effects.

Example 3(contd.) Among all cases (a1)−(e), only (a3), (c2), and (d1) meet the requirement that $p = 6$ with $t_i \geq 4$ for at least one ray of the star. The design proposed in Example 3, given by $S_1 = \langle A, B, C, D\rangle$ and $S_2 = \langle E, F, ABCD\rangle$, is an example of $St(8, 1; \pi_3, \pi_4; \pi_1)$ discussed in (a3). This star leads to sacrificing the assessment on the factorial effects corresponding to the pencil in the nucleus $\{ABCD\}$, which could be an issue if one wishes to construct a regular fractional factorial design with the added factor being in S_1 or S_2. However, if more stages of randomization are to be introduced with added factors in them, $St(8,1;\pi_3,\pi_4;\pi_1)$ can serve the purpose.

Alternate designs can be constructed using $St(4,1;\pi_4,\pi_5;\pi_3)$ or $St(3;\pi_5;\pi_4)$, the existence and construction of stars is discussed in (c2) and (d1), respectively. These two stars meet the size requirement and lead to construction of fractional factorial designs that allow assessment of all the factorial effects.

The use of a star in designing the overlapping structure among the RDCSSs turns out to be advantageous, but there is a tradeoff between number of effects that can be assessed and the variance of the effect estimates. The effects in the common overlap (π_{t_0}) are estimated with a relatively large variance compared to other effects. If the design under consideration is an unreplicated full factorial, one may prefer to sacrifice a few effects by minimizing the overlap. On the other hand, if unreplicated fractional factorial designs are required, sacrificing higher order interactions of the basic factors is not desirable, and stars with relatively large overlap as in Examples 2 and 3 (contd.) are more useful.

6. Balanced Stars and Minimal $(t-1)$-covers

We begin by establishing a connection between balanced stars and minimal covers, introduced in Section 3, and then indicate its applications.

Lemma 6. *Let $p = kt + s$, where $0 < s < t < p$ and $k \geq 1$. Then there exists a minimal $(t-1)$-cover $\mathcal{C}$ of $\mathcal{P} = PG(p-1,q)$ that consists of $q^s((q^{kt}-1)/(q^t-1)$ $-1)$ disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ and a star $St(q^s+1,\pi_t,\pi_{t-s})$ in $\mathcal{P}$.*

Proof. Let $\mathcal{U}$ be a $(t+s-1)$-dimensional subspace of $\mathcal{P}$. Following Corollary 2.3 in Eisfeld and Storme (2000), there exists a collection $\mathcal{S}$ of $q^s[(q^{kt}-1)/(q^t-1)-1]$ disjoint $(t-1)$-dimensional subspaces of $\mathcal{P}$ which do no intersect $\mathcal{U}$ and form a partition of $\mathcal{P}\backslash\mathcal{U}$. Moreover, by Lemma 5, there exists a star $\Omega = St(\mu,\pi_t,\pi_{t-s})$ in $\mathcal{U}$ that also covers $\mathcal{U}$, and the number of rays in Ω is $\mu = q^s+1$. Thus, the disjoint $(t-1)$-dimensional subspaces in $\mathcal{S}$, together with the rays of Ω form a minimal $(t-1)$-cover $\mathcal{C}$ of $\mathcal{P}$, as envisaged in Lemma 1.

In particular, for $k=1$, Lemma 6 implies the existence of a minimal $(t-1)$-cover of $\mathcal{P} = PG(t+s-1,q)$ which equals $St(q^s+1,\pi_t,\pi_{t-s})$ in $\mathcal{P}$. It is, however, important to note that in many practical situations, one can find stars that are not minimal covers but perform better in the present context than the stars that are. For instance, both $\Omega_1 = St(7,\pi_4,\pi_3)$ and $\Omega_2 = St(5,\pi_4,\pi_2)$ cover $PG(5,2)$.

Although Ω_2 is a minimal cover and Ω_1 is not, the nucleus of Ω_1 is large enough to allow an informative half-normal plot, while that of Ω_2 fails to do so.

Returning to Lemma 6, in the same spirit, one can consider replacing the star $\Omega = St(q^s + 1, \pi_t, \pi_{t-s})$ in the minimal cover $\mathcal{C}$ by a star of larger nucleus, whenever the nucleus of Ω is too small to allow an informative half normal plot. The resulting geometric structure, say $\mathcal{C}^*$, has more $(t-1)$-dimensional subspaces than $\mathcal{C}$ and can entail greater flexibility in the sense of accommodating more RDCSSs, if required. The idea of replacing the star in $\mathcal{C}$ has potential applications in areas such as microchip experiments where one can afford to have a reasonably large number of experimental units. For instance, if $p = 10$, $q = 2$, and $t = 4$, then the minimal 3-cover $\mathcal{C}$ in Lemma 6 consists of 64 disjoint 3-dimensional subspaces and a star $St(5, \pi_4, \pi_2)$. If we modify $\mathcal{C}$ by using a star $St(7, \pi_4, \pi_3)$ instead of $St(5, \pi_4, \pi_2)$, the resulting structure $\mathcal{C}^*$ would allow assessment of all the effects in $\mathcal{P} = PG(9, 2)$.

7. Discussion

We have proposed two classes of designs for efficient planning of full and fractional factorial experiments under multistage randomization: designs that adapt minimal $(t-1)$-covers, and designs obtained from stars. It is seen that, in contrast to minimal covers, stars enjoy considerable flexibility with regard to the size of the overlap and hence have much greater scope in assessing the significance of factorial effects.

As a practical guideline, if the assessment of all the effects is required, or a few of the RDCSSs are of unequal size, stars can be used to construct designs with multistage randomization. Whereas, since the effects in the common overlap (nucleus of the star) are estimated with a larger effect variance, if one can sacrifice the assessment of a few higher order interactions and the desired RDCSSs are of equal sizes, minimal $(t-1)$-covers can be used to construct designs.

Acknowledgement

The work of PR and DB were supported by grants from the Natural Sciences and Engineering Research Council of Canada. RM's research was supported by a grant from the Indian Institute of Management Calcutta.

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(Received November 2008; accepted July 2009)