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State bounding estimation for a linear continuous-time singular system with time-varying delay

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Abstract

This paper investigates the problem of a state bounding estimation for a linear continuous-time singular system with time-varying delay. By employing the maximal Lyapunov–Krasovskii functional and applying the new free-matrix-based integral inequality, some proper conditions are derived in terms of LMIs and a bounding estimation lemma and set are obtained for the studied singular system.

Keywords: State bounding estimation; Time delays; Singular system

1 Introduction

During the past years, state bounding estimation has been widely applied in control systems with actuator saturation, peak-to-peak gain minimization, and parameter estimation (see [1–5]). A state bounding estimation is meant to get the corresponding state bounding set which is limited by the inside and outside of the initial conditions. State bounding estimation is so important that it has aroused much attention in control research. Meanwhile, there have been several kinds of correlative applications of a state bounding estimation, such as reachable set estimation and design actuator saturation for control systems (see [1–53]).

Specially, by applying the S-procedure, an ellipsoid reachable set bounding was derived for linear systems without time delays in [19]. However, time delays cannot always be avoided in practice and they often cause the system's instability and poor performance. Recently, many researchers have studied many kinds of dynamic systems with time delays (see [3, 6–25]). Thus, some researchers naturally have devoted efforts to investigating the corresponding state bounding estimation for the dynamic systems with time delays. In [7], a delay-dependent criterion for an ellipsoid reachable bounding set was derived by Fridman and Shaked, applying a Lyapunov–Krasovskii functional. Later, in [11], a better ellipsoid reachable bounding set was proposed by Kim using a Lyapunov–Krasovskii functional with just one nonconvex scalar. Some new criteria for reachable bounding sets were established by employing the maximal Lyapunov–Krasovskii functional in [14].

On the other hand, singular systems have been more intensively studied than state-space ones because they can present a better description of the behavior for some systems. There have been many singular systems in lots of practical systems, such as circuit systems, aircraft modeling, chemical processes and economic systems. Leaving alone their practical performance, singular systems are worth investigating for their theoretical importance and have drawn many researchers' attention in recent years because their basis is different from state-space systems. However, many researchers have successfully extended the basic theory of state-space systems to the singular systems. Recently, there have been several contributions on the state bounding estimation for nonlinear singular systems by applying different methods (see [11–19]). Particulary, Feng and Lam in [17] obtained the reachable set estimation for singular systems with time delays by using a Lyapunov–Krasovskii functional but not the maximal Lyapunov–Krasovskii functional.

In this paper, we extend the state bounding estimation to a singular system with time-varying delay. By employing the maximal Lyapunov–Krasovskii functional and applying the new free-matrix-based integral inequality, some proper conditions are derived in terms of LMIs and a new bounding estimation lemma and set are obtained for the studied singular system.

Notations: Throughout this paper, R^n denotes n-dimensional Euclidean space and $R^{m\times n}$ is the set of $m \times n$ real matrices. $[\cdot,\cdot]$ represents an interval. The superscripts '-1' and 'T' stand for the inverse and transpose of a matrix, respectively. $\|\cdot\|$ refers to the Euclidean vector norm. $*$ denotes the symmetric block in a symmetric matrix. $\mathrm{Sym}(P)$ is defined as $P + P^T$. For a real number ϵ, use the notation $\epsilon^+ = \max\{\epsilon, 0\}$, which means $\epsilon^+ = \begin{cases} \epsilon, & \epsilon > 0, \\ 0, & \epsilon \le 0. \end{cases}$ Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

2 Problem statement and preliminaries

Consider the following linear continuous-time singular system with time-varying delay:

$$\begin{aligned} &E\dot{x}(t) = Ax(t) + Dx\bigl(t - d(t)\bigr) + Bu(t), \\ &x(t) = \phi(t), \quad t \in [-d_M, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector; the matrices E, A, D and B are constant matrices with appropriate dimensions and $\mathrm{rank}(E) = n_1$; $d(t)$ is the time-varying delay satisfying $0 \le d_m \le d(t) \le d_M$, $\dot{d}(t) < d$, $d > 0$; $u(t) \in R^m$ stands for a disturbance which satisfies $u^T(t)u(t) \le u^2 \le \|x_t\|^2$, where u is a real constant; $\phi(t)$ is the initial condition and continuously differentiable on $[-d_M, 0]$. We denote $\|\phi\| = \max_{t\in[-d_M,0]} \|\phi\|$.

Remark 1 The initial condition for the studied system in [17] must be zero. However, the initial condition in this paper either may be equal to zero or not equal to zero. It is obvious that our discussed system is more general than the one in [17].

To discuss the state bounding problem for a linear continuous-time singular system with time-varying delay, the next definitions are necessary.

Definition 2.1 If the singular system (1) is satisfied with one of the next conditions, respectively, the system (1) is said to be regular, impulse free, stable and admissible:

(1) The singular system in (1) is said to be regular if the matrix pair (E, A) is regular.

(2) The singular system in (1) is said to be impulse free if the matrix pair (E, A) is impulse free.

(3) The singular system in (1) is said to be stable, if for any $\delta > 0$, there exists a scalar $\varepsilon(\delta) > 0$, such that, for any compatible initial condition x_0 satisfying $\|x_0\| \leq \varepsilon(\delta)$, the solution $x(t)$ of (1) satisfies $\|x(t)\| \leq \delta$ for $t \geq 0$; furthermore, $x(t) \to 0$, when $t \to \infty$.

(4) The singular system in (1) is said to be admissible if it is regular, impulse free and stable.

Definition 2.2 For a given $\alpha > 0$, system (1) with $u(t) = 0$ ($t \geq 0$) is said to be α-exponentially stable if there exists a positive constant ρ such that every solution $x(t,\phi)$ of (1) satisfies the following inequality:

$$\left\|x(t,\phi)\right\| \leq \rho\|\phi\|e^{-\alpha t}, \quad \forall t \geq 0. \tag{2}$$

For $\gamma > 0$, let the ball $\mathcal{B}(\gamma)$ be defined by $\mathcal{B}(\gamma) = \{x \in R^n : \|x\| \leq \gamma\}$. By adopting the concept of ball convergence in [16], we have the following definition.

Definition 2.3 For a given $\gamma > 0$, the system (1) is said to be globally exponentially convergent within the ball $\mathcal{B}(\gamma)$ if there exist a constant $\alpha > 0$ and a non-decreasing function $K(\cdot)$ such that the following inequality holds:

$$\left\|x(K,\phi)\right\| \leq \gamma + K\left(\|\phi\|\right)e^{-\alpha t}, \quad \forall t \geq 0. \tag{3}$$

The main objective of this paper is to obtain delay-dependent conditions for the state bounding problem of singular system (1). First, the conditions are investigated for the existence of two balls, namely, $\mathcal{B}(\gamma_1)$ and $\mathcal{B}(\gamma_2)$ with the radii γ_1, γ_2 explicitly dependent on the upper bound u^2 of the disturbance such that: (i) for all initial conditions in $\mathcal{B}(\gamma_1)$, the corresponding state trajectories of the systems are bounded inside the ball $\mathcal{B}(\gamma_2)$ all the time, and (ii) for all the initial conditions that are outside $\mathcal{B}(\gamma_1)$, the corresponding state trajectories of the systems converge exponentially (with a convergence rate α) within $\mathcal{B}(\gamma_2)$. Then the conditions are derived for the reachable set bounding of (1) with zero initial condition and the α-exponential stability of (1) without any disturbance.

In the following, some necessary lemmas are introduced.

Lemma 2.1 ([18]) *Let x be a differentiable function: $[a,b] \to R^n$. For symmetric matrices $Q \in R^{n\times n}$, $Y_1 \in R^{3n\times 3n}$, $Y_3 \in R^{3n\times 3n}$ and any matrices $Y_2 \in R^{3n\times 3n}$, $M_1 \in R^{3n\times n}$, $M_2 \in R^{3n\times n}$ satisfying*

$$\Psi = \begin{bmatrix} Y_1 & Y_2 & M_1 \\ * & Y_3 & M_2 \\ * & * & Q \end{bmatrix} \geq 0,$$

the following integral inequality holds:

$$-\int_a^b \dot{x}^T(s)Q\dot{x}(s)\,ds \leq \varpi^T \Omega \varpi, \tag{4}$$

where

$$\varpi = \left[x^T(a), x^T(b), \frac{1}{b-a}\int_a^b \dot{x}^T(s)\,ds\right]^T,$$

$$\Omega = \left[(b-a)\left(Y_1 + \frac{1}{3}Y_3\right)\right] + \mathrm{Sym}\{M_1\Pi_1 + M_2\Pi_2\},$$
$$\Pi_1 = e_1 - e_2, \qquad \Pi_2 = 2e_3 - e_1 - e_2,$$
$$e_1 = [I,0,0], \qquad e_2 = [0,I,0], \qquad e_3 = [0,0,I].$$

Lemma 2.2 ([17]) *Let $0 \le \tau_m < \tau_M$, $0 < \lambda < 1$, $Q \ge 0$, $x(t)$ be a continuous vector-valued function on $[\tau_m, \tau_M]$. If $\|x(t)\| \le \lambda\|x(t-\tau(t))\| + Q$, $t \ge 0$, then $\|x(t)\| \le \sup_{-\tau_M \le t \le 0}\|x\| + \frac{Q}{1-\lambda}$.*

Lemma 2.3 ([17]) *If system (1) is regular and impulse free, there exist two nonsingular matrices M and N such that*

$$MEN = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad MAN = \begin{bmatrix} \hat{A}_1 & 0 \\ 0 & I \end{bmatrix}.$$

Let

$$\hat{x}(t) = N^{-1}x(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix},$$

where $\hat{x}_1(t) \in R^{n_1}$ and $\hat{x}_2(t) \in R^{n-n_1}$. Denote

$$MA_\tau N = \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{21} & \hat{D}_{22} \end{bmatrix}, \qquad MB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}.$$

Then system (1) is a restricted system, equivalent to the following one:

$$\begin{aligned} \dot{\hat{x}}_1(t) &= \hat{A}_1\hat{x}_1(t) + \hat{D}_{11}\hat{x}_1\big(t-\tau(t)\big) + \hat{D}_{12}\hat{x}_2\big(t-\tau(t)\big) + \hat{B}_1u(t), \\ 0 &= \hat{x}_2(t) + \hat{D}_{21}\hat{x}_1\big(t-\tau(t)\big) + \hat{D}_{22}\hat{x}_2\big(t-\tau(t)\big) + \hat{B}_2u(t). \end{aligned} \tag{5}$$

Lemma 2.4 *Assume that there exist a functional $V(x_t)$ and positive scalars $\lambda_1, \lambda_2, \alpha, \beta \in (0,1)$ such that*

$$(1) \quad \lambda_1\big\|x(t)\big\|^2 \le V(x_t) \le \lambda_2\|x_t\|^2, \quad t \ge 0, \tag{6}$$
$$(2) \quad \dot{V}(x_t) + \alpha V(x_t) \le \beta u^T(t)u(t), \tag{7}$$

where x_t represents the state trajectory $\{x(t+\theta) : \theta \in C[-d_M, 0]\}$. Then every solution $x(t,\phi)$ of (1) satisfies

$$\big\|x(t,\phi)\big\|^2 \le \frac{\beta u^2}{\alpha\lambda_1} + \left(\frac{\lambda_2}{\lambda_1}\|\phi\|^2 - \frac{\beta u^2}{\alpha\lambda_1}\right)^+ e^{-\alpha t}, \quad \forall t \ge 0. \tag{8}$$

Proof Notice

$$\dot{V}(x_t) + \alpha V(x_t) \le \beta u^T(t)u(t). \tag{9}$$

By multiplying both sides of the inequality in (7) with $e^{\alpha t}$, we have

$$e^{\alpha t}\dot{V}(x_t) + \alpha e^{\alpha t}V(x_t) = \frac{d}{dt}\big(e^{\alpha t}V(x_t)\big) \le \beta e^{\alpha t}u^T(t)u(t). \tag{10}$$

Then, by performing the integral of (10) from 0 to $T > 0$, it is not difficult to obtain

$$e^{\alpha t}V(x_T) - V(x_0) = \int_0^T \beta e^{\alpha t}u^T(t)u(t)\,dt \le \int_0^T \beta e^{\alpha t}u^2\,dt = \frac{\beta}{\alpha}u^2\big(e^{\alpha T} - 1\big). \tag{11}$$

By simple computation, it is easy to get

$$V(x_T) \le e^{-\alpha t}V(x_0)\frac{\beta}{\alpha}u^2\big(e^{\alpha T} - 1\big)e^{-\alpha T} = \frac{\beta}{\alpha}u^2 + \left[V(x_0) - \frac{\beta}{\alpha}u^2\right]e^{-\alpha T}. \tag{12}$$

Finally, for all $t \ge 0$, replacing T with t in (12), we get

$$V(x_t) \le \frac{\beta}{\alpha}u^2 + \left[V(x_0) - \frac{\beta}{\alpha}u^2\right]e^{-\alpha t}. \tag{13}$$

Taking (6) into account, we derive

$$\big\|x(t,\phi)\big\|^2 \le \frac{\beta u^2}{\alpha\lambda_1} + \left(\frac{\lambda_2}{\lambda_1}\|\phi\|^2 - \frac{\beta u^2}{\alpha\lambda_1}\right)^+ e^{-\alpha t}. \tag{14}$$

The proof is thus completed. □

Remark 2 Lemma 2.4 in this paper provides a basic tool for the problem of state bounding not only containing the state convergence within a ball but also including the reachable set bounding for a continuous-time singular system with bounded disturbance input. It is obvious that Lemma 2.4 in this paper can be regarded as an expansion of Lemma 3 in [17]. Particularly, taking $\beta = \frac{\alpha}{u^2}$ and $V(x_0)$, it is easy to see that Lemma 2.4 is reduced to Lemma 3 in [17], which was proved to be more useful for the reachable set estimation of singular systems with bounded disturbances. Note that Lemma 2.4 in this paper can also be applied to the case where there is no disturbance in system (1). In this case, the studied problem is reduced to the α-exponential stability analysis for a singular system with interval time-varying delay.

3 Main results

According to Lemma 2.3, we consider the following system's state bounding estimation:

$$\begin{aligned} &\dot{\hat{x}}_1(t) = \hat{A}_1\hat{x}_1(t) + \hat{D}_{11}\hat{x}_1\big(t-\tau(t)\big) + \hat{D}_{12}\hat{x}_2\big(t-\tau(t)\big) + \hat{B}_1u(t), \\ &0 = \hat{x}_2(t) + \hat{D}_{21}\hat{x}_1\big(t-\tau(t)\big) + \hat{D}_{22}\hat{x}_2\big(t-\tau(t)\big) + \hat{B}_2u(t). \end{aligned} \tag{15}$$

Assumption 1 The matrix pair (E, D) is regular and $\|\hat{D}_{22}\| < 1$.

Theorem 3.1 *Under Assumption 1, the singular system (1) is bounded by the ellipsoid*

$$\mathcal{B}(\varepsilon) = \left\{x \in R^n \big| x^T\hat{\hat{P}}_j x \le \frac{\beta u^2}{\alpha}\right\}$$

$$\left(\hat{P}_j = \big(S^{-T}P_jS^{-1}\big)_{n_1\times n_1}, \hat{\hat{P}}_j = T^{-T}\begin{bmatrix}\eta\hat{P}_j & 0 \\ 0 & \frac{(1-\eta)\beta u^2}{\epsilon_2^2\alpha}\end{bmatrix}T^{-1}\right)$$

if there exist positive $R^{n\times n}$ matrices P_j, Q_1, Q_2, Q_3, X_1, X_2, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, any $R^{3n\times 3n}$ symmetric matrices

$$Y_1 = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix}, \qquad Y_3 = \begin{bmatrix} \bar{Y}_{11} & \bar{Y}_{12} & \bar{Y}_{13} \\ \bar{Y}_{21} & \bar{Y}_{22} & \bar{Y}_{23} \\ \bar{Y}_{31} & \bar{Y}_{32} & \bar{Y}_{33} \end{bmatrix},$$

$$Z_1 = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}, \qquad Z_3 = \begin{bmatrix} \bar{Z}_{11} & \bar{Z}_{12} & \bar{Z}_{13} \\ \bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{23} \\ \bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33} \end{bmatrix},$$

any $R^{3n\times 3n}$ matrices Y_2, Z_2, and any $R^{3n\times n}$ matrices

$$N_1 = \begin{bmatrix} N_{11} \\ N_{12} \\ N_{13} \end{bmatrix}, \qquad N_2 = \begin{bmatrix} N_{21} \\ N_{22} \\ N_{23} \end{bmatrix}, \qquad M_1 = \begin{bmatrix} M_{11} \\ M_{12} \\ M_{13} \end{bmatrix}, \qquad M_2 = \begin{bmatrix} M_{21} \\ M_{22} \\ M_{23}, \end{bmatrix}$$

such that the following matrix inequalities hold:

$$\Omega = (\Omega_{ij})_{8\times 8} \quad (i,j = 1,2,\ldots,8) < 0, \tag{16}$$

$$\Omega_1 = \begin{bmatrix} Y_1 & Y_2 & M_1 \\ * & Y_3 & M_2 \\ * & * & E^T X_1 E \end{bmatrix} \geq 0, \tag{17}$$

$$\Omega_2 = \begin{bmatrix} Z_1 & Z_2 & N_1 \\ * & Z_3 & N_2 \\ * & * & E^T X_2 E \end{bmatrix} \geq 0, \tag{18}$$

where

$$\Omega_{ij} = \Omega_{ji} \quad (i,j = 1,2,\ldots,8),$$

$$\Omega_{11} = Q_1 + Q_2 + Q_3 + \alpha E^T P_j E - e^{-\alpha d_m} E^T X_2 E + \operatorname{sym}\bigl(G_1^T A\bigr),$$

$$\Omega_{12} = -e^{-\alpha d_m} E^T X_2 E + A^T G_2,$$

$$\Omega_{13} = G_1^T D + A^T G_3,$$

$$\Omega_{14} = G_4^T A,$$

$$\Omega_{15} = A^T G_5,$$

$$\Omega_{16} = A^T G_6,$$

$$\Omega_{17} = (P_j E + E_0 U)^T - G_1^T + A^T G_7,$$

$$\Omega_{18} = G_1^T B,$$

$$\Omega_{22} = -e^{-\alpha d_m} E^T X_2 E - e^{-\alpha d_m} Q_1 + (d_M - d_m)^2 e^{-\alpha d_M} \left(Y_{11} + \frac{\bar{Y}_{11}}{3} \right)$$

$$+ (d_M - d_m)^2 e^{-\alpha d_M} (M_{11} + M_{21}),$$

$$\begin{aligned}
\Omega_{23} &= G_2^T D + (d_M - d_m)^2 e^{-\alpha d_M}\left(Y_{12} + \frac{\bar{Y}_{12}}{3}\right) + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(M_{12} - M_{11}) \\
&\quad - \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(M_{21} + M_{22}), \\
\Omega_{25} &= (d_M - d_m)^2 e^{-\alpha d_M}\left(Y_{13} + \frac{\bar{Y}_{13}}{3}\right) + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(2M_{13} + M_{21} - M_{23}), \\
\Omega_{27} &= -G_2^T, \\
\Omega_{28} &= G_2^T B, \\
\Omega_{33} &= -(1-\mu) e^{-\alpha d_m} Q_3 + \operatorname{sym}\left(G_3^T D\right) + (d_M - d_m)^2 e^{-\alpha d_M}\left(Y_{22} + \frac{\bar{Y}_{22}}{3} + Z_{11} + \frac{\bar{Z}_{11}}{3}\right) \\
&\quad + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(2M_{12} - 2M_{22} + 2N_{12} - 2N_{21}), \\
\Omega_{34} &= D^T G_4 + (d_M - d_m)^2 e^{-\alpha d_M}\left(Z_{12} + \frac{\bar{Z}_{12}}{3}\right) \\
&\quad + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(N_{12} - N_{11} - N_{22} - N_{21}), \\
\Omega_{35} &= D^T G_5 + (d_M - d_m)^2 e^{-\alpha d_M}\left(Y_{23} + \frac{\bar{Y}_{23}}{3}\right) + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2} M_{13}, \\
\Omega_{36} &= D^T G_6 + (d_M - d_m)^2 e^{-\alpha d_M}\left(Z_{13} + \frac{\bar{Z}_{13}}{3}\right) + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(N_{13} + 2N_{21} - N_{23}), \\
\Omega_{37} &= D^T G_7 - G_3^T, \\
\Omega_{38} &= G_3^T B, \\
\Omega_{44} &= -e^{-\alpha d_M} Q_2 + (d_M - d_m)^2 e^{-\alpha d_M}\left(Z_{22} + \frac{\bar{Z}_{22}}{3}\right) - \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(N_{12} + 2N_{22}), \\
\Omega_{46} &= (d_M - d_m)^2 e^{-\alpha d_M}\left(Z_{23} + \frac{\bar{Z}_{23}}{3}\right) + \frac{(d_M - d_m)^2 e^{-\alpha d_M}}{2}(-N_{13} + 2N_{22} - N_{23}), \\
\Omega_{48} &= -G_4^T + G_4^T B, \\
\Omega_{55} &= (d_M - d_m)^2 e^{-\alpha d_M}\left(Y_{33} + \frac{\bar{Y}_{33}}{3}\right) + 2(d_M - d_m)^2 e^{-\alpha d_M} M_{23}, \\
\Omega_{57} &= -G_5^T, \\
\Omega_{58} &= G_5^T B, \\
\Omega_{66} &= (d_M - d_m)^2 e^{-\alpha d_M}\left(Z_{33} + \frac{\bar{Z}_{33}}{3}\right) + 2(d_M - d_m)^2 e^{-\alpha d_M} N_{23}, \\
\Omega_{67} &= -G_6^T, \\
\Omega_{68} &= G_6^T B, \\
\Omega_{77} &= d_m^2 X_2 + 2(d_M - d_m)^2 X_1 - \operatorname{sym}(G_7), \\
\Omega_{78} &= G_7^T B, \\
\Omega_{88} &= -\beta I + \operatorname{sym}\left(G_8^T B\right).
\end{aligned}$$

Proof Given a family of matrices $P_j > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $X_1 > 0$, $X_2 > 0$, the pointwise maximum Lyapunov–Krasovskii functional is defined as

$$\begin{aligned} V_{\max} &= \max\{V_{1,j}(x) + V_2(x_t) + V_3(x_t) + V_4(x_t)\} = \max\{V_j(x)\} \\ &= V_{1,\max}(x) + V_2(x_t) + V_3(x_t) + V_4(x_t), \quad j = 1, 2, \ldots, N, \end{aligned} \tag{19}$$

where

$$\begin{aligned} V_{1,j}(x) &= x^T(t)E^T P_j E x(t), \\ V_{1,\max}(x) &= \max V_{1,j}(x), \\ V_2(x_t) &= \int_{t-d_m}^{t} e^{\alpha\theta - t} x^T(\theta) Q_1 x(\theta)\, d\theta \\ &\quad + \int_{t-d_M}^{t} e^{\alpha\theta - t} x^T(\theta) Q_2 x(\theta)\, d\theta + \int_{t-d(t)}^{t} e^{\alpha\theta - t} x^T(\theta) Q_3 x(\theta)\, ds, \\ V_3(x_t) &= (d_M - d_m) \int_{-d_M}^{-d_m} \int_{t+s}^{t} e^{\alpha\theta - t} \dot{x}^T(\theta) E^T X_1 E \dot{x}(\theta)\, ds\, d\theta, \\ V_4(x_t) &= d_m \int_{-d_m}^{0} \int_{t+s}^{t} e^{\alpha\theta - t} \dot{x}^T(\theta) E^T X_2 E \dot{x}(\theta)\, ds\, d\theta. \end{aligned}$$

From Lemma 2.4, we have

$$\lambda_1 \|x(t)\|^2 \le V_{\max} \le \lambda_2 \|x_t\|^2, \quad t \ge 0, \tag{20}$$

where

$$\begin{aligned} \lambda_1 &= \min_{1 \le j \le N} \lambda(E^T P_j E), \\ \lambda_2 &= \max_{1 \le j \le N} \lambda(E^T P_j E) + \max_{1 \le j \le N} \lambda(Q_1) d_m + \max_{1 \le j \le N} \lambda(Q_2) d_M + \max_{1 \le j \le N} \lambda(Q_3) d_M \\ &\quad + \max_{1 \le j \le N} \lambda(E^T X_1 E)(d_M - d_m)^3 + \max_{1 \le j \le N} \lambda(E^T X_2 E) d_m^2. \end{aligned}$$

In order to better express the idea of our proof, define a set $M_{\max}(x) := \{j \in \{1, 2, \ldots, K\} : V_j(x) = V_{\max}(x)\}$ for any $x \ne 0$. Therefore, $V_j(x) < V_{\max}(x)$, if $j \ne M_{\max}(x)$. Without loss of generality, we assume that the first n ellipsoids intersect at x. In that case, $M_{\max} = \{1, 2, \ldots, n\}$. Thus one obtains for all $k \in \{1, 2, \ldots, K\}$, $x^T(P_k - P_j)x \le 0$, $\forall j \le n$.

Let $\eta_j = \{x \in R^n : V_j(x) \ge V_k(x), \forall k \ne j\}$. Observe that x is differentiable if $x \in \eta_j \setminus V_{k \ne j} \eta_k$, while x is nondifferentiable if $x \in \bigcap_{j=1}^{n} \eta_j \setminus \bigcup_{j=m+1}^{K} \eta_j$.

Since the derivative of a convex function at a differential point can be regarded as a special case of a directional derivative for the same convex function at a nondifferentiable point, we combine these two situations and only illustrate the proof for the situation of a nondifferentiable point in the following discussion. Therefore,

$$V_{\max} = x^T(t)E^T P_j E x(t) + V_2(x_t) + V_3(x_t) + V_4(x_t), \quad j \in M_{\max}(x), \tag{21}$$

$$\partial V_{1,\max}(x) = Co\{\partial P_j x\}, \quad j \in M_{\max}(x). \tag{22}$$

Calculating the derivative of $V_{\max}$, we have

$$\begin{aligned}\nabla_{\dot{x}}V_{1,\max}(x) &= 2\dot{x}^T(t)E^T(P_jE + E_0u)x(t)\\ &= 2\big(E\dot{x}(t)\big)^T P_jEx(t) + 2\big(E\dot{x}(t)\big)^T P_jE_0ux(t),\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{V}_2(x_t) &= -\alpha V_2 + x^T(t)(Q_1 + Q_2 + Q_3)x(t) - e^{-\alpha d_m}x^T(t-d_m)Q_1x(t-d_m)\\ &\quad - e^{-\alpha d_M}x^T(t-d_M)Q_2x(t-d_M)\\ &\quad - \big(1-\dot{\tau}(t)\big)e^{-\alpha d(t)}x^T\big(t-d(t)\big)Q_3x\big(t-d(t)\big)\\ &\le -\alpha V_2 + x^T(t)(Q_1 + Q_2 + Q_3)x(t) - e^{-\alpha d_m}x^T(t-d_m)Q_1x(t-d_m)\\ &\quad - e^{-\alpha d_M}x^T(t-d_M)Q_2x(t-d_M)\\ &\quad - (1-\mu)e^{-\alpha d(t)}x^T\big(t-d(t)\big)Q_3x\big(t-d(t)\big),\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{V}_3(x_t) &= -\alpha V_3 + (d_M-d_m)^2\dot{x}^T(t)E^TX_1E\dot{x}(t)\\ &\quad - (d_M-d_m)\int_{t-d_M}^{t-d_m} e^{\alpha(\theta-t)}\dot{x}^T(\theta)E^TX_1E\dot{x}(\theta)\,d\theta\\ &\le -\alpha V_3 + (d_M-d_m)^2\dot{x}^T(t)E^TX_1E\dot{x}(t)\\ &\quad - (d_M-d_m)e^{-\alpha d_M}\int_{t-d_M}^{t-d_m}\dot{x}^T(\theta)E^TX_1E\dot{x}(\theta)\,d\theta,\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{V}_4(x_t) &= -\alpha V_4 + d_m^2\dot{x}^T(t)E^TX_2E\dot{x}(t) - d_m\int_{t-d_m}^{t} e^{\alpha(\theta-t)}\dot{x}^T(\theta)E^TX_2E\dot{x}(\theta)\,d\theta\\ &\le -\alpha V_4 + d_m^2\dot{x}^T(t)E^TX_2E\dot{x}(t) - d_me^{-\alpha d_m}\int_{t-d_m}^{t}\dot{x}^T(\theta)E^TX_2E\dot{x}(\theta)\,d\theta.\end{aligned} \tag{26}$$

By employing Lemma 2.1, there exist matrices Y_1, Y_2, Y_3, Z_1, Z_2, $Z_3 \in R^{3n\times 3n}$, and any matrices $N_1, N_2, M_1, M_2 \in R^{3n\times n}$ satisfying

$$\Omega_1 = \begin{bmatrix} Y_1 & Y_2 & M_1\\ * & Y_3 & M_2\\ * & * & E^TX_1E\end{bmatrix} \ge 0, \qquad \Omega_2 = \begin{bmatrix} Z_1 & Z_2 & N_1\\ * & Z_3 & N_2\\ * & * & E^TX_2E\end{bmatrix} \ge 0$$

such that $d_m < d(t) < d_M$. We have

$$\begin{aligned}\dot{V}_3(x_t) &\le -\alpha V_3 + (d_M-d_m)^2\dot{x}^T(t)E^TX_1E\dot{x}(t)\\ &\quad - (d_M-d_m)e^{-\alpha d_M}\int_{t-d_M}^{t-d(t)}\dot{x}^T(\theta)E^TX_1E\dot{x}(\theta)\,d\theta\\ &\quad - (d_M-d_m)e^{-\alpha d_M}\int_{t-d(t)}^{t-d_m}\dot{x}^T(\theta)E^TX_1E\dot{x}(\theta)\,d\theta\\ &\le -\alpha V_3 + (d_M-d_m)^2\dot{x}^T(t)E^TX_1E\dot{x}(t)\\ &\quad + (d_M-d_m)e^{-\alpha d_M}\left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m}x^T(\theta)\,d\theta\right]\\ &\quad \times\left[\big(d(t)-d_m\big)\left(Y_1 + \frac{1}{3}Y_3\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\times \left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right]^T \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right] \\
&\times \operatorname{Sym}\big(M_1\big(x\big(t-d(t)\big)-x(t-d_m)\big)\big) \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right] \\
&\times \operatorname{Sym}\left(M_2\left(\frac{2}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta - x\big(t-d(t)\big)-x(t-d_m)\right)\right) \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T\big(t-d(t)\big), x^T(t-d_M), \frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)} x^T(\theta)\,d\theta\right] \\
&\times \left[\big(d_M-d(t)\big)\left(Z_1+\frac{1}{3}Z_3\right)\right] \\
&\times \left[x^T\big(t-d(t)\big), x^T(t-d_M), \frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)} x^T(\theta)\,d\theta\right]^T \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T\big(t-d(t)\big), x^T(t-d_M), \frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)} x^T(\theta)\,d\theta\right] \\
&\times \operatorname{Sym}\big(N_1\big(x(t-d_M)-x\big(t-d(t)\big)\big)\big) \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T\big(t-d(t)\big), x^T(t-d_M), \frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)} x^T(\theta)\,d\theta\right] \\
&\times \operatorname{Sym}\left(N_2\left(\frac{2}{d_M-d(t)}\int_{t-d_M}^{t-d(t)} x^T(\theta)\,d\theta - x\big(t-d(t)\big)-x(t-d_M)\right)\right) \\
\le{}& -\alpha V_3 + (d_M-d_m)^2\dot{x}^T(t)E^TX_1E\dot{x}(t) \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right] \\
&\times \left[(d_M-d_m)\left(Y_1+\frac{1}{3}Y_3\right)\right] \\
&\times \left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right]^T \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right] \\
&\times \operatorname{Sym}\big(M_1\big(x\big(t-d(t)\big)-x(t-d_m)\big)\big) \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T(t-d_m), x^T\big(t-d(t)\big), \frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta\right] \\
&\times \operatorname{Sym}\left(M_2\left(\frac{2}{d(t)-d_m}\int_{t-d(t)}^{t-d_m} x^T(\theta)\,d\theta - x\big(t-d(t)\big)-x(t-d_m)\right)\right) \\
&+ (d_M-d_m)e^{-\alpha d_M}\left[x^T\big(t-d(t)\big), x^T(t-d_M), \frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)} x^T(\theta)\,d\theta\right] \\
&\times \left[(d_M-d_m)\left(Z_1+\frac{1}{3}Z_3\right)\right]
\end{aligned}
$$

$$\begin{aligned}&\times\left[x^T\big(t-d(t)\big),x^T(t-d_M),\frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)}x^T(\theta)\,d\theta\right]^T\\&+(d_M-d_m)e^{-\alpha d_M}\left[x^T\big(t-d(t)\big),x^T(t-d_M),\frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)}x^T(\theta)\,d\theta\right]\\&\times\operatorname{Sym}\Big(N_1\big(x(t-d_M)-x\big(t-d(t)\big)\big)\Big)\\&+(d_M-d_m)e^{-\alpha d_M}\left[x^T\big(t-d(t)\big),x^T(t-d_M),\frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)}x^T(\theta)\,d\theta\right]\\&\times\operatorname{Sym}\left(N_2\left(\frac{2}{d_M-d(t)}\int_{t-d_M}^{t-d(t)}x^T(\theta)\,d\theta-x\big(t-d(t)\big)-x(t-d_M)\right)\right).\end{aligned}\tag{27}$$

By applying the Jensen inequality in $\dot{V}_4(x_t)$, we have

$$\begin{aligned}\dot{V}_4(x_t)\le{}&-\alpha V_4+d_m^2\dot{x}^T(t)E^TX_2E\dot{x}(t)\\&-d_me^{-\alpha d_m}\big(x(t)-x(t-d_m)\big)^TE^TX_2E\big(x(t)-x(t-d_m)\big).\end{aligned}\tag{28}$$

Introducing the free weighting matrix $\mathcal{G}$, we have

$$2\xi^T(t)\mathcal{G}^T\big[-E\dot{x}_i(t)+Ax(t)+Dx\big(t-d(t)\big)+Bu(t)\big]=0,\tag{29}$$

where

$$\begin{aligned}\xi(t)={}&\left[x^T(t),x^T(t-d_m),x^T\big(t-d(t)\big),x^T(t-d_M),\frac{1}{d(t)-d_m}\int_{t-d(t)}^{t-d_m}x^T(\theta)\,d\theta,\right.\\&\left.\frac{1}{d_M-d(t)}\int_{t-d_M}^{t-d(t)}x^T(\theta)\,d\theta,\big(E\dot{x}(t)\big)^T,u^T(t)\right]^T,\\\mathcal{G}={}&[G_1,G_2,G_3,G_4,G_5,G_6,G_7,G_8].\end{aligned}$$

Therefore, we have

$$\begin{aligned}\dot{V}\;=\;&\dot{V}_{\max}+\alpha V_{\max}(x_t)-\beta u^T(t)u(t)\\&+2\xi^T(t)\mathcal{G}^T\big[-E\dot{x}_i(t)+Ax(t)+Dx\big(t-d(t)\big)+Bu(t)\big]\\\le{}&\xi^T(t)\Omega\xi(t).\end{aligned}\tag{30}$$

Therefore, let $\Delta=\dot{V}_{\max}+\alpha V_{\max}(x_t)-\beta u^T(t)u(t)$. Obviously, we have

$$\Delta\le\sum_{k=1}^{M}\gamma_{jk}x^T(p_k-p_j)x\le 0.\tag{31}$$

According to Lemma 2.4, we have

$$\big\|x(t,\phi)\big\|\le\gamma_2+\left\{\left(\frac{\lambda_2}{\lambda_1}\|\phi\|^2-\frac{\beta u^2}{\alpha\lambda_1}\right)^+\right\}^{\frac{1}{2}}e^{-\alpha t},\quad\forall t\ge 0,\tag{32}$$

where

$$\gamma_2 = \sqrt{\frac{\beta u^2}{\alpha\lambda_1}}, \qquad \lambda_1 = \min_{1\leq j\leq N} \lambda\bigl(E^T P_j E\bigr),$$

$$\lambda_2 = \max_{1\leq j\leq N} \lambda\bigl(E^T P_j E\bigr) + \max_{1\leq j\leq N} \lambda(Q_1)d_m + \max_{1\leq j\leq N} \lambda(Q_2)d_M + \max_{1\leq j\leq N} \lambda(Q_3)d_M$$
$$+ \max_{1\leq j\leq N} \lambda\bigl(E^T X_1 E\bigr)(d_M - d_m)^3 + \max_{1\leq j\leq N} \lambda\bigl(E^T X_2 E\bigr)d_m^2. \qquad \square$$

Remark 3 Define $\gamma_1 = \frac{\lambda_1}{\lambda_2}\gamma_2$. It is obvious that λ_1 and λ_2 are positive constants due to the conditions from Theorem 1. Denote $\mathcal{B}_1 := \mathcal{B}(\gamma_1)$, $\mathcal{B}_2 := \mathcal{B}(\gamma_2)$, then $\mathcal{B}_1 \subset \mathcal{B}_2$. According to the above inequality, it is easy for us to obtain two cases as regards the state trajectory of the singular system (1). On the one hand, for any initial condition which belongs to $\mathcal{B}_1$, the corresponding state trajectory is bounded within the ball $\mathcal{B}_2$. On the other hand, for any initial condition which is outside $\mathcal{B}_1$, as t tends to infinity, the corresponding trajectory converges exponentially within the ball $\mathcal{B}_2$.

When taking $\phi(t) = 0$, the singular system (1) is translated into the form

$$\begin{aligned} &E\dot{x}_i(t) = Ax(t) + Dx\bigl(t - d(t)\bigr) + Bu(t), \\ &x(t) = 0 \quad t \in [-d_M, 0]. \end{aligned} \tag{33}$$

It is obvious that the new system (33) is just a special case of the system (1). The reachable set estimation for system (33) has been mainly studied by employing the common Lyapunov–Krasovskii functional in [17]. The corresponding bounding ellipsoids have also been derived. In this paper, we apply the maximal Lyapunov–Krasovskii functional to provide a non-ellipsoidal reachable set estimation for the system (1).

Define the reachable set of (1) as follows:

$$\mathcal{R}_x = \bigl\{x(t) \in R^n | x(t), u(t) \text{ satisfy (1) and } u^T(t)u(t) \leq u^2, t \geq 0\bigr\}. \tag{34}$$

From (34), it is easy to see that the reachable set of a system is regarded as a bounded set of all reachable states starting from the origin by input disturbances with constrained peak value. Since recently, the most interesting problem of estimation and control for dynamical systems has been to find an estimation of the bounds of the reachable sets. Therefore, many researchers have devoted efforts to investigating such conditions for deriving an ellipsoid or a non-ellipsoid which bounds the reachable set of the system.

Let Q be a symmetric positive definite matrix and a scalar $r \geq 0$, the ellipsoid is defined as follows:

$$\varepsilon(Q, r) = \bigl\{x \in R^n, x^T Q x \leq r\bigr\}. \tag{35}$$

By employing the Lyapunov–Krasovskii functional in (19), from Lemma 2.4, we obtain

$$V(x_t) \leq \frac{\beta u^2}{\alpha}\bigl(1 - e^{-\alpha t}\bigr) < \frac{\beta u^2}{\alpha}, \quad \forall t \geq 0. \tag{36}$$

Note that $V(x_t) \geq x^T(t)E^T P_j E x(t)$. Therefore, we are ready to get $x^T(t)E^T P_j E x(t) \leq \frac{\beta u^2}{\alpha}$. By combining the definitions about $\mathcal{R}_x, \varepsilon(Q, r)$, it is not difficult to get the reachable set $\mathcal{R}_x$

of system (1) under zero initial condition. The reachable set is bounded by the ellipsoid $\varepsilon(E^TP_jE, r^*)$, where $r^* = \frac{\beta u^2}{\alpha}$.

$x^T(t)E^TP_jEx(t) \le \frac{\beta u^2}{\alpha}$ stands for $\hat{x}^T(t)T^TE^TS^TS^{-T}P_jS^{-1}SET\hat{x}(t) \le \frac{\beta u^2}{\alpha}$, that is to say, $\hat{x}^T(t)\hat{P}_j\hat{x}(t) \le \frac{\beta u^2}{\alpha}$ with $\hat{P}_j = (S^{-T}P_jS^{-1})_{n_1\times n_1}$.

Thus, the following inequality holds:

$$\|\hat{x}_1(t)\| \le \epsilon_1 \tag{37}$$

with $\epsilon_1 = \frac{\beta u^2}{\alpha\sqrt{\min_{1\le j\le N}\lambda(\hat{P}_j)}}$. Obviously, $\|\hat{x}_1(t)\|$ is bounded.

On the other hand, from Assumption 1, it follows from Lemma 2.3 that

$$\begin{aligned}\|\hat{x}_2(t)\| &= \|\hat{D}_{21}\hat{x}_1(t-\tau(t)) + \hat{D}_{22}\hat{x}_2(t-\tau(t)) + \hat{B}_2u(t)\| \\ &\le \|\hat{D}_{21}\|\epsilon_1 + \|\hat{D}_{22}\|\|\hat{x}_2(t-\tau(t))\| + \|\hat{B}_2\|u.\end{aligned} \tag{38}$$

By employing Lemma 6 in [17], $\|\hat{x}_2(t)\| \le \epsilon_2$. That is, $\hat{x}_2^T(t)\frac{1}{\epsilon_2^2}\hat{x}_2(t) \le 1$.

It yields $\|\hat{x}_2(t)\| \le \sup_{-d_M\le t\le 0}\|x(t)\| + \frac{\|\hat{D}_{21}\|\epsilon_1+\|\hat{B}_2\|u}{1-\|\hat{D}_{22}\|} = \|\phi\| + \frac{\|\hat{D}_{21}\|\epsilon_1+\|\hat{B}_2\|u}{1-\|\hat{D}_{22}\|}$. We have

$$\hat{x}_2^T(t)\frac{1}{\epsilon_2^2}\hat{x}_2(t) \le 1 \tag{39}$$

with $\epsilon_2 = \|\phi\| + \frac{\|\hat{D}_{21}\|\epsilon_1+\|\hat{B}_2\|u}{1-\|\hat{D}_{22}\|}$.

Then adding the inequality in (37) times η and the inequality in (39) times $(1-\eta)\frac{\beta u^2}{\alpha}$, we obtain

$$\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}^T\begin{bmatrix}\eta\hat{P}_j & 0\\ 0 & \frac{(1-\eta)\beta u^2}{\epsilon_2^2\alpha}\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix} \le \frac{\beta u^2}{\alpha}. \tag{40}$$

That is,

$$x^T\hat{\hat{P}}_jx \le \frac{\beta u^2}{\alpha}, \qquad \hat{P}_j = (S^{-T}P_jS^{-1})_{n_1\times n_1}, \qquad \hat{\hat{P}}_j = T^{-T}\begin{bmatrix}\eta\hat{P}_j & 0\\ 0 & \frac{(1-\eta)\beta u^2}{\epsilon_2^2\alpha}\end{bmatrix}T^{-1}.$$

The next thing is to certify the regularity and non-impulsiveness characteristics of system (1). It is not easy to get

$$\tilde{E}^T\tilde{P}_j = \tilde{P}_j^T\tilde{E} = \begin{bmatrix}E^TP_jE & 0\\ 0 & 0\end{bmatrix} \ge 0, \tag{41}$$

$$\operatorname{sym}(\tilde{A}^T\tilde{P}_j) + \tilde{Q} - \tilde{E}^T\tilde{X}\tilde{E} = \begin{bmatrix}\Omega_{11} & \Omega_{17}\\ * & \Omega_{77}\end{bmatrix} < 0, \tag{42}$$

where

$$\tilde{A} = \begin{bmatrix}0 & I\\ A & -I\end{bmatrix}, \qquad \tilde{E} = \begin{bmatrix}E & 0\\ 0 & 0\end{bmatrix},$$

$$\tilde{X} = \begin{bmatrix} e^{-\alpha d_m} X_2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{P}_j = \begin{bmatrix} P_j E + E_0 U & 0 \\ G_1 & G_7 \end{bmatrix},$$

$$\tilde{Q} = \begin{bmatrix} Q_1 + Q_2 + Q_3 + \alpha E^T P_j E & 0 \\ 0 & d_m^2 X_2 + (d_M - d_m)^2 X_1 \end{bmatrix}.$$

From (41)–(42), we get

$$\operatorname{sym}(\tilde{A}^T \tilde{P}_j) - \tilde{E}^T \tilde{Z} \tilde{E} < 0. \tag{43}$$

Because $\operatorname{rank}(\tilde{E}) = \operatorname{rank}(E) = n_1 \leq n$, there exist nonsingular matrices $\tilde{S}$ and $\tilde{T}$ such that

$$\bar{E} = \tilde{S}\tilde{E}\tilde{T} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Denote

$$\bar{A} = \tilde{S}\tilde{A}\tilde{T} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \bar{P} = \tilde{S}^{-T}\tilde{P}\tilde{N} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

Taking (43) into account, we have $P_{12} = 0$ and $P_{11} > 0$. Then pre-multiplying and post-multiplying (43) by $\tilde{T}^T$ and T, respectively, it is easy to obtain $\operatorname{sym}(A_{22}^T P_{22}) < 0$ showing that A_{22} is nonsingular. Thus, the pair $(\tilde{E}, \tilde{A})$ is regular and impulse free. In addition, by simple computation, it is easy to certify that $\det(sE - A) = \det(s\tilde{E} - \tilde{A})$ and $\deg(\det(sE - A)) = \deg(\det(s\tilde{E} - \tilde{A}))$. Finally, it is obvious that the system (1) is regular and impulse free.

4 Numerical examples

In this section, a numerical simulation example will be presented to demonstrate the effectiveness of our obtained results.

Example 1 Consider the two dimensional system (1) with

$$E = \begin{bmatrix} -0.6 \\ 1.1 \end{bmatrix}, \qquad A = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$u(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}, \qquad D = 0.$$

We have given

$$P_1 = \begin{bmatrix} 0.4597 & 0.0270 \\ 0.0270 & 0.3100 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.8599 & 0.7400 \\ -0.7400 & 2.1083 \end{bmatrix}.$$

Figures 1 and 2 describe the reachable set estimation for the singular system (1) with $\phi(t) = 0$.

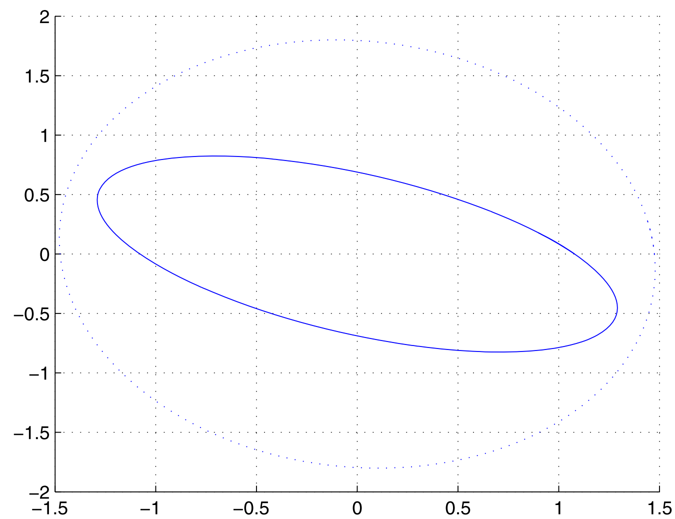

Figure 1 The reachable set estimation for the singular system (1) with $\phi(t) = 0$

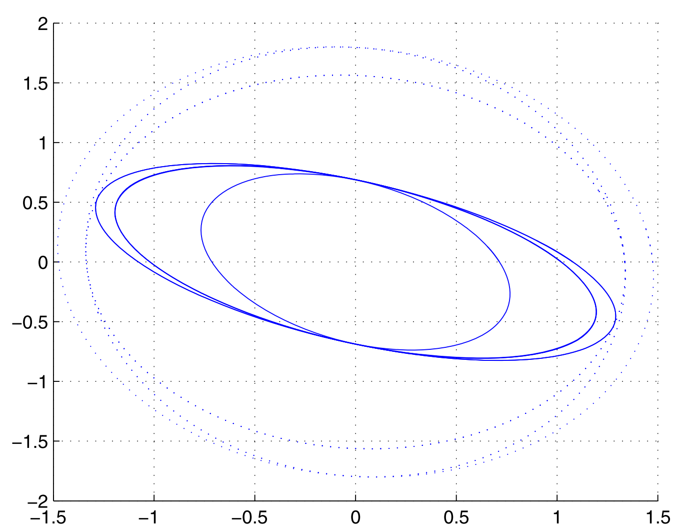

Figure 2 The reachable set estimation for the singular system (1) with $\phi(t) = 0$

5 Conclusions

The problem of a state bounding estimation of a continuous-time singular system with time delays has been investigated in this paper. Some proper conditions have been established to guarantee the state bounding set for the singular system with time delays by using the maximal Lyapunov–Krasovskii functional and employing the new free-matrix-based integral inequality. The above methods can be extended to our future studies, such as of fractional-order systems and memristor-based neural networks.

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