Order Number 8717647

State observers and state-feedback controllers for a class of nonlinear systems

Hauksdóttir, Anna Soffía, Ph.D.

The Ohio State University, 1987

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark ✓.

1. Glossy photographs or pages ______
2. Colored illustrations, paper or print ______
3. Photographs with dark background ______
4. Illustrations are poor copy ______
5. Pages with black marks, not original copy ✓
6. Print shows through as there is text on both sides of page ______
7. Indistinct, broken or small print on several pages ______
8. Print exceeds margin requirements ______
9. Tightly bound copy with print lost in spine ______
10. Computer printout pages with indistinct print ______
11. Page(s) ______ lacking when material received, and not available from school or author.
12. Page(s) ______ seem to be missing in numbering only as text follows.
13. Two pages numbered ______. Text follows.
14. Curling and wrinkled pages ______
15. Dissertation contains pages with print at a slant, filmed as received ______
16. Other ______

STATE OBSERVERS AND STATE-FEEDBACK CONTROLLERS FOR A CLASS OF NONLINEAR SYSTEMS

A Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

by

Anna Soffía Hauksdóttir, B.S.E.E., M.S.E.E.

* * * * *

The Ohio State University

1987

Dissertation Committee:

Prof. Robert E. Fenton

Assist. Prof. Kathleen A. K. Ossman

Assoc. Prof. Ümit Özgüner

Approved by:

Adviser

Department of Electrical Engineering

To Thorgeir, Haukur, Adalheidur, Jóhannes and Helga

ACKNOWLEDGEMENTS

I express sincere appreciation to my advisor, Dr. Robert E. Fenton, for his excellent guidance for the past six years. His technical expertise and dedication were a true inspiration. I also appreciate the helpful comments on my research by Drs. Kathleen A.K. Ossman and Ümit Özgüner, as well as the essential support provided by Dr. H.C. Ko, Chairman of The Department of Electrical Engineering at the Ohio State University, ZONTA International, NATO and The American-Scandinavian Foundation. My special thanks to the Electrical Engineering Department faculty for their practical and rigorous instruction. Very special thanks go to my husband, Thorgeir, whose endless support has been invaluable, and to my family in Iceland for their everpresent support.

VITA

June 7, 1958 .... Born—Reykjavík, Iceland

1981 .... B.S.E.E., The University of Iceland, Reykjavík, Iceland

1981-1986 .... Graduate Teaching Associate, The Ohio State University, Columbus, Ohio

1983 .... M.S.E.E., The Ohio State University, Columbus, Ohio

1986-Present .... Graduate Fellow, The Ohio State University, Columbus, Ohio

PUBLICATIONS

Hauksdottir A.S., Fenton R.E., "On the design of a vehicle longitudinal controller," *IEEE Trans. Veh. Tech.*, vol. VT-34, no. 4, pp. 182-187, Nov. 1985. Preliminary version presented at the IEEE 34th Vehicular Technology Conference, Pittsburgh, Pennsylvania, May 1984.

Fenton R.E., Hauksdottir A.S., Murthy S.S., "Automatic vehicle control using microprocessors," presented at the IEEE 1986 Workshop on Automotive Applications of Electronics.

FIELDS OF STUDY

Major Field: Control Systems (Drs. R.E. Fenton and Ü. Özgüner)

Minor Fields: Digital Systems (Dr. K. Breeding), Power Systems (Dr. S. Sebo), Mathematics (Dr. S. Smith)

TABLE OF CONTENTS

ACKNOWLEDGEMENTS iii

VITA iv

LIST OF FIGURES ix

I. INTRODUCTION 1

1.1 Control systems—a historical overview [1]-[9] 1

1.2 Controllability and observability of nonlinear systems 5

1.3 The design of observers and controllers for nonlinear systems . 6

1.4 A class of nonlinear systems 7

II. NONLINEAR STATE OBSERVERS AND STATE-FEEDBACK CONTROLLERS 10

2.1 State observers for nonlinear systems 10

2.1.1 Observability aspects 10

2.1.2 A nonlinear observer form 12

2.1.3 Observer design 13

2.2 State-feedback controllers for nonlinear systems 20

2.2.1 Controllability aspects 20

2.2.2 A nonlinear controller form 21

2.2.3 Controller design 22

2.3 Discussion 24

III. ON NONLINEAR TRANSFORMATIONS **25**

3.1 A nonlinear transformation 25

3.2 Transformations to observer and controller forms 29

3.2.1 Observer form 29

3.2.2 Controller form 36

3.3 Transform of nonlinear feedback gains 44

3.3.1 Output-to-$\dot{x}$ feedback 44

3.3.2 State-to-input feedback 46

3.3.3 Output-to-input feedback 48

3.4 Discussion 49

IV. COMPUTATIONAL ASPECTS **50**

4.1 Transformation to observer form 50

4.1.1 Case $n = 1$ 50

4.1.2 Case $n = 2$ 51

4.1.3 Case $n > 2$ 64

4.1.4 Discussion 69

4.2 Transformation to controller form 70

4.2.1 Discussion 75

V. SIMULATION STUDIES **76**

5.1 System in general form 76

5.2 Transformation from general form to observer form 80

5.3 Transformation from observer form to controller form 82

5.4 Observer/controller design 86

5.5 Simulation results 88
5.5.1 Observer—general case 88
5.5.2 Observer—special case 90
5.6 Discussion 96

VI. SUMMARY, CONCLUSIONS AND SUGGESTED FUTURE STUDIES 97
6.1 Summary 97
6.2 Conclusions 99
6.3 Suggested future studies 101

A. PREREQUISITES FOR CHAPTER II 103
A.1 Norms 103
A.2 Aspects of the nonlinear error dynamics 104

B. PREREQUISITES FOR CHAPTER III 109
B.1 Aspects of nonlinear independence 109
B.2 The notation of Su and Hunt 110

C. PROGRAM LISTINGS 112

D. TRANSFORMATIONS TO OTHER NONLINEAR FORMS 116
D.1 Observability form 116
D.2 Controllability form 121
D.3 Diagonal form 130
D.4 Discussion 134

E. STRUCTURAL OBSERVABILITY AND CONTROLLABILITY **136**

E.1 Observer form and observability form 136

E.2 Controller form and controllability form 147

E.3 Discussion 154

F. ADDITIONAL SIMULATION STUDIES **155**

F.1 System in observer form 155

F.2 Transformation from observer form to controller form 155

F.3 Observer/controller design 156

F.4 A comparable linear system 158

F.5 Simulation results 159

F.6 Discussion 159

REFERENCES **167**

LIST OF FIGURES

1 Output-to-$\dot{x}$ feedback 45

2 State-to-input feedback 47

3 Output-to-input feedback 48

4 Vehicle longitudinal control system 87

5 Position error for different observer initial conditions 89

6 State error 91

7 Position error for different observer-poles locations 92

8 Position error for different observer initial conditions 93

9 State error 94

10 Position error for different observer-poles locations 95

11 Control system 157

12 State error for nonlinear system, observer poles at -3 160

13 State error for linear system, observer poles at -3 161

14 State error for nonlinear system, observer poles at -0.5 162

15 State error for linear system, observer poles at -0.5 163

16 State error (e_1) for the nonlinear and the linear system, observer poles at -0.5 164

17 State error (e_2) for the nonlinear and the linear system, observer poles at -0.5 165

18 State error (e_3) for the nonlinear and the linear system, observer poles at -0.5 166

CHAPTER I

INTRODUCTION

1.1 Control systems—a historical overview [1]-[9]

the more you learn
the more you know
how little you know

Mankind has always searched for technological improvements, and in the 20th century this search has been especially successful. Further, probing new areas frequently draws attention to other unexplored ones. Thus, it can probably never be correctly stated that "everything that can be invented has been invented" (Charles H. Duell, Commissioner of The U.S. Patent and Trademark Office, recommending its closing in 1899.)

Control systems are a major component in many technological advancements and have a history that goes back as far as several centuries B.C.. Hero's device for opening the doors of a temple is among the earliest known open-loop control systems. It was based on heating air in a container half full of water; the expanding air caused a pumping of the water into a bucket which was attached to door spindles by ropes. The bucket then descended due to its increased weight and turned the spindles.

A turning point in the history of closed-loop control systems, in the sense that it is internationally accepted by the engineering community, was James Watt's 1788 flyball governor for speed control. This prompted widened interest in and intuitive inventions of other closed-loop control devices. Since these efforts frequently resulted in unstable systems, the development of control theory became imperative. An analytical study of the stability of the flyball governor was made by Maxwell in 1868 and a similar study by Wischnegradsky [10] in 1876. About 1922 Minorsky [11] showed how differential equations describing a system could be used to determine its stability. In 1932 Nyquist [12] developed a method for determining the stability of a closed-loop system on the basis of its open-loop response to steady-state sinusoidal inputs. Black [13] extended Nyquist's work and published a landmark paper in 1934. In that same year, Hazen [14] published a paper on a position control system that invoked intense interest. He called his invention a "servomechanism" where "servo" comes from the word servant (slave).

Other basic work was done during the next six years or until World War II, during which many publications were withheld due to security restrictions; however, individual companies and laboratories engaged in research motivated by military requirements. Much of this was published after the war; e.g., the famous Massachusettes Institute of Technology's 28-volume Radiation Laboratory Series dealt in part with control systems [15] and Bode [16] published a paper in which Nyquist's work was further expanded. Progress then accelerated with frequency-response analysis of closed-loop control systems being applied by Hall [17] and Harris [18] in 1946, and the root-locus method presented by Evans [19] in 1948.

The frequency-response and root-locus methods are considered the core of classical control theory, which deals with linear, time-invariant, single-input-single-output (LTISISO) systems. Thus, in a classical design relatively simple LTISISO models are employed. Satisfactory performance is achieved in many such applications, where the real system is not LTI, since well-designed classical control systems are generally robust. However, in many advanced systems, e.g., those in the aerospace domain, simplified models have been inadequate. Improved ones generally involve a large number of variables, and nonlinear and time-varying parameters, thus removing the basis for a classical design. (Some ad-hoc approaches, such as the phase-plane method and the describing-function method, were developed for nonlinear systems, but those tend to give satisfactory results for lower-order systems only.) Classical performance criteria such as percent-maximum overshoot, settling time and steady-state error are further not especially useful in many advanced applications where optimal criteria such as minimum time, minimum energy or minimum fuel are the primary design requirements.

Modern control theory, which can be applied to many situations in which classical control is inadequate, has been under development since the late 1950's. The resulting control algorithms are typically complex, i.e., nonlinear and/or time-varying, as compared to classical control algorithms; however, the development of digital computers, a system component in and essential design tool for most modern control systems, has made that complexity a minor drawback.

Applications of well-known methods from other fields have contributed greatly to the development of this theory. For example, the state-space approach, which was developed in the late 1950's, came from mathematicians such as Lefschetz [20], Pontryagin [21] and Bellman [22], using the theory of ordinary differential equations, matrix theory and linear algebra.

The state-space formulation led to new concepts in linear system design [23]. Linear controllability and observability were defined in their present form by Kalman in 1959-1960 [24] who also noted their duality. Kalman's paper stimulated a wide range of other work, e.g., results on jointly controllable and observable realizations by Gilbert [25], Kalman [26] and Popov [27]. In 1959, J. Bertram [28] realized by using root-locus techniques, that any desired characteristic polynomial could be obtained by full-state feedback, if the given system were controllable. A statement and a complete proof of this result was then first published in 1960 by Rissanen [29]. Popov independently deduced the same result for the multiple-input problem in 1964 [30,31] and subsequently Rosenbrock [32] discussed the improvement of system response achievable by using state feedback to relocate eigenvalues. The observer was apparently first introduced in unpublished work by Bertram in 1961 [33] and by Bass in 1963 [28]. An independent and slightly different approach was published by Luenberger in 1964 [34].

The foundations of optimal control theory were also developed through the state-space approach in the 1950's and early 1960's. Minimum-time control laws were obtained for low-order systems in the early 1950's through geometric and heuristic proofs. In the period 1953-1957, the basic theory regarding existence, uniqueness and other general properties of time-optimal control was developed by Bellman [35], Gamkrelidze, Krasovskii [36,37] and LaSalle [38], and the link between the calculus of variations and control problems was discovered. Pontryagin introduced the maximum principle to handle the "hard" constraints typically present in control problems [39], and this principle was proved by Pontryagin, Boltyanskii and Gamkrelidze [39] in 1962. Bellman et. al. [40] gave an explicit solution for linear systems with quadratic loss functions in 1958, and in 1960 Kalman [41] showed that the linear quadratic problem led to a Riccati equation. In the early 1960's a formulation of a stochastic variational problem assuming random disturbances led to the development of stochastic control theory and Linear Quadratic Gaussian (LQG) theory [42,43].

Another branch of modern control theory, which has been under development since the 1950's, is adaptive control. Such control aims at ensuring satisfactory performance when system dynamics are unknown and/or when changes in those dynamics occur due to nonlinearities or environmental disturbances. The earliest research was motivated by design requirement pertaining to high-performance aircraft. The dynamics of such aircraft vary substantially over their wide operational range of speeds and altitudes, and this variability must be accounted for in the design process. Several adaptive schemes were proposed in the 1950's [44,45,46,47] and then interest appeared to diminish. However, with the coming of the 60's, many advances in state-space theory, stability theory, stochastic control theory, system identification and parameter estimation, led to substantial advances in adaptive control systems. Dual control [48] and dynamic programming [35] increased the understanding of adaptive processes, and adaptive principles were incorporated into learning algorithms [49]. Since the 1970's, interest in this area has been vigorous. New adaptive control schemes were invented [50,51] and the synthesis of adaptive schemes, whose stability was guaranteed by using theorems of Lyapunov, Lure, Popov, et.al., was a major contribution. By the early 1980's some basic problem of identification and adaptive control for linear time-invariant systems were solved, assuming a known process order and some restrictions on inputs and disturbances [52,53].

Another area which has been of especial interest since the early 1970's is nonlinear control. The underlying theory is far from complete; however, its development is becoming imperative since nonlinear design approaches have become an essential key to the successful design of many high-performance control systems.

1.2 Controllability and observability of nonlinear systems

The observability and controllability of nonlinear systems received substantial attention in the early 1970's [54,55]. A nonlinear analog of linear controllability was developed by Hermann, Krener and others [56]-[63] in terms of the Lie Algebra of vector fields using methods of differential geometry. An approach to nonlinear observability, related to that of nonlinear controllability, was also derived [64]. In all of these works, rank conditions were derived for local weak controllability and local weak observability. Loosely put, an n-th order system is locally weakly controllable if one needs local coordinates of dimension n to distinguish the system trajectories from any initial point. Similarly a system is locally weakly observable if one can instantaneously distinguish each point in the state space from its neighbors.

One drawback of such local weak observability is that it depends on the input applied to the system. To surmount this restriction, a nonlinear analog of the linear observability form [23] was defined [65]. The existence of a nonlinear transformation to such a form gives rise to a necessary and sufficient condition of nonlinear observability for any input. Similarly, a nonlinear controllability form, which leads to a sufficient condition for controllability was defined [66].

1.3 The design of observers and controllers for nonlinear systems

Despite the interesting theoretical aspects of nonlinear observability and controllability, the link between these results and the actual design of observers or controllers has not been readily established. A few studies have been reported that have concentrated on the latter, which is the subject of this study. An asymptotic observer for nonlinear systems of the form

$$\dot{x} = f(x) \qquad x \epsilon \Re^n$$
$$y = h(x) \qquad y \epsilon \Re^1$$

has been defined and a Lyapunov-like method to design such observers has been derived [67]. A nonlinear controller form for nonlinear systems described by

$$\dot{x} = A(x,t) + B(x,t)u(t) \qquad x \epsilon \Re^n, u \epsilon \Re^m$$

has been defined and a control strategy that results in a linear, time-invariant, closed-loop system has been specified [68]. A nonlinear transformation to achieve this form was defined and the corresponding partial differential equations were solved by using an integrating factor. Similarly, a nonlinear observer form for nonlinear systems of the form

$$\begin{aligned} \dot{x} &= f(x,t) \quad x \in \Re^n \\ y &= h(x,t) \quad y \in \Re^1 \end{aligned} \tag{1.1}$$

and a corresponding observer which included nonlinear observer gains, were defined [69]. The partial differential equations involved were derived, but not solved. Solving those equations is crucial in determining the exact nonlinear observer gains; consequently the latter were approximated through linearization of the error equation about the observed states. Finally, a transformation to a nonlinear observer form for nonlinear systems described by

$$\begin{aligned} \dot{x} &= f(x,u) \quad x \in \Re^n \\ y &= h(x) \qquad y \in \Re^p \end{aligned} \tag{1.2}$$

has been presented [70,71]. In this form, the nonlinearities were functions of only the input and the output and thus perfectly reproducible in the defined observer, resulting in linear error dynamics.

1.4 A class of nonlinear systems

Generally, nonlinear systems are extremely hard to analyze and most of the developed observer/controller design methods are impractical. Therefore a restricted, yet widely applicable, class of nonlinear systems was chosen for this study, in the hope that a practical design methodology would result. This class is described by

$$\dot{x} = A(x)x + b(x)u \quad x \epsilon \Re^n$$
$$y = c(x)x \qquad u, y \epsilon \Re^1 \tag{1.3}$$

A time-varying version of this class was studied in the context of global and local controllability [72]. Further, this is a special class of the systems treated by Sommer [68] and Krener and Respondek [71]. A number of systems are naturally described by such equations, e.g., vehicle dynamics [97,74,75], ship dynamics [76,77] and aircraft dynamics [78,79]. Further, a nonlinear damped oscillator [80], any system described by Van der Pol's equation, and many pendulum problems show behavior of this form. Finally, many nonlinear systems of a more general form may be put into this form by employing a Taylor series expansion (including as many terms as feasible).

It is proposed to do the following for the class of systems defined by (1.3):

1. Define a nonlinear observer form, that facilitates an easy selection of nonlinear observer gains. Those gains should result in error dynamics that are bounded by a decaying exponential function.

2. Define a nonlinear controller form, that facilitates an easy selection of nonlinear feedback gains. Those gains should result in a linear closed-loop system, whose eigenvalues depend on the gains selected.

3. Investigate the nature of various nonlinear transformations with specific emphasis on those that result in the defined nonlinear observer and controller forms.

4. Special emphasis will be given to the practical use of these methods, i.e., determining the nonlinear transformations and the observer and feedback gains for this class of nonlinear systems.

5. Using the design methodology developed simulation studies will be conducted to demonstrate that:

   (a) If a nonlinear system can be transformed to a nonlinear controller form then it can be compensated by nonlinear state feedback such that linear closed-loop behavior results.

   (b) If a nonlinear system can be transformed to a nonlinear observer form then the states of a nonlinear system can be reconstructed using nonlinear observer gains.

   Further, the use of a combined nonlinear observer/controller will be examined by simulation.

The development of the described methodology would facilitate the design of observers/controllers for one class of nonlinear systems. This could result in considerable improvement over a linearization and classical-design approach—especially in situations where the nonlinear effects are pronounced and, as a result, less-than-desired robustness is realized.

CHAPTER II

NONLINEAR STATE OBSERVERS AND STATE-FEEDBACK CONTROLLERS

2.1 State observers for nonlinear systems

2.1.1 Observability aspects

Several definitions of observability for nonlinear systems have been presented in the literature. The most common ones are observability (O), local observability (LO), weak observability (WO) and local weak observability (LWO) [64]. A system is said to be O if an input can be found, such that when two different states result in the same output, this input will cause transitions to different outputs, thus distinguishing between the original two states. It must be possible to do this for any state pair which result in the same output. For example, for a system in $\mathcal{R}^2$ and $y = [\ 1 \quad 0\ ]x$, the states $[\ 2 \quad 4\ ]^T$ and $[\ 2 \quad 6\ ]^T$ result in the same output and an input must be found so that the corresponding transitions (e.g., to $[\ 7 \quad 3\ ]^T$ and $[\ 4 \quad 1\ ]^T$, respectively) result in different outputs. The system is LO if only a "short" (or local) transition by each state is necessary to distinguish the two states. It is WO if any two neighboring states can be distinguised after a transition from each state. Practically speaking this may be sufficient; e.g. $[\ 2 \quad 4\ ]^T$ and $[\ 2 \quad 6\ ]^T$ may have to be distinguishable, whereas the distinguishability of $[\ 2 \quad 4\ ]^T$ and $[\ 2 \quad 5674\ ]^T$ may not matter[1]. Finally, a system is LWO if any two neighboring

[1]This depends on the norm used for distinguishability.

states can be distinguished after a short transition from each state. Note that LO is the strongest concept here, as it implies both O and LWO. The latter both imply WO.[2]

If a system is analytic and controllable in some sense then it is WO iff it is LWO iff an "observability rank condition" is satisfied [64]. The latter refers to the Jacobian of a vector, composed of the output and its derivatives, being of full rank (this condition allows one to solve for the states given the input, the output and their derivatives). A similar definition of observability is made in [82], i.e., a system is observable if, for every pair of different states of the system, there exists an input which distinguishes these states, that is gives rise to different corresponding outputs. A system is simply defined as observable in [70] and [71] when the previously discussed observability rank condition is satisfied.

One drawback in these definitions is that observability generally depends on an input. It would be desirable to have a definition which is dependent only on the mathematical structure of the state equations. Such a definition was given in [65], and it was demonstrated that the existence of a transformation to a structurally observable form gives a sufficient condition for observability of the original system[3]. This observability is discussed in detail in Appendix E.

In contrast to the linear, time-invariant case, these definitions neither provide insights into how an observer can be designed nor insure this can be done [71]. Since observer design is of primary interest here, the following functional definition of state observability is adopted (this definition is related to that of an exponential observer in [67]):

[2]For LTI systems all of these concepts are equivalent.

[3]LTI systems which can be formulated in either the observer or observability forms (see [23] for definitions of these forms) are examples of structurally observable forms.

Definition 2.1.1 *A time-invariant system is state observable, if each system state can be accurately estimated by an asymptotic observer, whose state error is exponentially bounded.*

Note that systems that satisfy this definition may also satisfy some of the others and vice versa. The link between these, however, has not been established.

2.1.2 A nonlinear observer form

A nonlinear observer form for the class of systems (1.3) is defined by analogy to the linear observer form (e.g., in [23]) as[4]

$$\begin{aligned} \dot{x}^o &= A^o(x^o)\,x^o + b^o(x^o)\,u \\ y &= c^o x^o \end{aligned} \tag{2.1}$$

where

$$A^o(x^o) = \begin{bmatrix} a_{11}^o(x^o) & 1 & 0 & \cdots & 0 \\ a_{21}^o(x^o) & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ a_{n1}^o(x^o) & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad b^o(x^o) = \begin{bmatrix} b_1^o(x^o) \\ b_2^o(x^o) \\ \vdots \\ \vdots \\ b_n^o(x^o) \end{bmatrix} \tag{2.2}$$

$$c^o = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$x^o = \begin{bmatrix} x_1^o & x_2^o & \cdots & \cdots & x_n^o \end{bmatrix}^T.$$

[4]The superscript "o" denotes observer form.

Here it is assumed that the entries in $A^o(x^o)$ and $b^o(x^o)$ are "well behaved", i.e., belong to $\mathcal{C}^{\infty}$[5]. Note that this is not the only possible definition. This particular form is chosen since it is as general as possible and facilitates the selection of nonlinear observer gains that result in asymptotically stable error dynamics provided some sufficiency conditions are met (see Section 2.1.3).

2.1.3 Observer design

The general case—state-dependent nonlinearities

In the general case of (2.1)-(2.2) an observer is, in keeping with the usual practice for linear systems, chosen of the form

$$\begin{aligned} \dot{\hat{x}}^o &= A^o(\hat{x}^o)\hat{x}^o + b^o(\hat{x}^o)u + l^o(\hat{x}^o)(y - c^o\hat{x}^o) \\ \hat{y} &= c^o\hat{x}^o \end{aligned} \tag{2.3}$$

where

$$l^o(\hat{x}^o) = \begin{bmatrix} l_1^o(\hat{x}^o) & l_2^o(\hat{x}^o) & \cdots & l_n^o(\hat{x}^o) \end{bmatrix}^T$$

and

$$l_i^o(\hat{x}^o) \,\epsilon\, \mathcal{C}^{\infty} \qquad\qquad i = 1, 2, \ldots, n$$

are the nonlinear observer gains to be determined. The observation error e^o is defined by

$$e^o \equiv x^o - \hat{x}^o. \tag{2.4}$$

[5] $\mathcal{C}^{\infty}$ encompasses the class of continuous functions that are infinitely differentiable. This assumption is not essential, but invoked to avoid counting the degree of differentiability needed in the different arguments (for example, the sufficient conditions for observability in Theorem 2.1.1 require that a Lipschitz condition be satisfied, which in turn requires uniform continuity and bounded variation of the entries [83]). This convenient assumption is frequently invoked in other studies for these same reasons; e.g., see [64].

Differentiating (2.4) and then substituting (2.1) and (2.3) for $\dot{x}^o$ and $\dot{\hat{x}}^o$ respectively, results in the error dynamics

$$\dot{e}^o = A^o(x^o)x^o + b^o(x^o)u - A^o(\hat{x}^o)\hat{x}^o - b^o(\hat{x}^o)u - l^o(\hat{x}^o)(y - c^o\hat{x}^o). \tag{2.5}$$

Adding the term $A^o(\hat{x}^o)x^o$ to, and subtracting it from, the right-hand side (R.H.S.) of (2.5) and rearranging gives

$$\begin{aligned}\dot{e}^o &= (A^o(\hat{x}^o) - l^o(\hat{x}^o)c^o)e^o \\ &\quad + (A^o(x^o) - A^o(\hat{x}^o))x^o + (b^o(x^o) - b^o(\hat{x}^o))u.\end{aligned} \tag{2.6}$$

Choosing $l^o(\hat{x}^o)$ as[6]

$$l^o(\hat{x}^o) = \begin{bmatrix} a^o_{11}(\hat{x}^o) + l^o_1 & a^o_{21}(\hat{x}^o) + l^o_2 & \cdots & a^o_{n1}(\hat{x}^o) + l^o_n \end{bmatrix}^T, \tag{2.7}$$

where $l^o_1, l^o_2, \ldots, l^o_n$ are constants, results in

$$A^o(\hat{x}^o) - l^o(\hat{x}^o)c^o = A^o_e = \begin{bmatrix} -l^o_1 & 1 & 0 & \cdots & 0 \\ -l^o_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ -l^o_n & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{2.8}$$

Further[7],

$$(A^o(x^o) - A^o(\hat{x}^o))x^o = (a^o_{\cdot 1}(x^o) - a^o_{\cdot 1}(\hat{x}^o))x_1 = (a^o_{\cdot 1}(x^o) - a^o_{\cdot 1}(\hat{x}^o))y. \tag{2.9}$$

Thus the error dynamics (2.6) may be rewritten using (2.8) and (2.9) as

$$\dot{e}^o = A^o_e e^o + (a^o_{\cdot 1}(x^o) - a^o_{\cdot 1}(\hat{x}^o))y + (b^o(x^o) - b^o(\hat{x}^o))u. \tag{2.10}$$

[6]Note that by this choice, $l^o_i(\hat{x}^o) \in C^\infty$, $i = 1, 2, \ldots, n$, since $a^o_{i1}(x^o) \in C^\infty$, $i = 1, 2, \ldots, n$.

[7]In general, the notation $g_{\cdot j}$ $j = 1, \ldots, m$ will denote the j-th column in an $n \times m$ matrix G, similarly, $g_{i\cdot}$ $i = 1, \ldots, n$ denotes the i-th row.

The conditions under which these dynamics are asymptotically stable, resulting in sufficient conditions for state observability of the observer form, are specified in the following theorem.

Theorem 2.1.1 *If the error dynamics are described by (2.10) and if the following are satisfied for* $0 \leq t < \infty$[8]*:*

1. *The Lipschitz condition*[9]

$$\left\|a^o_{\cdot 1}(x^o) - a^o_{\cdot 1}(\hat{x}^o)\right\|_1 \leq a\left\|e^o\right\|_1 \tag{2.11a}$$

where $0 \leq a < \infty$*; further if* $a > 0$ *then*

$$0 \leq \left\|y\right\|_1 \leq Y < \infty \tag{2.11b}$$

must also be satisfied.

2. *The Lipschitz condition*[10]

$$\left\|b^o(x^o) - b^o(\hat{x}^o)\right\|_1 \leq b\left\|e^o\right\|_1 \tag{2.11c}$$

where $0 \leq b < \infty$*; further if* $b > 0$ *then*

$$0 \leq \left\|u\right\|_1 \leq U < \infty \tag{2.11d}$$

must also be satisfied.

Then A^o_e *can always be chosen such that*

[8] See Appendix A for a definition of the norm $\|\cdot\|_1$.

[9] This Lipschitz condition is satisfied, since the a^o_{i1} are differentiable and the derivatives are bounded [83].

[10] This Lipschitz condition is satisfied, since the b^o_i are differentiable and the derivatives are bounded [83].

$$\|\Phi^o(t,\tau)\|_1 = \left\|e^{A_e^o(t-\tau)}\right\|_1 \leq Me^{-k(t-\tau)} \quad \forall t \geq \tau, \quad \forall \tau \geq t_0, \tag{2.12}$$

where M, k are positive finite constants satisfying

$$aY + bU < k/M,$$

which gives

$$\lim_{t\to\infty} e^o(t) = 0$$

and thus the system is state observable. Further, $e^o(t)$ is bounded by a decaying exponential function, whose speed of decay can be controlled by $l_1^o, l_2^o, \ldots, l_n^o$.

Proof: This proof follows a similar one in [5], pp. 275-276. Using (A.4) from Appendix A there results

$$\begin{aligned}\|e^o(t)\|_1 \;\leq\; & \|\Phi^o(t,t_0)\|_1 \|e^o(t_0)\|_1 \\ & + \int_{t_0}^t \|\Phi^o(t,\tau)\|_1 \|(a_{\cdot 1}^o(x^o(\tau)) - a_{\cdot 1}^o(\hat{x}^o(\tau)))\|_1 \|y(\tau)\|_1 \, d\tau \\ & + \int_{t_0}^t \|\Phi^o(t,\tau)\|_1 \|(b^o(x^o(\tau)) - b^o(\hat{x}^o(\tau)))\|_1 \|u(\tau)\|_1 \, d\tau\end{aligned}$$

or

$$\begin{aligned}\|e^o(t)\|_1 \;\leq\; & Me^{-k(t-t_0)} \|e^o(t_0)\|_1 \\ & + \int_{t_0}^t Me^{-k(t-\tau)} a \|e(\tau)\|_1 Y d\tau \\ & + \int_{t_0}^t Me^{-k(t-\tau)} b \|e(\tau)\|_1 U d\tau.\end{aligned}$$

Multiplying through by the positive term e^{kt} and rearranging gives

$$e^{kt} \|e^o(t)\|_1 \leq Me^{kt_0} \|e^o(t_0)\|_1 + \int_{t_0}^t (aY + bU) Me^{k\tau} \|e^o(\tau)\|_1 \, d\tau. \tag{2.13}$$

Define $R(t)$ as the R.H.S. of (2.13)

$$R(t) = Me^{kt_0} \|e^o(t_0)\|_1 + \int_{t_0}^t (aY + bU) Me^{k\tau} \|e^o(\tau)\|_1 \, d\tau, \tag{2.14}$$

then

$$\dot{R}(t) = (aY + bU)Me^{kt}\left\|e^{o}(t)\right\|_1 . \tag{2.15}$$

Using (2.15) in the left-hand side (L.H.S.) and (2.14) in the R.H.S. of (2.13) there results

$$\frac{\dot{R}(t)}{(aY + bU)M} \leq R(t),$$

or

$$\frac{\dot{R}(t)}{R(t)} \leq (aY + bU)M. \tag{2.16}$$

Integrating both sides of (2.16) from t_0 to t gives

$$\int_{t_0}^{t} \frac{\dot{R}(t)}{R(t)} dt \leq \int_{t_0}^{t} (aY + bU)M dt$$

$$\ln(R(t))\Big|_{t_0}^{t} \leq (aY + bU)Mt\Big|_{t_0}^{t}$$

$$\ln(R(t)/R(t_0)) \leq (aY + bU)M(t - t_0). \tag{2.17}$$

Rearranging (2.17) results in

$$R(t) \leq R(t_0)e^{(aY+bU)M(t-t_0)}. \tag{2.18}$$

$R(t_0)$ is from (2.14)

$$R(t_0) = Me^{kt_0}\left\|e^{o}(t_0)\right\|_1 . \tag{2.19}$$

Using (2.19) in (2.18) and then (2.13) and (2.18) gives

$$e^{kt}\left\|e^{o}(t)\right\|_1 \leq Me^{kt_0}\left\|e^{o}(t_0)\right\|_1 e^{(aY+bU)M(t-t_0)}$$

or

$$\|e^o(t)\|_1 \leq M e^{-(k-(aY+bU)M)(t-t_0)} \|e^o(t_0)\|_1 . \tag{2.20}$$

Thus from (2.20)

$$\lim_{t \to \infty} \|e^o(t)\|_1 = 0,$$

or equivalently,

$$\lim_{t \to \infty} e^o(t) = 0$$

if

$$k - (aY + bU)M > 0$$

or

$$aY + bU < k/M.$$

Since the error goes to zero, the estimates asymptotically approach the system states and thus the system is state observable by Definition 2.1.1.

□

Note that Theorem 2.1.1 gives bounds that can be related to the observer pole locations[11]; however, although useful for proving asymptotic stability, these bounds may be too conservative to use as a design guideline when selecting the poles.

A special case—output-dependent nonlinearities

If the nonlinearities in the observer form are functions of only one state variable, i.e., $x_1 = y$, the observer design becomes especially simple [69], [71]. Eqn. (2.1) simplifies to

[11] A theorem that gives a relation between $e^{A_c^o(t-\tau)}$ and the eigenvalues of A_c^o is stated and proven in Appendix A.

$$\dot{x}^o = A^o(y)x^o + b^o(y)u$$
$$y = c^o x^o. \tag{2.21}$$

Here the observer is chosen as

$$\dot{\hat{x}}^o = A^o(y)\hat{x}^o + b^o(y)u + l^o(y)(y - c^o\hat{x}^o)$$
$$\hat{y} = c^o\hat{x}^o \tag{2.22}$$

where

$$l^o(y) = \begin{bmatrix} l_1^o(y) & l_2^o(y) & \cdots & l_n^o(y) \end{bmatrix}^T$$

are the observer gains to be determined. Notice the difference between (2.3) and (2.22) is that $A^o(y)$, $b^o(y)$ and $l^o(y)$ are used as opposed to $A^o\left(\hat{x}_1^o\right)$, $b^o\left(\hat{x}_1^o\right)$ and $l^o\left(\hat{x}_1^o\right)$. This results in the considerably simpler nonlinear error dynamics

$$\dot{e}^o = A^o(y)x^o + b^o(y)u - A^o(y)\hat{x}^o - b^o(y)u - l^o(y)(y - c^o\hat{x}^o)$$

or

$$\dot{e}^o \;=\; (A^o(y) - l^o(y)c^o)e^o. \tag{2.23}$$

Choosing $l^o(y)$ as

$$l^o(y) = \begin{bmatrix} a_{11}^o(y) + l_1^o & a_{21}^o(y) + l_2^o & \cdots & a_{n1}^o(y) + l_n^o \end{bmatrix}^T$$

where $l_1^o, l_2^o, \ldots, l_n^o$ are constants results in

$$A^o(y) - l^o(y)c^o = A_e^o = \begin{bmatrix} -l_1^o & 1 & 0 & \cdots & 0 \\ -l_2^o & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ -l_n^o & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{2.24}$$

and

$$\dot{e}^o = A_e^o e^o.$$

Note that linear error dynamics are obtained and $l_1^o, l_2^o, \ldots, l_n^o$ can be chosen to select the desired observer poles. Thus this form is state observable by Definition 2.1.1.

2.2 State-feedback controllers for nonlinear systems

2.2.1 Controllability aspects

Just as in the case of observability, several definitions of controllability for nonlinear systems have been presented in the literature. The most common ones are controllability (C), local controllability (LC), weak controllability (WC) and local weak controllability (LWC) [64]. A system is C if one can find an input that will cause a state transition from any initial point x_0 to any other final point x_1. A system is LC if only a short distance (thus " local") must be traversed to reach x_1 when it is in the neighborhood of x_0, $\forall x_0, x_1$. A system is WC if one can *either* reach x_1 from x_0 *or* vice versa, $\forall x_1, x_0$ (this is weaker than C where one has to be able to reach x_1 *from* x_0, $\forall x_0, x_1$). Finally a system is LWC if one can either reach x_1 from x_0 or vice versa and only a short distance is traversed when x_0 and x_1 are in the same neighborhood, $\forall x_0, x_1$. Note that LC is the strongest concept here as it implies C and LWC which, in turn, imply WC.[12]

If a system is analytic then it is WC iff it is LWC iff a "controllability rank condition" is satisfied [64]. The latter refers to the dimension of a space of tangent vectors being n (if the tangent vectors span an n-dimensional space, then the states will also span an n-dimensional space). A system is defined as controllable in [84] if there exists a measurable input that takes the system from x_0 to x_1, $\forall x_0, x_1$. Similarly a system is defined controllable in the large in [68] if there exists an

[12]For LTI systems all of these concepts are equivalent.

input that will take the system from x_0 to x_1, $\forall x_0, x_1$, and the trajectory stays in a simply connected region.

Just as in the case of observability, it would be desirable to have a definition which is dependent only on the mathematical structure of the state equations. Such a definition was given in [66], and it was demonstrated that the existence of a transformation to a structurally controllable form gives a sufficient condition for controllability of the original system [13]. This controllability is discussed in detail in Appendix E.

The main drawback in these definitions is that, in contrast to linear systems, the satisfying of some controllability property may neither provide insights into how a controller can be designed nor insure that this can be done. Since controller design is of primary interest here, the following restrictive, operational definition of state controllability is adopted:

Definition 2.2.1 *A time-invariant system is state controllable, if a controller can be designed such that linear closed-loop dynamics with any desired pole locations result.*

Note that systems that satisfy this definition may satisfy some of the others and vice versa. The link between these, however, has not been established.

2.2.2 A nonlinear controller form

A nonlinear controller form for the class of systems (1.3) is defined by analogy to the linear controller form (e.g., in [23]) as[14]

[13] LTI systems which can be formulated in either the observer or observability forms (see [23] for definitions of these forms) are examples of structurally observable forms.

[14] The superscript "c" denotes observer form.

$$\dot{x}^c = A^c(x^c)\,x^c + b^c u$$
$$y = c^c(x^c)\,x^c \tag{2.25}$$

where

$$A^c(x^c) = \begin{bmatrix} a^c_{11}(x^c) & a^c_{12}(x^c) & \cdots & \cdots & a^c_{1n}(x^c) \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad b^c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tag{2.26}$$

$$c^c(x^c) = \begin{bmatrix} c^c_1(x^c) & c^c_2(x^c) & \cdots & \cdots & c^c_n(x^c) \end{bmatrix}$$

and

$$x^c = \begin{bmatrix} x^c_1 & x^c_2 & \cdots & \cdots & x^c_n \end{bmatrix}.$$

Here it is assumed that the entries in $A^c(x^c)$ and $b^c(x^c)$ are in C^∞. Note that this is not the only possible definition. This particular form is chosen since it is as general as possible and facilitates the selection of nonlinear feedback gains that result in linear closed-loop dynamics (see Section 2.2.3).

2.2.3 Controller design

Using nonlinear full-state feedback (assuming all the states are available), analogous to the usual practice for linear systems, u is selected as

$$u = -k^c(x^c)\,x^c + v \tag{2.27}$$

where

$$k^c(x^c) = \begin{bmatrix} k^c_1(x^c) & k^c_2(x^c) & \cdots & \cdots & k^c_n(x^c) \end{bmatrix}$$

and

$$k_i^c(x^c)\,\epsilon\, C^\infty \qquad i = 1, 2, \ldots, n$$

are the nonlinear feedback gains to be determined. The resulting state equation is

$$\dot{x}^c = A^c(x^c)\,x^c - b^c k^c(x^c)\,x^c + b^c v = (A^c(x^c) - b^c k^c(x^c))\,x^c + b^c v.$$

Choosing $k^c(x^c)$ as[15]

$$k^c(x^c) = \left[\begin{array}{cccccc} a_{11}^c(x^c) + k_1^c & a_{12}^c(x^c) + k_2^c & \cdots & \cdots & a_{1n}^c(x^c) + k_n^c \end{array}\right] \tag{2.28}$$

where $k_1^c, k_2^c, \ldots, k_n^c$ are constants gives

$$A^c(x^c) - b^c k^c(x^c) = A_c^c = \left[\begin{array}{ccccc} -k_1^c & -k_2^c & \cdots & \cdots & -k_n^c \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

which results in

$$\dot{x}^c = A_c^c x^c + b^c v. \tag{2.29}$$

Thus, linear closed-loop dynamics are achieved, whose eigenvalues can be placed arbitrarily by choosing $k_1^c, k_2^c, \ldots, k_n^c$. Therefore, a system which can be transformed to controller form is state controllable by Definition 2.2.1.

[15]Note that by this choice, $k_i^c(x^c)\,\epsilon\, C^\infty$, $i = 1, 2, \ldots, n$, since $a_{1i}^c(x^c)\,\epsilon\, C^\infty$, $i = 1, 2, \ldots, n$.

2.3 Discussion

It has been outlined in this chapter how an observer can be designed for systems in the nonlinear observer form (2.1)-(2.2) (or (2.21)) and how a controller can be designed for systems in the nonlinear controller form (2.25)-(2.26). However, these two forms are subclasses of the general class (1.3)[16]. Thus, it is of interest to investigate how a nonlinear system (1.3) can be transformed into these forms. This is the subject of the next chapter.

[16]Other subclasses, which are analogous to other "standard forms" are examined in Appendix D.

CHAPTER III

ON NONLINEAR TRANSFORMATIONS

3.1 A nonlinear transformation

In the class of systems (1.3) (rewritten here as (3.1))

$$\begin{aligned} \dot{x} &= A(x)x + b(x)u \quad x\epsilon\Re^n \\ y &= c(x)x \qquad\qquad\quad u, y\epsilon\Re^1 \end{aligned} \tag{3.1}$$

the $A(x)$, $b(x)$ and $c(x)$ are of the general form

$$A(x) = \begin{bmatrix} a_{11}(x) & \cdots & \cdots & \cdots & a_{1n}(x) \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n1}(x) & \cdots & \cdots & \cdots & a_{nn}(x) \end{bmatrix}, \qquad b(x) = \begin{bmatrix} b_1(x) \\ \vdots \\ \vdots \\ \vdots \\ b_n(x) \end{bmatrix}, \tag{3.2}$$

$$c(x) = \begin{bmatrix} c_1(x) & \cdots & \cdots & \cdots & c_n(x) \end{bmatrix},$$

and their entries are assumed to belong to $\mathcal{C}^\infty$. This system can be transformed to

$$\begin{aligned} \dot{x}^q &= A^q(x^q)\,x^q + b^q(x^q)\,u \\ y &= c^q(x^q)\,x^q \end{aligned} \tag{3.3}$$

by employing a nonlinear, one-to-one transformation of the form

$$x^q = Q^q(x)x \tag{3.4}$$

where $Q^q(x)x$ is partially differentiable[1] w.r.t. x. The superscript q is a designator which will be used in this section for any one of five specific forms discussed in Section 3.2 and Appendix D. Using (3.4) in (3.3) gives

$$\begin{aligned} \dot{x}^q &= A^q(Q^q(x)x)Q^q(x)x + b^q(Q^q(x)x)u \\ y &= c^q(Q^q(x)x)Q^q(x)x. \end{aligned} \tag{3.6}$$

Differentiating (3.4) w.r.t. t gives

$$\dot{x}^q = \frac{\partial(Q^q(x)x)}{\partial x}\dot{x},$$

and substituting for $\dot{x}$ from (3.1) gives

$$\begin{aligned} \dot{x}^q &= \frac{\partial(Q^q(x)x)}{\partial x}A(x)x + \frac{\partial(Q^q(x)x)}{\partial x}b(x)u \\ y &= c(x)x. \end{aligned} \tag{3.7}$$

Upon equating corresponding coefficients in (3.6) and (3.7), one obtains

$$\begin{aligned} \frac{\partial(Q^q(x)x)}{\partial x}A(x) &= A^q(Q^q(x)x)Q^q(x) \\ \frac{\partial(Q^q(x)x)}{\partial x}b(x) &= b^q(Q^q(x)x) \\ c(x) &= c^q(Q^q(x)x)Q^q(x). \end{aligned} \tag{3.8}$$

where

[1]The assumption of a one-to-one transformation implies that both $Q^q(x)$ and $\frac{\partial(Q^q(x)x)}{\partial x}$ w.r.t. x are nonsingular (see Appendix B for a definition of this term for nonlinear matrices) and the inverse transformation

$$x = Q\left(x^q\right)x^q \tag{3.5}$$

exists for all x^q.

$$
\begin{aligned}
A^q(Q^q(x)x) &= A^q(x^q)\Big|_{x^q=Q^q(x)x} \\
b^q(Q^q(x)x) &= b^q(x^q)\Big|_{x^q=Q^q(x)x} \\
c^q(Q^q(x)x) &= c^q(x^q)\Big|_{x^q=Q^q(x)x}.
\end{aligned}
$$

The notation may be simplified by defining [2]

$$
\begin{aligned}
\bar{A}^q(x) &= A^q(Q^q(x)x) = A^q(x^q)\Big|_{x^q=Q^q(x)x} \\
\bar{b}^q(x) &= b^q(Q^q(x)x) = b^q(x^q)\Big|_{x^q=Q^q(x)x} \\
\bar{c}^q(x) &= c^q(Q^q(x)x) = c^q(x^q)\Big|_{x^q=Q^q(x)x}.
\end{aligned} \tag{3.9}
$$

Then (3.8) may be rewritten as

$$\frac{\partial(Q^q(x)x)}{\partial x}A(x) = \bar{A}^q(x)Q^q(x) \tag{3.10a}$$

$$\frac{\partial(Q^q(x)x)}{\partial x}b(x) = \bar{b}^q(x) \tag{3.10b}$$

$$c(x) = \bar{c}^q(x)Q^q(x). \tag{3.10c}$$

The $\bar{A}^q(x)$, $\bar{b}^q(x)$, $\bar{c}^q(x)$ are typically defined to be of some specific form of interest, e.g., the nonlinear observer form (2.1)-(2.2) or the nonlinear controller form (2.25)-(2.26). In general, the $Q^q(x)$ matrix and the unknown nonlinear elements $\bar{a}^q_{ij}(x)$ of $\bar{A}^q(x)$ are determined from (3.10a)[3]. Further, depending on how $\bar{b}^q(x)$ and $\bar{c}^q(x)$ are chosen, additional information about $Q^q(x)$ and/or the $\bar{A}^q(x)$ may be obtained from (3.10b) and/or (3.10c). Normally, the determination of $Q^q(x)$

[2]The notation $A^q(x) = A^q(Q^q(x)x)$ is ambiguous since it incorrectly suggests $A^q(x) = A^q(x^q)\Big|_{x^q=x}$, thus $\bar{A}^q(x) = A^q(Q^q(x)x)$ is used.

[3]Note that in the linear case $\det(sI - A) = \det(sI - A^q)$; however, the nonlinear analog, $\det(sI - A(x)) = \det(sI - \bar{A}^q(x))$ is not valid.

and $\bar{A}^q(x)$ cannot be separated, i.e., a set of nonlinear equations with terms from both embedded results, which makes the problem difficult to solve. In special cases, however, some simplification can be achieved.

In all cases, the L.H.S. of (3.10a) gives

$$\frac{\partial(Q^q(x)x)}{\partial x}A(x) = \begin{bmatrix} \partial(q^q_{1\cdot}(x)x)/\partial x \\ \partial(q^q_{2\cdot}(x)x)/\partial x \\ \vdots \\ \vdots \\ \partial(q^q_{1\cdot}(x)x)/\partial x \end{bmatrix} A(x). \tag{3.11}$$

This can be expressed in a simpler form by defining the linear operator $\mathcal{L}$: $\Re^{1\times n} \longrightarrow \Re^{1\times n}$ operating on $v(x)$ as

$$\mathcal{L}\left[v(x)\right] \equiv \left[\partial(v(x)x)/\partial x\right] A(x). \tag{3.12}$$

Further,

$$\mathcal{L}^0\left[v(x)\right] = v(x)$$
$$\mathcal{L}^1\left[v(x)\right] = \mathcal{L}\left[v(x)\right]$$
$$\mathcal{L}^2\left[v(x)\right] = \mathcal{L}\left[\mathcal{L}^1\left[v(x)\right]\right] = \mathcal{L}\left[\mathcal{L}\left[v(x)\right]\right]$$

or in general

$$\mathcal{L}^i\left[v(x)\right] = \mathcal{L}\left[\mathcal{L}^{i-1}\left[v(x)\right]\right] = \underbrace{\mathcal{L}\left[\mathcal{L}\left[\cdots\mathcal{L}\left[\right.\right.\right.}_{\text{i times}}\ v(x)]\cdots]]. \tag{3.13}$$

Thus (3.11) becomes from (3.13)

$$\frac{\partial(Q^q(x)x)}{\partial x}A(x) = \begin{bmatrix} \mathcal{L}^1\left[q_{1\cdot}(x)\right] \\ \mathcal{L}^1\left[q_{2\cdot}(x)\right] \\ \vdots \\ \vdots \\ \mathcal{L}^1\left[q_{n\cdot}(x)\right] \end{bmatrix}. \tag{3.14}$$

Once $Q^q(x)$ has been determined, $\bar{A}^q(x)$, $\bar{b}^q(x)$ and $\bar{c}^q(x)$ may be calculated using (3.10a), (3.10b) and (3.10c). Then the inverse transformation (3.5) can be determined and used to find

$$A^q(x^q) = \bar{A}^q(x)\Big|_{x=Q(x^q)x^q} = \bar{A}^q(Q(x^q)x^q)$$
$$b^q(x^q) = \bar{b}^q(x)\Big|_{x=Q(x^q)x^q} = \bar{b}^q(Q(x^q)x^q)$$
$$c^q(x^q) = \bar{c}^q(x)\Big|_{x=Q(x^q)x^q} = \bar{c}^q(Q(x^q)x^q).$$

3.2 Transformations to observer and controller forms

3.2.1 Observer form

In the case of the observer form (2.1)-(2.2), which has been shown to be useful for observer design, the R.H.S. of (3.10a) becomes (q is specified as o for this form)

$$\begin{aligned}
\bar{A}^o(x)Q^o(x) &= \begin{bmatrix} \bar{a}^o_{11}(x) & 1 & 0 & \cdots & 0 \\ \bar{a}^o_{21}(x) & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ \bar{a}^o_{n1}(x) & 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} q^o_{1\cdot}(x) \\ q^o_{2\cdot}(x) \\ \vdots \\ \vdots \\ q^o_{n\cdot}(x) \end{bmatrix} \\
&= \begin{bmatrix} \bar{a}^o_{11}(x)q^o_{1\cdot}(x) + q^o_{2\cdot}(x) \\ \bar{a}^o_{21}(x)q^o_{1\cdot}(x) + q^o_{3\cdot}(x) \\ \vdots \\ \vdots \\ \bar{a}^o_{n1}(x)q^o_{1\cdot}(x) \end{bmatrix}. \qquad (3.15)
\end{aligned}$$

Combining (3.14) and (3.15) results in the following:

$$\left.\begin{array}{rcl} q^o_{2\cdot}(x) & = & \mathcal{L}^1\left[q^o_{1\cdot}(x)\right] - \bar{a}^o_{11}(x)q^o_{1\cdot}(x) \\ q^o_{3\cdot}(x) & = & \mathcal{L}^1\left[q^o_{2\cdot}(x)\right] - \bar{a}^o_{21}(x)q^o_{1\cdot}(x) \\ & \vdots & \\ & \vdots & \\ q^o_{n\cdot}(x) & = & \mathcal{L}^1\left[q^o_{(n-1)\cdot}(x)\right] - \bar{a}^o_{(n-1)1}(x)q^o_{1\cdot}(x) \end{array}\right\} \tag{3.16a}$$

and

$$\mathcal{L}^1\left[q^o_{n\cdot}(x)\right] = \bar{a}^o_{n1}(x)q^o_{1\cdot}(x). \tag{3.16b}$$

Now upon employing (3.10c)

$$c(x) = \bar{c}^o Q^o(x) = q^o_{1\cdot}(x). \tag{3.16c}$$

Expressing $q^o_{2\cdot}(x), q^o_{3\cdot}(x), \ldots, q^o_{n\cdot}(x)$ in terms of $c(x)$ using (3.16c) and (3.16a) results in

$$\begin{array}{rcl} q^o_{2\cdot}(x) & = & \mathcal{L}^1\left[c(x)\right] - \bar{a}^o_{11}(x)c(x) = \mathcal{L}^1\left[c(x)\right] - \mathcal{L}^0\left[\bar{a}^o_{11}(x)c(x)\right] \\ q^o_{3\cdot}(x) & = & \mathcal{L}^1\left[\mathcal{L}^1\left[c(x)\right] - \mathcal{L}^0\left[\bar{a}^o_{11}(x)c(x)\right]\right] - \bar{a}^o_{21}(x)c(x) \\ & = & \mathcal{L}^2\left[c(x)\right] - \mathcal{L}^1\left[\bar{a}^o_{11}(x)c(x)\right] - \mathcal{L}^0\left[\bar{a}^o_{21}(x)c(x)\right] \\ & \vdots & \\ q^o_{n\cdot}(x) & = & \mathcal{L}^{n-1}\left[c(x)\right] - \mathcal{L}^{n-2}\left[\bar{a}^o_{11}(x)c(x)\right] - \cdots \\ & & \cdots - \mathcal{L}^1\left[\bar{a}^o_{(n-2)1}(x)c(x)\right] - \mathcal{L}^0\left[\bar{a}^o_{(n-1)1}(x)c(x)\right]. \end{array} \tag{3.17}$$

Using (3.17) and (3.16c) in (3.16b) gives

$$\begin{array}{rcl} \mathcal{L}^n\left[c(x)\right] & - & \mathcal{L}^{n-1}\left[\bar{a}^o_{11}(x)c(x)\right] - \cdots \\ \cdots & - & \mathcal{L}^2\left[\bar{a}^o_{(n-2)1}(x)c(x)\right] - \mathcal{L}^1\left[\bar{a}^o_{(n-1)1}(x)c(x)\right] = \bar{a}^o_{n1}(x)c(x) \end{array}$$

or

$$\mathcal{L}^n\left[c(x)\right] = \sum_{i=1}^{n} \mathcal{L}^{i-1}\left[\bar{a}^o_{(n+1-i)1}(x)c(x)\right]. \tag{3.18}$$

Thus, determining $Q^o(x)$ can be done in two steps:

1. Solve for $\bar{a}^o_{11}(x), \bar{a}^o_{21}(x), \ldots, \bar{a}^o_{n1}(x)$ from (3.18).

2. Calculate $Q^o(x)$ from (3.16a) and (3.16c) (rewritten here as (3.19))

$$\left.\begin{aligned}
q^o_{1\cdot}(x) &= c(x)\\
q^o_{2\cdot}(x) &= \mathcal{L}^1\left[q^o_{1\cdot}(x)\right] - \bar{a}^o_{11}(x)c(x)\\
q^o_{3\cdot}(x) &= \mathcal{L}^1\left[q^o_{2\cdot}(x)\right] - \bar{a}^o_{21}(x)c(x)\\
&\vdots\\
&\vdots\\
q^o_{n\cdot}(x) &= \mathcal{L}^1\left[q^o_{(n-1)\cdot}(x)\right] - \bar{a}^o_{(n-1)1}(x)c(x).
\end{aligned}\right\} \tag{3.19}$$

Comments:

1. Here the $\bar{a}^o_{\cdot 1}(x)$ can be obtained without any knowledge of $Q^o(x)$; the latter can be obtained from these quantities.

2. Eq. (3.18) represents a system of n $n-1$-st-order linear partial-differential equations in n variables and n unknowns—a system which is generally very difficult to solve[4].

3. Once this system has been solved, however, the calculation of $Q^o(x)$ from (3.19) is relatively simple.

[4]Computational aspects of this system of equations are discussed in Chapter IV.

Example 3.2.1 A simple, nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Here (3.18) becomes

$$\mathcal{L}^2\left[c(x)\right] = \mathcal{L}^1\left[\bar{a}_{11}^o(x)c(x)\right] + \mathcal{L}^0\left[\bar{a}_{21}^o(x)c(x)\right]. \tag{3.20}$$

Calculating each term gives

$$\begin{aligned}
\mathcal{L}^2\left[c(x)\right] &= \mathcal{L}\left[\mathcal{L}\left[c(x)\right]\right] = \mathcal{L}\left[\frac{\partial}{\partial x}\left(c(x)x\right)A(x)\right] = \mathcal{L}\left[\frac{\partial}{\partial x}\left(x_1\right)A(x)\right] \\
&= \mathcal{L}\left[\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix}\right] = \mathcal{L}\left[\begin{bmatrix} -1 & 1 \end{bmatrix}\right] \\
&= \begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} = \begin{bmatrix} 1 & -1-x_1 \end{bmatrix}, \qquad (3.21a) \\
\mathcal{L}^1\left[\bar{a}_{11}^o(x)c(x)\right] &= \frac{\partial}{\partial x}\left(\bar{a}_{11}^o(x)c(x)x\right)A(x) = \frac{\partial}{\partial x}\left(\bar{a}_{11}^o(x)x_1\right)A(x) \\
&= \begin{bmatrix} \frac{\partial}{\partial x_1}\left(\bar{a}_{11}^o(x)x_1\right) & \frac{\partial}{\partial x_2}\left(\bar{a}_{11}^o(x)x_1\right) \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} \\
&= \left[-\frac{\partial}{\partial x_1}\left(\bar{a}_{11}^o(x)x_1\right)\right. \\
&\qquad \left.\frac{\partial}{\partial x_1}\left(\bar{a}_{11}^o(x)x_1\right) - x_1^2\frac{\partial}{\partial x_2}\left(\bar{a}_{11}^o(x)\right)\right], \qquad (3.21b) \\
\mathcal{L}^o\left[\bar{a}_{21}^o(x)c(x)\right] &= \bar{a}_{21}^o(x)c(x) = \begin{bmatrix} \bar{a}_{21}^o(x) & 0 \end{bmatrix}. \qquad (3.21c)
\end{aligned}$$

Substituting (3.21a)-(3.21c) in (3.20) results in two equations

$$\begin{aligned}
1 &= -\frac{\partial}{\partial x_1}\left(\bar{a}_{11}^o(x)x_1\right) + \bar{a}_{21}^o(x) \qquad (3.22a) \\
-1 - x_1 &= \frac{\partial}{\partial x_1}\left(\bar{a}_{11}^o(x)x_1\right) - x_1^2\frac{\partial}{\partial x_2}\left(\bar{a}_{11}^o(x)\right). \qquad (3.22b)
\end{aligned}$$

In this case, it is possible to find $\bar{a}^o_{11}(x)$ to satisfy (3.22b) and then calculate $\bar{a}^o_{21}(x)$ from (3.22a). One possible solution to (3.22b), as may easily be verified by direct substitution, is

$$\bar{a}^o_{11}(x) = \bar{a}^o_{11}(x_1) = -\frac{1}{2}x_1 - 1.$$

The corresponding $\bar{a}^o_{21}(x)$ is

$$\bar{a}^o_{21}(x) = 1 + \frac{\partial}{\partial x_1}\left(\bar{a}^o_{11}(x)x_1\right) = 1 + \frac{\partial}{\partial x_1}\left(-\frac{1}{2}x_1^2 - x_1\right) = -x_1.$$

Now $Q^o(x)$ can be calculated using (3.19)

$$q^o_{1\cdot}(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$q^o_{2\cdot}(x) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} - \left(-\tfrac{1}{2}x_1 - 1\right)\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}x_1 & 1 \end{bmatrix}$$

or

$$Q^o(x) = \begin{bmatrix} 1 & 0 \\ \tfrac{1}{2}x_1 & 1 \end{bmatrix}.$$

Further,

$$Q^o(x)^{-1} = \begin{bmatrix} 1 & 0 \\ -\tfrac{1}{2}x_1 & 1 \end{bmatrix}$$

and

$$\frac{\partial\left(Q^o(x)x\right)}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} x_1 \\ \tfrac{1}{2}x_1^2 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x_1 & 1 \end{bmatrix}.$$

Now $\bar{A}^o(x)$, $\bar{b}^o(x)$ and $\bar{c}^o(x)$ can be calculated using (3.10a)-(3.10c). Thus,

$$\bar{A}^o(x) = \begin{bmatrix} 1 & 0 \\ x_1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}x_1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ x_1 & 1 \end{bmatrix} \begin{bmatrix} -1-\frac{1}{2}x_1 & 1 \\ \frac{1}{2}x_1^2 & -x_1 \end{bmatrix} = \begin{bmatrix} -1-\frac{1}{2}x_1 & 1 \\ -x_1 & 0 \end{bmatrix},$$

$$\bar{b}^o(x) = \begin{bmatrix} 1 & 0 \\ x_1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\bar{c}^o(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2}x_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Note that the entries in $\bar{A}^o(x)$ are consistent with the previously calculated $\bar{a}_{11}^o(x)$ and $\bar{a}_{21}^o(x)$.

Example 3.2.2 A linear, n-th-order system.

$$\dot{x} = Ax + bu$$
$$y = cx.$$

Here (3.18) becomes ($\bar{a}_{i1}^o = a_{i1}^o$, $i = 1, \ldots, n$ from (3.9))

$$\mathcal{L}^n[c] = \sum_{i=1}^{n} \mathcal{L}^{i-1}\left[a_{(n+1-i)1}^o c\right]. \tag{3.23}$$

For constant A and c, there results from (3.12) and (3.13)

$$\mathcal{L}^n[c] = cA^n$$
$$\mathcal{L}^{i-1}\left[a_{(n+1-i)1}^o c\right] = a_{(n+1-i)1}^o cA^{i-1} \quad i = 1, \ldots, n,$$

thus (3.23) becomes

$$cA^n = \sum_{i=1}^{n} a_{(n+1-i)1}^o cA^{i-1}$$

or

$$c\left(A^n - a^o_{11}A^{n-1} - a^o_{21}A^{n-2} - \cdots - a^o_{n1}I\right) = 0. \tag{3.24}$$

Since c is arbitrary, then

$$A^n - a^o_{11}A^{n-1} - a^o_{21}A^{n-2} - \cdots - a^o_{n1}I = 0.$$

A comparison of this result with the Cayley-Hamilton theorem, which is valid for LTI systems, reveals that $-a^o_{n1}, -a^o_{(n-1)1}, \ldots, -a^o_{11}$ are the coefficients of the characteristic equation

$$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \tag{3.25}$$

i.e., $a_1 = -a^o_{11}$, $a_2 = -a^o_{21}$, $\ldots$, $a_n = -a^o_{n1}$. Further, (3.19) gives

$$\begin{aligned}
q^o_{1\cdot} &= c \\
q^o_{2\cdot} &= cA - a^o_{11}c \\
q^o_{3\cdot} &= cA^2 - a^o_{11}cA - a^o_{21}c \\
&\vdots \\
q^o_{n\cdot} &= cA^{n-1} - a^o_{11}cA^{n-2} - a^o_{21}cA^{n-3} - \cdots - a^o_{(n-1)1}c
\end{aligned}$$

or

$$Q^o = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -a^o_{11} & \ddots & \ddots & & \vdots \\ -a^o_{21} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ -a^o_{(n-1)1} & \cdots & -a^o_{21} & -a^o_{11} & 1 \end{bmatrix} \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix} = \Xi\mathcal{O}$$

where Ξ is a lower triangular Toeplitz matrix with first column $\begin{bmatrix} 1 & -a^o_{11} & -a^o_{21} & \cdots & -a^o_{(n-1)1} \end{bmatrix}$ and $\mathcal{O}$ is the observability matrix. Since Ξ has full rank, the required transformation exists if and only if $\mathcal{O}$ has full rank. This result is, of course, consistent with that in any standard text on linear systems, e.g. see Figure 2.4-3 in [23].

3.2.2 Controller form

In the case of the controller form (2.25)-(2.26), which has been shown to be useful for controller design, the R.H.S. of (3.10a) becomes (q is specified as c for this form)

$$\bar{A}^c(x)Q^c(x) = \begin{bmatrix} \bar{a}^c_{11}(x) & \bar{a}^c_{12}(x) & \cdots & \cdots & \bar{a}^c_{1n}(x) \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} q^c_{1\cdot}(x) \\ q^c_{2\cdot}(x) \\ \vdots \\ \vdots \\ q^c_{n\cdot}(x) \end{bmatrix}.$$

$$= \begin{bmatrix} \bar{a}^c_{11}(x)q^c_{1\cdot}(x) + \bar{a}^c_{12}(x)q^c_{2\cdot}(x) + \cdots + \bar{a}^c_{1n}(x)q^c_{n\cdot}(x) \\ q^c_{1\cdot}(x) \\ q^c_{2\cdot}(x) \\ \vdots \\ q^c_{(n-1)\cdot}(x) \end{bmatrix}. \quad (3.26)$$

Combining (3.14) and (3.26) results in the following:

$$\left.\begin{aligned} q^c_{(n-1)\cdot}(x) &= \mathcal{L}^1\left[q^c_{n\cdot}(x)\right] \\ q^c_{(n-2)\cdot}(x) &= \mathcal{L}^1\left[q^c_{(n-1)\cdot}(x)\right] \\ &\vdots \\ q^c_{2\cdot}(x) &= \mathcal{L}^1\left[q^c_{3\cdot}(x)\right] \\ q^c_{1\cdot}(x) &= \mathcal{L}^1\left[q^c_{2\cdot}(x)\right] \end{aligned}\right\} \quad (3.27a)$$

and

$$\bar{a}^c_{11}(x)q^c_{1\cdot}(x) + \bar{a}^c_{12}(x)q^c_{2\cdot}(x) + \cdots + \bar{a}^c_{1n}(x)q^c_{n\cdot}(x) = \mathcal{L}^1\left[q^c_{1\cdot}(x)\right]. \quad (3.27b)$$

Now upon employing (3.10b)

$$\begin{bmatrix} \partial(q^c_{1\cdot}(x)x)/\partial x \\ \partial(q^c_{2\cdot}(x)x)/\partial x \\ \vdots \\ \vdots \\ \partial(q^c_{n\cdot}(x)x)/\partial x \end{bmatrix} b(x) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{3.27c}$$

Expressing $q^c_{1\cdot}(x), q^c_{2\cdot}(x), \ldots, q^c_{(n-1)\cdot}(x)$ in terms of $q^c_{n\cdot}(x)$ using (3.27a), and substituting in (3.27c) gives

$$\begin{bmatrix} \partial(\mathcal{L}^{n-1}\left[q^c_{n\cdot}(x)\right]x)/\partial x \\ \partial(\mathcal{L}^{n-2}\left[q^c_{n\cdot}(x)\right]x)/\partial x \\ \vdots \\ \vdots \\ \partial(\mathcal{L}^{0}\left[q^c_{n\cdot}(x)\right]x)/\partial x \end{bmatrix} b(x) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{3.28}$$

The next part was inspired by Zak [85] who obtained a transformation to controller form for a general class of nonlinear systems. His notation which will also be used here, is defined in Appendix B. Su's identity [86]

$$\langle dT, [f, g,] \rangle = \langle d \langle dT, g, \rangle, f \rangle - \langle d \langle dT, f \rangle, g \rangle$$

where T is a scalar field (function), f and g are vector fields and d is the differential operator $\partial/\partial x$, will be used repeatedly, but in the form

$$\langle d \langle dT, g \rangle, f \rangle = \langle dT, [f, g] \rangle + \langle d \langle dT, f \rangle, g \rangle.$$

In this notation the last equation in (3.28) becomes

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\mathcal{L}^0\left[q^c_{n\cdot}(x)\right]x\right)b(x) &= \frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)b(x) \\
&= \left\langle d\left(q^c_{n\cdot}(x)x\right), b(x)\right\rangle \\
&= \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^0 A(x)x, b(x)\right)\right\rangle \\
&= 0.
\end{aligned} \tag{3.29}
$$

The second last equation gives

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\mathcal{L}^1\left[q^c_{n\cdot}(x)\right]x\right)b(x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)A(x)x\right)b(x) \\
&= \left\langle d\left\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\right\rangle, b(x)\right\rangle \\
&= 0.
\end{aligned}
$$

Using Su's identity gives ($T = q^c_{n\cdot}(x)x$, $g = A(x)x$, $f = b(x)$)

$$
\begin{aligned}
&\left\langle d\left\langle d\,\overbrace{\left(q^c_{n\cdot}(x)x\right)}^{T}, \overbrace{A(x)x}^{g}\right\rangle, \overbrace{b(x)}^{f}\right\rangle \\
&= \left\langle d\overbrace{\left(q^c_{n\cdot}(x)x\right)}^{T}, \left[\overbrace{b(x)}^{f}, \overbrace{A(x)x}^{g}\right]\right\rangle + \left\langle d\underbrace{\left\langle d\overbrace{\left(q^c_{n\cdot}(x)x\right)}^{T}, \overbrace{b(x)}^{f}\right\rangle}_{=0 \text{ from } (3.29)}, \overbrace{A(x)x}^{g}\right\rangle \\
&= -\left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^1 A(x)x, b(x)\right)\right\rangle \\
&= 0.
\end{aligned} \tag{3.30}
$$

In a similar fashion, the third last equation gives

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\mathcal{L}^2\left[q^c_{n\cdot}(x)\right]x\right)b(x) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)A(x)x\right)A(x)x\right)b(x) \\
&= \left\langle d\left\langle d\left\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\right\rangle, A(x)x\right\rangle, b(x)\right\rangle \\
&= 0
\end{aligned}
$$

and employing $T = \left\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\right\rangle$, $g = A(x)x$, $f = b(x)$, there results

$$
\begin{aligned}
&\left\langle d\left\langle d\overbrace{\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\rangle}^{T}, \overbrace{A(x)x}^{g}\right\rangle, \overbrace{b(x)}^{f}\right\rangle \\
&= \left\langle d\overbrace{\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\rangle}^{T}, \left[\overbrace{b(x)}^{f}, \overbrace{A(x)x}^{g}\right]\right\rangle \\
&+ \left\langle d\underbrace{\left\langle d\overbrace{\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\rangle}^{T}, \overbrace{b(x)}^{f}\right\rangle}_{=0 \text{ from } (3.30)}, \overbrace{A(x)x}^{g}\right\rangle \\
&= \left\langle d\left\langle d\left(q^c_{n\cdot}(x)x\right), A(x)x\right\rangle, -\left(ad^1 A(x)x, b(x)\right)\right\rangle \\
&= 0.
\end{aligned}
\tag{3.31}
$$

Applying Su's identity again gives ($T = q^c_{n\cdot}(x)x$, $g = A(x)x$, $f = -\left(ad^1 A(x)x, b(x)\right)$)

$$
\begin{aligned}
&\left\langle d\left\langle d\overbrace{\left(q^c_{n\cdot}(x)x\right)}^{T}, \overbrace{A(x)x}^{g}\right\rangle, \overbrace{-\left(ad^1 A(x)x, b(x)\right)}^{f}\right\rangle \\
&= \left\langle d\overbrace{\left(q^c_{n\cdot}(x)x\right)}^{T}, \left[\overbrace{-\left(ad^1 A(x)x, b(x)\right)}^{f}, \overbrace{A(x)x}^{g}\right]\right\rangle \\
&+ \left\langle d\underbrace{\left\langle \overbrace{d\left(q^c_{n\cdot}(x)x\right)}^{T}, \overbrace{-\left(ad^1 A(x)x, b(x)\right)}^{f}\right\rangle}_{=0 \text{ from } (3.30)}, \overbrace{A(x)x}^{g}\right\rangle \\
&= \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^2 A(x)x, b(x)\right)\right\rangle \\
&= 0.
\end{aligned}
\tag{3.32}
$$

Repeating the same process for all the equations in (3.28) results in

$$
\begin{aligned}
(-1)^0 \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^0 A(x)x, b(x)\right)\right\rangle &= 0 \\
(-1)^1 \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^1 A(x)x, b(x)\right)\right\rangle &= 0 \\
(-1)^2 \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^2 A(x)x, b(x)\right)\right\rangle &= 0 \\
&\vdots \\
(-1)^{n-2} \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^{n-2} A(x)x, b(x)\right)\right\rangle &= 0 \\
(-1)^{n-1} \left\langle d\left(q^c_{n\cdot}(x)x\right), \left(ad^{n-1} A(x)x, b(x)\right)\right\rangle &= 1.
\end{aligned}
$$

This may be rewritten as

$$\begin{aligned}
(-1)^0 \tfrac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)\left(ad^0 A(x)x, b(x)\right) &= 0\\
(-1)^1 \tfrac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)\left(ad^1 A(x)x, b(x)\right) &= 0\\
(-1)^2 \tfrac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)\left(ad^2 A(x)x, b(x)\right) &= 0\\
&\vdots\\
(-1)^{n-2} \tfrac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)\left(ad^{n-2} A(x)x, b(x)\right) &= 0\\
(-1)^{n-1} \tfrac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)\left(ad^{n-1} A(x)x, b(x)\right) &= 1.
\end{aligned}$$

or

$$\frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)\mathcal{C}(x) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

where

$$\begin{aligned}
\mathcal{C}(x) = \Big[& (-1)^0\left(ad^0 A(x)x, b(x)\right) \quad (-1)^1\left(ad^1 A(x)x, b(x)\right) \quad \cdots \\
& \cdots \quad (-1)^{n-1}\left(ad^{n-1} A(x)x, b(x)\right) \Big].
\end{aligned} \tag{3.33}$$

Assuming $\rho\mathcal{C}(x) = n$,

$$\frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}\mathcal{C}^{-1}(x). \tag{3.34}$$

Thus, determining $Q^c(x)$ can be done in two steps:

1. Solve for $q^c_{n\cdot}(x)$ from (3.34)

2. Calculate $Q^c(x)$ from (3.27a) (rewritten here as (3.35))

$$\left.\begin{aligned}
q^c_{(n-1)\cdot}(x) &= \mathcal{L}^1\left[q^c_{n\cdot}(x)\right]\\
q^c_{(n-2)\cdot}(x) &= \mathcal{L}^1\left[q^c_{(n-1)\cdot}(x)\right]\\
&\vdots\\
q^c_{2\cdot}(x) &= \mathcal{L}^1\left[q^c_{3\cdot}(x)\right]\\
q^c_{1\cdot}(x) &= \mathcal{L}^1\left[q^c_{2\cdot}(x)\right].
\end{aligned}\right\} \tag{3.35}$$

Comments:

1. Here the rows of $Q^c(x)$ can be obtained without any knowledge of $\bar{a}^c_{1\cdot}(x)$.

2. Eq. (3.34) represents a system of n first-order linear partial differential equations in n variables and one unknown $(q^c_{n\cdot}(x)x)$. Solution methods for such equations are well documented in the literature[5].

3. Once (3.34) has been solved, the calculation of $Q^c(x)$ from (3.35) is relatively simple.

Example 3.2.3 A simple, nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Here $(-1)^0 \left(ad^0 A(x)x, b(x)\right)$ and $(-1)^1 \left(ad^1 A(x)x, b(x)\right)$ become

$$(-1)^0 \left(ad^0 A(x)x, b(x)\right) = b(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} (-1)^1 \left(ad^1 A(x)x, b(x)\right) &= -\left[A(x)x, b(x)\right] \\ &= -\tfrac{\partial}{\partial x}\left(b(x)\right) A(x)x + \tfrac{\partial}{\partial x}\left(A(x)x\right) b(x) \\ &= \frac{\partial}{\partial x} \begin{bmatrix} -x_1 + x_2 \\ -x_1 x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} - & 1 \\ - & -x_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -x_1 \end{bmatrix} \end{aligned}$$

[5] Computational aspects of this system of equations are discussed in Chapter IV.

Thus

$$C(x) = \begin{bmatrix} 0 & 1 \\ 1 & x_1 \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & 1 \end{bmatrix} C^{-1}(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} - & - \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Now (3.34) becomes

$$\frac{\partial}{\partial x}\left(q^c_{2\cdot}(x)x\right) = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and it is easily verified that

$$q^c_{2\cdot}(x) = q^c_{2\cdot} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

is a solution. Now using (3.35)

$$q^c_{1\cdot}(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix},$$

and

$$Q^c(x) = Q^c = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Further,

$$Q^{c-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and

$$\frac{\partial (Q^c x)}{\partial x} = Q^c.$$

Now $\bar{A}^c(x)$, $\bar{b}^c(x)$ and $\bar{c}^c(x)$ can be calculated using (3.10a)-(3.10c). Thus,

$$\begin{aligned}\bar{A}^c(x) &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -x_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -x_1 & -x_1 \end{bmatrix} = \begin{bmatrix} -1-x_1 & -x_1 \\ 1 & 0 \end{bmatrix},\end{aligned}$$

$$\bar{b}^c(x) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\bar{c}^c(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Example 3.2.4 A linear, n-th-order system.

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx.\end{aligned}$$

Here

$$\mathcal{C}(x) = \mathcal{C},$$

i.e., the well-known controllability matrix. Here (3.34) results in

$$\frac{\partial}{\partial x}(q^c_{n\cdot}(x)x) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}$$

and

$$q^c_{n\cdot}(x) = q^c_{n\cdot} = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}$$

is a solution. Then (3.35) gives

$$\begin{aligned} q^c_{(n-1)\cdot} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} C^{-1}A \\ q^c_{(n-2)\cdot} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} C^{-1}A^2 \\ &\vdots \\ q^c_{2\cdot} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} C^{-1}A^{n-2} \\ q^c_{1\cdot} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} C^{-1}A^{n-1}. \end{aligned}$$

The required transformation exists if and only if C has full rank. This result is, of course, consistent with that in any standard text on linear systems.

3.3 Transform of nonlinear feedback gains

It is of interest to investigate the transformation rules for nonlinear feedback gains, i.e., how these gains transform when the system states are transformed according to (3.4). The three main cases of interest are: output-to-$\dot{x}$ feedback, state-to-input feedback and output-to-input feedback.

3.3.1 Output-to-$\dot{x}$ feedback

A typical system where output-to-$\dot{x}$ feedback is used is shown in Fig. 1. In this case the state equation becomes

$$\begin{aligned} \dot{x} &= A(x)x + b(x)u + h(x)y \\ y &= c(x)x, \end{aligned} \tag{3.36}$$

where

$$h(x) = \begin{bmatrix} h_1(x) & h_2(x) & \cdots & \cdots & h_n(x) \end{bmatrix}^T.$$

Substituting for y in (3.36) results in

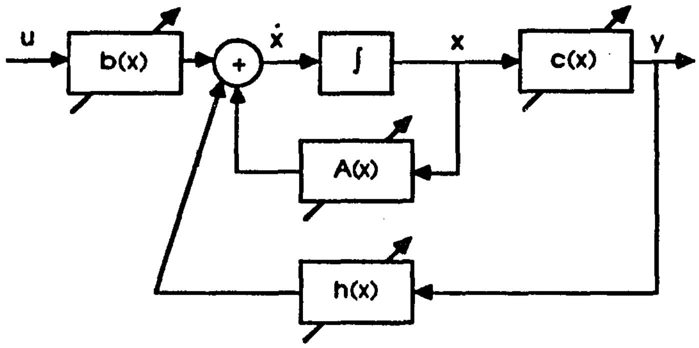

Figure 1: Output-to-$\dot{x}$ feedback

$$\dot{x} = (A(x) + h(x)c(x))\,x + b(x)u.$$

Thus, $h(x)c(x)$ transforms according to (3.10a)

$$\frac{\partial\left(Q^q(x)x\right)}{\partial x}h(x)c(x) = \bar{h}^q(x)\bar{c}^q(x)Q^q(x) \tag{3.37}$$

Comparing (3.10c) and (3.37) shows that $h(x)$ transforms according to

$$\frac{\partial\left(Q^q(x)x\right)}{\partial x}h(x) = \bar{h}^q(x). \tag{3.38}$$

In a typical system the $\dot{x}$ node is not accessible and thus this type of feedback is not practical. One case, however, where the $\dot{x}$ node is accessible is in an observer, e.g., the observer discussed in Section 2.1. Here

$$\bar{h}^q(\hat{x}) = \bar{l}^o(\hat{x}) = l^o(\hat{x}^o)\Big|_{\hat{x}^o = Q^o(\hat{x})\hat{x}},$$

thus $\bar{l}^o(\hat{x})$ transforms as in (3.38) or

$$\frac{\partial\left(Q^o(\hat{x})\hat{x}\right)}{\partial \hat{x}}l(\hat{x}) = \bar{l}^o(\hat{x}). \tag{3.39}$$

The observer is selected corresponding to (2.3) as

$$\dot{\hat{x}} = A(\hat{x})\,\hat{x} + b(\hat{x})\,u + l(\hat{x})\,(y - c(\hat{x})\,\hat{x}).$$

and the resulting error dynamics are

$$\dot{e} = (A(\hat{x}) - l(\hat{x})\,c(\hat{x}))\,e + (A(x) - A(\hat{x}))\,x + (b(x) - b(\hat{x}))\,u$$

where

$$e = x - \hat{x}.$$

If $\bar{l}^o(\hat{x})$ is chosen as described in Section 2.1 and $l(\hat{x})$ calculated using (3.39) then,

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{t\to\infty} (Q(x^o)\,x^o - Q(\hat{x}^o)\,\hat{x}^o) \\ &= \lim_{t\to\infty} (Q(x^o)\,(x^o - \hat{x}^o) + (Q(x^o) - Q(\hat{x}^o))\,\hat{x}^o) = 0.\end{aligned}$$

since $\lim_{t\to\infty} e^o(t) = 0$.

3.3.2 State-to-input feedback

A typical system where state-to-input feedback is used is shown in Fig. 2. In this case

$$u = h(x)x + v, \tag{3.40}$$

where

$$h(x) = \begin{bmatrix} h_1(x) & h_2(x) & \cdots & \cdots & h_n(x) \end{bmatrix}.$$

Substituting (3.40) in (3.1) results in

$$\dot{x} = (A(x) + b(x)h(x))\,x + b(x)v. \tag{3.41}$$

Thus, $b(x)h(x)$ transforms according to (3.10a)

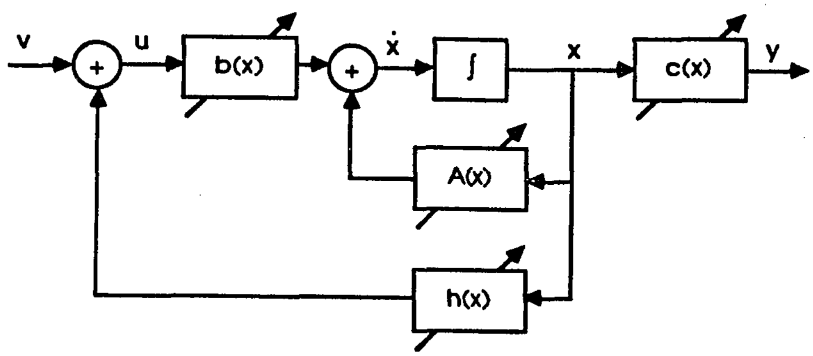

Figure 2: State-to-input feedback

$$\frac{\partial\left(Q^q(x)x\right)}{\partial x} b(x)h(x) = \bar{b}^q(x)\bar{h}^q(x)Q^q(x). \tag{3.42}$$

Comparing (3.10b) and (3.42) shows that $h(x)$ transforms according to

$$h(x) = \bar{h}^q(x)Q^q(x). \tag{3.43}$$

The typical full-state feedback falls into this category, e.g., the state-feedback controller discussed in Section 2.2. Here

$$\bar{h}^q(x) = -\bar{k}^c(x) = -k^c\left(x^c\right)\Big|_{x^c=Q^c(x)x},$$

and $\bar{k}^c(x)$ transforms as in (3.43) or

$$k(x) = \bar{k}^c(x)Q^c(x).$$

The state equation (3.41) becomes

$$\dot{x} = \left(A(x) - b(x)k(x)\right)x + b(x)v. \tag{3.44}$$

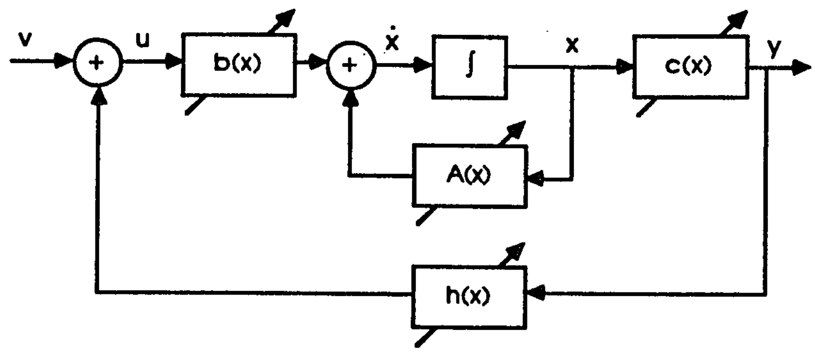

Figure 3: Output-to-input feedback

Note that this equation (3.44) is nonlinear, however, the closed-loop dynamics are the same as those of (2.29) and thus linear. Only the relationship between v and x^c is linear; any other state x which has a nonlinear relationship with x^c (i.e., $x = Q(x^c)\, x^c$) also has a nonlinear relationship with v.

3.3.3 Output-to-input feedback

A typical system where output-to-input feedback is used is shown in Fig. 3. In this case

$$u = h(x)y + v, \tag{3.45}$$

where $h(x)$ is a scalar function. Substituting (3.45) in (3.1) results in

$$\dot{x} = (A(x) + b(x)h(x)c(x))\, x + b(x)v$$
$$y = c(x)x.$$

Thus $b(x)h(x)c(x)$ transforms according to (3.10a)

$$\frac{\partial (Q^q(x)\, x)}{\partial x} b(x)h(x)c(x) = \bar{b}^q(x)\bar{h}^q(x)\bar{c}^q(x)Q^q(x). \tag{3.46}$$

Comparing (3.10b) and (3.10c) with (3.46) shows that $h(x)$ transforms according to

$$h(x) = \bar{h}^q(x).$$

3.4 Discussion

In this chapter the equations that must be solved to find the nonlinear transformation to the observer and the controller form were derived. These equations may be difficult to solve, especially in the observer case. Various computational aspects of (3.18) and (3.34) are considered in the next chapter.

CHAPTER IV

COMPUTATIONAL ASPECTS

4.1 Transformation to observer form

In this case the most difficult aspect is solving (3.18) rewritten here as (4.1)

$$\mathcal{L}^n\left[c(x)\right] = \sum_{i=1}^{n} \mathcal{L}^{i-1}\left[\bar{a}^o_{(n+1-i)1}(x)c(x)\right]. \tag{4.1}$$

In general this constitutes a system of n $(n-1)$-st-order linear partial-differential equations (PDEs) in n variables and n unknowns, which can be reduced to $n-1$ $(n-1)$-st-order linear PDEs in n variables and $n-1$ unknowns, namely $\bar{a}^o_{11}(x)$, $\bar{a}^o_{21}(x)$, $\ldots$, $\bar{a}^o_{(n-1)1}(x)$. Once these unknowns have been obtained, $\bar{a}^o_{n1}(x)$ can be determined from (4.1); however, knowledge of $\bar{a}^o_{n1}(x)$ is not necessary in order to find $Q^o(x)$.

4.1.1 Case $n = 1$

Here $x = x_1$ and (4.1) reduces to

$$\mathcal{L}^1\left[c\left(x_1\right)\right] = \mathcal{L}^0\left[\bar{a}^o_{11}\left(x_1\right)c\left(x_1\right)\right]$$

or

$$\frac{\partial}{\partial x_1}\left(c_1\left(x_1\right)x_1\right)a_{11}\left(x_1\right) = \bar{a}^o_{11}\left(x_1\right)c_1\left(x_1\right).$$

Assuming $c_1\left(x_1\right) \neq 0$, $\bar{a}^o_{11}\left(x_1\right)$ is given by

$$\bar{a}^o_{11}\left(x_1\right) = \frac{1}{c_1\left(x_1\right)}\frac{\partial}{\partial x_1}\left(c_1\left(x_1\right)x_1\right)a_{11}\left(x_1\right).$$

4.1.2 Case $n = 2$

Here $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and Eqs. (4.1) become for $n = 2$

$$\mathcal{L}^2 [c(x)] = \mathcal{L}^1 [\bar{a}^o_{11}(x)c(x)] + \mathcal{L}^0 [\bar{a}^o_{21}(x)c(x)] .$$

The quantity $\mathcal{L}^1 [c(x)]$ becomes

$$\mathcal{L}^1 [c(x)] = \frac{\partial}{\partial x}(c(x)x)A(x) = \begin{bmatrix} c^1_1(x) & c^1_2(x) \end{bmatrix} = c^1(x)$$

where

$$\begin{aligned} c^1_1(x) &= \tfrac{\partial}{\partial x_1}(c(x)x)a_{11}(x) + \tfrac{\partial}{\partial x_2}(c(x)x)a_{21}(x) \\ c^1_2(x) &= \tfrac{\partial}{\partial x_1}(c(x)x)a_{12}(x) + \tfrac{\partial}{\partial x_2}(c(x)x)a_{22}(x). \end{aligned} \tag{4.2}$$

Then $\mathcal{L}^2 [c(x)]$ is

$$\mathcal{L}^2 [c(x)] = \mathcal{L}^1 \left[\begin{bmatrix} c^1_1(x) & c^1_2(x) \end{bmatrix}\right] = \begin{bmatrix} c^2_1(x) & c^2_2(x) \end{bmatrix} = c^2(x)$$

where

$$\begin{aligned} c^2_1(x) &= \tfrac{\partial}{\partial x_1}\left(c^1(x)x\right)a_{11}(x) + \tfrac{\partial}{\partial x_2}\left(c^1(x)x\right)a_{21}(x) \\ c^2_2(x) &= \tfrac{\partial}{\partial x_1}\left(c^1(x)x\right)a_{12}(x) + \tfrac{\partial}{\partial x_2}\left(c^1(x)x\right)a_{22}(x). \end{aligned} \tag{4.3}$$

Finally,

$$\begin{aligned} &\mathcal{L} [\bar{a}^o_{11}(x)c(x)] \\ &= \tfrac{\partial}{\partial x}\left(\bar{a}^o_{11}(x)c(x)x\right) A(x) \\ &= \Big[c(x)xa_{11}(x)\tfrac{\partial}{\partial x_1}\left(\bar{a}^o_{11}(x)\right) + \tfrac{\partial}{\partial x_1}(c(x)x)\, a_{11}(x)\bar{a}^o_{11}(x) \\ &\quad +c(x)xa_{21}(x)\tfrac{\partial}{\partial x_2}\left(\bar{a}^o_{11}(x)\right) + \tfrac{\partial}{\partial x_2}(c(x)x)\, a_{21}(x)\bar{a}^o_{11}(x) \\ &\quad c(x)xa_{12}(x)\tfrac{\partial}{\partial x_1}\left(\bar{a}^o_{11}(x)\right) + \tfrac{\partial}{\partial x_1}(c(x)x)\, a_{12}(x)\bar{a}^o_{11}(x) \\ &\quad +c(x)xa_{22}(x)\tfrac{\partial}{\partial x_2}\left(\bar{a}^o_{11}(x)\right) + \tfrac{\partial}{\partial x_2}(c(x)x)\, a_{22}(x)\bar{a}^o_{11}(x) \Big], \end{aligned}$$

or

$$\begin{aligned}
&\mathcal{L}\left[\bar{a}_{11}^{o}(x)c(x)\right] \\
&= \Big[c(x)xa_{11}(x)\tfrac{\partial}{\partial x_1}\left(\bar{a}_{11}^{o}(x)\right) + c(x)xa_{21}(x)\tfrac{\partial}{\partial x_2}\left(\bar{a}_{11}^{o}(x)\right) \\
&\qquad + c_1^1(x)\bar{a}_{11}^{o}(x) \\
&\quad c(x)xa_{12}(x)\tfrac{\partial}{\partial x_1}\left(\bar{a}_{11}^{o}(x)\right) + c(x)xa_{22}(x)\tfrac{\partial}{\partial x_2}\left(\bar{a}_{11}^{o}(x)\right) \\
&\qquad + c_2^1(x)\bar{a}_{11}^{o}(x) \Big]
\end{aligned} \tag{4.4}$$

and

$$\mathcal{L}^0\left[\bar{a}_{21}^{o}(x)c(x)\right] = \left[\begin{array}{cc} c_1(x)\bar{a}_{21}^{o}(x) & c_2(x)\bar{a}_{21}^{o}(x) \end{array}\right].$$

Thus (4.1) results in a set of equations

$$\begin{aligned}
c_1^2(x) &= c(x)xa_{11}(x)\frac{\partial}{\partial x_1}\left(\bar{a}_{11}^{o}(x)\right) + c(x)xa_{21}(x)\frac{\partial}{\partial x_2}\left(\bar{a}_{11}^{o}(x)\right) \\
&\quad + c_1^1(x)\bar{a}_{11}^{o}(x) + c_1(x)\bar{a}_{21}^{o}(x) && (4.5a) \\
c_2^2(x) &= c(x)xa_{12}(x)\frac{\partial}{\partial x_1}\left(\bar{a}_{11}^{o}(x)\right) + c(x)xa_{22}(x)\frac{\partial}{\partial x_2}\left(\bar{a}_{11}^{o}(x)\right) \\
&\quad + c_2^1(x)\bar{a}_{11}^{o}(x) + c_2(x)\bar{a}_{21}^{o}(x). && (4.5b)
\end{aligned}$$

Eliminating $\bar{a}_{21}^{o}(x)$ from these equations results in a first-order linear[1] PDE in two variables and one unknown, $\bar{a}_{11}^{o}(x)$, of the form

$$d_1(x)\frac{\partial}{\partial x_1}\left(\bar{a}_{11}^{o}(x)\right) + d_2(x)\frac{\partial}{\partial x_2}\left(\bar{a}_{11}^{o}(x)\right) + d(x)\bar{a}_{11}^{o}(x) + h(x) = 0 \tag{4.6}$$

where

$$\begin{aligned}
d_1(x) &= c_2(x)c(x)xa_{11}(x) - c_1(x)c(x)xa_{12}(x) \\
d_2(x) &= c_2(x)c(x)xa_{21}(x) - c_1(x)c(x)xa_{22}(x) \\
d(x) &= c_2(x)c_1^1(x) - c_1(x)c_2^1(x) \\
h(x) &= -c_2(x)c_1^2(x) + c_1(x)c_2^2(x)
\end{aligned} \tag{4.7}$$

[1]A PDE is linear if the unknown and all its derivatives appear linearly.

Alternatively, (4.6) can be written in the compact form

$$d_1(x)u_{x_1} + d_2(x)u_{x_2} + d(x)u + h(x) = 0, \tag{4.8}$$

where

$$u = \bar{a}_{11}^o(x),$$

$$u_{x_1} = \frac{\partial}{\partial x_1}\left(\bar{a}_{11}^o(x)\right)$$

and

$$u_{x_2} = \frac{\partial}{\partial x_2}\left(\bar{a}_{11}^o(x)\right).$$

Once (4.6) (or (4.8)) has been solved for $\bar{a}_{11}^o(x)$, (4.5a) or (4.5b) may be used to solve for $\bar{a}_{21}^o(x)$ if desired.

A general solution method for $n = 2$

Solution methods for equations of the form (4.8) are well documented in the literature (see e.g. pp. 205-212 in [88]—here $h(x) = 0$ but the same method also applies to $h(x) \neq 0$). The following briefly describes the approach taken in [89] (pp. 133-137).

This approach, wherein it is assumed that $d_1(x)$ and $d_2(x)$ do not vanish simultaneously for any x, is based on finding a change of coordinates such that (4.8) becomes an ordinary differential equation (ODE) in a new coordinate system. The new coordinates, ξ and η, are related to the original ones, x_1 and x_2, by

$$\xi = \xi(x_1, x_2)$$

and

$$\eta = \eta(x_1, x_2).$$

Since this transformation must be invertible, one must require that

$$J \equiv \frac{\partial(\xi,\eta)}{\partial(x_1,x_2)} = \frac{\partial\xi}{\partial x_1}\frac{\partial\eta}{\partial x_2} - \frac{\partial\xi}{\partial x_2}\frac{\partial\eta}{\partial x_1} = \xi_{x_1}\eta_{x_2} - \xi_{x_2}\eta_{x_1} \neq 0.$$

The relation between u_{x_1}, u_{x_2} and u_ξ, u_η is given by

$$\begin{aligned} u_{x_1} &= \tfrac{\partial u}{\partial\xi}\tfrac{\partial\xi}{\partial x_1} + \tfrac{\partial u}{\partial\eta}\tfrac{\partial\eta}{\partial x_1} = u_\xi\xi_{x_1} + u_\eta\eta_{x_1} \\ u_{x_2} &= \tfrac{\partial u}{\partial\xi}\tfrac{\partial\xi}{\partial x_2} + \tfrac{\partial u}{\partial\eta}\tfrac{\partial\eta}{\partial x_2} = u_\xi\xi_{x_2} + u_\eta\eta_{x_2}. \end{aligned}$$

Substituting in (4.8) gives

$$d_1(x)\left(u_\xi\xi_{x_1} + u_\eta\eta_{x_1}\right) + d_2(x)\left(u_\xi\xi_{x_2} + u_\eta\eta_{x_2}\right) + d(x)u + h(x) = 0,$$

$$\left(d_1(x)\xi_{x_1} + d_2(x)\xi_{x_2}\right)u_\xi + \left(d_1(x)\eta_{x_1} + d_2(x)\eta_{x_2}\right)u_\eta + d(x)u + h(x) = 0$$

or

$$E(x)u_\xi + F(x)u_\eta + d(x)u + h(x) = 0 \tag{4.9}$$

where

$$\begin{aligned} E(x) &= d_1(x)\xi_{x_1} + d_2(x)\xi_{x_2} \\ F(x) &= d_1(x)\eta_{x_1} + d_2(x)\eta_{x_2}. \end{aligned} \tag{4.10}$$

Then η is chosen such that

$$F(x) = d_1(x)\eta_{x_1} + d_2(x)\eta_{x_2} = 0 \tag{4.11}$$

is satisfied. Eqn. (4.11) has infinitely many solutions of the form

$$\eta = f\left(v\left(x_1,x_2\right)\right)$$

where $v(x_1,x_2)$ is the general solution of

$$\frac{dx_1}{d_1(x_1,x_2)} = \frac{dx_2}{d_2(x_1,x_2)}.$$

One of these $\eta(x_1,x_2)$ is selected and then some $\xi(x_1,x_2)$ is chosen such that $J \neq 0$[2]. Then the inverse transformation

[2]For example, if $d_1\left(x_1^0,x_2^0\right) \neq 0$ then taking

$$x_1 = x_1(\xi, \eta)$$
$$x_2 = x_2(\xi, \eta) \tag{4.12}$$

can be determined.

Since $F(x) = 0$ then (4.9) becomes

$$E(x)u_\xi + d(x)u + h(x) = 0. \tag{4.13}$$

When $J \neq 0$ and $d_1(x)$ and $d_2(x)$ do not vanish simultaneously, then $E(x) \neq 0$ [89] and (4.13) can be rewritten as

$$u_\xi + \gamma(x)u + \delta(x) = 0 \tag{4.14}$$

where

$$\gamma(x) = d(x)/E(x)$$
$$\delta(x) = h(x)/E(x).$$

Then using (4.12) one can write

$$u_\xi + \gamma(\xi, \eta)u + \delta(\xi, \eta) = 0, \tag{4.15}$$

i.e., an ODE in ξ where η can be treated as a constant. Eqn. (4.15) is referred to as the canonical form of (4.8) and its solution is given by (p. 390, [83])

$$u(\xi, \eta) = \frac{1}{M(\xi, \eta)} \left(- \int \delta(\xi, \eta) M(\xi, \eta) d\xi + c \right) \tag{4.16}$$

where

$$M(\xi, \eta) = \exp^{\int \gamma(\xi, \eta) d\xi}. \tag{4.17}$$

$$\eta\left(x_1^0, x_2\right) = x_2$$

guarantees the existence of a unique solution to (4.11). Then $\eta_{x_2}\left(x_1^0, x_2^0\right) = 1$ and choosing $\xi\left(x_1, x_2\right) = x_1$ gives $J = 1$ in the neighborhood of $\left(x_1^0, x_2^0\right)$.

The general solution of (4.8) is then obtained by returning to the x_1, x_2 coordinates.

Example 4.1.1 A nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} 1 & x_2^2 \\ -1 & -1 \end{bmatrix} x$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Here (4.2) and (4.3) give

$$c_1^1(x) = 1$$
$$c_2^1(x) = x_2^2$$
$$c_1^2(x) = 1 - 3x_2^2$$
$$c_2^2(x) = -2x_2^2.$$

Then (4.7) results in

$$\begin{aligned} d_1(x) &= -x_1x_2^2 \\ d_2(x) &= x_1 \\ d(x) &= -x_2^2 \\ h(x) &= -2x_2^2. \end{aligned}$$

Thus (4.8) becomes

$$-x_1x_2^2u_{x_1} + x_1u_{x_2} - x_2^2u - 2x_2^2 = 0. \tag{4.18}$$

In this case $d_1(x)$ and $d_2(x)$ vanish simultaneously at $x_1 = 0$. However, (4.18) can be divided by x_1 to avoid this problem, resulting in

$$-x_2^2u_{x_1} + u_{x_2} - x_1^{-1}x_2^2u - 2x_1^{-1}x_2^2 = 0.$$

Now

$$\begin{aligned} d_1(x) &= -x_2^2 \\ d_2(x) &= 1 \\ d(x) &= -x_1^{-1}x_2^2 \\ h(x) &= -2x_1^{-1}x_2^2. \end{aligned}$$

Eqn. (4.11) now becomes

$$-x_2^2\eta_{x_1} + \eta_{x_2} = 0,$$

thus

$$\frac{dx_1}{-x_2^2} = \frac{dx_2}{1}$$

or

$$dx_1 + x_2^2 dx_2 = 0$$

must be solved. Integrating gives

$$x_1 + \frac{1}{3}x_2^3 = C,$$

where C is an arbitrary constant. Thus $\eta(x_1, x_2)$ may be taken as

$$\eta(x_1, x_2) = x_1 + \frac{1}{3}x_2^3. \tag{4.19}$$

Now J becomes

$$J = \xi_{x_1}x_2^2 - \xi_{x_2} \cdot 1.$$

Choosing

$$\xi(x_1, x_2) = x_2 \tag{4.20}$$

results in $J = -1$. The inverse transformation can be found from (4.19) and (4.20) as

$$x_1(\xi,\eta) = \eta - \tfrac{1}{3}\xi^3$$
$$x_2(\xi,\eta) = \xi.$$

Now (4.10) gives

$$E(x) = d_2(x) = 1,$$

and thus $\gamma(x_1, x_2)$ and $\delta(x_1, x_2)$ are given by

$$\gamma(x_1,x_2) = -x_1^{-1}x_2^2$$
$$\delta(x_1,x_2) = -2x_1^{-1}x_2^2$$

or

$$\gamma(\xi,\eta) = -\left(\eta - \tfrac{1}{3}\xi^3\right)^{-1}\xi^2$$
$$\delta(\xi,\eta) = -2\left(\eta - \tfrac{1}{3}\xi^3\right)^{-1}\xi^2.$$

Employing (4.17) gives

$$M(\xi,\eta) = \exp^{-\int\left(\eta - \frac{1}{3}\xi^3\right)^{-1}\xi^2 d\xi} = \exp^{\ln\left(\eta - \frac{1}{3}\xi^3\right)} = \eta - \frac{1}{3}\xi^3,$$

and from (4.16)

$$\begin{aligned} u(\xi,\eta) &= \left(\eta - \tfrac{1}{3}\xi^3\right)^{-1}\left(\int 2\left(\eta - \frac{1}{3}\xi^3\right)^{-1}\xi^2\left(\eta - \frac{1}{3}\xi^3\right)d\xi + C\right) \\ &= \left(\eta - \tfrac{1}{3}\xi^3\right)^{-1}\left(\int 2\xi^2 d\xi + C\right) \\ &= \left(\eta - \tfrac{1}{3}\xi^3\right)^{-1}\left(\tfrac{2}{3}\xi^3 + C\right), \end{aligned}$$

where C is an arbitrary constant. Inverse transforming gives

$$u(x_1,x_2) = x_1^{-1}\left(\frac{2}{3}x_2^3 + C\right) = \frac{2}{3}x_1^{-1}x_2^3,$$

taking $C = 0$. Thus

$$\bar{a}_{11}^o(x) = u(x_1,x_2) = \frac{2}{3}x_1^{-1}x_2^3.$$

Then it can be easily verified from (3.19), (3.10a) and (3.10c) that

$$Q^o(x) = \begin{bmatrix} 1 & 0 \\ 1 - \frac{2}{3}x_1^{-1}x_2^3 & x_2^2 \end{bmatrix},$$

$$\bar{A}^o(x) = \begin{bmatrix} \frac{2}{3}x_1^{-1}x_2^3 & 1 \\ -x_2^2 + 1 & 0 \end{bmatrix}$$

and

$$\bar{c}^o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Special case 1

Since there are no boundary conditions on u (or $\bar{a}_{11}^o(x)$), one has the freedom to choose any u that satisfies (4.8). An especially simple case arises if (4.8) can be put in the form

$$d_1(x)u_{x_1} + d_2(x)u_{x_2} + du + h = 0 \tag{4.21}$$

where d and h are constants. Then u can be taken as a constant (thus u_{x_1} and u_{x_2} are zero), given by

$$u = -h/d.$$

A number of nonlinear systems and all second-order LTI systems fall into this category.

Example 4.1.2 A nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} 0 & x_1^2 \\ -1 & -1 \end{bmatrix} x$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

From (4.2), (4.3) and (4.7) one has

$$\begin{aligned} c_1^1(x) &= -1 \\ c_2^1(x) &= -1 \\ c_1^2(x) &= 1 \\ c_2^2(x) &= 1 - x_1^2 \end{aligned}$$

$$\begin{aligned} d_1(x) &= 0 \\ d_2(x) &= -x_2 \\ d(x) &= -1 \\ h(x) &= -1. \end{aligned}$$

Thus (4.8) becomes

$$-x_2 u_{x_2} - u - 1 = 0$$

which is in the form of (4.21), and thus one solution is

$$u = -1.$$

Clearly

$$\bar{a}_{11}^o(x) = -1.$$

It can be easily verified from (3.19), (3.10a) and (3.10c) that here

$$Q^o(x) = Q^o = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\bar{A}^o(x) = \begin{bmatrix} -1 & 1 \\ -x_1^2 & 0 \end{bmatrix}$$

and

$$\bar{c}^o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Special case 2

Theorem 4.1.1 *If*

$$A(x) = \begin{bmatrix} a_{11}(x) & a_{12} \\ a_{21}(x) & a_{22}(x_1) \end{bmatrix}$$

$$c = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

where a_{12} *is a constant, then*

$$\bar{a}^o_{11}(x) = a_{11}(x) + \frac{1}{x_1}\int a_{22}(x_1)\,dx_1 \tag{4.22}$$

$$\bar{a}^o_{21}(x) = -\det A(x) \tag{4.23}$$

is a solution of (4.1)[3].

Proof: (by direct substitution)

$$\begin{aligned}
c^1_1(x) &= a_{11}(x) \\
c^1_2(x) &= a_{12} \\
c^2_1(x) &= \frac{\partial}{\partial x_1}(a_{11}(x)x_1)\,a_{11}(x) + a_{12}a_{21}(x) + \frac{\partial}{\partial x_2}(a_{11}(x))\,x_1a_{21}(x) \\
c^2_2(x) &= \frac{\partial}{\partial x_1}(a_{11}(x)x_1)\,a_{12} + a_{12}a_{22}(x_1) + \frac{\partial}{\partial x_2}(a_{11}(x))\,x_1a_{22}(x_1)
\end{aligned}$$

$$\begin{aligned}
d_1(x) &= -a_{12}x_1 \\
d_2(x) &= -x_1a_{22}(x_1) \\
d(x) &= -a_{12} \\
h(x) &= \frac{\partial}{\partial x_1}(a_{11}(x)x_1)\,a_{12} + a_{12}a_{22}(x_1) + x_1a_{22}(x_1)\frac{\partial}{\partial x_2}(a_{11}(x)).
\end{aligned}$$

[3]Note the similarity to the linear case where

$$\bar{a}^o_{11} = \operatorname{trace} A = a_{11} + a_{22}$$

$$\bar{a}^o_{21} = -\det A.$$

Thus (4.8) becomes

$$-a_{12}x_1u_{x_1} - x_1a_{22}(x_1)u_{x_2} - a_{12}u + \frac{\partial}{\partial x_1}(a_{11}(x)x_1)a_{12}$$
$$+a_{12}a_{22}(x_1) + x_1a_{22}(x_1)\frac{\partial}{\partial x_2}(a_{11}(x)) = 0. \qquad (4.24)$$

Substituting

$$\begin{aligned} u &= \bar{a}^o_{11}(x) = a_{11}(x) + \tfrac{1}{x_1}\int a_{22}(x_1)\,dx_1 \\ u_{x_1} &= \tfrac{\partial}{\partial x_1}(a_{11}(x)) - \tfrac{1}{x_1^2}\int a_{22}(x_1)\,dx_1 + \frac{1}{x_1}a_{22}(x_1) \\ u_{x_2} &= \tfrac{\partial}{\partial x_2}(a_{11}(x)) \end{aligned}$$

in (4.24) results in

$$\begin{aligned} &-a_{12}x_1\frac{\partial}{\partial x_1}(a_{11}(x)) - a_{12}x_1\left(-\frac{1}{x_1^2}\int a_{22}(x_1)\,dx_1\right) - a_{12}x_1\left(\frac{1}{x_1}a_{22}(x_1)\right) \\ &\quad -x_1a_{22}(x_1)\tfrac{\partial}{\partial x_2}(a_{11}(x)) - a_{12}a_{11}(x) - a_{12}\tfrac{1}{x_1}\int a_{22}(x_1)\,dx_1 \\ &\quad +\tfrac{\partial}{\partial x_1}(a_{11}(x)x_1)a_{12} + a_{12}a_{22}(x_1) + x_1a_{22}(x_1)\tfrac{\partial}{\partial x_2}(a_{11}(x)) \\ &= -a_{12}x_1\frac{\partial}{\partial x_1}(a_{11}(x)) + a_{12}\frac{1}{x_1}\int a_{22}(x_1)\,dx_1 - a_{12}a_{22}(x_1) \\ &\quad -a_{12}a_{11}(x) - a_{12}\tfrac{1}{x_1}\int a_{22}(x_1)\,dx_1 + a_{12}x_1\frac{\partial}{\partial x_1}(a_{11}(x)) \\ &\quad +a_{12}a_{11}(x) + a_{12}a_{22}(x_1) \\ &= 0, \end{aligned}$$

thus (4.22) is a solution of (4.1).

Then (4.5a) gives

$$\begin{aligned} \bar{a}^o_{21}(x) &= c_1^2(x) - c(x)xa_{11}(x)\tfrac{\partial}{\partial x_1}\left(\bar{a}^o_{11}(x)\right) \\ &\quad -c(x)xa_{21}(x)\tfrac{\partial}{\partial x_2}\left(\bar{a}^o_{11}(x)\right) - c_1^1(x)\bar{a}^o_{11}(x). \end{aligned}$$

Substituting

$$
\begin{aligned}
\bar{a}^o_{21}(x) &= \tfrac{\partial}{\partial x_1}\left(a_{11}(x)x_1\right)a_{11}(x) + a_{12}a_{21}(x) + \tfrac{\partial}{\partial x_2}\left(a_{11}(x)\right)x_1 a_{21}(x) \\
&\quad - x_1 a_{11}(x)\tfrac{\partial}{\partial x_1}\left(a_{11}(x)\right) - x_1 a_{11}(x)\left(-\tfrac{1}{x_1^2}\int a_{22}(x_1)\,dx_1\right) \\
&\quad - x_1 a_{11}(x)\tfrac{1}{x_1}a_{22}(x_1) - x_1 a_{21}(x)\tfrac{\partial}{\partial x_2}\left(a_{11}(x)\right) - a_{11}(x)a_{11}(x) \\
&\quad - a_{11}(x)\tfrac{1}{x_1}\int a_{22}(x_1)\,dx_1 \\
&= x_1 a_{11}(x)\tfrac{\partial}{\partial x_1}\left(a_{11}(x)\right) + a_{11}^2(x) + a_{12}a_{21}(x) \\
&\quad - x_1 a_{11}(x)\tfrac{\partial}{\partial x_1}\left(a_{11}(x)\right) + a_{11}(x)\tfrac{1}{x_1}\int a_{22}(x_1)\,dx_1 \\
&\quad - a_{11}(x)a_{22}(x_1) - a_{11}^2(x) - a_{11}(x)\tfrac{1}{x_1}\int a_{22}(x_1)\,dx_1 \\
&= a_{12}a_{21}(x) - a_{11}(x)a_{22}(x_1) \\
&= -\det A(x),
\end{aligned}
$$

thus verifying (4.23) is a solution of (4.1).

□

Example 4.1.3 A nonlinear, second-order system.

$$
\dot{x} = \begin{bmatrix} -x_1^2 - x_2^2 & 1 \\ -x_2^2 & -x_1^2 \end{bmatrix} x
$$

$$
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
$$

Here (4.22) and (4.23) give

$$
\bar{a}^o_{11}(x) = -x_1^2 - x_2^2 + \frac{1}{x_1}\int\left(-x_1^2\right)dx_1 = -x_1^2 - x_2^2 - \frac{1}{3}x_1^2 = -\frac{4}{3}x_1^2 - x_2^2
$$

$$
\bar{a}^o_{21}(x) = 1\cdot\left(-x_2^2\right) - \left(-x_1^2 - x_2^2\right)\left(-x_1^2\right) = -x_2^2 - x_1^4 - x_1^2 x_2^2.
$$

It can be easily verified from (3.19), (3.10a) and (3.10c) that

$$
Q^o(x) = \begin{bmatrix} 1 & 0 \\ \frac{1}{3}x_1^2 & 1 \end{bmatrix},
$$

$$\bar{A}^o(x) = \begin{bmatrix} -\frac{4}{3}x_1^2 - x_2^2 & 1 \\ -x_2^2 - x_1^4 - x_1^2 x_2^2 & 0 \end{bmatrix}$$

and

$$\bar{c}^o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

4.1.3 Case $n > 2$

The notation D^α attributed to Schwartz (pp. 54-55 [90]) will be useful in the following. Here

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

where α_i, $i = 1, 2, \ldots, n$ are nonnegative integers. D^α is defined by

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

The order of D^α is denoted by

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Before expressing (4.1) in terms of this notation, the following observations will be helpful:

> Applying the operator $\mathcal{L}$ on some $\bar{a}^o_{(n+1-i)1}(x)c(x)$ generates a 1 x n vector whose elements are composed of $\bar{a}^o_{(n+1-i)1}(x)$ and all its first derivatives $\partial\left(\bar{a}^o_{(n+1-i)1}(x)\right)/\partial x_{l_1}$, $l_1 = 1, 2, \ldots, n$ (see (4.4) in the case $n = 2$). Applying $\mathcal{L}$ again generates a 1 x n vector whose elements are composed of $\bar{a}^o_{(n+1-i)1}(x)$, all its first derivatives $\partial\left(\bar{a}^o_{(n+1-i)1}(x)\right)/\partial x_{l_1}$, $l_1 = 1, 2, \ldots, n$ and all its second derivatives $\partial^2\left(\bar{a}^o_{(n+1-i)1}(x)\right)/\partial x_{l_1}\partial x_{l_2}$, $l_1 = 1, 2, \ldots, n$, $l_2 = l_1 + 1, l_1 + 2, \ldots, n$. Applying the operator $i - 1$

times on $\bar{a}^o_{(n+1-i)1}(x)c(x)$ generates a 1 x n vector whose elements are composed of $\bar{a}^o_{(n+1-i)1}(x)$, all its first derivatives, all its second derivatives, ..., all its $(i-2)$-th derivatives and all its $(i-1)$-th derivatives.

Thus $\bar{a}^o_{11}(x)$, $\bar{a}^o_{21}(x)$, ..., $\bar{a}^o_{n1}(x)$, the first derivatives of $\bar{a}^o_{11}(x)$, $\bar{a}^o_{21}(x)$, ..., $\bar{a}^o_{(n-1)1}(x)$, the second derivatives of $\bar{a}^o_{11}(x)$, $\bar{a}^o_{21}(x)$, ..., $\bar{a}^o_{(n-2)1}(x)$, etc., the $(n-2)$-th derivatives of $\bar{a}^o_{11}(x)$ and $\bar{a}^o_{21}(x)$ and the $(n-1)$-th derivative of $\bar{a}^o_{11}(x)$ will be present in (4.1), with all of these appearing linearly since $\mathcal{L}$ is a linear operator. Then (4.1) can be rewritten as

$$\begin{aligned}
\mathcal{L}^n[c(x)] &= \sum_{|\alpha|=0} D^\alpha \left[\begin{array}{cccccc} \bar{a}^o_{11}(x) & \bar{a}^o_{21}(x) & \cdots & \cdots & \bar{a}^o_{n1}(x) \end{array}\right] A_\alpha(x) \\
&\quad + \sum_{|\alpha|=1} D^\alpha \left[\begin{array}{ccccc} \bar{a}^o_{11}(x) & \bar{a}^o_{21}(x) & \cdots & \bar{a}^o_{(n-1)1}(x) & 0 \end{array}\right] A_\alpha(x) \\
&\quad + \cdots + \sum_{|\alpha|=n-2} D^\alpha \left[\begin{array}{ccccc} \bar{a}^o_{11}(x) & \bar{a}^o_{21}(x) & 0 & \cdots & 0 \end{array}\right] A_\alpha(x) \\
&\quad + \sum_{|\alpha|=n-1} D^\alpha \left[\begin{array}{ccccc} \bar{a}^o_{11}(x) & 0 & \cdots & \cdots & 0 \end{array}\right] A_\alpha(x).
\end{aligned} \tag{4.25}$$

Here each $A_\alpha(x)$ is an n x n matrix, where the last k rows in $A_\alpha(x)\Big|_{|\alpha|=k}$, $k = 1, 2, \ldots, n-1$, are zero. This represents a system of n $(n-1)$-st-order linear PDEs in n variables and n unknowns.

Note that $\bar{a}^o_{n1}(x)$ only appears in the first term where $|\alpha| = 0$. Thus one of the n equations in (4.25) can be used to express $\bar{a}^o_{n1}(x)$ in terms of the other unknowns and their derivatives. This expression can then be used to eliminate $\bar{a}^o_{n1}(x)$ from the other $n-1$ equations, resulting in a system of $n-1$ $(n-1)$-st-order linear PDEs in n variables and $n-1$ unknowns of the form

$$
\begin{aligned}
D(x) \;=\; & \sum_{|\alpha|=0} D^{\alpha}\left[\begin{array}{cccccc} \bar{a}^{o}_{11}(x) & \bar{a}^{o}_{21}(x) & \cdots & \cdots & \bar{a}^{o}_{(n-1)1}(x) \end{array}\right] B_{\alpha}(x) \\
& + \sum_{|\alpha|=1} D^{\alpha}\left[\begin{array}{cccccc} \bar{a}^{o}_{11}(x) & \bar{a}^{o}_{21}(x) & \cdots & \cdots & \bar{a}^{o}_{(n-1)1}(x) \end{array}\right] B_{\alpha}(x) \\
& + \sum_{|\alpha|=2} D^{\alpha}\left[\begin{array}{ccccc} \bar{a}^{o}_{11}(x) & \bar{a}^{o}_{21}(x) & \cdots & \bar{a}^{o}_{(n-2)1}(x) & 0 \end{array}\right] B_{\alpha}(x) \qquad (4.26) \\
& + \cdots + \sum_{|\alpha|=n-2} D^{\alpha}\left[\begin{array}{ccccc} \bar{a}^{o}_{11}(x) & \bar{a}^{o}_{21}(x) & 0 & \cdots & 0 \end{array}\right] B_{\alpha}(x) \\
& + \sum_{|\alpha|=n-1} D^{\alpha}\left[\begin{array}{ccccc} \bar{a}^{o}_{11}(x) & 0 & \cdots & \cdots & 0 \end{array}\right] B_{\alpha}(x)
\end{aligned}
$$

Here each $B_{\alpha}(x)$ is an $(n-1)$ x $(n-1)$ matrix, where the last $k-1$ rows in $B_{\alpha}(x)\Big|_{|\alpha|=k}$, $k = 1, 2, \ldots, n-1$, are zero and $D(x)$ is a 1 x n vector. Since deriving the elements of $B_{\alpha}(x)$ becomes tedious for higher-order systems, it is recommended that a program with a symbolic mathematics capability (e.g., MACSYMA [95]) be used in such cases.

In general equations of the type (4.26) are quite "rich" in nature, and the corresponding theory covers a wide field; thus a complete coverage is impractical here. The interested reader is therefore referred to the literature, where studies of such equations have been reported; e.g., see Chapter 3 in [90]. Another excellent reference is Courant and Hilbert's classical text, *Methods of Mathematical Physics*, vol. II [96]. For topics related to the reduction of a (system of) higher-order PDE(s) to a (larger) system of first-order PDEs, see Chapter I, §2 and Appendix 2 to Chapter I. Further, see Chapter III and and §3 in Part I and §15 in Part II of Chapter VI, regarding solution approaches.

Special case

An especially simple case of (4.26) arises if $B_{\alpha}(x) = B_{\alpha}$ for $|\alpha| = 0$, a constant matrix, and $D(x) = D$, a constant vector. Then, since one has the freedom to choose any $\bar{a}^o_{i1}(x)$'s, $i = 1, 2, \ldots, n-1$, that satisfy these equations, these quantities can be chosen as constants $\bar{a}^o_{i1}(x) = \bar{a}^o_{i1}$[4] satisfying

$$\begin{bmatrix} \bar{a}^o_{11} & \bar{a}^o_{21} & \cdots & \cdots & \bar{a}^o_{(n-1)1} \end{bmatrix} B_\alpha \Big|_{|\alpha|=0} = D.$$

Then if

$$\rho B_\alpha \Big|_{|\alpha|=0} = n$$

one has

$$\begin{bmatrix} \bar{a}^o_{11} & \bar{a}^o_{21} & \cdots & \cdots & \bar{a}^o_{(n-1)1} \end{bmatrix} = D \left(B_\alpha \Big|_{|\alpha|=0} \right)^{-1}.$$

A number of nonlinear systems and all LTI systems fall into this category (thus the assumption $\bar{a}^o_{i1}(x) = \bar{a}^o_{i1}$ in Example 3.2.2 was justified).

Example 4.1.4 A nonlinear, third-order system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & x_2^2 & 0 \end{bmatrix} x$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x.$$

Here (4.1) becomes

$$\mathcal{L}^3 [c(x)] = \mathcal{L}^2 [\bar{a}^o_{11}(x) c(x)] + \mathcal{L}^1 [\bar{a}^o_{21}(x) c(x)] + \mathcal{L}^0 [\bar{a}^o_{31}(x) c(x)].$$

---

[4] Then

$$D^\alpha \begin{bmatrix} \bar{a}^o_{11}(x) & \bar{a}^o_{21}(x) & \cdots & \cdots & \bar{a}^o_{n1}(x) \end{bmatrix} = 0$$

for $|\alpha| > 0$.

Calculating each term gives

$$\begin{aligned}
\mathcal{L}^0\left[\bar{a}^o_{31}(x)c(x)\right] &= \begin{bmatrix} 0 & \bar{a}^o_{31}(x) & 0 \end{bmatrix},\\
\mathcal{L}^1\left[\bar{a}^o_{21}(x)c(x)\right] &= \frac{\partial}{\partial x}\left(\bar{a}^o_{21}(x)x_2\right)A(x)\\
&= \bar{a}^o_{21}(x)\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}A(x)+\ldots\\
&= \bar{a}^o_{21}(x)\begin{bmatrix} -1 & -1 & 0 \end{bmatrix}+\ldots,
\end{aligned}$$

where "..." denotes terms including first- and higher-order derivatives of $\bar{a}^o_{21}(x)$,

$$\begin{aligned}
\mathcal{L}^2\left[\bar{a}^o_{11}(x)c(x)\right] &= \mathcal{L}^1\left[\frac{\partial}{\partial x}\left(\bar{a}^o_{11}(x)x_2\right)A(x)\right]\\
&= \mathcal{L}^1\left[\bar{a}^o_{11}(x)\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}A(x)+\ldots\right]\\
&= \mathcal{L}^1\left[\bar{a}^o_{11}(x)\begin{bmatrix} -1 & -1 & 0 \end{bmatrix}+\ldots\right]\\
&= \frac{\partial}{\partial x}\left(\bar{a}^o_{11}(x)(-x_1-x_2)\right)A(x)+\ldots\\
&= \bar{a}^o_{11}(x)\begin{bmatrix} -1 & -1 & 0 \end{bmatrix}A(x)+\ldots\\
&= \bar{a}^o_{11}(x)\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}+\ldots.
\end{aligned}$$

Finally $\mathcal{L}^3[c(x)]$ can easily be calculated as

$$\mathcal{L}^3[c(x)] = \begin{bmatrix} 0 & 1-x_2^2 & 1 \end{bmatrix}.$$

Thus the first and the third equation in (4.1) give

$$\begin{aligned}
0 &= \bar{a}^o_{11}(x)-\bar{a}^o_{21}(x)+\ldots\\
1 &= -\bar{a}^o_{11}(x)+\ldots,
\end{aligned}$$

giving

$$B_\alpha\Big|_{|\alpha|=0} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \bar{a}^o_{11}(x) & \bar{a}^o_{21}(x) \end{bmatrix} = \begin{bmatrix} \bar{a}^o_{11} & \bar{a}^o_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \right)^{-1}$$
$$= \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

It can be easily verified from (3.19), (3.10a) and (3.10c) that this gives

$$Q^o(x) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\bar{A}^o(x) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -x_2^2 & 0 & 0 \end{bmatrix}$$

and

$$\bar{c}^o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

4.1.4 Discussion

In this section various computational aspects of (4.1) were discussed. A solution was given for the case $n = 1$. For $n = 2$, a first-order linear PDE in two variables and one unknown $\bar{a}^o_{11}(x)$ was derived and a general solution method was described. Then two special cases were discussed where finding a solution is especially easy. For $n > 2$, a system of $n-1$ $(n-1)$-st-order linear PDEs in n variables and $n-1$ unknowns was derived. However, in this case, the variety of situations that can arise is large; thus a complete coverage is beyond the scope of this work. A special case where a solution can be found easily was discussed.

4.2 Transformation to controller form

In this case (3.34) must be solved, rewritten here as (4.27)

$$\frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} C^{-1}(x) \tag{4.27}$$

or

$$\frac{\partial}{\partial x}(T(x)) = h(x) \tag{4.28}$$

where $T(x)$ is a function

$$T(x) = q^c_{n\cdot}(x)x \tag{4.29}$$

and $h(x)$ is a vector

$$h(x) = \begin{bmatrix} 0 & \cdots & \cdots 0 & 1 \end{bmatrix} C^{-1}(x) = \begin{bmatrix} h_1(x) & h_2(x) & \cdots & \cdots & h_n(x) \end{bmatrix}.$$

Eqn. (4.28) represents a system of n first-order, linear PDEs in n variables and one unknown of the form

$$\frac{\partial(T(x))}{\partial x_i} = h_i(x) \qquad i = 1, 2, \ldots, n. \tag{4.30}$$

If such a $T(x)$ exists then the $\frac{1}{2}n(n-1)$ conditions

$$\frac{\partial^2(T(x))}{\partial x_i \partial x_j} = \frac{\partial^2(T(x))}{\partial x_j \partial x_i} \tag{4.31}$$

for $i = 1, 2, \ldots, n$, $j = i+1, i+2, \ldots, n$ or equivalently

$$\frac{\partial\left(h_j(x)\right)}{\partial x_i} = \frac{\partial\left(h_i(x)\right)}{\partial x_j} \tag{4.32}$$

must hold. These conditions are necessary and sufficient conditions for (4.28) to have a solution (p. 45, [91]). This solution is given by

$$T(x) = \int_{x_0}^{x} \left(h_1(x)dx_1 + h_2(x)dx_2 + \cdots + h_n(x)dx_n\right) + C, \tag{4.33}$$

where C is an arbitrary constant. This integration is performed by first finding all $T_1(x), T_2(x), \ldots, T_n(x)$ that satisfy (4.30) or equivalently

$$\begin{aligned} T_1(x) &= \int h_1(x) dx_1 \\ T_2(x) &= \int h_2(x) dx_1 \\ &\vdots \\ T_n(x) &= \int h_n(x) dx_1. \end{aligned}$$

Then the $T(x)$ is formed from those $T_i(x)$ that also satisfy (4.31). Once $T(x)$ is found, $q^c_{n\cdot}(x)$ can be found that satisfies (4.29). This $q^c_{n\cdot}(x)$ can be chosen in many different ways, however, it must be chosen such that $Q^c(x)$ and $\frac{\partial}{\partial x}\left(Q^c(x)x\right)$ are of full rank.

Example 4.2.1 A nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} -1 & -x_1^2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Here $\mathcal{C}(x)$ becomes from (3.33)

$$\mathcal{C}(x) = \begin{bmatrix} 0 & -x_1^2 \\ 1 & -1 \end{bmatrix},$$

thus

$$h(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1}(x) = \begin{bmatrix} -x_2^{-2} & 0 \end{bmatrix}.$$

Here (4.32) is satisfied since

$$\frac{\partial (h_1(x))}{\partial x_2} = \frac{\partial (h_2(x))}{\partial x_1} = 0.$$

Thus a solution $T(x)$ exists and

$$T_1(x) = x_1^{-1} + C_1$$

$$T_2(x) = C_2$$

where C_1 and C_2 are arbitrary constants. Here

$$\frac{\partial^2}{\partial x_2 \partial x_1} T_1(x) = 0$$

and

$$\frac{\partial^2}{\partial x_1 \partial x_2} T_2(x) = 0.$$

Thus

$$T(x) = x_1^{-1} + C,$$

where C is an arbitrary constant. One possible $q_{2\cdot}^c(x)$ is (taking $C = 1$)

$$q_{2\cdot}^c(x) = \left[\; x_1^{-2} + x_1^{-1} \quad 0 \;\right].$$

Then it can easily be verified from (3.35), (3.10a) and (3.10b) that

$$Q^c(x) \begin{bmatrix} x_1^{-2} & 1 \\ x_1^{-2} + x_1^{-1} & 0 \end{bmatrix},$$

$$\bar{A}^c(x) = \begin{bmatrix} 0 & \frac{x_1^2+1}{x_1+1} \\ 1 & 0 \end{bmatrix}$$

and

$$\bar{b}^c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In cases where (4.32) does not hold, it may be possible to find an integrating factor, $\mu(x)$, such that

$$\frac{\partial\left(\mu(x)h_j(x)\right)}{\partial x_i}=\frac{\partial\left(\mu(x)h_i(x)\right)}{\partial x_j} \tag{4.34}$$

for $i=1,2,\ldots,n$, $j=i+1,i+2,\ldots,n$. This is equivalent to solving the problem

$$\frac{\partial}{\partial x}\left(q^c_{n\cdot}(x)x\right)=\left[\begin{array}{cccccc}0 & \cdots & \cdots & 0 & \mu(x)\end{array}\right]C^{-1}(x)$$

or equivalently, if b^c in (2.26) is replaced by

$$b^c\left(x^c\right)=\left[\begin{array}{c}\mu\left(x^c\right)\\ 0\\ \vdots\\ 0\end{array}\right]. \tag{4.35}$$

This is acceptable and (2.29) still results if (2.27) is replaced by

$$u=-\mu^{-1}\left(x^c\right)k^c\left(x^c\right)x^c+v$$

assuming $\mu^{-1}\left(x^c\right)$ does not have singularities in critical areas of the state space. This is the approach taken by Sommer [68].

If one such integrating factor exists, then an infinite number of integrating factors exists [92]. Further, if the following $\frac{1}{6}n(n-1)(n-2)$ identities are satisfied, the existence of integrating factors is guaranteed (p. 4-6 [93]),

$$\begin{aligned}h_k(x)&\left(\frac{\partial h_i(x)}{\partial x_j}-\frac{\partial h_j(x)}{\partial x_i}\right)\\ &+h_i(x)\left(\frac{\partial h_j(x)}{\partial x_k}-\frac{\partial h_k(x)}{\partial x_j}\right)\\ &+h_j(x)\left(\frac{\partial h_k(x)}{\partial x_i}-\frac{\partial h_i(x)}{\partial x_k}\right)=0,\end{aligned} \tag{4.36}$$

where $i=1,2,\ldots,n$, $j=i+1,i+2,\ldots,n$, $k=j+1,j+2,\ldots,n$. Only $\frac{1}{2}(n-1)(n-2)$ of these are independent. Note that when $n=2$ integrating factors always exist (pp. 56-58, [94]).

Example 4.2.2 A nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} -1 & -1 - x_2^2 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Here $\mathcal{C}(x)$ becomes

$$\mathcal{C}(x) = \begin{bmatrix} 0 & -1 - 3x_2^2 \\ 1 & 1 \end{bmatrix},$$

thus

$$h(x) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1}(x) = \begin{bmatrix} \frac{-1}{1+3x_2^2} & 0 \end{bmatrix}.$$

Here

$$\frac{\partial (h_1(x))}{\partial x_2} = \left(1 + 3x_2^2\right)^{-2} 6x_2$$

and

$$\frac{\partial (h_2(x))}{\partial x_1} = 0.$$

However, if

$$\mu(x) = 1 + 3x_2^2$$

then

$$\frac{\partial (\mu(x) h_1(x))}{\partial x_2} = 0$$

and

$$\frac{\partial (\mu(x) h_2(x))}{\partial x_1} = 0.$$

Then one may proceed to solve for $T(x)$ as before.

4.2.1 Discussion

In this section computational aspects of (4.27) were discussed. Necessary and sufficient conditons for the existence of a solution and an explicit solution were stated. Further, the possibility of solving (4.27) through the use of an integrating factor and conditions that guarantee the existence of such factors were discussed. In cases when (4.27) can not be solved directly and no integrating factors exist, it may be concluded that neither a transformation to the defined controller form nor the modified one (where $b^c(x^c)$ is given by (4.35)), exists.

CHAPTER V

SIMULATION STUDIES

Simulation studies, undertaken to demonstrate the effectiveness of the methods developed in this work, are described in this chapter. For this purpose, many abstract examples are possible; however, in view of the rather theoretical nature of this document, it was considered preferable to choose one that has practical applicability. Thus, an empirically derived, nonlinear, model of a vehicle's longitudinal dynamics [97] is employed, and a nonlinear observer/controller is designed. The latter has the same closed-loop dynamics as in [98][1], where nonlinear compensation, which employed parameter-scheduling and a "linear" observer/controller, was employed.

5.1 System in general form

The longitudinal dynamics of an automobile can be represented as [97][2]

[1]This choice was made for comparison purposes.

[2]Strictly speaking, since the model is nonlinear, the differential operator $p = d/dt$ should be used in place of the Laplace variable s. However, it is assumed that velocity-dependent parameters vary slowly as functions of velocity and therefore the approximation

$$p(a(V)V) = a(V)pV + Vpa(V) \approx a(V)pV$$

can be made, justifying the use of s rather than p.

$$\frac{V_w(s)}{V_i(s)} = \frac{k_p(V)}{\left(t_p(V)s+1\right)(s+k_l(V))} \tag{5.1}$$

and

$$\frac{V(s)}{V_w(s)} = \frac{\xi(V)}{s+\xi(V)}. \tag{5.2}$$

Here

V_i : a voltage applied to an electrohydraulic actuator, which controls the position of the throttle valve,

$k_p(V)$: a function associated with the throttle actuator and the propulsion system,

$t_p(V)$: a function associated with the propulsion system and its interaction with the roadway interface,

$k_l(V)$: a function associated with aerodynamic drag and vehicle mass,

V_w : speed of the rear wheels,

$\xi(V)$: a function associated with the tire-roadway interface,

V : vehicle speed in an inertial frame of reference.

It has been shown experimentally [97] that

$$\begin{aligned} k_l(V) &\approx 0.05 \\ \frac{k_p(V)}{t_p(V)} &\approx 1 \end{aligned} \tag{5.3}$$

for $0 \leq V \leq 30.5m/s$—the speed range of interest. Further, a functional representation is needed for $t_p(V)$ and $\xi(V)$. Here these were chosen as

$$\xi(V) = \frac{12}{1+0.25V} \tag{5.4}$$

and

$$t_p(V) = \frac{1.2}{1 + 0.50V}. \tag{5.5}$$

These functions approximate the experimental curves [99] adequately for the purpose of this study, further their simplicity is to advantage here[3].

Combining (5.1) and (5.2) and using (5.3) results in

$$\frac{V(s)}{V_i(s)} = \frac{\xi(V)}{\left(s + 1/t_p(V)\right)(s + 0.05)(s + \xi(V))}.$$

or

$$\frac{V(s)}{V_i(s)} = \frac{\xi(V)}{s^3 + a_1(V)s^2 + a_2(V)s + a_3(V)}, \tag{5.6}$$

where

$$\begin{aligned} a_1(V) &= 1/t_p(V) + \xi(V) + 0.05 \\ a_2(V) &= 0.05\xi(V) + 1/t_p(V)\left(\xi(V) + 0.05\right) \\ a_3(V) &= 0.05\xi(V)/t_p(V). \end{aligned}$$

Then using (5.4) and (5.5) these become

$$\begin{aligned} a_1(V) &= 0.4167\tfrac{V^2+6.12V+123.68}{V+4} \\ a_2(V) &= 0.0208\tfrac{V^2+966V+2043.2}{V+4} \\ a_3(V) &= \tfrac{V+2}{V+4}. \end{aligned} \tag{5.7}$$

Since an observer/controller configuration does not affect the "zeros" of a system, the nonlinear term $\xi(V)$ in the nominator of $V(s)/V_i(s)$ was cancelled by inverse compensation $\beta/\xi(V)$ so that the same closed-loop dynamics as in [98] could be achieved. This results in

$$\frac{V(s)}{u(s)} = \frac{\beta}{s^3 + a_1(V)s^2 + a_2(V)s + a_3(V)}, \tag{5.8}$$

[3]One could get a more accurate approximation, e.g., by a least-square error technique, however, most certainly at the cost of increased analytical complexity.

where

$$u = \frac{\beta}{\xi(V)} V_i$$

and

$$\beta = \frac{1}{2}\xi(V)_{max} \approx 6.$$

Now an arbitrary state-space representation is chosen[4] e.g., (here $y = V = x_1$)

$$\begin{aligned} \dot{x} &= A(x)x + b(x)u = \begin{bmatrix} -a_1(x_1) & 0 & 1 \\ -a_3(x_1) & 0 & 0 \\ -a_2(x_1) & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} u \\ y &= c(x)x = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x. \end{aligned} \tag{5.9}$$

It can easily be shown that this state-space representation corresponds to (5.8) by calculating[5]

$$\begin{aligned} \frac{Y(s)}{U(s)} &= c(x)(sI - A(x))^{-1} b(x) \\ &= \frac{1}{s^3 + a_1(x_1)s^2 + a_2(x_1)s + a_3(x_1)} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & 1 & - \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} 0 \\ \beta \\ 0 \end{bmatrix} \\ &= \frac{\beta}{s^3 + a_1(x_1)s^2 + a_2(x_1)s + a_3(x_1)}. \end{aligned}$$

---

[4]In this case it is possible to find the observer form directly from (5.8), since the nonlinearities are functions of the output, which is one of the state variables in that form. Here, for demonstration purposes, it was opted to begin with the system in some general form which was sufficiently "close" to the observer form so that the transformation to that form could be easily found.

[5]This formula does not hold for nonlinear systems unless, as is the case here, the nonlinear, state-space representation was derived from a transfer function.

5.2 Transformation from general form to observer form

Here (3.18) becomes

$$\mathcal{L}^3\left[c(x)\right] = \mathcal{L}^2\left[\bar{a}^o_{11}(x)c(x)\right] + \mathcal{L}^1\left[\bar{a}^o_{21}(x)c(x)\right] + \mathcal{L}^0\left[\bar{a}^o_{31}(x)c(x)\right].$$

Generally—even though the elements of $A(x)$ are functions of e.g. x_1 alone, the elements of $\bar{A}^o(x)$ will not be functions of x_1 alone. However, the assumption

$$\begin{aligned}
\bar{a}^o_{11}(x) &= \bar{a}^o_{11}(x_1) \\
\bar{a}^o_{21}(x) &= \bar{a}^o_{21}(x_1) \\
\bar{a}^o_{31}(x) &= \bar{a}^o_{31}(x_1)
\end{aligned}$$

is made, since it results in considerable simplification[6]. Calculating each term gives:

$$\begin{aligned}
&\mathcal{L}^3\left[c(x)\right] \\
&= \Bigg[-a_1(x_1)\frac{\partial}{\partial x_1}\left\{\left(a_1(x_1)\frac{\partial}{\partial x_1}(a_1(x_1)x_1) - a_2(x_1)\right)x_1\right. \\
&\qquad \left. -\frac{\partial}{\partial x_1}(a_1(x_1)x_1)x_3\right\} - a_3(x_1) + a_2(x_1)\frac{\partial}{\partial x_1}(a_1(x_1)x_1) \\
&\quad -\frac{\partial}{\partial x_1}(a_1(x_1)x_1) \\
&\quad \frac{\partial}{\partial x_1}\left\{\left(a_1(x_1)\frac{\partial}{\partial x_1}(a_1(x_1)x_1) - a_2(x_1)\right)x_1 - \frac{\partial}{\partial x_1}(a_1(x_1)x_1)x_3\right\}\Bigg],
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}^2\left[\bar{a}^o_{11}(x_1)c(x)\right] \\
&= \Bigg[-a_1(x_1)\frac{\partial}{\partial x_1}\left\{\left(-a_1(x_1)\frac{\partial}{\partial x_1}(\bar{a}^o_{11}(x_1)x_1)\right)x_1\right. \\
&\qquad \left. +\frac{\partial}{\partial x_1}(a^o_{11}(x_1)x_1)x_3\right\} - a_2(x_1)\frac{\partial}{\partial x_1}(\bar{a}^o_{11}(x_1)x_1) \\
&\quad \frac{\partial}{\partial x_1}(\bar{a}^o_{11}(x_1)x_1) \\
&\quad \frac{\partial}{\partial x_1}\left\{\left(-a_1(x_1)\frac{\partial}{\partial x_1}(\bar{a}^o_{11}(x_1)x_1)\right)x_1 + \frac{\partial}{\partial x_1}(\bar{a}^o_{11}(x_1)x_1)x_3\right\}\Bigg],
\end{aligned}$$

[6]If a solution cannot be found with this assumption, it must be removed.

$$\mathcal{L}^1\left[\bar{a}^o_{21}(x_1)\,c(x)\right]$$
$$= \begin{bmatrix} -a_1(x_1)\dfrac{\partial}{\partial x_1}\left(\bar{a}^o_{21}(x_1)\,x_1\right) & 0 & \dfrac{\partial}{\partial x_1}\left(\bar{a}^o_{21}(x_1)\,x_1\right) \end{bmatrix},$$

and

$$\mathcal{L}^0\left[\bar{a}^o_{31}(x_1)\,c(x)\right]$$
$$= \begin{bmatrix} \bar{a}^o_{31}(x_1) & 0 & 0 \end{bmatrix}.$$

This results in three equations, the second of which is

$$-\frac{\partial}{\partial x_1}\left(a_1(x_1)\,x_1\right) = \frac{\partial}{\partial x_1}\left(\bar{a}^o_{11}(x_1)\,x_1\right),$$

thus

$$\bar{a}^o_{11}(x_1) = -a_1(x_1). \tag{5.10}$$

The third equation then becomes

$$\begin{aligned} &\frac{\partial}{\partial x_1}\left\{\left(a_1(x_1)\frac{\partial}{\partial x_1}\left(a_1(x_1)\,x_1\right) - a_2(x_1)\right)x_1 - \frac{\partial}{\partial x_1}\left(a_1(x_1)\,x_1\right)x_3\right\} \\ &= \frac{\partial}{\partial x_1}\left\{a_1(x_1)\frac{\partial}{\partial x_1}\left(a_1(x_1)\,x_1\right)x_1 - \frac{\partial}{\partial x_1}\left(a_1(x_1)\,x_1\right)x_3\right\} \\ &\quad + \frac{\partial}{\partial x_1}\left(\bar{a}^o_{21}(x_1)\,x_1\right), \end{aligned}$$

or

$$-\frac{\partial}{\partial x_1}\left(a_2(x_1)\,x_1\right) = \frac{\partial}{\partial x_1}\left(\bar{a}^o_{21}(x_1)\,x_1\right);$$

thus

$$\bar{a}^o_{21}(x_1) = -a_2(x_1). \tag{5.11}$$

Similarly the first equation becomes, using (5.10) and (5.11),

$$\bar{a}^o_{31}(x_1) = -a_3(x_1).$$

Then it can easily be verified using (3.19), (3.10a), (3.10b) and (3.10c) and $x_1^o = x_1$ that

$$Q^o(x) = Q^o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$A^o(x^o) = \begin{bmatrix} -a_1(x_1^o) & 1 & 0 \\ -a_2(x_1^o) & 0 & 1 \\ -a_3(x_1^o) & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11}^o(x_1^o) & 1 & 0 \\ a_{21}^o(x_1^o) & 0 & 1 \\ a_{31}^o(x_1^o) & 0 & 0 \end{bmatrix},$$

$$b^o(x^o) = \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix} \tag{5.12}$$

and

$$c^o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Here, from (5.7) ($V = x_1$)

$$\begin{aligned} a_{11}^o(x_1^o) &= -a_1(x_1^o) = -0.4167\frac{x_1^{o^2}+6.12x_1^o+123.68}{x_1^o+4} \\ a_{21}^o(x_1^o) &= -a_2(x_1^o) = -0.0208\frac{x_1^{o^2}+966x_1^o+2043.2}{x_1^o+4} \\ a_{31}(x_1^o) &= -a_3(x_1^o) = -\frac{x_1^o+2}{x_1^o+4}. \end{aligned} \tag{5.13}$$

These equations, (5.12) and (5.13), specify the desired observer form.

5.3 Transformation from observer form to controller form

This transformation can be effected by first employing (3.34) in the form

$$\frac{\partial}{\partial x}(q_{3\cdot}^c(x)x) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} C^{-1}(x),$$

which using (3.33) and some calculation results in

$$\frac{\partial}{\partial x}\left(q_{3\cdot}^{c}(x)x\right) = \beta^{-1}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

One possible solution is

$$q_{3\cdot}^{c}\left(x^{o}\right) = q_{3\cdot}^{c} = \beta^{-1}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then $q_{2\cdot}^{c}\left(x^{o}\right)$ and $q_{1\cdot}^{c}\left(x^{o}\right)$ can be calculated from (3.35) resulting in

$$q_{2\cdot}^{c}\left(x^{o}\right) = \beta^{-1}\begin{bmatrix} a_{11}^{o}\left(x_{1}^{o}\right) & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &q_{1\cdot}^{c}\left(x^{o}\right) \\ &= \beta^{-1}\begin{bmatrix} a_{11}^{o}\left(x_{1}^{o}\right)\frac{\partial}{\partial x_{1}}\left(a_{11}^{o}\left(x_{1}^{o}\right)x_{1}^{o}\right) + a_{21}^{o}\left(x_{1}^{o}\right) & \frac{\partial}{\partial x_{1}}\left(a_{11}^{o}\left(x_{1}^{o}\right)x_{1}^{o}\right) & 1 \end{bmatrix}. \end{aligned}$$

Thus

$$Q^{c}\left(x^{o}\right) = \beta^{-1}\begin{bmatrix} g_{1}\left(x_{1}^{o}\right) & g_{3}\left(x_{1}^{o}\right) & 1 \\ g_{2}\left(x_{1}^{o}\right) & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{5.14}$$

where

$$\begin{aligned} g_{1}\left(x_{1}^{o}\right) &= a_{11}^{o}\left(x_{1}^{o}\right)\frac{\partial}{\partial x_{1}}\left(a_{11}^{o}\left(x_{1}^{o}\right)x_{1}^{o}\right) + a_{21}^{o}\left(x_{1}^{o}\right) \\ g_{2}\left(x_{1}^{o}\right) &= a_{11}^{o}\left(x_{1}^{o}\right) \\ g_{3}\left(x_{1}^{o}\right) &= \frac{\partial}{\partial x_{1}}\left(a_{11}^{o}\left(x_{1}^{o}\right)x_{1}^{o}\right). \end{aligned}$$

Using (5.13), $g_{1}\left(x_{1}^{o}\right)$ and $g_{3}\left(x_{1}^{o}\right)$ become

$$g_{1}\left(x_{1}^{o}\right) = 0.3472\frac{{x_{1}^{o}}^{5}+15.12{x_{1}^{o}}^{4}+145.167{x_{1}^{o}}^{3}+930.487{x_{1}^{o}}^{2}+2633.434x_{1}^{o}+28632.01}{\left(x_{1}^{o}+4\right)^{3}}$$

and (5.15)

$$g_{3}\left(x_{1}^{o}\right) = -0.8333\frac{{x_{1}^{o}}^{3}+9.06{x_{1}^{o}}^{2}+24.48x_{1}^{o}+247.36}{\left(x_{1}^{o}+4\right)^{2}}.$$

Calculating $Q^{c^{-1}}(x^o)$ and $\frac{\partial(Q^c(x^o)x^o)}{\partial x^o}$ gives

$$Q^{c^{-1}}(x^o) = \beta \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -g_2\left(x_1^o\right) \\ 1 & -g_3\left(x_1^o\right) & -a_{21}^o\left(x_1^o\right) \end{bmatrix}$$

and

$$\frac{\partial\left(Q^c(x^o)\,x^o\right)}{\partial x^o} = \beta^{-1} \begin{bmatrix} \frac{\partial}{\partial x_1}\left(g_1\left(x_1^o\right)x_1^o + g_3\left(x_1^o\right)x_2^o\right) & g_3\left(x_1^o\right) & 1 \\ \frac{\partial}{\partial x_1}\left(g_2\left(x_1^o\right)x_1^o\right) & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then using (3.10a), (3.10b) and (3.10c) gives

$$\bar{A}^c(x^o) = \begin{bmatrix} \bar{a}_{11}^c(x^o) & \bar{a}_{12}^c(x^o) & \bar{a}_{13}^c(x^o) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\bar{b}^c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and

$$\bar{c}^c(x^o) = \begin{bmatrix} 0 & 0 & \beta \end{bmatrix},$$

where

$$\bar{a}_{11}^c(x^o) = g_3\left(x_1^o\right)$$
$$\bar{a}_{12}^c(x^o) = \frac{\partial}{\partial x_1}\left(g_1\left(x_1^o\right)x_1^o + g_3\left(x_1^o\right)x_2^o\right) - g_3^2\left(x_1^o\right)$$
$$\bar{a}_{13}^c(x^o) = a_{31}^o\left(x_1^o\right).$$

Using (5.13) and (5.15) here results in

$$\begin{aligned}
\bar{a}^c_{11}(x^o) &= -0.8333\frac{x_1^{o^3} + 9.06x_1^{o^2} + 24.48x_1^o + 247.36}{\left(x_1^o + 4\right)^2}, \\
\bar{a}^c_{12}(x^o) &= 0.3472\left(x_1^{o^6} + 18x_1^{o^5} + 185.48x_1^{o^4} + 466.08x_1^{o^3} - 1630.46x_1^{o^2}\right. \\
&\qquad \left.-60418.22x_1^o - 7845.89\right) / \left(x_1^o + 4\right)^4 \\
&\quad -0.8333\frac{x_1^{o^3} + 12x_1^{o^2} + 48x_1^o - 396.8}{\left(x_1^o + 4\right)^3}x_2^o,
\end{aligned} \tag{5.16}$$

and

$$\bar{a}^c_{13}(x^o) = -\frac{x_1^o + 2}{x_1^o + 4}.$$

Using (5.14) in $x^c = Q^c(x^o)\, x^o$ gives

$$x_3^c = \beta^{-1}x_1^o,$$

and

$$x_2^c = \beta^{-1}\left(g_2\left(x_1^o\right)x_1^o + x_2^o\right);$$

thus,

$$x_1^o = \beta x_3^c = 6x_3^c$$

and

$$x_2^o = \beta x_2^c - g_2\left(\beta x_3^c\right)\beta x_3^c = 6x_2^c + 15\frac{x_3^{c^3} + 1.02x_3^{c^2} + 3.44x_3^c}{x_3^c + 0.67}.$$

Substituting these quantities in (5.16) gives

$$A^c(x^c) = \begin{bmatrix} a^c_{11}(x^c) & a^c_{12}(x^c) & a^c_{13}(x^c) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$b^c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{5.17}$$

and

$$c^c(x^c) = \begin{bmatrix} 0 & 0 & 6 \end{bmatrix},$$

where

$$\begin{aligned} a^c_{11}(x^c) &= -5\frac{{x^c_3}^3 + 1.51{x^c_3}^2 + 0.68x^c_3 + 1.145}{(x^c_3+0.67)^2}, \\ a^c_{12}(x^c) &= -5x^c_2\frac{{x^c_3}^3 + 2{x^c_3}^2 + 1.33x^c_3 - 1.837}{(x^c_3+0.67)^3} \\ &\quad -0.25\frac{{x^c_3}^5 + 82.85{x^c_3}^4 + 216.48{x^c_3}^3 + 198.29{x^c_3}^2 + 72.94x^c_3 + 8.41}{(x^c_3+0.67)^4}, \end{aligned} \tag{5.18}$$

and

$$a^c_{13}(x^c) = -\frac{x^c_3+0.33}{x^c_3+0.67}.$$

These equations, (5.17) and (5.18) specify the desired controller form.

5.4 Observer/controller design

The control system to be designed is shown in Fig. 4, where the nonlinear vehicle dynamics/inverse compensator correspond to the state-space representation of Eqn. (5.9). This is a position controller in which X_r is the reference position and X is the actual vehicle position. The position error, X_e, is the difference between these positions and is to remain as small as possible.

The pole-placement was such that

$$\frac{V(s)}{C(s)} = \frac{1600}{s^3 + 30.034s^2 + 241.384s + 809.68}, \tag{5.19}$$

where C is defined in Fig. 4. The feedback gains were chosen as in (2.28), where

$$k^c_1 = 30.034$$
$$k^c_2 = 241.384$$
$$k^c_3 = 809.68.$$

The observer gains were chosen as in (2.7), where l^o_1, l^o_2 and l^o_3 were selected corresponding to observer poles at $-6, -6, -6$ (a similar choice was made in [98]

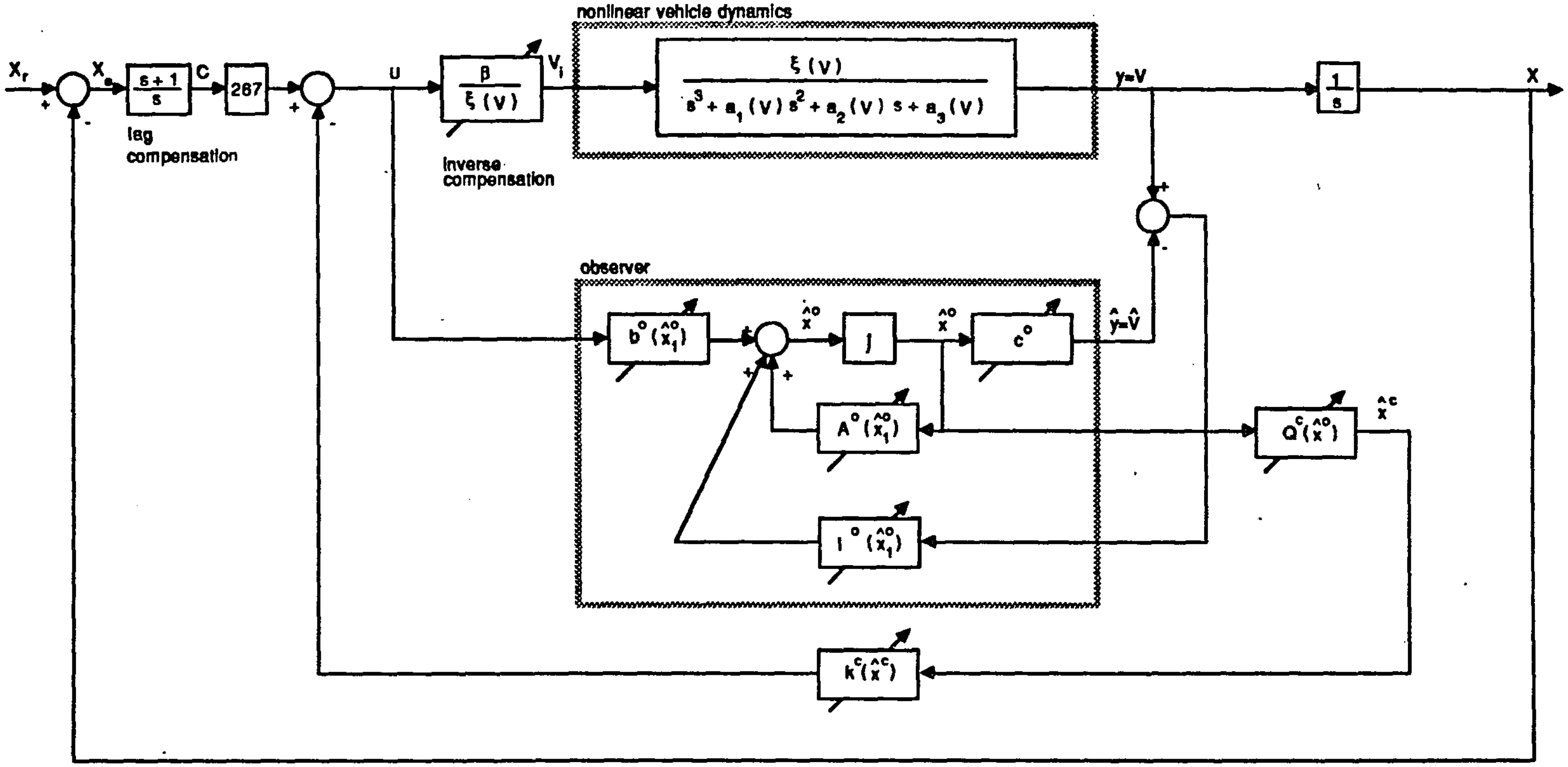

Figure 4: Vehicle longitudinal control system

where a reduced-order observer had 2 poles at -6). Finally, the "outer loop" of the control system was chosen as in [98], i.e., a lag compensator and a gain.

5.5 Simulation results

The performance of the system was evaluated by a digital simulation (see Appendix C for a program listing). The system of Fig. 4 was excited with the command input given by

$$X_r = 0.65t^2 - 0.0045t^3 \, m \qquad 0 \leq t \leq 30s$$
$$X_r = 463.5 + 26.85(t - 30.) \, m \quad t > 30s.$$

This large-signal input is used when merging a vehicle from standstill into mainline traffic and encompasses virtually the entire speed range of interest. For this input several different cases were examined and $X_e(t)$ was obtained.

5.5.1 Observer—general case

Here the observer was constructed as in (2.3), i.e., $\hat{x}^o$ was used in $A^o(\hat{x}^o)$, $b^o(\hat{x}^o)$ and $l^o(\hat{x}^o)$. Simulation results for two different sets of observer initial conditions:

Case I. Observer poles: $-6, -6, -6$; $\hat{x}_1^o(0) = 0$, $\hat{x}_2^o(0) = 0$, $\hat{x}_3^o(0) = 0$,

and

Case II. Observer poles: $-6, -6, -6$; $\hat{x}_1^o(0) = 2$, $\hat{x}_2^o(0) = 4$, $\hat{x}_3^o(0) = -3$,

are shown in Fig. 5. In Case I, the observer reconstructs the system states (whose initial conditions are also zero) perfectly throughout the simulation. Effectively, this is equivalent to using the real system states for state-feedback. In Case II, the peak-position error is slightly higher than in Case I but the system recovers in 2-3 seconds and thereafter follows the trajectory of that case. These results compare

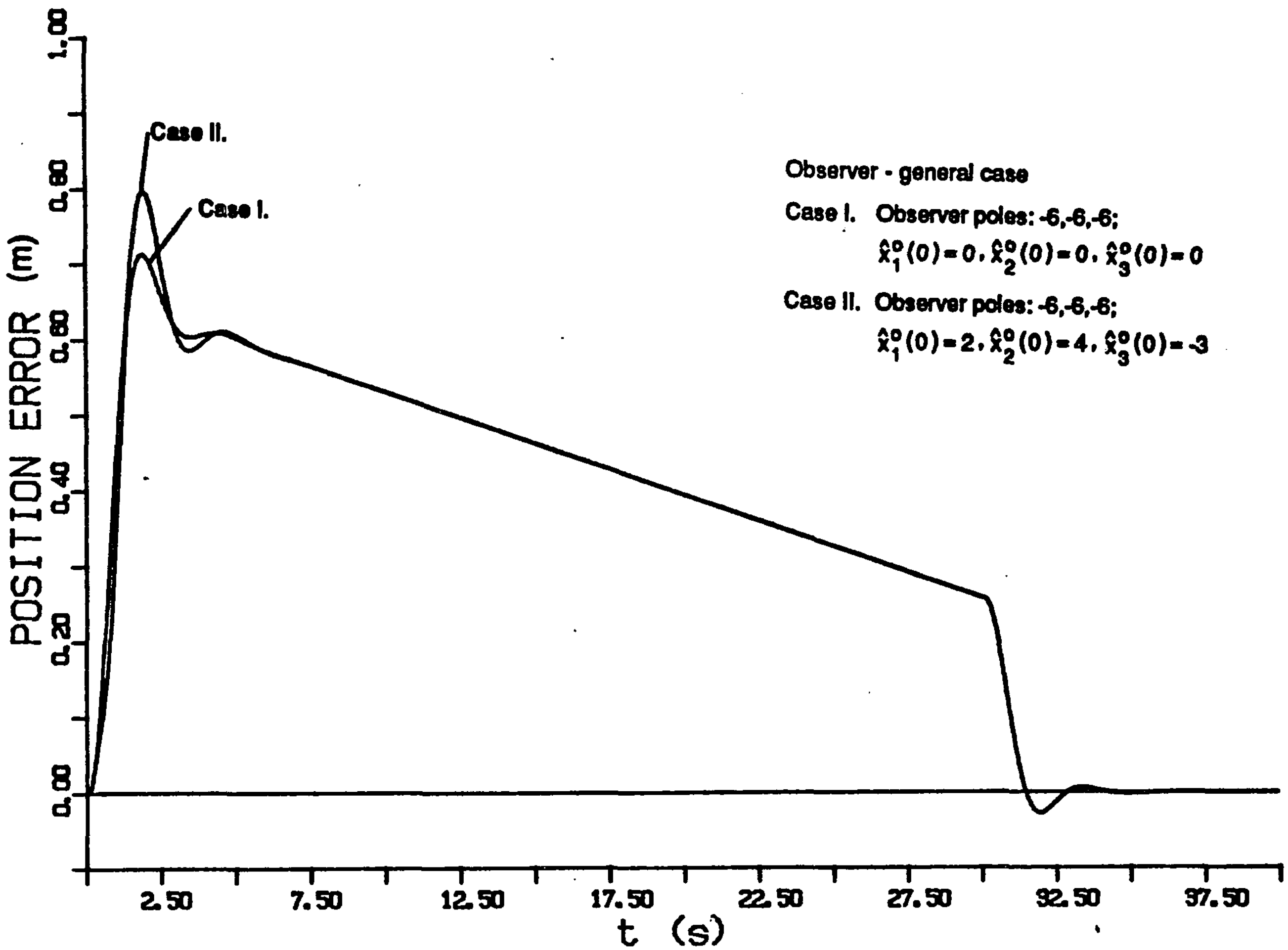

Figure 5: Position error for different observer initial conditions

favorably to those in [98][7]; however, the latter show discretized effects because of the less-accurate, parameter scheduling technique employed. These results are also in close correspondence with those obtained from field studies [99], where a classical design with approximately the same poles as the present one, was evaluated. Thus the present design is judged to be physically realizable.

The state errors ($e^o = x^o - \hat{x}^o$) in Case II are shown on an expanded time-scale in Fig. 6, from which may be observed that these errors die out in less than two seconds.

The effect of moving the observer poles closer to the $j\omega$-axis while keeping the initial conditions the same as in Case II, were evaluated in Case III.

Case III. Observer poles: $-1, -1, -1$; $\hat{x}_1^o(0) = 2$, $\hat{x}_2^o(0) = 4$, $\hat{x}_3^o(0) = -3$.

The simulation results for this case are is shown in Fig. 7, where Case II is also depicted. Note the performance degradation in Case III due to the slower reconstruction of the states[8]; however, the system recovers within 10s and thereafter follows the desired trajectory.

5.5.2 Observer—special case

Since the system falls into the category of output-dependent nonlinearities, the observer can be constructed using (2.22), i.e., y is used in $A^o(y)$, $b^o(y)$ and $l^o(y)$. The cases discussed in 5.5.1 are shown in Figs. 8, 9 and 10, respectively. Generally the results were identical when the observer poles were $-6, -6, -6$ (Fig. 8 is identical to Fig. 5, Fig. 9 is identical to Fig. 6). In Case III, which is shown

[7] I.e., see Fig. 2.11, p. 18 (digital simulation results) and Fig. 5.1b, p.44 (hybrid simulation results).

[8] Here the observer poles are approximately the same "speed" as the system closed-loop dominant poles ($-1.04 \pm j2.16$).

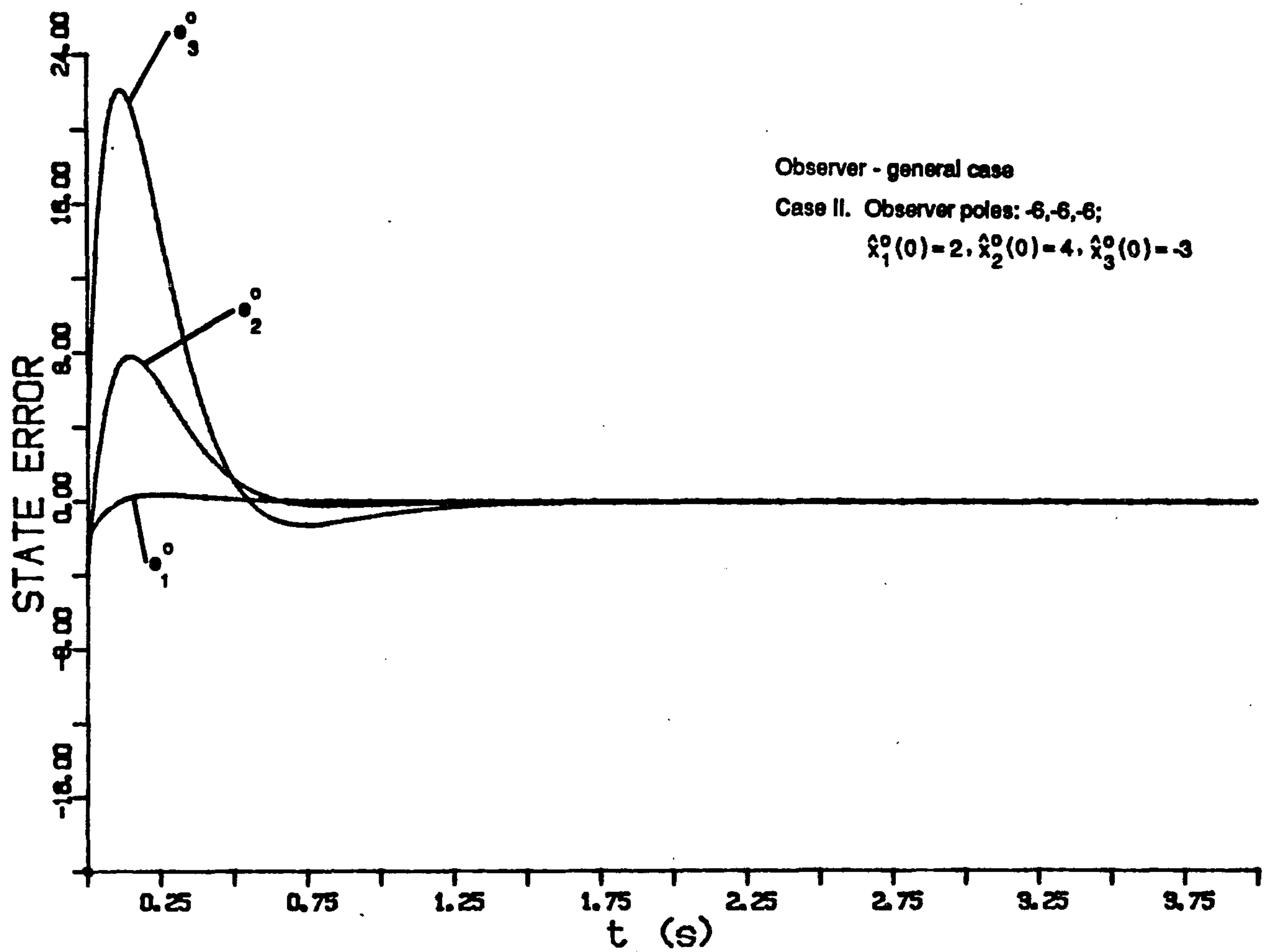

Figure 6: State error

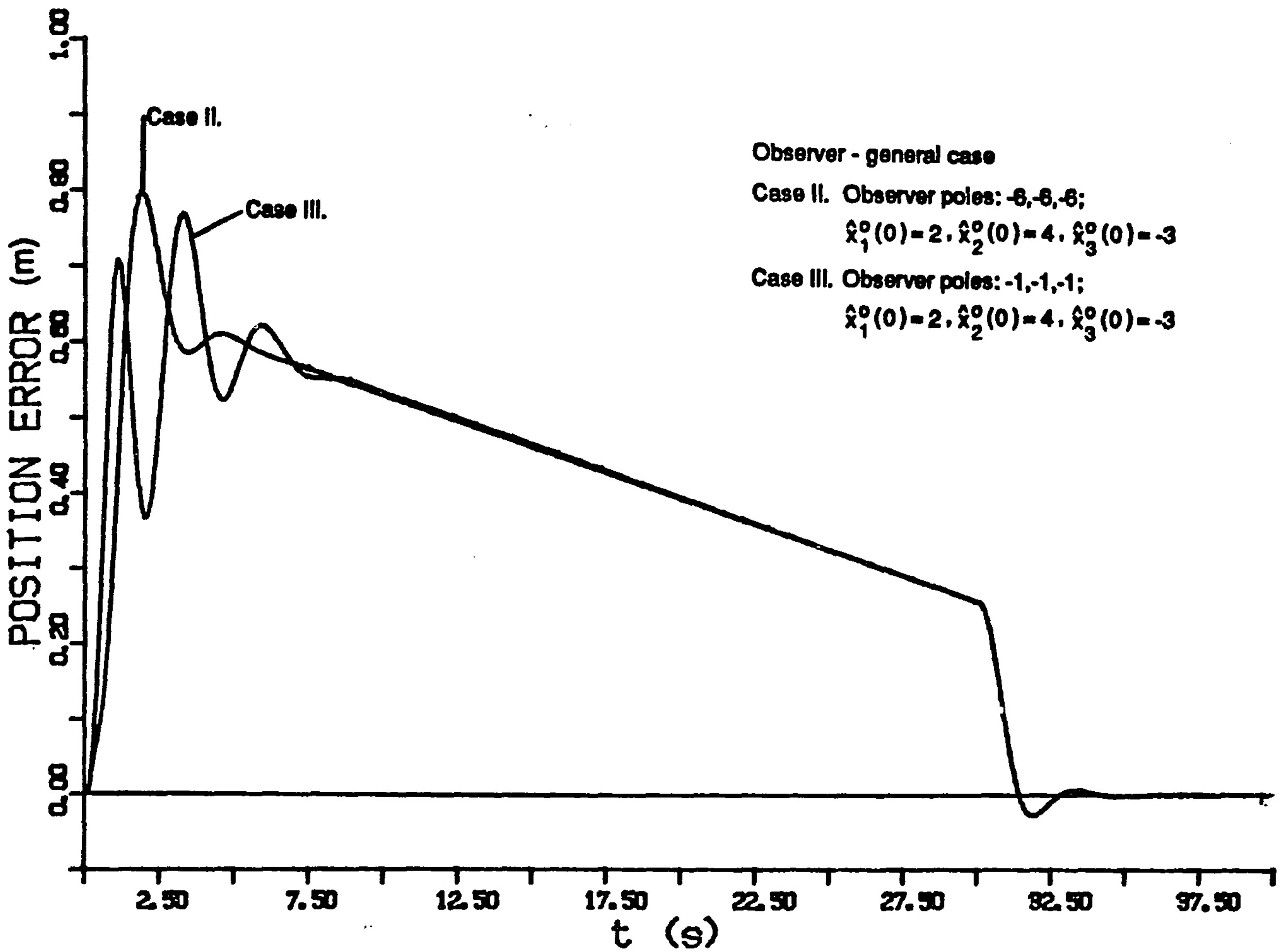

Figure 7: Position error for different observer-poles locations

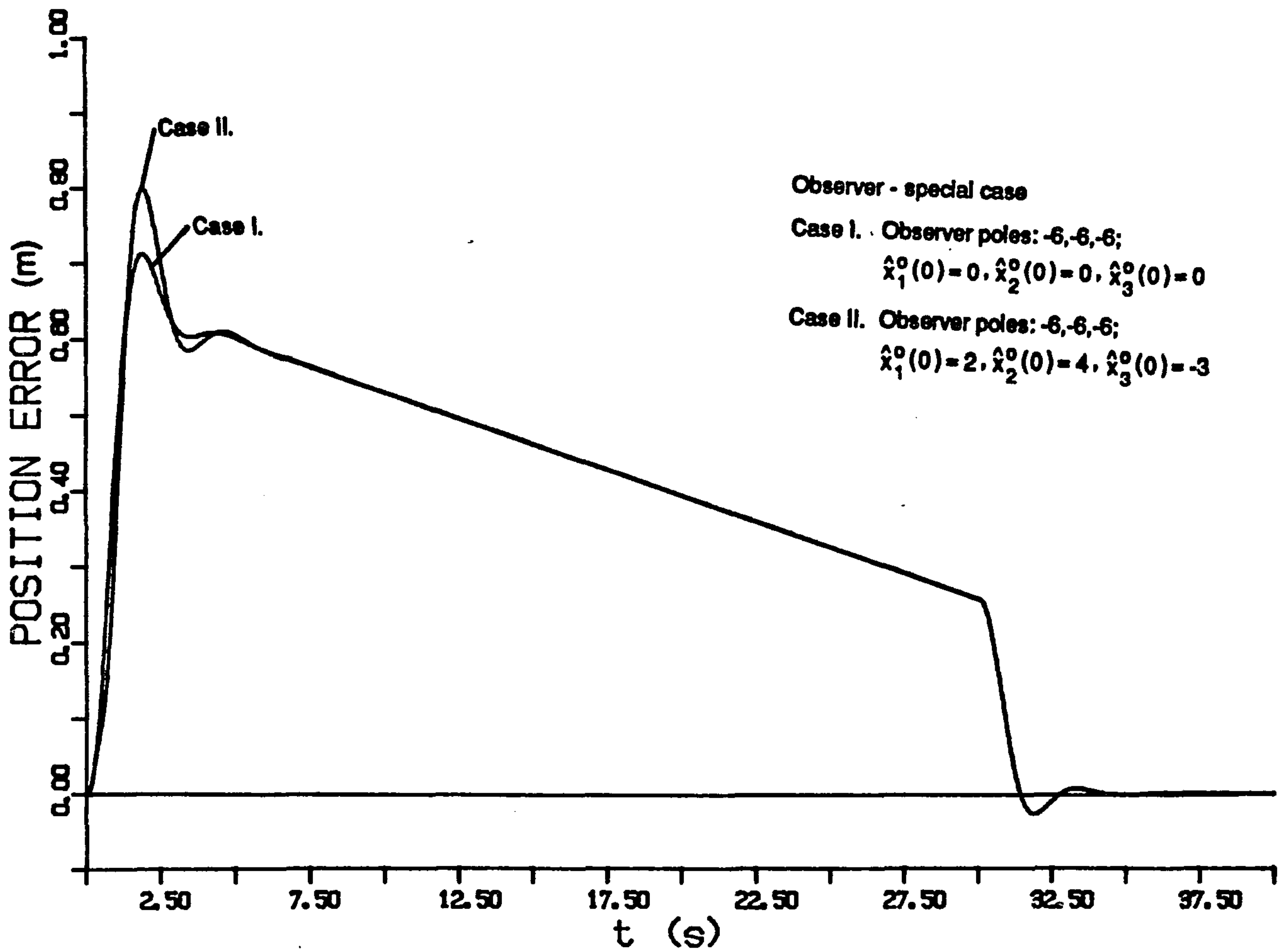

Figure 8: Position error for different observer initial conditions

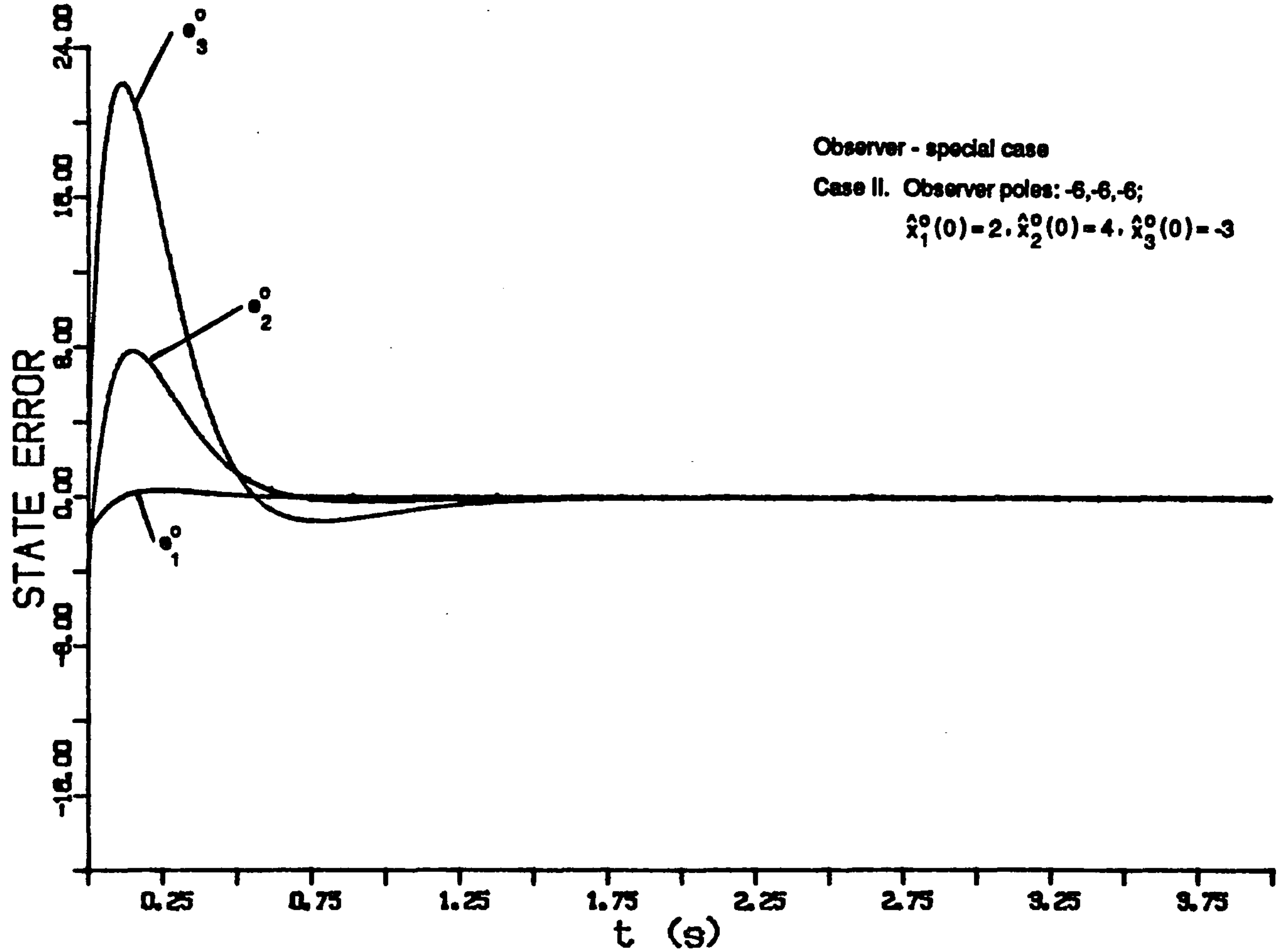

Figure 9: State error

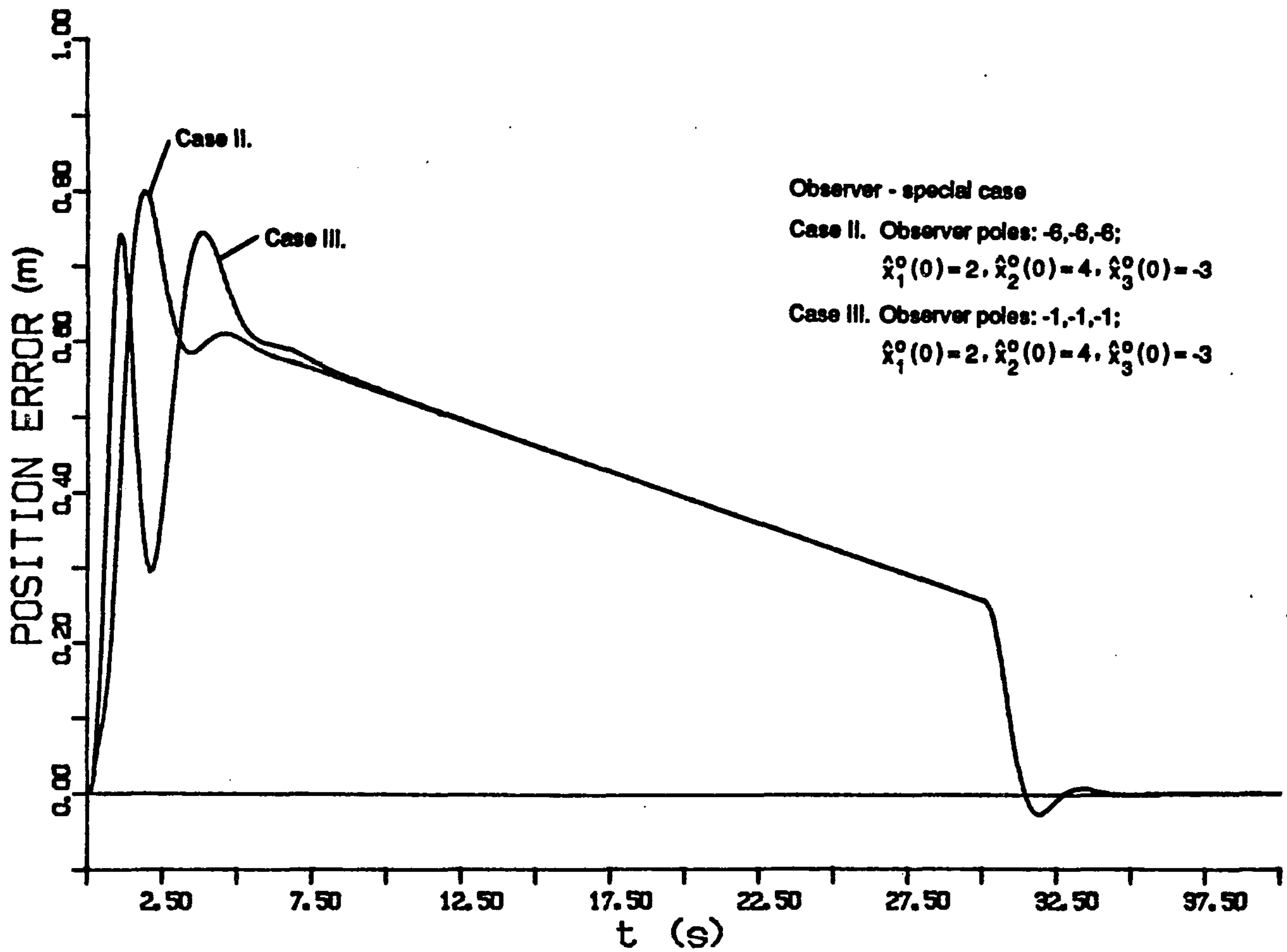

Figure 10: Position error for different observer-poles locations

in Fig. 10, there is slightly less oscillation than in the corresponding case in Fig. 7, however, the oscillations die out in about 10 s in both cases.

5.6 Discussion

Generally the simulation results were as expected and compared well to those previously reported. The results were also in close correspondence with those obtained from field studies [99] which indicates that the present controller would be physically realizable.

It is interesting how quickly the state error died out for sufficiently fast observer poles, even in the general case (where $\hat{x}$ is used in the observer). In fact all the results were practially identical for the general case and the special case (where y is used in the observer) for sufficiently fast observer poles. This suggests that the nonlinearities in the error dynamics are not causing a serious problem, nor do they require large observer pole values for good system performance.

Additional simulation studies where the system nonlinearities are functions of all the system states are reported in Appendix F.

CHAPTER VI

SUMMARY, CONCLUSIONS AND SUGGESTED FUTURE STUDIES

6.1 Summary

The purpose of this work was to develop a design methodology for observers and controllers for a class of nonlinear systems, where the elements of the system matrices (the state matrix ($A(x)$), the input matrix ($b(x)$) and the output matrix ($c(x)$)) are C^{∞}-functions of the system states (referred to as the general form).

In Chapter II various observability aspects were briefly discussed, a nonlinear observer form was defined, and a corresponding observer using nonlinear observer gains was specified. The observer states were used to calculate the nonlinear elements of the state and input matrices in the observer (referred to as the "general case"), resulting in nonlinear error dynamics. These matrices satisfy a Lipschitz condition[1] since their entries are C^{∞} functions and then, by a proper choice of nonlinear observer gains, the error dynamics can be made asymptotically stable. In a special case, where the state and input matrices are functions of the output only (referred to as the "special case"), the output can be used to calculate the nonlinear elements of these matrices in the observer, resulting in linear error dynamics.

Various aspects of controllability were also discussed and a nonlinear controller

[1] If the state matrix is nonlinear, then the output must be bounded. If the input matrix is nonlinear, then the input must be bounded.

form was defined. A controller, using nonlinear controller gains, was devised for this form, which resulted in linear closed-loop dynamics with arbitrary eigenvalues.

Transformations from the general form to the defined observer and controller forms were derived in Chapter III. In the observer case, this led to a system of n, $(n-1)$-st-order linear PDEs in n variables and n unknowns, generally a difficult problem to solve. In the controller case, a much simpler system of linear PDEs was obtained; i.e., n, first-order equations in n variables and one "unknown". The transformations were applied to simple nonlinear examples in both cases; further, when applied to a general n-th-order linear system, well-known controllability and observability conditions were obtained. A method for transforming nonlinear feedback gains from the defined forms to the general form was also specified.

Various computational aspects of the PDEs derived in Chapter III were considered in Chapter IV. In the observer case a general solution was given for $n = 1$. In the case $n = 2$, the system of two equations was reduced to a single first-order PDE in two variables and one unknown, a problem widely documented in the literature. A general solution method for this case was described (pp. 133-137, [89]) and an example was given. Two special cases, where the solution can be found easily, were stated and examples given. In the case $n > 2$, the system of n equations was reduced to a system of $n-1$, $(n-1)$-st-order PDEs in n variables and $n-1$ unknowns. In general, such a system of equations is quite "rich" in nature, and a complete coverage was beyond the scope of this work; thus, the interested reader is referred to the literature. However, a special case was solved and an example was given.

Solution methods for the equations that arose in the controller case are well documented in the literature. Necessary and sufficient conditions (p. 45, [91]) were given for the existence of a solution, an explicit solution for the general case was stated (p. 45, [91]) and an example was given. In the case when the necessary and sufficient conditions for the existence of a solution are not satisfied, it may be possible to find a solution through the use of an integrating factor [68]. The conditions that guarantee the existence of such factors (p. 4-6 [93]) were discussed together with an example.

Simulation studies were reported in Chapter V. A third-order nonlinear system, which was based on empirically derived, vehicle longitudinal dynamics, was chosen as an example. Transformations from the general form to the observer form and from the observer form to the controller form were found. A nonlinear observer/controller was designed such that the resulting linear closed-loop dynamics and observer dynamics matched those of previous studies for comparison purposes. Results were shown for different observer initial conditions and different observer poles for both the general case and the special case.

6.2 Conclusions

The observer (controller) design methodology, once the system is in the observer (controller) form, is basically very simple and analogous to the linear case. However, nonlinear systems are frequently difficult to analyze and/or design as is exemplified here in finding the nonlinear transformation to the observer form. This involves solving a complex system of PDEs and can get extremely involved, especially for higher-order systems. On the other hand, a much simpler system of PDEs results for the controller form. Further, finding the solution when the existence conditions are satisfied, is relatively simple. In both cases, finding the coefficients of the PDEs and, once the PDEs have been solved, calculating the final observer/controller form is, albeit straightforward[2], often tedious.

[2]This generally involves only differentiation and algebraic manipulation.

Simulation results showed excellent correlation with previous studies employing the same closed-loop dynamics. Since the latter included a field study, the specified design should be physically realizable. All results were practically identical for both the general case and the special case when the observer poles were 2-3 times larger than the dominant closed-loop poles. This suggests that the nonlinearites in the error dynamics do not require large observer pole values for adequate state-error decay, this is also supported by a simulation study reported in Appendix F. The state error for initial conditions of 2,4,-3 and observer poles at $-6, -6, -6$ died out in less than 2 seconds. Finally, response deterioration caused by slowing down the observer poles to approximately the same speed as the dominant closed-loop poles was observed.

Note that this system, while chosen for its practicality, is sufficiently complex so that it exercises all aspects of the design methodology developed. The methodology should also work well for other systems of the class treated in this work[3] and further, it has definite advantages over parameter-scheduling, especially for systems where the nonlinearities are functions of many state variables resulting in a large number of operating points[4].

The major goal of this work has been achieved and, despite some practical difficulties that may be diminished through future studies, a successful design methodology for the chosen class of nonlinear systems has been developed.

[3]This has also been indicated by other simulation studies, one of which is reported in Appendix F.

[4]For example, if the nonlinearities are functions of three state variables and there are 30 operating points for each state variable, then the total number of operating points is $30^3 = 27,000$.

6.3 Suggested future studies

Nonlinear control theory is still relatively young and far from having reached maturity, and there are many topics that are candidates for future studies. One, related to this study, concerns establishing a strong theoretical basis between the observability (controllability) of a nonlinear system and the design of an observer (controller). As opposed to the case of linear systems, such a link has not been clearly established.

Several topics directly related to this work need further investigation and are summarized as follows:

1. The necessity/feasibility of using Theorems 2.1.1 and A.2 as design guidelines for selecting observer poles is of interest. The former results in bounds that can be related to the observer poles. Although these bounds are useful for proving asymptotic stability of the error dynamics, they may be too conservative and thus result in a choice of unnecessarily fast observer poles, thus enhancing the system's susceptibility to noise. If the use of these theorems were feasible for observer design, software programs that ease that task should be developed. On the other hand, selecting the observer poles from a simulation study may be more attractive due to its simplicity; further, simulation results already obtained have indicated that the nonlinearities in the error dynamics do not seriously affect the speed of error decay.

2. Deriving the coefficients of the systems of PDEs in both the observer and controller case, while straightforward, is quite cumbersome for higher-order systems. This task could, however, possibly be automated using MACSYMA[5].

[5]MACSYMA is a program with a symbolic mathematics capability.

3. The system of PDEs for the observer case is, especially for higher-order systems, very difficult to solve. This area needs more investigation and even though some work is contained in the literature on such systems, the situations that arise are so diverse, that each becomes an interesting challenge.

4. Many of the algebraic tasks involved in calculating the nonlinear observer and controller forms, once the PDEs have been solved, may be extremely cumbersome. These tasks could also possibly be automated using MACSYMA.

5. A number of preliminary findings involving structural observability (controllability) of the nonlinear observer and observability (controller and controllability) forms[6] are reported in Appendix E. These were not fully investigated here since this subject[7] was beyond the scope of this work. This seems a very profitable area for future investigations.

6. Simulation studies have indicated very favorable performance of a combined observer/controller configuration. However, a proof of the overall stability of such systems should be worked out.

Looking towards topics that were not addressed here; two seem of immediate interest. First, it seems relatively straightforward to incorporate the concepts developed in this work, toward the design of a reduced-order observer. Second, the multiple input-multiple output case was not investigated and should provide an interesting challenge.

---

[6]Transformations to the nonlinear observability and controllability forms are derived in Appendix D.

[7]This subject is related to the lack of a theoretical basis between the observability (controllability) of nonlinear systems and the design of an observer (controller).

APPENDIX A

PREREQUISITES FOR CHAPTER II

A.1 Norms

The norm definitions below follow that of [81].

Definition A.1 *The norm of a matrix A is defined as*

$$\|A\| \equiv \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|. \tag{A.1}$$

This is an induced norm since it is defined through the norm of x. The following vector norm, $\|x\|_1$, was chosen here from several possible ones, as it leads to a convenient matrix norm, $\|A\|_1$.

Definition A.2 *The norm of a vector x is defined as*

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|. \tag{A.2}$$

For $\|x\|_1$, the corresponding matrix norm, $\|A\|_1$, becomes (using (A.2) in (A.1))

$$\|A\|_1 = \max_j \left(\sum_{i=1}^{n} |a_{ij}|\right). \tag{A.3}$$

A.2 Aspects of the nonlinear error dynamics

Theorem A.1 *If e^o satisfies (2.10) then it also satisfies*

$$\begin{aligned} e^o(t) &= \Phi^o(t,t_0)e^o(t_0) + \int_{t_0}^{t} \Phi^o(t,\tau)(a^o_{\cdot 1}(x^o(\tau)) - a^o_{\cdot 1}(\hat{x}^o(\tau)))y(\tau)d\tau \\ &\quad + \int_{t_0}^{t} \Phi^o(t,\tau)(b^o(x^o(\tau)) - b^o(\hat{x}^o(\tau)))u(\tau)d\tau \end{aligned} \tag{A.4}$$

where $\Phi^o(t,\tau) = e^{A^o_e(t-\tau)}$ satisfies

$$\frac{\partial}{\partial t}\Phi^o(t,\tau) = A^o_e\Phi^o(t,\tau).$$

Proof: This proof follows a similar one in [81] p. 139. Differentiating (A.4) with respect to (w.r.t.) t gives

$$\begin{aligned} \frac{d}{dt}e^o(t) &= \frac{d}{dt}\Phi^o(t,t_0)e^o(t_0) \\ &\quad + \frac{d}{dt}\int_{t_0}^{t} \Phi^o(t,\tau)(a^o_{\cdot 1}(x^o(\tau)) - a^o_{\cdot 1}(\hat{x}^o(\tau)))y(\tau)d\tau \\ &\quad + \frac{d}{dt}\int_{t_0}^{t} \Phi^o(t,\tau)(b^o(x^o(\tau)) - b^o(\hat{x}^o(\tau)))u(\tau)d\tau \\ &= \frac{\partial}{\partial t}\Phi^o(t,t_0)e^o(t_0) \\ &\quad + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi^o(t,\tau)(a^o_{\cdot 1}(x^o(\tau)) - a^o_{\cdot 1}(\hat{x}^o(\tau)))y(\tau)d\tau \\ &\quad + \frac{\partial}{\partial t}\int_{t_0}^{t} \Phi^o(t,\tau)(b^o(x^o(\tau)) - b^o(\hat{x}^o(\tau)))u(\tau)d\tau \\ &= A^o_e\Phi^o(t,t_0)e^o(t_0) \\ &\quad + \Phi^o(t,t)(a^o_{\cdot 1}(x^o(t)) - a^o_{\cdot 1}(\hat{x}^o(t)))y(t) \\ &\quad + \int_{t_0}^{t} \frac{\partial}{\partial t}\Phi^o(t,\tau)(a^o_{\cdot 1}(x^o(\tau)) - a^o_{\cdot 1}(\hat{x}^o(\tau)))y(\tau)d\tau \\ &\quad + \Phi^o(t,t)(b^o(x^o(t)) - b^o(\hat{x}^o(t)))u(t) \\ &\quad + \int_{t_0}^{t} \frac{\partial}{\partial t}\Phi^o(t,\tau)(b^o(x^o(\tau)) - b^o(\hat{x}^o(\tau)))u(\tau)d\tau \end{aligned}$$

$$\begin{aligned}
&= A_e^o\Big[\Phi^o(t,t_0)e^o(t_0) \\
&\quad + \int_{t_0}^{t}\Phi^o(t,\tau)(a_{\cdot 1}^o(x^o(\tau)) - a_{\cdot 1}^o(\hat{x}^o(\tau)))y(\tau)d\tau \\
&\quad + \int_{t_0}^{t}\Phi^o(t,\tau)(b^o(x^o(\tau)) - b^o(\hat{x}^o(\tau)))u(\tau)d\tau\Big] \\
&\quad +(a_{\cdot 1}^o(x^o(t)) - a_{\cdot 1}^o(\hat{x}^o(t)))y(t) + (b^o(x^o(t)) - b^o(\hat{x}^o(t)))u(t) \\
&= A_e^o e^o(t) \\
&\quad +(a_{\cdot 1}^o(x^o(t)) - a_{\cdot 1}^o(\hat{x}^o(t)))y(t) + (b^o(x^o(t)) - b^o(\hat{x}^o(t)))u(t)
\end{aligned}$$

at $t = t_0$ one has

$$e^o(t_0) = \Phi^o(t_0,t_0)e^o(t_0) + \int_{t_0}^{t_0}\cdots d\tau + \int_{t_0}^{t_0}\cdots d\tau = e^o(t_0)$$

□

Theorem A.2 *A relation between $e^{A_e^o(t-\tau)}$ and the eigenvalues of A_e^o, λ_1, λ_2, ..., λ_n, is given by*

$$e^{A_e^o(t-\tau)} = Q_d^o e^{A_e^d(t-\tau)} Q_d^{o^{-1}} \qquad (A.5)$$

where

$$A_e^d = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

and

$$Q_d^o = \Xi\Lambda,$$

Ξ is a lower triangular Toeplitz matrix, with first column $[1 \; l_1^o \; l_2^o \cdots l_{n-1}^o]$,
Λ is a Vandermode matrix with columns $[1 \; \lambda_i \; \lambda_i^2 \cdots \lambda_i^{n-1}]$, $i = 1, \ldots, n$.

Proof: If

$$A_e^o = Q_d^o A_e^d Q_d^{o-1}, \tag{A.6}$$

then Q_d^o is composed of the eigenvectors of A_e^o. Further, multiplying (A.6) by the scalar $t - \tau$ gives

$$A_e^o(t-\tau) = Q_d^o A_e^d(t-\tau) Q_d^{o-1}. \tag{A.7}$$

Then from (A.7)

$$e^{A_e^o(t-\tau)} = Q_d^o e^{A_e^d(t-\tau)} Q_d^{o-1} \tag{A.8}$$

is valid (easily proved using the Cayley-Hamilton theorem). Thus it suffices to show that $\Xi [1 \; \lambda \; \lambda^2 \cdots \lambda^{n-1}]^T$ is an eigenvector of A_e^o (a similar result was stated for a matrix in controller form in [23] pp. 54-55), i.e.,

$$A_e^o \Xi \begin{bmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^{n-1} \end{bmatrix}^T = \lambda \Xi \begin{bmatrix} 1 & \lambda & \lambda^2 & \ldots & \lambda^{n-1} \end{bmatrix}^T. \tag{A.9}$$

The L.H.S. of (A.9) gives

$$A_e^o \Xi \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}$$

$$
= \begin{bmatrix} -l_1^o & 1 & & & \\ -l_2^o & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ -l_{n-1}^o & & & \ddots & 1 \\ -l_n^o & & & & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & & & \\ l_1^o & 1 & \ddots & & \\ l_2^o & l_1^o & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n-1}^o & l_{n-2}^o & \cdots & l_1^o & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} -l_1^o & 1 & & & \\ -l_2^o & 0 & \ddots & & \\ \vdots & & \ddots & \ddots & \\ -l_{n-1}^o & & & \ddots & 1 \\ -l_n^o & & & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ l_1^o + \lambda \\ l_2^o + \lambda l_1^o + \lambda^2 \\ \vdots \\ l_{n-1}^o + \lambda l_{n-2}^o + \cdots + \lambda^{n-2} l_1^o + \lambda^{n-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} \lambda \\ \lambda l_1^o + \lambda^2 \\ \vdots \\ \lambda l_{n-2}^o + \cdots + \lambda^{n-2} l_1^o + \lambda^{n-1} \\ -l_n^o \end{bmatrix} \tag{A.10}
$$

$$
= \begin{bmatrix} \lambda \\ \lambda l_1^o + \lambda^2 \\ \vdots \\ \lambda l_{n-2}^o + \cdots + \lambda^{n-2} l_1^o + \lambda^{n-1} \\ \lambda l_{n-1}^o + \cdots + \lambda^{n-1} l_1^o + \lambda^n \end{bmatrix}.
$$

The R.H.S. of (A.9) gives

$$\lambda \Xi \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ l_1^o + \lambda \\ l_2^o + \lambda l_1^o + \lambda^2 \\ \vdots \\ l_{n-1}^o + \lambda l_{n-2}^o + \cdots + \lambda^{n-2} l_1^o + \lambda^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda \\ \lambda l_1^o + \lambda^2 \\ \vdots \\ \lambda l_{n-2}^o + \cdots + \lambda^{n-2} l_1^o + \lambda^{n-1} \\ \lambda l_{n-1}^o + \cdots + \lambda^{n-1} l_1^o + \lambda^n \end{bmatrix}. \tag{A.11}$$

Comparing (A.11) and (A.10) shows that (A.9) holds.

□

Note that Theorem 2.1.1 can be used to obtain an upper bound on $\left\| e^{A_e^o(t-\tau)} \right\|_1$. Then Theorem A.2 and Eq. (A.3) can be used to relate $\left\| e^{A_e^o(t-\tau)} \right\|_1$ to the eigenvalues of A_e^o. This would typically be done by selecting some eigenvalues, calculating $e^{A_e^o(t-\tau)}$ using (A.5), then using (A.3) to find $\left\| e^{A_e^o(t-\tau)} \right\|_1$ and checking whether (2.12) is satisfied. For higher-order systems, this procedure should be accomplished using a digital computer.

APPENDIX B

PREREQUISITES FOR CHAPTER III

B.1 Aspects of nonlinear independence

The concept of independence for function-valued vectors is defined as follows (analogous to independence for constant vectors):

Definition B.1 *A set $\mathcal{P}$ of n function-valued row or column vectors*

$$\{p_1(x), p_2(x), \ldots, p_m(x)\}$$

in $\mathcal{R}^n$ will be said to be nonlinearly dependent if there exist some functions, possibly complex, $\{\alpha_1(x), \ldots, \alpha_m(x)\}$ in $\mathcal{R}^n$ and not all zero for which

$$\alpha_1(x)p_1(x) + \alpha_2(x)p_2(x) + \cdots + \alpha_m(x)p_m(x) = 0$$

for all x. Otherwise, $\mathcal{P}$ will be said to be nonlinearly independent.

Example B.1 The row vectors $\begin{bmatrix} x_1 & 0 \end{bmatrix}$ and $\begin{bmatrix} x_2 & 0 \end{bmatrix}$ are nonlinearly dependent since $\begin{bmatrix} x_1 & 0 \end{bmatrix} = \frac{x_1}{x_2}\begin{bmatrix} x_2 & 0 \end{bmatrix}$. On the other hand $\begin{bmatrix} x_1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & x_2 \end{bmatrix}$ are nonlinearly independent, since $\begin{bmatrix} x_1 & 0 \end{bmatrix} \neq \alpha(x)\begin{bmatrix} 0 & x_2 \end{bmatrix}$, $\alpha(x) \neq 0$ unless $x_1 = x_2 = 0$.

The following four statements are equivalent:

1. The set $\mathcal{P}$ is nonlinearly independent in $\mathcal{R}^n$.

2. $$\rho \begin{bmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ \vdots \\ p_n(x) \end{bmatrix} = n$$

($p_k(x)$ row vectors).

$$\rho \begin{bmatrix} p_1(x) & p_2(x) & \cdots & \cdots & p_n(x) \end{bmatrix} = n$$

($p_k(x)$ column vectors).

3. The set $\mathcal{P}$ forms a basis for $\Re^n$.

4. The set $\mathcal{P}$ spans $\Re^n$.

Then the following holds: If the set $\mathcal{P}$ forms a basis for $\Re^n$, then any vector $q(x)\epsilon\Re^n$ can be expressed as

$$q(x) = \sum_{i=1}^{n} \alpha_i(x) p_i(x), \qquad ,$$

i.e., as a nonlinear combination of the $p_k(x)$'s, $k = 1, \ldots, n$.

Definition B.2 *A nonlinear matrix $A(x)$ is nonsingular if its columns (rows) are nonlinearly independent.*

B.2 The notation of Su and Hunt

The following notation [85] was inspired by Su [86] and Hunt [87].

Definition B.3 *For two vector fields f and g in $\mathcal{R}^n$, the Lie bracket $[f,g]$ is a vector field in $\mathcal{R}^n$ defined as*

$$[f,g] \equiv \frac{\partial g}{\partial x}f - \frac{\partial f}{\partial x}g.$$

Further, the following notation is introduced

$$\begin{aligned}\left(ad^0 f, g\right) &= g \\ \left(ad^k f, g\right) &= \left[f, \left(ad^{k-1} f, g\right)\right], \qquad k = 1, 2, \ldots.\end{aligned}$$

Thus

$$\begin{aligned}\left(ad^0 f, g\right) &= g \\ \left(ad^1 f, g\right) &= [f, g] \\ \left(ad^2 f, g\right) &= [f, [f, g]] \\ \left(ad^3 f, g\right) &= [f, [f, [f, g]]]\end{aligned}$$

etc.

Definition B.4 *For a scalar field (function) h and a vector field f in $\mathcal{R}^n$, the Lie derivative of h with respect to f is defined by*

$$\langle dh, f\rangle \equiv \frac{\partial h}{\partial x}f.$$

APPENDIX C

PROGRAM LISTINGS

This appendix contains sample program listings of the software developed for the simulations studies in this work.

```
*       main program for simulation of vehicle longitudinal dynamics
*        nonlinear observer/controller
*       programmer: Anna Soffia Hauksdottir
*                   OSU 1987

        dimension x(8)
        dimension p(3),q(3),xlo(3)
        dimension xx(4002),yy(4002)

        common xr,xlo

10      write (5,*) 'input observer poles'
        read (5,*) p
          xlo(1)=p(1)+p(2)+p(3)
          xlo(2)=p(1)*p(2)+p(1)*p(3)+p(2)*p(3)
          xlo(3)=p(1)*p(2)*p(3)

        write (5,*) 'input observer initial conditions'
        read (5,*) q
          x(6)=q(1)                     !observer state xhat_1^o
          x(7)=q(2)                     !observer state xhat_2^o
          x(8)=q(3)                     !observer state xhat_3^o


c       initialize for simulation
        sec=40.                         !total simulation time
        t=0.                            !real simulation time
        h=0.001                         !time interval for rk4 integ.
        nd=8                            !number of state variables
        loopcnt=sec/h                   !no. times thru simulation-loop
        do j=1,5                        !initialize state variables
                x(j) = 0.               ! x(1): position
        end do                          ! x(2): int. of pos. error
                                        ! x(3): system state x_1
                                        ! x(4): system state x_2
                                        ! x(5): system state x_3
```

```
c       main simulation starts
        do j=1,loopcnt

c               load into vectors for plotter
                xx(jj) = t
                yy(jj) = xr-x(1)

c               reference position
                if (t .lt. 30.) then
                        xr = 0.65*t**2-0.0045*t**3
                else
                        xr = 463.5+26.85*(t-30.)
                end if

c               call Runge-Kutta integration algorithm
                call rk4(x,t,h,nd)

        enddo

c       call plotting routine
        n=4002
        call hpplot(xx,yy,n,1,0.,40.,-.1,1.)

        write (5,*) 'do you want another run?'
        read (5,*) l
        if (l.eq.1) goto 10

        stop
        end


*       this is the Runge-Kutta integration algorithm

        subroutine rk4(yy,t,h,nd)

        dimension yy(8),xk1(8),xk2(8),xk3(8),xk4(8)
        dimension y1(8),dy(8)
        dimension xlo(3)

        common xr,xlo

        call derfun(yy,dy,t)
        do i=1,nd
                xk1(i)=h*dy(i)
                y1(i)=yy(i)+xk1(i)/2.
        enddo
        t=t+.5*h
        call derfun(y1,dy,t)
        do i=1,nd
                xk2(i)=h*dy(i)
                y1(i)=yy(i)+xk2(i)/2.
        enddo
        call derfun(y1,dy,t)
        do i=1,nd
                xk3(i)=h*dy(i)
                y1(i)=yy(i)+xk3(i)
        enddo
        t=t+.5*h
        call derfun(y1,dy,t)
        do i=1,nd
                xk4(i)=h*dy(i)
        enddo
        do i=1,nd
                yy(i)=yy(i)+(xk1(i)+2.*xk2(i)+2.*xk3(i)+xk4(i))/6.
        enddo

        return
        end
```

```
*       this is the subroutine that calculates the finite differences
*        for the Runge-Kutta integration

        subroutine derfun(y,dy,t)

        dimension y(8),dy(8),xlo(3)

        common xr,xlo

c       calculate parameters for general form
        a11 = -0.4166664*(y(3)**2+6.12*y(3)+123.68)/(y(3)+4.)
        a21 = -(y(3)+2.)/(y(3)+4.)
        a31 = -0.0208332*(y(3)**2+966.*y(3)+2043.2)/(y(3)+4.)

c       calculate parameters for observer form
        ao11 = -0.4166664*(y(6)**2+6.12*y(6)+123.68)/(y(6)+4.)
        ao21 = -0.0208332*(y(6)**2+966.*y(6)+2043.2)/(y(6)+4.)
        ao31 = -(y(6)+2.)/(y(6)+4.)

c       calculate nonlinear parameters for controller form
        g1 = 0.3472217778*(y(6)**5+15.12*y(6)**4+145.1672187*y(6)**3
     x  +930.4865879*y(6)**2+2633.434211*y(6)+28632.01343)
     x  /(y(6)+4.)**3
        g2 = -0.4166664*(y(6)**2+6.12*y(6)+123.68)/(y(6)+4.)
        g3 = -0.8333328*(y(6)**3+9.06*y(6)**2+24.48*y(6)+247.36)
     x  /(y(6)+4.)**2
        xc1 = (g1*y(6)+g3*y(7)+y(8))/6.
        xc2 = (g2*y(6)+y(7))/6.
        xc3 = y(6)/6.
        ac11 = -4.9999968*(xc3**3+1.51*xc3**2
     x  +0.68*xc3+1.145185185)/(xc3+0.666666667)**2
        ac12 = -4.9999968*xc2*(xc3**3+2.*xc3**2
     x  +1.333333333*xc3-1.837037037)/(xc3+0.6666666667)**3
     x  -0.24995651*(xc3**5+82.847321*xc3**4+216.48164*xc3**3
     x  +198.2861*xc3**2+72.94256897*xc3+8.409689867)
     x  /(xc3+0.6666666667)**4
        ac13 = -(xc3+0.333333333)/(xc3+0.6666666667)

c       calculate input into nonlinear observer/controller
        u = 266.6666667*(xr-y(1)+y(2))-(ac11+30.034)*xc1
     x  -(ac12+241.384)*xc2-(ac13+809.68)*xc3

        dy(1) = y(3)                    !pos. fin. diff.
        dy(2) = xr-y(1)                 !int. pos. err. fin. diff.

c       calculate system finite differences
        dy(3) = a11*y(3)+y(5)
        dy(4) = a21*y(3)+6.*u
        dy(5) = a31*y(3)+y(4)

c       calculate observer finite differences
        dy(6) = ao11*y(6)+y(7)+(ao11+xlo(1))*(y(3)-y(6))
        dy(7) = ao21*y(6)+y(8)+(ao21+xlo(2))*(y(3)-y(6))
        dy(8) = ao31*y(6)+6.*u+(ao31+xlo(3))*(y(3)-y(6))

        return
        end
```

```
* plot data as x vs y on a hp plotter
* n = n+2 = # of data points plus two
* iline = 0 automatic scaling
* 1 fixed scaling

        subroutine hpplot(x,y,n,iline,xmin,xmax,ymin,ymax)

        dimension x(n),y(n)

        call hpinit(0, 0, 0, 1, 3)

        call scale(x, 16.0, n-2, 1)
        call scale(y, 11.0, n-2, 1)

        if (iline.eq.0) goto 10
                x(n-1)=xmin
                x(n)=(xmax-xmin)/16.
                y(n-1)=ymin
                y(n)=(ymax-ymin)/11.

10      write (5,*) 'input noax'
        read (5,*) noax
        if (noax.eq.1) goto 20
                call axis(4., 4.,5ht (s),-5, 16.0, 0.0, x(n-1), x(n))
                call axis(4., 4.,11hSTATE ERROR,11,11.0,90.0,y(n-1),y(n))

20      call plot(4., 4., -2)

        call line(x, y, n-2, 1, 0, 10)

        call plot(0.0,0.0,999)

        call hpinit(1, 0, 0, 1, 3)

        return
        end
```

APPENDIX D

TRANSFORMATIONS TO OTHER NONLINEAR FORMS

Transformations to nonlinear forms, which are analogous to the linear "standard forms," i.e., the observability, controllability and diagonal forms [23], are discussed in this appendix.

D.1 Observability form

A nonlinear observability form for the class of systems (3.1)-(3.2) is defined by analogy to the linear observability form (e.g., in [23]) as

$$\begin{aligned} \dot{x}^{ob} &= A^{ob}\left(x^{ob}\right)x^{ob} + b^{ob}\left(x^{ob}\right)u \\ y &= c^{ob}x^{ob} \end{aligned} \tag{D.1}$$

where

$$A^{ob}\left(x^{ob}\right) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_{n1}^{ob}\left(x^{ob}\right) & a_{n2}^{ob}\left(x^{ob}\right) & \cdots & \cdots & a_{nn}^{ob}\left(x^{ob}\right) \end{bmatrix},$$

$$b^{ob}\left(x^{ob}\right) = \begin{bmatrix} b_1^{ob}\left(x^{ob}\right) \\ b_2^{ob}\left(x^{ob}\right) \\ \vdots \\ \vdots \\ b_n^{ob}\left(x^{ob}\right) \end{bmatrix}, \qquad c^{ob} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \tag{D.2}$$

and q is specified as ob.

In this case, the R.H.S. of (3.10a) becomes

$$\begin{aligned} \bar{A}^{ob}(x)Q^{ob}(x) &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \bar{a}_{n1}^{ob}(x) & \bar{a}_{n2}^{ob}(x) & \cdots & \cdots & \bar{a}_{nn}^{ob}(x) \end{bmatrix} \begin{bmatrix} q_{1\cdot}^{ob}(x) \\ q_{2\cdot}^{ob}(x) \\ \vdots \\ \vdots \\ q_{n\cdot}^{ob}(x) \end{bmatrix} \\ &= \begin{bmatrix} q_{2\cdot}^{ob}(x) \\ q_{3\cdot}^{ob}(x) \\ \vdots \\ q_{n\cdot}^{ob}(x) \\ \bar{a}_{n1}^{ob}(x)q_{1\cdot}^{ob}(x) + \bar{a}_{n2}^{ob}(x)q_{2\cdot}^{ob}(x) + \cdots + \bar{a}_{nn}^{ob}(x)q_{n\cdot}^{ob}(x) \end{bmatrix}. \end{aligned} \tag{D.3}$$

Combining (3.14) and (D.3) results in the following:

$$\left.\begin{aligned} q_{2\cdot}^{ob}(x) &= \mathcal{L}^1\left[q_{1\cdot}^{ob}(x)\right] \\ q_{3\cdot}^{ob}(x) &= \mathcal{L}^1\left[q_{2\cdot}^{ob}(x)\right] \\ &\vdots \\ q_{n\cdot}^{ob}(x) &= \mathcal{L}^1\left[q_{(n-1)\cdot}^{ob}(x)\right] \end{aligned}\right\} \tag{D.4a}$$

and

$$\bar{a}_{n1}^{ob}(x)q_{1\cdot}^{ob}(x) + \bar{a}_{n2}^{ob}(x)q_{2\cdot}^{ob}(x) + \cdots + \bar{a}_{nn}^{ob}(x)q_{n\cdot}^{ob}(x) = \mathcal{L}^1\left[q_{n\cdot}^{ob}(x)\right]. \tag{D.4b}$$

Now upon employing (3.10c)

$$c(x) = \bar{c}^{ob} Q^{ob}(x) = q_{1\cdot}^{ob}(x). \tag{D.4c}$$

Thus, determining $Q^{ob}(x)$ can be done in one step:

1. Calculate $Q^{ob}(x)$ from (D.4c) and (D.4a) (rewritten here as (D.5))

$$\begin{aligned} q_{1\cdot}^{ob}(x) &= c(x) \\ q_{2\cdot}^{ob}(x) &= \mathcal{L}^1\left[q_{1\cdot}^{ob}(x)\right] \\ q_{3\cdot}^{ob}(x) &= \mathcal{L}^1\left[q_{2\cdot}^{ob}(x)\right] \\ &\vdots \\ q_{n\cdot}^{ob}(x) &= \mathcal{L}^1\left[q_{(n-1)\cdot}^{ob}(x)\right]. \end{aligned} \tag{D.5}$$

Comments:

1. Here the rows of $Q^{ob}(x)$ can be obtained without any knowledge of $\bar{a}_{n\cdot}^{ob}(x)$.

2. Calculating $Q^{ob}(x)$ is relatively simple.

Example D.1.1 A simple, nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} -x_1 & x_1^2 \\ x_1 x_2 & -x_1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} x_1 & x_2 \end{bmatrix} x.$$

Here (D.5) gives

$$\begin{aligned} q_{1\cdot}^{ob}(x) &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \\ q_{2\cdot}^{ob}(x) &= \frac{\partial}{\partial x}\left(x_1^2 + x_2^2\right) A(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} -x_1 & x_1^2 \\ x_1 x_2 & -x_1 \end{bmatrix} \\ &= \begin{bmatrix} -2x_1^2 + 2x_1 x_2^2 & 2x_1^3 - 2x_1 x_2 \end{bmatrix}. \end{aligned}$$

Thus

$$Q^{ob}(x) = \begin{bmatrix} x_1 & x_2 \\ -2x_1^2 + 2x_1x_2^2 & 2x_1^3 - 2x_1x_2 \end{bmatrix},$$

$$Q^{ob}(x)^{-1} = \frac{1}{\Delta} \begin{bmatrix} 2x_1^3 - 2x_1x_2 & -x_2 \\ 2x_1^2 - 2x_1x_2^2 & x_1 \end{bmatrix}$$

where

$$\Delta = 2x_1^4 - 2x_1x_2^3$$

and

$$\begin{aligned} \frac{\partial\left(Q^{ob}(x)x\right)}{\partial x} &= \frac{\partial}{\partial x}\begin{bmatrix} x_1^2 + x_2^2 \\ -2x_1^3 + 2x_1^2x_2^2 + 2x_1^3x_2 - 2x_1x_2^2 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & 2x_2 \\ -6x_1^2 + 4x_1x_2^2 + 6x_1^2x_2 - 2x_2^2 & 4x_1^2x_2 + 2x_1^3 - 4x_1x_2 \end{bmatrix}. \end{aligned}$$

Now $\bar{A}^{ob}(x)$, $\bar{b}^{ob}(x)$ and $\bar{c}^{ob}(x)$ can be calculated using (3.10a)-(3.10c). Thus,

$$\bar{A}^{ob}(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ -6x_1^2 + 4x_1x_2^2 + 6x_1^2x_2 - 2x_2^2 & 4x_1^2x_2 + 2x_1^3 - 4x_1x_2 \end{bmatrix}$$

$$\begin{bmatrix} -x_1 & x_1^2 \\ x_1x_2 & -x_1 \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} 2x_1^3 - 2x_1x_2 & -x_2 \\ 2x_1^2 - 2x_1x_2^2 & x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 & 2x_2 \\ -6x_1^2 + 4x_1x_2^2 + 6x_1^2x_2 - 2x_2^2 & 4x_1^2x_2 + 2x_1^3 - 4x_1x_2 \end{bmatrix}$$

$$\frac{1}{\Delta} \begin{bmatrix} 2x_1^2x_2 - 2x_1^3x_2^2 & x_1x_2 + x_1^3 \\ 2x_1^4x_2 - 2x_1^3 & -x_1x_2^2 - x_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{\Delta}\left(2x_1^4 - 2x_1x_2^3\right) \\ \bar{a}_{21}^{ob}(x) & \bar{a}_{22}^{ob}(x) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \bar{a}_{21}^{ob}(x) & \bar{a}_{22}^{ob}(x) \end{bmatrix},$$

where

$$\begin{aligned} \bar{a}_{21}^{ob}(x) &= \frac{1}{\Delta}\left(4x_1^7x_2 - 4x_1^6 + 8x_1^6x_2^2 - 8x_1^5x_2 + 12x_1^5x_2^2 - 8x_1^5x_2 - 12x_1^5x_2^3\right. \\ &\quad \left. - 4x_1^4x_2 + 12x_1^4x_2^2 - 8x_1^4x_2^4 + 8x_1^3x_2^3 + 4x_1^3x_2^4 - 4x_1^2x_2^3\right), \end{aligned}$$

$$\begin{aligned} \bar{a}_{22}^{ob}(x) &= \frac{1}{\Delta}\left(-8x_1^5 + 6x_1^5x_2 + 2x_1^4x_2^2 - 4x_1^4x_2 - 2x_1^3x_2\right. \\ &\quad \left. + 4x_1^3x_2^2 - 4x_1^3x_2^3 + 8x_1^2x_2^3 - 2x_1x_2^3\right), \end{aligned}$$

$$\begin{aligned} \bar{b}^{ob}(x) &= \begin{bmatrix} 2x_1 & 2x_2 \\ -6x_1^2 + 4x_1x_2^2 + 6x_1^2x_2 - 2x_2^2 & 4x_1^2x_2 + 2x_1^3 - 4x_1x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 \\ -6x_1^2 + 4x_1x_2^2 + 6x_1^2x_2 - 2x_2^2 \end{bmatrix}, \end{aligned}$$

and

$$\bar{c}^{o}(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \frac{1}{\Delta} \begin{bmatrix} 2x_1^3 - 2x_1x_2 & -x_2 \\ 2x_1^2 - 2x_1x_2^2 & x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Example D.1.2 A linear, n-th-order system.

$$\dot{x} = Ax + bu$$
$$y = cx.$$

Here, (D.5) gives

$$\begin{aligned} q_{1\cdot}^{ob} &= c \\ q_{2\cdot}^{ob} &= cA \\ q_{3\cdot}^{ob} &= cA^2 \\ &\vdots \\ q_{n\cdot}^{ob} &= cA^{n-1} \end{aligned}$$

or

$$Q^{ob} = \mathcal{O}.$$

The required transformation exists if and only if $\mathcal{O}$ has full rank. This result is, of course, consistent with that in any standard text on linear systems.

D.2 Controllability form

A nonlinear controllability form of the class of systems (3.1)-(3.2) is defined by analogy to the linear controllability form (e.g., in [23]) as

$$\begin{aligned} \dot{x}^{cb} &= A^{cb}\left(x^{cb}\right)x^{cb} + b^{cb}u \\ y &= c^{cb}\left(x^{cb}\right)x^{cb} \end{aligned} \tag{D.6}$$

where

$$A^{cb}\left(x^{cb}\right)=\begin{bmatrix}0 & \cdots & \cdots & 0 & a_{1n}^{cb}\left(x^{cb}\right)\\ 1 & \ddots & & \vdots & a_{2n}^{cb}\left(x^{cb}\right)\\ 0 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & 1 & a_{nn}^{cb}\left(x^{cb}\right)\end{bmatrix}, \qquad b^{cb}=\begin{bmatrix}1\\0\\ \vdots\\ \vdots\\0\end{bmatrix},$$

$$c^{cb}\left(x^{cb}\right)=\left[\begin{array}{cccccc} c_1^{cb}\left(x^{cb}\right) & c_2^{cb}\left(x^{cb}\right) & \cdots & \cdots & c_n^{cb}\left(x^{cb}\right)\end{array}\right], \tag{D.7}$$

and q is specified as cb.

In this case, the R.H.S. of (3.10a) becomes

$$\begin{aligned}\bar{A}^{cb}(x)Q^{cb}(x) &= \begin{bmatrix}0 & \cdots & \cdots & 0 & \bar{a}_{1n}^{cb}(x)\\ 1 & \ddots & & \vdots & \bar{a}_{2n}^{cb}(x)\\ 0 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & 1 & \bar{a}_{nn}^{cb}(x)\end{bmatrix}\begin{bmatrix}q_{1\cdot}^{cb}(x)\\ q_{2\cdot}^{cb}(x)\\ \vdots\\ \vdots\\ q_{n\cdot}^{cb}(x)\end{bmatrix}\\ &= \begin{bmatrix}\bar{a}_{1n}^{cb}(x)q_{n\cdot}^{cb}(x)\\ q_{1\cdot}^{cb}(x)+\bar{a}_{2n}^{cb}(x)q_{n\cdot}^{cb}(x)\\ q_{2\cdot}^{cb}(x)+\bar{a}_{3n}^{cb}(x)q_{n\cdot}^{cb}(x)\\ \vdots\\ \vdots\\ q_{(n-1)\cdot}^{cb}(x)+\bar{a}_{nn}^{cb}(x)q_{n\cdot}^{cb}(x)\end{bmatrix}.\end{aligned} \tag{D.8}$$

Combining (3.14) and (D.8) results in the following:

$$\left.\begin{aligned} q_{(n-1)\cdot}^{cb}(x) &= \mathcal{L}^1\left[q_{n\cdot}^{cb}(x)\right]-\bar{a}_{nn}^{cb}(x)q_{n\cdot}^{cb}(x)\\ q_{(n-2)\cdot}^{cb}(x) &= \mathcal{L}^1\left[q_{(n-1)\cdot}^{cb}(x)\right]-\bar{a}_{(n-1)n}^{cb}(x)q_{n\cdot}^{cb}(x)\\ &\vdots\\ q_{1\cdot}^{cb}(x) &= \mathcal{L}^1\left[q_{2\cdot}^{cb}(x)\right]-\bar{a}_{2n}^{cb}(x)q_{n\cdot}^{cb}(x)\end{aligned}\right\} \tag{D.9a}$$

and

$$\mathcal{L}^1\left[q_{1\cdot}^{cb}(x)\right] = \bar{a}_{1n}^{cb}(x) q_{n\cdot}^{cb}(x). \tag{D.9b}$$

Now upon employing (3.10b)

$$\begin{bmatrix} \partial(q_{1\cdot}^{cb}(x)x)/\partial x \\ \partial(q_{2\cdot}^{cb}(x)x)/\partial x \\ \vdots \\ \vdots \\ \partial(q_{n\cdot}^{cb}(x)x)/\partial x \end{bmatrix} b(x) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{D.9c}$$

Expressing $q_{(n-1)\cdot}^{cb}(x), q_{(n-2)\cdot}^{cb}(x), \ldots, q_{1\cdot}^{cb}(x)$ in terms of $q_{n\cdot}^{cb}(x)$ using (D.9a) results in

$$\left.\begin{aligned} q_{(n-1)\cdot}^{cb} &= \mathcal{L}^1\left[q_{n\cdot}^{cb}(x)\right] - \bar{a}_{nn}^{cb}(x) q_{n\cdot}^{cb}(x) \\ &= \mathcal{L}^1\left[q_{n\cdot}^{cb}(x)\right] - \textstyle\sum_{i=1}^{1} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-1)n}^{cb}(x) q_{n\cdot}^{cb}(x)\right] \\ q_{(n-2)\cdot}^{cb} &= \mathcal{L}^2\left[q_{n\cdot}^{cb}(x)\right] - \textstyle\sum_{i=2}^{2} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-2)n}^{cb}(x) q_{n\cdot}^{cb}(x)\right] \\ &\quad -\bar{a}_{(n-1)n}^{cb}(x) q_{n\cdot}^{cb}(x) \\ &= \mathcal{L}^2\left[q_{n\cdot}^{cb}(x)\right] - \textstyle\sum_{i=1}^{2} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-2)n}^{cb}(x) q_{n\cdot}^{cb}(x)\right] \\ &\ \vdots \\ q_{1\cdot}^{cb} &= \mathcal{L}^{n-1}\left[q_{n\cdot}^{cb}(x)\right] - \textstyle\sum_{i=2}^{n-1} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-(n-1))n}^{cb}(x) q_{n\cdot}^{cb}(x)\right] \\ &\quad -\bar{a}_{2n}^{cb}(x) q_{n\cdot}^{cb}(x) \\ &= \mathcal{L}^{n-1}\left[q_{n\cdot}^{cb}(x)\right] - \textstyle\sum_{i=1}^{n-1} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-(n-1))n}^{cb}(x) q_{n\cdot}^{cb}(x)\right]. \end{aligned}\right\} \tag{D.10}$$

Substituting(D.10) in (D.9b) gives

$$\mathcal{L}^n\left[q_{n\cdot}^{cb}(x)\right] - \sum_{i=2}^{n} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-(n))n}^{cb}(x) q_{n\cdot}^{cb}(x)\right] = \bar{a}_{1n}^{cb}(x) q_{n\cdot}^{cb}(x)$$

or

$$\mathcal{L}^n\left[q_{n\cdot}^{cb}(x)\right] = \sum_{i=1}^{n} \mathcal{L}^{i-1}\left[\bar{a}_{in}^{cb}(x) q_{n\cdot}^{cb}(x)\right]. \tag{D.11}$$

Using (D.10) in (D.9c) gives

$$\begin{bmatrix} \partial \ \left(\left(\mathcal{L}^{n-1}\left[q_{n\cdot}^{cb}(x)\right] \right.\right. \\ \left.\left. \quad - \sum_{i=1}^{n-1} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-(n-1))n}^{cb}(x) q_{n\cdot}^{cb}(x)\right]\right) x\right) / \partial x \\ \partial \ \left(\left(\mathcal{L}^{n-2}\left[q_{n\cdot}^{cb}(x)\right] \right.\right. \\ \left.\left. \quad - \sum_{i=1}^{n-2} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-(n-2))n}^{cb}(x) q_{n\cdot}^{cb}(x)\right]\right) x\right) / \partial x \\ \vdots \\ \partial \ \left(\left(\mathcal{L}^{1}\left[q_{n\cdot}^{cb}(x)\right] \right.\right. \\ \left.\left. \quad - \sum_{i=1}^{1} \mathcal{L}^{i-1}\left[\bar{a}_{(n+i-1)n}^{cb}(x) q_{n\cdot}^{cb}(x)\right]\right) x\right) / \partial x \\ \partial \ \left(\left(\mathcal{L}^{0}\left[q_{n\cdot}^{cb}(x)\right]\right) x\right) / \partial x \end{bmatrix} b(x) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}. \tag{D.12}$$

Proceeding as in 3.2.2, (D.12) can be rewritten in the form

$$\begin{aligned} &\frac{\partial}{\partial x}\left(q_{n\cdot}^{cb}(x)x\right)\Big[\ (-1)^0\left(ad^0 A(x)x, b(x)\right) \ \ (-1)^1\left(ad^1 A(x)x, b(x)\right) \ \cdots \\ &\quad \cdots \ (-1)^{n-1}\left(ad^{n-1}A(x)x, b(x)\right)\ \Big] \\ &-\frac{\partial}{\partial x}\left(\bar{a}_{nn}^{cb}(x) q_{n\cdot}^{cb}(x)x\right)\Big[\ 0 \ \ (-1)^0\left(ad^0 A(x)x, b(x)\right) \ \cdots \\ &\quad \cdots \ (-1)^{n-2}\left(ad^{n-2}A(x)x, b(x)\right)\ \Big] \\ &\vdots \\ &-\frac{\partial}{\partial x}\left(\bar{a}_{2n}^{cb}(x) q_{n\cdot}^{cb}(x)x\right)\Big[\ 0 \ \ 0 \ \cdots \ (-1)^0\left(ad^0 A(x)x, b(x)\right)\ \Big] \\ &= \Big[\ 0 \ \cdots \ 0 \ \ 1\ \Big] \end{aligned} \tag{D.13}$$

Thus, determining $Q^{cb}(x)$ can be done in two steps:

1. Solve (D.11) and (D.13) for $\bar{a}_{1n}^{cb}(x), \bar{a}_{2n}^{cb}(x), \ldots, \bar{a}_{nn}^{cb}(x)$ and $q_{n\cdot}^{cb}$.

2. Calculate $Q^{cb}(x)$ from (D.9a) (rewritten here as (D.14)).

$$
\left.\begin{aligned}
q^{cb}_{(n-1)\cdot}(x) &= \mathcal{L}^1\left[q^{cb}_{n\cdot}(x)\right] - \bar{a}^{cb}_{nn}(x)q^{cb}_{n\cdot}(x) \\
q^{cb}_{(n-2)\cdot}(x) &= \mathcal{L}^1\left[q^{cb}_{(n-1)\cdot}(x)\right] - \bar{a}^{cb}_{(n-1)n}(x)q^{cb}_{n\cdot}(x) \\
&\vdots \\
q^{cb}_{1\cdot}(x) &= \mathcal{L}^1\left[q^{cb}_{2\cdot}(x)\right] - \bar{a}^{cb}_{2n}(x)q^{cb}_{n\cdot}(x).
\end{aligned}\right\} \tag{D.14}
$$

Comments:

1. Here $\bar{a}^{cb}_{\cdot n}(x)$ and $q^{cb}_{n\cdot}(x)$ must be obtained simultaneously.

2. Eq. (D.11) represents n, n-th-order linear PDEs in n variables and $2n$ unknowns ($q^{cb}_{n\cdot}(x)x$ (one unknown), $\bar{a}^{cb}_{1n}(x)q^{cb}_{n\cdot}(x)$ (n unknowns), $\bar{a}^{cb}_{in}(x)q^{cb}_{n\cdot}(x)x$, $i = 2, 3, \ldots, n$ ($n-1$ unknowns)). Eq. (D.13) represents n, first-order linear PDEs in n variables and n unknowns ($q^{cb}_{n\cdot}(x)x$ (one unknown), $\bar{a}^{cb}_{in}(x)q^{cb}_{n\cdot}(x)x$, $i = 2, 3, \ldots, n$ ($n-1$ unknowns)). Together these equations constitute a system which is generally very difficult to solve.

3. Once this system has been solved, the calculation of $Q^{cb}(x)$ from (D.14) is relatively simple.

Example D.2.1 A simple, nonlinear, second-order system.

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} -x_2 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x.
\end{aligned}
$$

Here (D.13) becomes

$$
\begin{aligned}
&\begin{bmatrix} \frac{\partial}{\partial x_1}\left(q^{cb}_{2\cdot}(x)x\right) & -x_2\frac{\partial}{\partial x_1}\left(q^{cb}_{2\cdot}(x)x\right) + \frac{\partial}{\partial x_2}\left(q^{cb}_{2\cdot}(x)x\right) \end{bmatrix} \\
&-\begin{bmatrix} 0 & \frac{\partial}{\partial x_1}\left(\bar{a}^{cb}_{22}(x)q^{cb}_{2\cdot}(x)x\right) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix}.
\end{aligned}
$$

The first equation gives

$$\frac{\partial}{\partial x_1}\left(q_{2\cdot}^{cb}(x)x\right) = 0$$

or

$$q_{2\cdot}^{cb}(x) = \begin{bmatrix} 0 & q_{22}^{cb}(x_2) \end{bmatrix}.$$

Taking $\bar{a}_{22}^{cb}(x) = \bar{a}_{22}^{cb}(x_2)$ and $q_{22}^{cb}(x_2) = 1$ then gives one solution to the second equation, thus

$$q_{2\cdot}^{cb}(x_2) = q_{2\cdot}^{cb} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Eq. (D.11) here becomes

$$\mathcal{L}^2\left[q_{2\cdot}^{cb}(x)\right] = \mathcal{L}^1\left[\bar{a}_{22}^{cb}(x)q_{2\cdot}^{cb}(x)\right] + \mathcal{L}^0\left[\bar{a}_{12}^{cb}(x)q_{2\cdot}^{cb}(x)\right].$$

Calculating each term gives

$$\begin{aligned}
\mathcal{L}\left[\mathcal{L}\left[q_{2\cdot}^{cb}(x)\right]\right] &= \mathcal{L}\left[\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -x_2 & 0 \\ 1 & -1 \end{bmatrix}\right] \\
&= \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} -x_2 & 0 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -x_2 - 1 & 1 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^1\left[\bar{a}_{22}^{cb}(x)q_{2\cdot}^{cb}(x)\right] &= \frac{\partial}{\partial x}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right)A(x) \\
&= \begin{bmatrix} 0 & \frac{\partial}{\partial x_2}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right) \end{bmatrix}\begin{bmatrix} -x_2 & 0 \\ 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial}{\partial x_2}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right) & -\frac{\partial}{\partial x_2}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right) \end{bmatrix},
\end{aligned}$$

and

$$\mathcal{L}^0\left[\bar{a}_{12}^{cb}(x)q_{2\cdot}^{cb}(x)\right] = \begin{bmatrix} 0 & \bar{a}_{12}^{cb}(x) \end{bmatrix}.$$

These result in two equations

$$-x_2 - 1 \;=\; \frac{\partial}{\partial x_2}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right)$$

and

$$1 \;=\; -\frac{\partial}{\partial x_2}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right) + \bar{a}_{12}^{cb}(x).$$

Solving the former for $\bar{a}_{22}^{cb}(x_2)$ gives

$$\bar{a}_{22}^{cb}(x_2) = -\frac{1}{2}x_2 - 1.$$

Then the latter gives

$$\bar{a}_{12}^{cb}(x) = 1 + \frac{\partial}{\partial x_2}\left(\bar{a}_{22}^{cb}(x_2)\,x_2\right) = -x_2.$$

Now $q_{1\cdot}^{cb}(x)$ becomes from (D.14)

$$\begin{aligned} q_{1\cdot}^{cb}(x) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -x_2 & 0 \\ 1 & -1 \end{bmatrix} - \left(-\tfrac{1}{2}x_2 - 1\right)\begin{bmatrix} 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \tfrac{1}{2}x_2 \end{bmatrix}. \end{aligned}$$

Thus

$$Q^{cb}(x) = \begin{bmatrix} 1 & \tfrac{1}{2}x_2 \\ 0 & 1 \end{bmatrix},$$

$$Q^{cb}(x)^{-1} = \begin{bmatrix} 1 & -\tfrac{1}{2}x_2 \\ 0 & 1 \end{bmatrix}$$

and

$$\frac{\partial\left(Q^{cb}(x)x\right)}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} x_1 + \tfrac{1}{2}x_2^2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix}.$$

Now $\bar{A}^{cb}(x)$, $\bar{b}^{cb}(x)$ and $\bar{c}^{cb}(x)$ can be calculated using (3.10a)-(3.10c). Thus,

$$\bar{A}^{cb}(x) = \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -x_2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}x_2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -x_2 & \frac{1}{2}x_2^2 \\ 1 & -\frac{1}{2}x_2 - 1 \end{bmatrix} = \begin{bmatrix} 0 & -x_2 \\ 1 & -\frac{1}{2}x_2 - 1 \end{bmatrix}.$$

Note that this result is in agreement with the values previously obtained for $\bar{a}_{12}^{cb}(x)$ and $\bar{a}_{22}^{cb}(x)$. Further,

$$\bar{b}^{cb}(x) = \begin{bmatrix} 1 & x_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$\bar{c}^{cb}(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2}x_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}x_2 \end{bmatrix}.$$

Example D.2.2 A linear, n-th-order system.

$$\dot{x} = Ax + bu$$

$$y = cx$$

Eq. (D.13) gives ($\bar{a}_{in}^{cb} = a_{in}^{cb}$, $i = 1, \ldots, n$ from (3.9))

$$\begin{aligned} & q_{n\cdot}^{cb} \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix} \\ & -\bar{a}_{nn}^{cb} q_{n\cdot}^{cb} \begin{bmatrix} 0 & b & Ab & \cdots & A^{n-2}b \end{bmatrix} \\ & \vdots \\ & -\bar{a}_{2n}^{cb} q_{n\cdot}^{cb} \begin{bmatrix} 0 & \cdots & \cdots & 0 & b \end{bmatrix} \\ & = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \end{aligned}$$

or

$$\begin{bmatrix} q_{n\cdot}^{cb} \\ q_{n\cdot}^{cb}\left(A - a_{nn}^{cb} I\right) \\ \vdots \\ q_{n\cdot}^{cb}\left(A^{n-2} - a_{nn}^{cb}A^{n-3} - a_{(n-1)n}^{cb}A^{n-4} - \cdots - a_{3n}^{cb} I\right) \\ q_{n\cdot}^{cb}\left(A^{n-1} - a_{nn}^{cb}A^{n-2} - a_{(n-1)n}^{cb}A^{n-3} - \cdots - a_{2n}^{cb} I\right) \end{bmatrix} b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & & & \\ -a_{nn}^{cb} & 1 & \ddots & & \\ -a_{(n-1)n}^{cb} & -a_{nn}^{cb} & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{2n}^{cb} & -a_{3n}^{cb} & \cdots & -a_{nn}^{cb} & 1 \end{bmatrix} \begin{bmatrix} q_{n\cdot}^{cb} I \\ q_{n\cdot}^{cb} A \\ \vdots \\ q_{n\cdot}^{cb} A^{n-2} \\ q_{n\cdot}^{cb} A^{n-1} \end{bmatrix} b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

or

$$\begin{bmatrix} q_{n\cdot}^{cb} I \\ q_{n\cdot}^{cb} A \\ \vdots \\ q_{n\cdot}^{cb} A^{n-2} \\ q_{n\cdot}^{cb} A^{n-1} \end{bmatrix} b = \begin{bmatrix} 1 & 0 & & & \\ -a_{nn}^{cb} & 1 & \ddots & & \\ -a_{(n-1)n}^{cb} & -a_{nn}^{cb} & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_{2n}^{cb} & -a_{3n}^{cb} & \cdots & -a_{nn}^{cb} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

Thus,

$$q_{n\cdot}^{cb} \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{n-1} b \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

or

$$q_{n\cdot}^{cb} C = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

or

$$q_{n\cdot}^{cb} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} C^{-1}. \qquad \text{(D.15)}$$

Eq. (D.11) becomes

$$q_{n\cdot}^{cb}A^n = a_{1n}^{cb}q_{n\cdot}^{cb} + a_{2n}^{cb}q_{n\cdot}^{cb}A + \ldots + a_{nn}^{cb}q_{n\cdot}^{cb}A^{n-1}$$

or

$$q_{n\cdot}^{cb}\left(A^n - a_{nn}^{cb}A^{n-1} - \cdots - a_{2n}^{cb}A - a_{1n}^{cb}I\right) = 0. \tag{D.16}$$

Since $q_{n\cdot}^{cb}$ is arbitrary (from (D.15) since A and b are arbitrary), then

$$A^n - a_{nn}^{cb}A^{n-1} - \cdots - a_{2n}^{cb}A - a_{1n}^{cb}I = 0.$$

A comparison of this result with the Cayley-Hamilton theorem, which is valid for LTI sytems, reveals that $-a_{1n}^{cb}, -a_{2n}^{cb}, \ldots, -a_{nn}^{cb}$ are the coefficients of the characteristic equation (3.25), i.e., $a_1 = -a_{nn}^{cb}$, $a_2 = -a_{(n-1)n}^{cb}, \ldots, a_n = -a_{1n}^{cb}$. Eq. (D.14) finally gives

$$\begin{aligned}
q_{(n-1)\cdot}^{cb} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}\left(A - a_{nn}^{cb}I\right) \\
q_{(n-2)\cdot}^{cb} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}\left(A^2 - a_{nn}^{cb}A - a_{(n-1)n}^{cb}I\right) \\
&\vdots \\
q_{1\cdot}^{cb} &= \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}\left(A^{n-1} - a_{nn}^{cb}A^{n-2} - \cdots - a_{2n}^{cb}I\right).
\end{aligned}$$

The required transformation exists if and only if $\mathcal{C}$ has full rank. This result is, of course, consistent with that in any standard text on linear systems.

D.3 Diagonal form

A nonlinear diagonal form for the class of systems (3.1)-(3.2) is defined by analogy to the linear diagonal form (e.g., in [23]) as

$$\begin{aligned}
\dot{x}^d &= A^d\left(x^d\right)x^d + b^d\left(x^d\right)u \\
y &= c^d\left(x^d\right)x^d
\end{aligned} \tag{D.17}$$

where

$$A^d\left(x^d\right) = \begin{bmatrix} a_{11}^d\left(x^d\right) & 0 & \cdots & \cdots & 0 \\ 0 & a_{22}^d\left(x^d\right) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a_{nn}^d\left(x^d\right) \end{bmatrix},$$

$$b^d\left(x^d\right) = \begin{bmatrix} b_1^d\left(x^d\right) \\ b_2^d\left(x^d\right) \\ \vdots \\ \vdots \\ b_n^d\left(x^d\right) \end{bmatrix}, \tag{D.18}$$

$$c^d\left(x^d\right) = \begin{bmatrix} c_1^d\left(x^d\right) & c_2^d\left(x^d\right) & \cdots & \cdots & c_n^d\left(x^d\right) \end{bmatrix},$$

and q is specified as d. In this case, the R.H.S. of (3.10a) becomes

$$\begin{aligned} \bar{A}^d(x)Q^d(x) &= \begin{bmatrix} \bar{a}_{11}^d(x) & 0 & \cdots & \cdots & 0 \\ 0 & \bar{a}_{22}^d(x) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \bar{a}_{nn}^d(x) \end{bmatrix} \begin{bmatrix} q_{1\cdot}^d(x) \\ q_{2\cdot}^d(x) \\ \vdots \\ \vdots \\ q_{n\cdot}^d(x) \end{bmatrix} \\ &= \begin{bmatrix} \bar{a}_{11}^d(x)q_{1\cdot}^d(x) \\ \bar{a}_{22}^d(x)q_{2\cdot}^d(x) \\ \vdots \\ \vdots \\ \bar{a}_{nn}^d(x)q_{n\cdot}^d(x) \end{bmatrix}. \end{aligned} \tag{D.19}$$

Combining (3.14) and (D.19) results in

$$\left.\begin{array}{rcl}\mathcal{L}^1\left[q_{1\cdot}^d(x)\right] & = & \bar{a}_{11}^d(x)q_{1\cdot}^d(x) \\ \mathcal{L}^1\left[q_{2\cdot}^d(x)\right] & = & \bar{a}_{22}^d(x)q_{2\cdot}^d(x) \\ & \vdots & \\ \mathcal{L}^1\left[q_{n\cdot}^d(x)\right] & = & \bar{a}_{nn}^d(x)q_{n\cdot}^d(x).\end{array}\right\} \tag{D.20}$$

Thus, the $q_{i\cdot}^d(x)$'s are the "eigenvectors" and the $\bar{a}_{ii}^d(x)$'s, $i = 1,\ldots,n$ are the "eigenvalues" of the operator $\mathcal{L}$, and $Q^d(x)$ can be determined as follows:

1. Find n nonlinearly independent vectors $q_{1\cdot}^d(x), q_{2\cdot}^d(x), \ldots, q_{n\cdot}^d(x)$ and values $\bar{a}_{11}^d(x)$, $\bar{a}_{22}^d(x)$, ..., $\bar{a}_{nn}^d(x)$ such that each pair $q_{i\cdot}^d(x)$, $\bar{a}_{ii}^d(x)$ satisfies

$$\mathcal{L}^1\left[q_{i\cdot}^d(x)\right] = \bar{a}_{ii}^d(x)q_{i\cdot}^d(x) \quad i = 1,\ldots,n. \tag{D.21}$$

Comments:

1. Here $q_{1\cdot}^d(x)$, $\bar{a}_{11}^d(x)$ must be obtained simultaneously; $q_{2\cdot}^d(x)$, $\bar{a}_{22}^d(x)$ must be obtained simultaneously, etc.

2. The equations (D.21) to accomplish 1. are generally hard to solve.

3. Note that the state equations in this form are not decoupled, since in general $a_{ii}^d\left(x^d\right) \neq a_{ii}^d\left(x_i^d\right)$, $b_i^d\left(x^d\right) \neq b_i^d\left(x_i^d\right)$, $c_i^d\left(x^d\right) \neq c_i^d\left(x_i^d\right)$.

Example D.3.1 A simple, nonlinear, second-order system.

$$\dot{x} = \begin{bmatrix} -x_1 & x_1x_2 \\ x_1x_2 & -x_1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Here (D.21) becomes

$$\begin{bmatrix} \frac{\partial}{\partial x_1}\left(q_{i\cdot}^d(x)x\right) & \frac{\partial}{\partial x_2}\left(q_{i\cdot}^d(x)x\right) \end{bmatrix} \begin{bmatrix} -x_1 & x_1x_2 \\ x_1x_2 & -x_1 \end{bmatrix} = \bar{a}_{ii}^d(x)q_{i\cdot}^d(x).$$

This results in two equations

$$-x_1\frac{\partial}{\partial x_1}\left(q_{i\cdot}^d(x)x\right)+x_1x_2\frac{\partial}{\partial x_2}\left(q_{i\cdot}^d(x)x\right) = \bar{a}_{ii}^d(x)q_{i1}^d(x) \tag{D.22a}$$

and

$$x_1x_2\frac{\partial}{\partial x_1}\left(q_{i\cdot}^d(x)x\right)-x_1\frac{\partial}{\partial x_2}\left(q_{i\cdot}^d(x)x\right) = \bar{a}_{ii}^d(x)q_{i2}^d(x). \tag{D.22b}$$

By inspection $q_{1\cdot} = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $\bar{a}_{11}^d(x) = -x_1 + x_1x_2$ satisfies (D.22a) and (D.22b). By a similar inspection $q_{2\cdot}^d(x) = \begin{bmatrix} 1 & -1 \end{bmatrix}$ and $\bar{a}_{22}^d(x) = -x_1 - x_1x_2$ also satisfies (D.22a) and (D.22b). Since these $q_{1\cdot}^d(x)$ and $q_{2\cdot}^d(x)$ are independent, $Q^d(x)$ now becomes

$$Q^d(x) = Q^d = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$Q^{d^{-1}} = \left(-\frac{1}{2}\right)\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$\frac{\partial\left(Q^d x\right)}{\partial x} = Q^d$$

Now $\bar{A}^d(x)$, $\bar{b}^d(x)$ and $\bar{c}^d(x)$ can be calculated using (3.10a)-(3.10c). Thus,

$$\begin{aligned}
\bar{A}^d(x) &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -x_1 & x_1x_2 \\ x_1x_2 & -x_1 \end{bmatrix}\left(-\tfrac{1}{2}\right)\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\left(-\tfrac{1}{2}\right)\begin{bmatrix} x_1 - x_1x_2 & x_1 + x_1x_2 \\ -x_1x_2 + x_1 & -x_1x_2 - x_1 \end{bmatrix} \\
&= \begin{bmatrix} -x_1 + x_1x_2 & 0 \\ 0 & -x_1 - x_1x_2 \end{bmatrix}.
\end{aligned}$$

Note that this result is in agreement with the values previously obtained for $\bar{a}^d_{11}(x)$ and $\bar{a}^d_{22}(x)$. Further,

$$\bar{b}^d(x) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$\bar{c}^d(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(-\frac{1}{2}\right) \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Example D.3.2 A linear, n-th-order system.

$$\dot{x} = Ax + bu$$

$$y = cx.$$

Here (D.21) becomes ($\bar{a}^d_{ii} = a^d_{ii}$ from (3.9))

$$q^d_{i\cdot}A = a^d_{ii}q^d_{i\cdot} \quad i = 1, \ldots, n,$$

thus Q^d is composed of the (left) eigenvectors of A. The required transformation exists if and only if n linearly independent eigenvectors exist. This result is, of course, consistent with that in any standard text on linear systems.

D.4 Discussion

The main advantage of the observability and controllability forms in the case of LTI systems, is that their observability and controllability, respectively, is guaranteed structurally. Likewise the main advantages of the linear observer (controller) form are: a) the ease of constucting an observer (controller) b) its observability (controllability) is guaranteed structurally. Structural observability (controllability) will be defined in the next appendix and necessary and sufficient conditions for the structural observability (controllability) will be derived for the nonlinear observer and observability (controller and controllability) forms.

It is interesting to notice that the elements of and the transformation to the diagonal form are given by the eigenvalues and the eigenvectors, respectively, of the operator $\mathcal{L}$. Here, however, as opposed to the case of LTI systems the state equations are not decoupled.

APPENDIX E

STRUCTURAL OBSERVABILITY AND CONTROLLABILITY

E.1 Observer form and observability form

Structural observability is defined as follows (this definition is related to complete uniform observability in [65]):

Definition E.1 *A time-invariant system is structurally observable, if given one measurement at time t of the input and its first $n-2$ derivatives and the output and its first $n-1$ derivatives, one can calculate all the states at that time and this capability is independent of the system parameters.*

LTI systems which can be formulated in either the observer or observability forms (see [23] for definitions of these forms) are examples of structurally observable systems.

Theorem E.1 *The observer form (2.1)-(2.2) is structurally observable*[1] *iff*

$$\left.\begin{array}{l} a_{11}^{o}\left(x^{o}\right)=a_{11}^{o}\left(x_{1}^{o}\right) \\ a_{21}^{o}\left(x^{o}\right)=a_{21}^{o}\left(x_{1}^{o}, x_{2}^{o}\right) \\ \vdots \\ a_{n1}^{o}\left(x^{o}\right)=a_{n1}^{o}\left(x_{1}^{o}, x_{2}^{o}, \ldots, x_{n}^{o}\right) \end{array}\right\} \tag{E.1}$$

and

[1]Note that in the case of output dependent nonlinearities, $A^{o}(y)$ and $b^{o}(y)$ are of the special form (E.1) and (E.2) and thus the form (2.21) is structurally observable.

$$\left.\begin{aligned}
&b_1^o(x^o) = b_1^o\left(x_1^o\right)\\
&b_2^o(x^o) = b_2^o\left(x_1^o, x_2^o\right)\\
&\vdots\\
&b_n^o(x^o) = b_n^o\left(x_1^o, x_2^o, \ldots, x_n^o\right).
\end{aligned}\right\} \qquad (E.2)$$

Proof: First assume (E.1) and (E.2) hold. Here the $n-1$ first state equations and the output equation of (2.1)-(2.2) are given by

$$\begin{aligned}
\dot{x}_1^o &= a_{11}^o\left(x_1^o\right)x_1^o + x_2^o + b_1^o\left(x_1^o\right)u\\
\dot{x}_2^o &= a_{21}^o\left(x_1^o, x_2^o\right)x_1^o + x_3^o + b_2^o\left(x_1^o, x_2^o\right)u\\
&\vdots\\
\dot{x}_{n-1}^o &= a_{(n-1)1}^o\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right)x_1^o + x_n^o + b_{n-1}^o\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right)u\\
y &= x_1^o.
\end{aligned}$$

Thus, given one measurement of the input and its first $n-2$ derivatives and the output and its first $n-1$ derivatives, one can determine $x_1^o, x_2^o, \ldots, x_n^o$ by calculating

$$\begin{aligned}
x_1^o &= y = g_1(y)\\
x_2^o &= \dot{x}_1^o - a_{11}^o\left(x_1^o\right)x_1^o - b_1^o\left(x_1^o\right)u = g_2(y, \dot{y}, u)\\
x_3^o &= \dot{x}_2^o - a_{21}^o\left(x_1^o, x_2^o\right)x_1^o - b_2^o\left(x_1^o, x_2^o\right)u\\
&= \ddot{x}_1^o - \frac{\partial}{\partial x_1}\left(a_{11}^o\left(x_1^o\right)x_1^o\right)\dot{x}_1^o - \frac{\partial}{\partial x_1}\left(b_1^o\left(x_1^o\right)\right)\dot{x}_1^o u - b_1^o\left(x_1^o\right)\dot{u}\\
&\quad - a_{21}^o\left(x_1^o, x_2^o\right)x_1^o - b_2^o\left(x_1^o, x_2^o\right)u\\
&= g_3(y, \dot{y}, \ddot{y}, u, \dot{u})\\
&\vdots\\
x_n^o &= \dot{x}_{n-1}^o - a_{(n-1)1}^o\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right)x_1^o - b_{n-1}^o\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right)u\\
&= g_n\left(y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-2)}\right).
\end{aligned}$$

Now assume that the form (2.1)-(2.2) is structurally observable. Then it is structurally observable for $u = 0$, and the $n-1$ first state equations become

$$\begin{aligned}
\dot{x}_1^o &= a_{11}^o(x^o)\,x_1^o + x_2^o \\
\dot{x}_2^o &= a_{21}^o(x^o)\,x_1^o + x_3^o \\
&\vdots \\
\dot{x}_{(n-1)1}^o &= a_{(n-1)1}^o(x^o)\,x_1^o + x_n^o.
\end{aligned} \tag{E.3}$$

The output equation and the first $n-1$ derivatives of y give

$$\begin{aligned}
y &= x_1^o \\
\dot{y} &= a_{11}^o(x^o)\,x_1^o + x_2^o = f_1(x^o) + x_2^o \\
\ddot{y} &= \tfrac{d}{dt}\left(a_{11}^o(x^o)\,x_1^o\right) + a_{21}^o(x^o)\,x_1^o + x_3^o = f_2(x^o) + x_3^o \\
y^{(3)} &= \tfrac{d^2}{dt^2}\left(a_{11}^o(x^o)\,x_1^o\right) + \tfrac{d}{dt}\left(a_{21}^o(x^o)\,x_1^o\right) + a_{31}^o(x^o)\,x_1^o + x_4^o \\
&= f_3(x^o) + x_4^o \\
&\vdots \\
y^{(n-1)} &= \tfrac{d^{n-2}}{dt^{n-2}}\left(a_{11}^o(x^o)\,x_1^o\right) + \tfrac{d^{n-3}}{dt^{n-3}}\left(a_{21}^o(x^o)\,x_1^o\right) + \cdots \\
&\quad \cdots + a_{(n-1)1}^o(x^o)\,x_1^o + x_n^o = f_{n-1}(x^o) + x_n^o
\end{aligned} \tag{E.4}$$

Since the system is observable, there must be information on each state in at least one $y^{(k)}$, $k = 0, 1, \ldots, n-1$. Thus at least one $y^{(k)}$ must depend on x_n^o. Now $\dot{y}, \ddot{y}, \ldots, y^{(n-2)}$ may or may not depend on x_n^o ($\dot{y}$ depends on x_n^o if $f_1(x^o)$ does, $\ddot{y}$ depends on x_n^o if $f_2(x^o)$ does, etc.). If, however, one requires $f_{n-1}(x^o) = f_{n-1}\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right)$ then it is structurally guaranteed that $y^{(n-1)}$ depends on x_n , since there is nothing in $f_{n-1}\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right)$ that can cancel the effect of x_n. One then has

$$f_{n-1}\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right) = \sum_{i=0}^{n-2} \frac{d^i}{dt^i}\left(a_{(n-1-i)1}^o(x^o)\,x_1^o\right),$$

and thus must require that each of these terms is independent of x_n^o (otherwise it is not structurally guaranteed that f_{n-1} will be independent of x_n^o), i.e.,

$$\frac{d^i}{dt^i}\left(a_{(n-1-i)1}^o(x^o)\,x_1^o\right) = f_{(i)(n-1-i)}\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right) \tag{E.5}$$

for $i = 0, 1, \ldots, n-2$. Using Lemma E.1 this gives

$$\begin{aligned}
a_{11}^o(x^o) &= a_{11}^o\left(x_1^o\right) \\
a_{21}^o(x^o) &= a_{21}^o\left(x_1^o, x_2^o\right) \\
&\vdots \\
a_{(n-1)1}^o(x^o) &= a_{(n-1)1}^o\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right) \\
a_{n1}^o(x^o) &= a_{n1}^o\left(x_1^o, x_2^o, \ldots, x_n^o\right).
\end{aligned}$$

Using this in (E.4) gives

$$\begin{aligned}
f_1(x^o) &= f_1\left(x_1^o\right) \\
f_2(x^o) &= f_2\left(x_1^o, x_2^o\right) \\
&\vdots \\
f_{n-1}(x^o) &= f_{n-1}\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right) \\
f_n(x^o) &= f_n\left(x_1^o, x_2^o, \ldots, x_n^o\right).
\end{aligned}$$

and thus from (E.4) it is guaranteed that there is information on each state in at least one $y^{(k)}$, $k = 0, 1, \ldots, n-1$.

Now suppose $u \neq 0$, then the $n-1$ first state equations are given by

$$\begin{aligned}
\dot{x}_1^o &= a_{11}^o\left(x_1^o\right) x_1^o + x_2^o + b_1^o(x^o) u \\
\dot{x}_2^o &= a_{21}^o\left(x_1^o, x_2^o\right) x_1^o + x_3^o + b_2^o(x^o) u \\
&\vdots \\
\dot{x}_{n-1}^o &= a_{(n-1)1}^o\left(x_1^o, x_2^o, \ldots, x_{n-1}^o\right) x_1^o + x_n^o + b_{n-1}^o(x^o) u
\end{aligned}$$

The next part of the proof follows an idea from [65]. Here the output equation and the first $n-1$ derivatives of y give

$$
\begin{aligned}
y &= x_1^o \\
\dot{y} &= a_{11}^o\left(x_1^o\right)x_1^o + x_2^o + b_1^o\left(x^o\right)u \\
&= h_{11}\left(x_1^o, x_2^o\right) + b_1^o\left(x^o\right)u \\
\ddot{y} &= \frac{\partial}{\partial x_1^o}\left(h_{11}\left(x_1^o, x_2^o\right)\right)\dot{x}_1^o + \frac{\partial}{\partial x_2^o}\left(h_{11}\left(x_1^o, x_2^o\right)\right)\dot{x}_2^o \\
&\quad + \frac{\partial}{\partial x^o}\left(b_1^o\left(x^o\right)\right)\dot{x}^o u + b_1^o\left(x^o\right)\dot{u} \\
&= h_{12}\left(x^o, u\right) + b_1^o\left(x^o\right)\dot{u} \qquad \text{(E.6)} \\
y^{(3)} &= \frac{\partial}{\partial x^o}\left(h_{12}\left(x^o, u\right)\right)\dot{x}^o + \frac{\partial}{\partial u}\left(h_{12}\left(x^o, u\right)\right)\dot{u} \\
&\quad + \frac{\partial}{\partial x^o}\left(b_1^o\left(x^o\right)\right)\dot{x}^o\dot{u} + b_1^o\left(x^o\right)\ddot{u} \\
&= h_{13}\left(x^o, u, \dot{u}\right) + b_1^o\left(x^o\right)\ddot{u} \\
&\vdots \\
y^{(n-1)} &= h_{1(n-1)}\left(x^o, u, \dot{u}, \ldots, u^{(n-3)}\right) + b_1^o\left(x^o\right)u^{(n-2)}
\end{aligned}
$$

From an examination of the triangular structure of (E.6), relative to the successive derivatives, it can be seen that $u, \dot{u}, \ldots, u^{(n-2)}$ can be successively selected in such a way that no $y^{(k)}$ is depending upon x_j^o , $j > 1$ at some point $x^o(t)$. Thus $\partial b_1^o\left(x^o\right)/\partial x_j^o = 0$ or $b_1^o\left(x^o\right) = b_1^o\left(x_1^o\right)$. Now suppose $b_1^o\left(x^o\right)$ depends on $x_1^o, x_2^o, \ldots, x_k^o$ only for $k = 1, 2, \ldots, l-1$. Then the $n-1$ first state equations are

$$
\begin{aligned}
\dot{x}_1^o &= a_{11}^o\left(x_1^o\right)x_1^o + x_2^o + b_1^o\left(x_1^o\right)u \\
\dot{x}_2^o &= a_{21}^o\left(x_1^o, x_2^o\right)x_1^o + x_3^o + b_2^o\left(x_1^o, x_2^o\right)u \\
&\vdots \\
\dot{x}_{l-1}^o &= a_{(l-1)1}^o\left(x_1^o, x_2^o, \ldots, x_{l-1}^o\right)x_1^o + x_l^o + b_{l-1}^o\left(x_1^o, x_2^o, \ldots, x_{l-1}^o\right)u \\
\dot{x}_l^o &= a_{l1}^o\left(x_1^o, x_2^o, \ldots, x_l^o\right)x_1^o + x_{l+1}^o + b_l^o\left(x^o\right)u \\
&\vdots \\
\dot{x}_{n-1}^o &= a_{(n-1)1}^o\left(x^o\right)x_1^o + x_n^o + b_{n-1}^o\left(x^o\right)u.
\end{aligned}
$$

Here y and the first $n-1$ derivatives of y give

$$
\begin{aligned}
y &= x_1^o \\
\dot{y} &= a_{11}^o\left(x_1^o\right)x_1^o + x_2^o + b_1^o\left(x_1^o\right)u \\
&= h_{l1}\left(x_1^o, u\right) + x_2^o \\
\ddot{y} &= \frac{\partial}{\partial x_1^o}\left(h_{l1}\left(x_1^o, u\right)\right)\dot{x}_1^o + \frac{\partial}{\partial u}\left(h_{l1}\left(x_1^o, u\right)\right)\dot{u} + a_{21}^o\left(x_1^o, x_2^o\right)x_1^o \\
&\quad + x_3^o + b_2^o\left(x_1^o, x_2^o\right)u \\
&= h_{l2}\left(x_1^o, x_2^o, u, \dot{u}\right) + x_3^o \qquad \text{(E.7)} \\
&\vdots \\
y^{(l-1)} &= h_{l(l-1)}\left(x_1^o, x_2^o \ldots, x_{l-1}^o, u, \dot{u}, \ldots, u^{(l-2)}\right) + x_l^o \\
y^{(l)} &= h_{ll}\left(x_1^o, x_2^o \ldots, x_l^o, u, \dot{u}, \ldots, u^{(l-1)}\right) + x_{l+1}^o + b_l^o\left(x^o\right)u \\
&\vdots \\
y^{(n-1)} &= h_{l(n-1)}\left(x^o, u, \dot{u}, \ldots, u^{(n-2)}\right) + b_l^o\left(x^o\right)u^{(n-(l+1))}
\end{aligned}
$$

From examination of the triangular structure of (E.7) relative to the successive derivatives of u, it can be seen that $u, \dot{u}, \ldots, u^{(n-(l-1))}$ can be successively selected in such a way that no $y^{(k)}$ is depending upon x_j, $j > l$ at some point $x^o(t)$. Thus $\partial b_l^o\left(x^o\right)/\partial x_j^o = 0$ or equivalently

$$b_l^o\left(x^o\right) = b_l^o\left(x_1^o, x_2^o, \ldots, x_l^o\right).$$

Thus

$$
\begin{aligned}
b_1^o\left(x^o\right) &= b_1^o\left(x_1^o\right) \\
b_2^o\left(x^o\right) &= b_2^o\left(x_1^o, x_2^o\right) \\
&\vdots \\
b_n^o\left(x^o\right) &= b_n^o\left(x_1^o, x_2^o, \ldots, x_n^o\right).
\end{aligned}
$$

This concludes the proof of Theorem E.1.

□

Lemma E.1 *If (E.3) and (E.5) hold then*

$$a^o_{(n-1-i)1}(x^o) = a^o_{(n-1-i)1}\left(x^o_1, x^o_2, \ldots, x^o_{n-1-i}\right) \qquad i = 0, 1, \ldots, n-2$$

Proof: The following is given:

$$\frac{d^i}{dt^i}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right) = f_{(i)(n-1-i)}\left(x^o_1, x^o_2, \ldots, x^o_{n-1}\right),$$

$i = 0, 1, \ldots, n-2$. However, one has

$$\begin{aligned}
\frac{\partial}{\partial x^o_{n-1}} & \left(\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)\right) \dot{x}^o_{n-1} \\
& = \frac{\partial}{\partial x^o_{n-1}}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)\right)\left(a^o_{(n-1)1}(x^o)\, x^o_1 + x^o_n\right) \\
\frac{\partial}{\partial x^o_n} & \left(\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)\right) \dot{x}^o_n \\
& = \frac{\partial}{\partial x^o_n}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)\right) a^o_{n1}(x^o)\, x^o_1;
\end{aligned}$$

thus $\frac{d^i}{dt^i}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)$ is not guaranteed to be independent of x^o_n unless

$$\frac{\partial}{\partial x^o_{n-1}}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)\right) = 0$$

and

$$\frac{\partial}{\partial x^o_n}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right)\right) = 0.$$

This gives

$$\frac{d^{i-1}}{dt^{i-1}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right) = f_{(i-1)(n-1-i)}\left(x^o_1, x^o_2, \ldots, x^o_{n-2}\right).$$

Now suppose

$$\frac{d^{i-l}}{dt^{i-l}}\left(a^o_{(n-1-i)1}(x^o)\, x^o_1\right) = f_{(i-l)(n-1-i)}\left(x^o_1, x^o_2, \ldots, x^o_{n-1-l}\right)$$

for $l = 1, 2, \ldots, k-1$, then

$$\frac{\partial}{\partial x^o_{n-1-l+m}} \left(\frac{d^{i-l-1}}{dt^{i-l-1}} \left(a^o_{(n-1-i)1}(x^o)\, x^o_1 \right) \right) \dot{x}^o_{n-1-l+m}$$
$$= \frac{\partial}{\partial x^o_{n-1-l+m}} \left(\frac{d^{i-l-1}}{dt^{i-l-1}} \left(a^o_{(n-1-i)1}(x^o)\, x^o_1 \right) \right)$$
$$\left(a^o_{(n-1-l+m)1}(x^o)\, x^o_1 + x^o_{n-l+m} \right)$$
$$m = 0, 1, \ldots, l+1$$

thus $\frac{d^{i-l}}{dt^{i-l}} \left(a^o_{(n-1-i)1}(x^o)\, x^o_1 \right)$ is not guaranteed independent of x^o_{n-l+m} unless

$$\frac{\partial}{\partial x^o_{n-1-l+m}} \left(\frac{d^{i-l-1}}{dt^{i-l-1}} \left(a^o_{(n-1-i)1}(x^o)\, x^o_1 \right) \right) = 0.$$

This gives

$$\frac{d^{i-l-1}}{dt^{i-l-1}} \left(a^o_{(n-1-i)1}(x^o)\, x^o_1 \right) = f_{(i-l-1)(n-1-i)} \left(x^o_1, x^o_2, \ldots, x^o_{n-2-l} \right). \qquad \text{(E.8)}$$

Therefore it has been proven that (substitute $l = i-1$ in (E.8))

$$a^o_{(n-1-i)1}(x^o) = a^o_{(n-1-i)1} \left(x^o_1, x^o_2, \ldots, x^o_{n-1-i} \right)$$

for $i = 0, 1, \ldots, n-2$.

□

Theorem E.2 *The observability form (D.1)-(D.2) is structurally observable iff*

$$\left.\begin{array}{l} b^{ob}_1\left(x^{ob}\right) = b^{ob}_1\left(x^o_1\right) \\ b^{ob}_2\left(x^{ob}\right) = b^{ob}_2\left(x^{ob}_1, x^{ob}_2\right) \\ \vdots \\ b^{ob}_n\left(x^{ob}\right) = b^{ob}_n\left(x^{ob}_1, x^{ob}_2, \ldots, x^{ob}_n\right). \end{array}\right\} \qquad \textit{(E.9)}$$

Note that as a consequence of Theorem E.2, all systems in observability form with no input are structurally observable.

Proof: Proceeding similarly as when proving Theorem E.1, first assume (E.9) holds. Here the $n-1$ first state equations and the output equation of (D.1)-(D.2) are given by

$$\begin{aligned}
\dot{x}_1^{ob} &= x_2^{ob} + b_1^{ob}\left(x_1^{ob}\right)u \\
\dot{x}_2^{ob} &= x_3^{ob} + b_2^{ob}\left(x_1^{ob}, x_2^{ob}\right)u \\
&\vdots \\
\dot{x}_{n-1}^{ob} &= x_n^{ob} + b_{n-1}^{ob}\left(x_1^{ob}, x_2^{ob}, \ldots, x_{n-1}^{ob}\right)u \\
y &= x_1^{ob}.
\end{aligned}$$

Thus, given one measurement of the input and its first $n-2$ derivatives and the output and its first $n-1$ derivatives, one can determine $x_1^{ob}, x_2^{ob}, \ldots, x_n^{ob}$ by calculating

$$\begin{aligned}
x_1^{ob} &= y = g_1(y) \\
x_2^{ob} &= \dot{x}_1^{ob} - b_1^{ob}\left(x_1^{ob}\right)u = g_2\left(y, \dot{y}, u\right) \\
x_3^{ob} &= \dot{x}_2^{ob} - b_2^{ob}\left(x_1^{ob}, x_2^{ob}\right)u \\
&= \ddot{x}_1^{ob} - \frac{\partial}{\partial x_1}\left(b_1^{ob}\left(x_1^{ob}\right)\right)\dot{x}_1^{ob}u - b_1^{ob}\left(x_1^{ob}\right)\dot{u} - b_2^{ob}\left(x_1^{ob}, x_2^{ob}\right)u \\
&= g_3\left(y, \dot{y}, \ddot{y}, u, \dot{u}\right) \\
&\vdots \\
x_n^{ob} &= \dot{x}_{n-1}^{ob} - b_{n-1}^{ob}\left(x_1^{ob}, x_2^{ob}, \ldots, x_{n-1}^{ob}\right)u \\
&= g_n\left(y, \dot{y}, \ddot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-2)}\right).
\end{aligned}$$

Now assume that the form (D.1)-(D.2) is structurally observable. The $n-1$ first state equations are given by

$$\begin{aligned}
\dot{x}_1^{ob} &= x_2^{ob} + b_1^{ob}\left(x^{ob}\right)u \\
\dot{x}_2^{ob} &= x_3^{ob} + b_2^{ob}\left(x^{ob}\right)u \\
&\vdots \\
\dot{x}_{n-1}^{ob} &= x_n^{ob} + b_{n-1}^{ob}\left(x^{ob}\right)u
\end{aligned}$$

The next part of the proof follows a similar one in [65] for a different class of systems. Here the output equation and the first $n-1$ derivatives of y give

$$\begin{aligned}
y &= x_1^{ob} \\
\dot{y} &= x_2^{ob} + b_1^{ob}\left(x^{ob}\right)u \\
\ddot{y} &= x_3^{ob} + b_2^{ob}\left(x^{ob}\right)u + \frac{\partial}{\partial x^{ob}}\left(b_1^{ob}\left(x^{ob}\right)\right)u\dot{x}^{ob} + b_1^{ob}\left(x^{ob}\right)\dot{u} \\
&= h_{12}\left(x^{ob},u\right) + b_1^{ob}\left(x^{ob}\right)\dot{u} \\
y^{(3)} &= \frac{\partial}{\partial x^{ob}}\left(h_{12}\left(x^{ob},u\right)\right)\dot{x}^{ob} + \frac{\partial}{\partial u}\left(h_{12}\left(x^{ob},u\right)\right)\dot{u} \\
&\quad + \frac{\partial}{\partial x^{ob}}\left(b_1^{ob}\left(x^{ob}\right)\right)\dot{x}^{ob}\dot{u} + b_1^{ob}\left(x^{ob}\right)\ddot{u} \\
&= h_{13}\left(x^{ob},u,\dot{u}\right) + b_1^{ob}\left(x^{ob}\right)\ddot{u} \\
&\vdots \\
y^{(n-1)} &= h_{1(n-1)}\left(x^{ob},u,\dot{u},\ldots,u^{(n-3)}\right) + b_1^{ob}\left(x^{ob}\right)u^{(n-2)}
\end{aligned} \tag{E.10}$$

From examination of the triangular structure of (E.10) relative to the successive derivatives, it can be seen that $u, \dot{u}, \ldots, u^{(n-2)}$ can be successively selected in such a way that no $y^{(k)}$ is depending upon x_j^o , $j > 1$ at some point $x^o(t)$. Thus $\partial b_1^o(x^o)/\partial x_j^o = 0$ and $b_1^o(x^o) = b_1^o\left(x_1^o\right)$. Now suppose $b_1^o(x^o)$ depends on $x_1^o, x_2^o, \ldots, x_i^o$ only for $i = 1, 2, \ldots, l-1$. Then the $n-1$ first state equations are

$$\begin{aligned}
\dot{x}_1^{ob} &= x_2^{ob} + b_1^{ob}\left(x_1^{ob}\right)u \\
\dot{x}_2^{ob} &= x_3^{ob} + b_2^{ob}\left(x_1^{ob},x_2^{ob}\right)u \\
&\vdots \\
\dot{x}_{l-1}^{ob} &= x_l^{ob} + b_{l-1}^{ob}\left(x_1^{ob},x_2^{ob},\ldots,x_{l-1}^{ob}\right)u \\
\dot{x}_l^{ob} &= x_{l+1}^{ob} + b_l^{ob}\left(x^{ob}\right)u \\
&\vdots \\
\dot{x}_{n-1}^{ob} &= x_n^{ob} + b_{n-1}^{ob}\left(x^{ob}\right)u.
\end{aligned}$$

Here y and the first $n-1$ derivatives of y give

$$
\begin{aligned}
y &= x_1^{ob} \\
\dot{y} &= x_2^{ob} + b_1^{ob}\left(x_1^{ob}\right) u \\
&= h_{l1}\left(x_1^{ob}, u\right) + x_2^{ob} \\
\ddot{y} &= \frac{\partial}{\partial x_1^{ob}}\left(h_{l1}\left(x_1^{ob}, u\right)\right) \dot{x}_1^{ob} + \frac{\partial}{\partial u}\left(h_{l1}\left(x_1^{ob}, u\right)\right) \dot{u} \\
&\quad + x_3^{ob} + b_2^{ob}\left(x_1^{ob}, x_2^{ob}\right) u \\
&= h_{l2}\left(x_1^{ob}, x_2^{ob}, u, \dot{u}\right) + x_3^{ob} \qquad \text{(E.11)} \\
&\vdots \\
y^{(l-1)} &= h_{l(l-1)}\left(x_1^{ob}, x_2^{ob} \ldots, x_{l-1}^{ob}, u, \dot{u}, \ldots, u^{(l-2)}\right) + x_l^{ob} \\
y^{(l)} &= h_{ll}\left(x_1^{ob}, x_2^{ob} \ldots, x_l^{ob}, u, \dot{u}, \ldots, u^{(l-1)}\right) + x_{l+1}^{ob} + b_l^{ob}\left(x^{ob}\right) u \\
&\vdots \\
y^{(n-1)} &= h_{l(n-1)}\left(x^{ob}, u, \dot{u}, \ldots, u^{(n-2)}\right) + b_l^{ob}\left(x^{ob}\right) u^{(n-(l+1))}
\end{aligned}
$$

From an examination of the triangular structure of (E.11) relative to the successive derivatives of u, it can be seen that $u, \dot{u}, \ldots, u^{(n-(l-1))}$ can be successively selected in such a way that no $y^{(k)}$ is depending upon x_j, $j > l$ at some point $x^{ob}(t)$. Thus $\partial b_l^{ob}\left(x^{ob}\right)/\partial x_j^{ob} = 0$ or equivalently

$$
b_l^{ob}\left(x^{ob}\right) = b_l^{ob}\left(x_1^{ob}, x_2^{ob}, \ldots, x_l^{ob}\right).
$$

Thus

$$
\begin{aligned}
b_1^{ob}\left(x^{ob}\right) &= b_1^{ob}\left(x_1^{ob}\right) \\
b_2^{ob}\left(x^{ob}\right) &= b_2^{ob}\left(x_1^{ob}, x_2^{ob}\right) \\
&\vdots \\
b_n^{ob}\left(x^{ob}\right) &= b_n^{ob}\left(x_1^{ob}, x_2^{ob}, \ldots, x_n^{ob}\right).
\end{aligned}
$$

This concludes the proof of Theorem E.2.

□

E.2 Controller form and controllability form

Structural controllability is defined as follows [66]:

Definition E.2 *A time-invariant system is structurally controllable on the interval* $\left[t_0, t_f\right]$*, if for every*

$$x(t_0) = \begin{bmatrix} p_1 & p_2 & \cdots & \cdots & p_n \end{bmatrix}^T \qquad (E.12)$$

and any other

$$x\left(t_f\right) = \begin{bmatrix} q_1 & q_2 & \cdots & \cdots & q_n \end{bmatrix}^T \qquad (E.13)$$

there exists a calculable control function $u(t)$*,* $t \epsilon \left[t_0, t_f\right]$ *that takes the system from* $x(t_0)$ *to* $x\left(t_f\right)$ *and the existence of* $u(t)$ *is independent of system parameters.*

LTI systems which can be formulated in either the controller or controllability forms (see [23] for definitions of these forms) are examples of structurally controllable systems.

Theorem E.3 *The controller form (2.25)-(2.26) is structurally controllable.*

Proof: This proof follows an idea from [66]. Define an n-times differentiable function

$$\phi \epsilon C^n\left(\left[t_0, t_f\right]\right)$$

where

$$\phi^{(i)}(t_0) = \alpha_i \qquad \phi^{(i)}\left(t_f\right) = \beta_i \qquad i = 0, 1, \ldots, n-1. \qquad (E.14)$$

Then assume that

$$\phi(t) = x_n^c(t) \qquad t \epsilon \left[t_0, t_f\right]. \qquad (E.15)$$

The $n-1$ last state equations of (2.25)-(2.26) give

$$\begin{aligned}
\dot{x}_n^c &= x_{n-1}^c = \phi^{(1)} \\
\dot{x}_{n-1}^c &= x_{n-2}^c = \phi^{(2)} \\
&\vdots \\
\dot{x}_3^c &= x_2^c = \phi^{(n-2)} \\
\dot{x}_2^c &= x_1^c = \phi^{(n-1)}.
\end{aligned} \tag{E.16}$$

Thus the boundary values (E.14) are functions of the given initial and final values (E.13) of the state trajectory $x(t)$, $t\epsilon\left[t_0, t_f\right]$ and are thereby determined as

$$\begin{aligned}
\alpha_i &= p_{i+1} \\
\beta_i &= q_{i+1} \qquad\qquad i = 0, 1, \ldots, n-1.
\end{aligned}$$

The control $u(t)$ can be calculated from the first state equation of (2.25)-(2.26) as

$$u = \dot{x}_1^c - a_{11}^c\left(x^c\right)x_1^c - a_{12}^c\left(x^c\right)x_2^c - \cdots - a_{1n}^c\left(x^c\right)x_n^c \tag{E.17}$$

and using (E.15) and (E.16) in (E.17), it can be expressed in terms of ϕ as

$$u(t) = \phi^{(n)}(t) - \sum_{i=1}^{n} a_{1(n+1-i)}^c\left(\phi^{(n-1)}(t), \phi^{(n-2)}(t), \ldots, \phi^1(t)\right)\phi(t).$$

Thus, the required control function exists independently of the system structure, and therefore, the form (2.25)-(2.26) is structurally controllable.

□

Theorem E.4 *The controllability form (D.6)-(D.7) is structurally controllable iff*

$$\left.\begin{aligned}
a_{1n}^{cb}\left(x^{cb}\right) &= a_{1n}^{cb}\left(x_1^{cb}, x_2^{cb}, \ldots, x_n^{cb}\right)\\
a_{2n}^{cb}\left(x^{cb}\right) &= a_{2n}^{cb}\left(x_2^{cb}, x_3^{cb}, \ldots, x_n^{cb}\right)\\
a_{3n}^{cb}\left(x^{cb}\right) &= a_{3n}^{cb}\left(x_3^{cb}, x_4^{cb}, \ldots, x_n^{cb}\right)\\
&\vdots\\
a_{(n-1)n}^{cb}\left(x^{cb}\right) &= a_{(n-1)n}^{cb}\left(x_{n-1}^{cb}, x_n^{cb}\right)\\
a_{nn}^{cb}\left(x^{cb}\right) &= a_{nn}^{cb}\left(x_n^{cb}\right).
\end{aligned}\right\} \tag{E.18}$$

Proof: First assume (E.18) holds. This part of the proof follows a similar one in [66] for a different class of systems. Here again define an n-times n-times differentiable function

$$\phi \epsilon C^n\left(\left[t_0, t_f\right]\right)$$

where

$$\phi^{(i)}\left(t_0\right) = \alpha_i \qquad \phi^{(i)}\left(t_f\right) = \beta_i \qquad i = 0, 1, \ldots, n-1 \tag{E.19}$$

and assume

$$\phi(t) = x_n^c(t) \qquad t \epsilon \left[t_0, t_f\right].$$

The $n-1$ last state equations of (D.6)-(D.7) give

$$\begin{aligned}
\dot{x}_n^{cb} &= x_{n-1}^{cb} + a_{nn}^{cb}\left(x_n^{cb}\right) x_n^{cb}\\
\dot{x}_{n-1}^{cb} &= x_{n-2}^{cb} + a_{(n-1)n}^{cb}\left(x_{n-1}^{cb}, x_n^{cb}\right) x_n^{cb}\\
\dot{x}_{n-2}^{cb} &= x_{n-3}^{cb} + a_{(n-2)n}^{cb}\left(x_{n-2}^{cb}, x_{n-1}^{cb}, x_n^{cb}\right) x_n^{cb}\\
&\vdots\\
\dot{x}_3^{cb} &= x_2^{cb} + a_{3n}^{cb}\left(x_3^{cb}, x_4^{cb}, \ldots, x_n^{cb}\right) x_n^{cb}\\
\dot{x}_2^{cb} &= x_1^{cb} + a_{2n}^{cb}\left(x_2^{cb}, x_3^{cb}, \ldots, x_n^{cb}\right) x_n^{cb}
\end{aligned}$$

Rewriting the first equation as

$$x_{n-1}^{cb} = \dot{x}_n^{cb} - a_{nn}^{cb}\left(x_n^{cb}\right)x_n^{cb} = \phi^{(1)} - \gamma_{n-1,n-1}(\phi) = f_{n-1}\left(\phi, \phi^{(1)}\right). \qquad \text{(E.20)}$$

The second equation gives

$$x_{n-2}^{cb} = \dot{x}_{n-1}^{cb} - a_{(n-1)n}^{cb}\left(x_{n-1}^{cb}, x_n^{cb}\right)x_n^{cb},$$

or substituting for $\dot{x}_{n-1}^{cb}$ by differentiating (E.20)

$$x_{n-2}^{cb} = \ddot{x}_n^{cb} - \frac{d}{dt}\left(a_{nn}^{cb}\left(x_n^{cb}\right)x_n^{cb}\right) - a_{(n-1)n}^{cb}\left(x_{n-1}^{cb}, x_n^{cb}\right)x_n^{cb}.$$

Then substituting for $x_{n-1}^{cb} = f_{n-1}\left(\phi, \phi^{(1)}\right)$ this may be written as

$$x_{n-2}^{cb} = \phi^{(2)} - \gamma_{n-1,n-2}\left(\phi, \phi^{(1)}\right) - \gamma_{n-2,n-2}\left(\phi, \phi^{(1)}\right) = f_{n-2}\left(\phi, \phi^{(1)}, \phi^{(2)}\right).$$

Proceeding in a similar manner, one has in general

$$\begin{aligned} x_k^{cb} &= x_n^{cb(n-k)} - \sum_{i=k}^{n-1} \frac{d^{i-k}}{dt^{i-k}}\left(a_{(i+1)n}^{cb}\left(x_{i+1}^{cb}, x_{i+2}^{cb}, \ldots, x_n^{cb}\right)x_n^{cb}\right) \\ &= \phi^{(n-k)} - \sum_{i=k}^{n-1}\gamma_{i,k}\left(\phi, \phi^{(1)}, \ldots, \phi^{(n-1-k)}\right) \\ &= f_k\left(\phi, \phi^{(1)}, \ldots, \phi^{(n-k)}\right) \\ & \quad k = 1, 2, \ldots, n-1. \end{aligned} \qquad \text{(E.21)}$$

Then at $t = t_0$ one has from (E.21)

$$\alpha_{n-k} = p_k + \sum_{i=k}^{n-1}\gamma_{i,k}\left(\alpha_0, \alpha_1, \ldots, \alpha_{n-1-k}\right),$$

and at $t = t_f$,

$$\beta_{n-k} = q_k + \sum_{i=k}^{n-1}\gamma_{i,k}\left(\beta_0, \beta_1, \ldots, \beta_{n-1-k}\right)$$

for $k = 1, 2, \ldots, n-1$, i.e., a recurrent formula to determine the boundary values (E.19).

Now u can be written from the first state equation as

$$u = \dot{x}_1^{cb} - a_{1n}^{cb}\left(x_1^{cb}, x_2^{cb}, \ldots, x_n^{cb}\right) x_n^{cb}.$$

Here x_1^{cb} is from (E.21)

$$x_1^{cb} = x_n^{cb(n-1)} - \sum_{i=1}^{n-1} \frac{d^{i-1}}{dt^{i-1}} \left(a_{(i+1)n}^{cb}\left(x_{i+1}^{cb}, x_{i+2}^{cb}, \ldots, x_n^{cb}\right) x_n^{cb}\right).$$

Using this, u becomes

$$\begin{aligned} u &= x_n^{cb(n)} - \sum_{i=1}^{n-1} \frac{d^i}{dt^i} \left(a_{(i+1)n}^{cb}\left(x_{i+1}^{cb}, x_{i+2}^{cb}, \ldots, x_n^{cb}\right) x_n^{cb}\right) \\ &\quad - a_{1n}^{cb}\left(x_1^{cb}, x_2^{cb}, \ldots, x_n^{cb}\right) x_n^{cb} \\ &= x_n^{cb(n)} - \sum_{i=0}^{n-1} \frac{d^i}{dt^i} \left(a_{(i+1)n}^{cb}\left(x_{i+1}^{cb}, x_{i+2}^{cb}, \ldots, x_n^{cb}\right) x_n^{cb}\right) \end{aligned} \tag{E.22}$$

or

$$u = \phi^{(n)} - \sum_{i=0}^{n-1} \gamma_{i,0}\left(\phi, \phi^{(1)}, \ldots, \phi^{(n-1)}\right) \tag{E.23}$$

and is thus uniquely determined for all initial and final conditions.

Now assume that the form (D.6)-(D.7) is structurally controllable. Here u can be written as (derived similar to (E.22))

$$u = x_n^{cb(n)} - \sum_{i=0}^{n-1} \frac{d^i}{dt^i} \left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right) \tag{E.24}$$

Since the form (D.6)-(D.7) is structurally controllable, i.e., the existence of u for all initial and final conditons is structurally guaranteed, the R.H.S. of (E.24) must be independent of u and its derivatives. Thus one must have

$$\frac{d^i}{dt^i} \left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right) = f_{(i)(i+1)}\left(x_1^{cb}, x_2^{cb}, \ldots, x_n^{cb}\right), \tag{E.25}$$

for $i = 1, 2, \ldots, n-1$, i.e., independent of u and its derivatives. Using Lemma E.2 (which is the dual of Lemma E.1) this gives

$$
\begin{aligned}
a_{1n}^{cb}\left(x^{cb}\right) &= a_{1n}^{cb}\left(x_1^{cb}, x_2^{cb}, \ldots, x_n^{cb}\right) \\
a_{2n}^{cb}\left(x^{cb}\right) &= a_{2n}^{cb}\left(x_2^{cb}, x_3^{cb}, \ldots, x_n^{cb}\right) \\
a_{3n}^{cb}\left(x^{cb}\right) &= a_{3n}^{cb}\left(x_3^{cb}, x_4^{cb}, \ldots, x_n^{cb}\right) \\
&\vdots \\
a_{(n-1)n}^{cb}\left(x^{cb}\right) &= a_{(n-1)n}^{cb}\left(x_{n-1}^{cb}, x_n^{cb}\right) \\
a_{nn}^{cb}\left(x^{cb}\right) &= a_{nn}^{cb}\left(x_n^{cb}\right).
\end{aligned}
$$

This concludes the proof of Theorem E.4.

□

Lemma E.2 *If (D.6)-(D.7) and (E.25) hold then*

$$
a_{(i+1)n}^{cb}\left(x^{cb}\right) = a_{(i+1)n}^{cb}\left(x_{i+1}^{cb}, x_{i+2}^{cb}, \ldots, x_n^{cb}\right) \quad i = 1, 2, \ldots, n-1.
$$

Proof: One has

$$
\frac{d^i}{dt^i}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right) = f_{(i)(i+1)}\left(x_1^{cb}, x_2^{cb}, \ldots, x_n^{cb}\right) \quad i = 1, 2, \ldots, n-1.
$$

However,

$$
\begin{aligned}
&\frac{\partial}{\partial x_1^{cb}}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right)\right) \dot{x}_1^{cb} \\
&= \frac{\partial}{\partial x_1^{cb}}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right)\right)\left(a_{1n}^{cb}\left(x^{cb}\right) x_n^{cb} + u\right).
\end{aligned}
$$

Thus one must have

$$
\frac{\partial}{\partial x_1^{cb}}\left(\frac{d^{i-1}}{dt^{i-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right)\right) = 0
$$

such that the R.H.S. of (E.24) is independent of u or equivalently

$$
\frac{d^{i-1}}{dt^{i-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right) x_n^{cb}\right) = f_{(i-1)(i+1)}\left(x_2^{cb}, x_3^{cb}, \ldots, x_n^{cb}\right).
$$

Now suppose

$$\frac{d^{i-l}}{dt^{i-l}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right) = f_{(i-l)(i+1)}\left(x_{l+1}^{cb}, x_{l+2}^{cb}, \dots, x_n^{cb}\right)$$

for $l = 1, 2, \dots, i-1$. Here one has

$$\begin{aligned}
&\frac{\partial}{\partial x_1^{cb}}\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right)\right)\dot{x}_1^{cb} \\
&\quad = \frac{\partial}{\partial x_1^{cb}}\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right)\right)\left(a_{1n}^{cb}\left(x^{cb}\right)x_n^{cb} + u\right).
\end{aligned}$$

Thus

$$\frac{\partial}{\partial x_1^{cb}}\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right)\right) = 0 \tag{E.26}$$

such that the R.H.S. of (E.24) is independent of u. Further,

$$\begin{aligned}
&\frac{\partial}{\partial x_m^{cb}}\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right)\right)\dot{x}_m^{cb} \\
&\quad = \frac{\partial}{\partial x_m^{cb}}\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right)\right)\left(x_{m-1} + a_{mn}^{cb}\left(x^{cb}\right)x_n^{cb}\right)
\end{aligned}$$

for $m = 2, 3, \dots, l+1$. Thus

$$\frac{\partial}{\partial x_m^{cb}}\left(\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right)\right) = 0 \tag{E.27}$$

such that the R.H.S. of (E.24) is independent of the derivatives of u. Thus from (E.26) and (E.27)

$$\frac{d^{i-l-1}}{dt^{i-l-1}}\left(a_{(i+1)n}^{cb}\left(x^{cb}\right)x_n^{cb}\right) = f_{(i-l-1)(i+1)}\left(x_{l+2}^{cb}, x_{l+3}^{cb}, \dots, x_n^{cb}\right). \tag{E.28}$$

Therefore it has been proven that (substitute $l = i-1$ in (E.28))

$$a_{(i+1)n}^{cb}\left(x^{cb}\right) = a_{(i+1)n}^{cb}\left(x_{i+1}^{cb}, x_{i+2}^{cb}, \dots, x_n^{cb}\right)$$

for $i = 1, 2, \dots, n-1$.

□

E.3 Discussion

In this appendix necessary and sufficient conditions[2] that guarantee the structural observability (controllability) of the defined nonlinear observer and observability (controller and controllability) forms were specified. These conditions should be incorporated in the nonlinear transformations since the existence of such transformations to structurally observable (controllable) forms gives sufficient conditions of nonlinear observability (controllability) of the original system. However, this subject is related to the lack of a theoretical basis between the observability (controllability) of nonlinear systems and the design of an observer (controller) and will be left for future investigation since, as such, it is beyond the scope of this work.

[2]As opposed to the case of LTI systems, only the no-input observability form (the controller form) is structurally observable (controllable).

APPENDIX F

ADDITIONAL SIMULATION STUDIES

F.1 System in observer form

An abstract system, whose nonlinearities are functions of all the state variables, was chosen for an additional simulation study. The system is given in observer form by

$$\dot{x}^o = \begin{bmatrix} a^o_{11}(x^o) & 1 & 0 \\ a^o_{21}(x^o) & 0 & 1 \\ a^o_{31}(x^o) & 0 & 0 \end{bmatrix} x^o + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x^o,$$

where

$$\begin{aligned} a^o_{11}(x^o) &= -1 - x^{o^2}_1 \\ a^o_{21}(x^o) &= -1 - x^{o^2}_2 \\ a^o_{31}(x^o) &= -1 - x^{o^2}_3. \end{aligned}$$

F.2 Transformation from observer form to controller form

Here (3.34) gives

$$\frac{\partial}{\partial x^o}\left(q^c_{3\cdot}(x^o)\, x^o\right) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix};$$

thus, $q_{3\cdot}^{c}(x^{o})$ may be taken as

$$q_{3\cdot}^{c}(x^{o}) = q_{3\cdot}^{c} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then calculating the remaining rows of $Q^{c}(x^{o})$ from (3.35) gives

$$Q^{c}(x^{o}) = \begin{bmatrix} 3x_1^{o^4} + 4x_1^{o^2} - x_2^{o^2} & -1 - 3x_1^{o^2} & 1 \\ -1 - x_1^{o^2} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Now $\bar{A}^{c}(x^{o})$ can be calculated using (3.10a) with the result

$$\bar{A}^{c}(x^{o}) = \begin{bmatrix} \bar{a}_{11}^{c}(x^{o}) & \bar{a}_{12}^{c}(x^{o}) & \bar{a}_{13}^{c}(x^{o}) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where

$$\bar{a}_{11}^{c}(x^{o}) = -2x_1^{o}x_2^{o} - 1 - 3x_1^{o^2},$$

$$\bar{a}_{12}^{c}(x^{o}) = 6x_1^{o^4} + 6x_1^{o^2} - 6x_1^{o^3}x_2^{o} - 8x_1^{o}x_2^{o} - x_2^{o^2} - 1,$$

and

$$\bar{a}_{13}^{c}(x^{o}) = -1 - x_3^{o^2}.$$

F.3 Observer/controller design

A block diagram of the desired system is shown in Figure 11[1]. The pole-placement was such that the resulting closed-loop poles were -1, -5 and -6, corresponding to a characteristic equation of

[1]Note that in general, as is done here, it is possible to use $\bar{k}^{c}(\hat{x}^{o})$ in place of $k^{c}(\hat{x}^{c})$ since $\hat{x}^{o}$ is available. This saves some calculation, in that $A^{c}(x^{c})$ does not have to be determined.

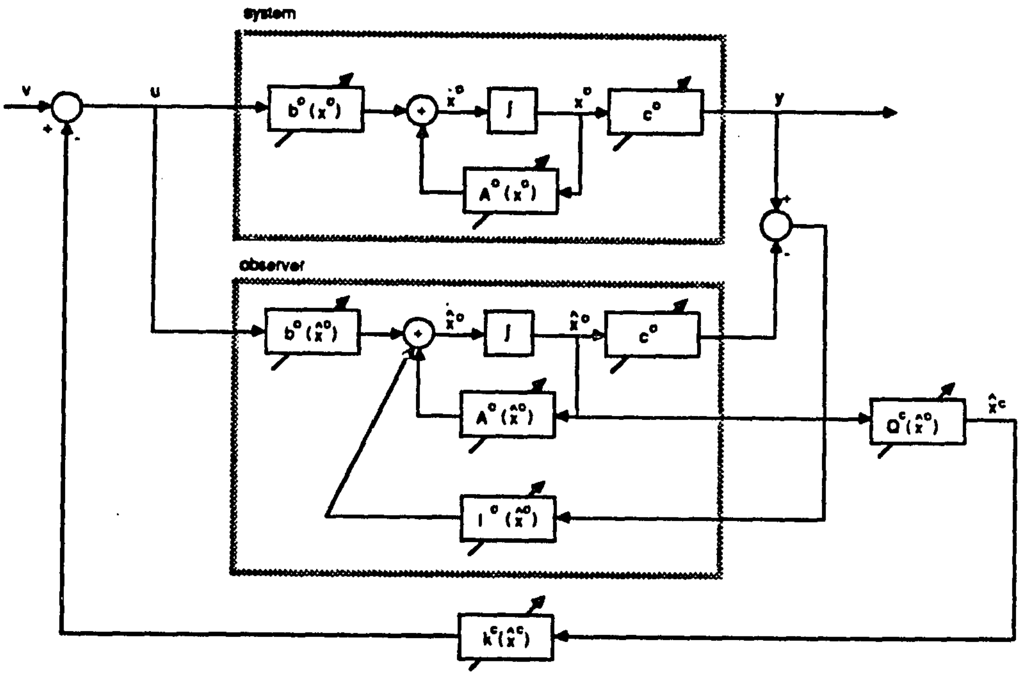

Figure 11: Control system

$$s^3 + 12s^2 + 41s + 30 = 0.$$

The observer poles were initially chosen as three times the dominant closed-loop pole or a triple pole at -3.

F.4 A comparable linear system

A "comparable" linear system was chosen with the purpose of comparing the speed of error decay. This system was chosen to have approximately the same response time as the original system (which is slow) and is given by

$$\dot{x}^o = \begin{bmatrix} a^o_{11} & 1 & 0 \\ a^o_{21} & 0 & 1 \\ a^o_{31} & 0 & 0 \end{bmatrix} x^o + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x^o,$$

where

$$a^o_{11} = -2$$
$$a^o_{21} = -2.25$$
$$a^o_{31} = -1.25.$$

Here (3.34) gives

$$\frac{\partial}{\partial x^o}(q^c_{3\cdot}x^o) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

thus, $q^c_{3\cdot}$ may be taken as

$$q^c_{3\cdot} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then calculating the rest of Q^c from (3.35) gives

$$Q^c = \begin{bmatrix} 1.75 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The closed-loop poles and the observer poles were chosen as in the nonlinear case.

F.5 Simulation results

The speed-of-error decay for the two systems was evaluated by a digital simulation, using a step input of unit magnitude. The state errors, resulting from initial conditions of $0.3, -0.2$ and 0.4 on both the nonlinear and comparable linear system, are shown in Figs. 12 and 13, respectively. These are, for all practical purposes, identical.

The state errors for a triple observer pole at -0.5 and the same initial conditions on both systems are shown in Figs. 14 and 15, respectively. Note that slight differences are observed in the error responses in this case. For easier comparison, e_1, e_2 and e_3 are shown in Figs. 16, 17 and 18, respectively, for both systems. Note that the speed-of-error decay is faster for the nonlinear system than the linear one. Additional simulations for different initial conditions indicated similar behavior for this system.

F.6 Discussion

These simulation results show excellent behavior of the error dynamics of the nonlinear system when compared to those of the linear system. This again indicates that the nonlinearities in the error dynamics do not degrade the speed-of-error decay and that faster decay can be expected in some cases.

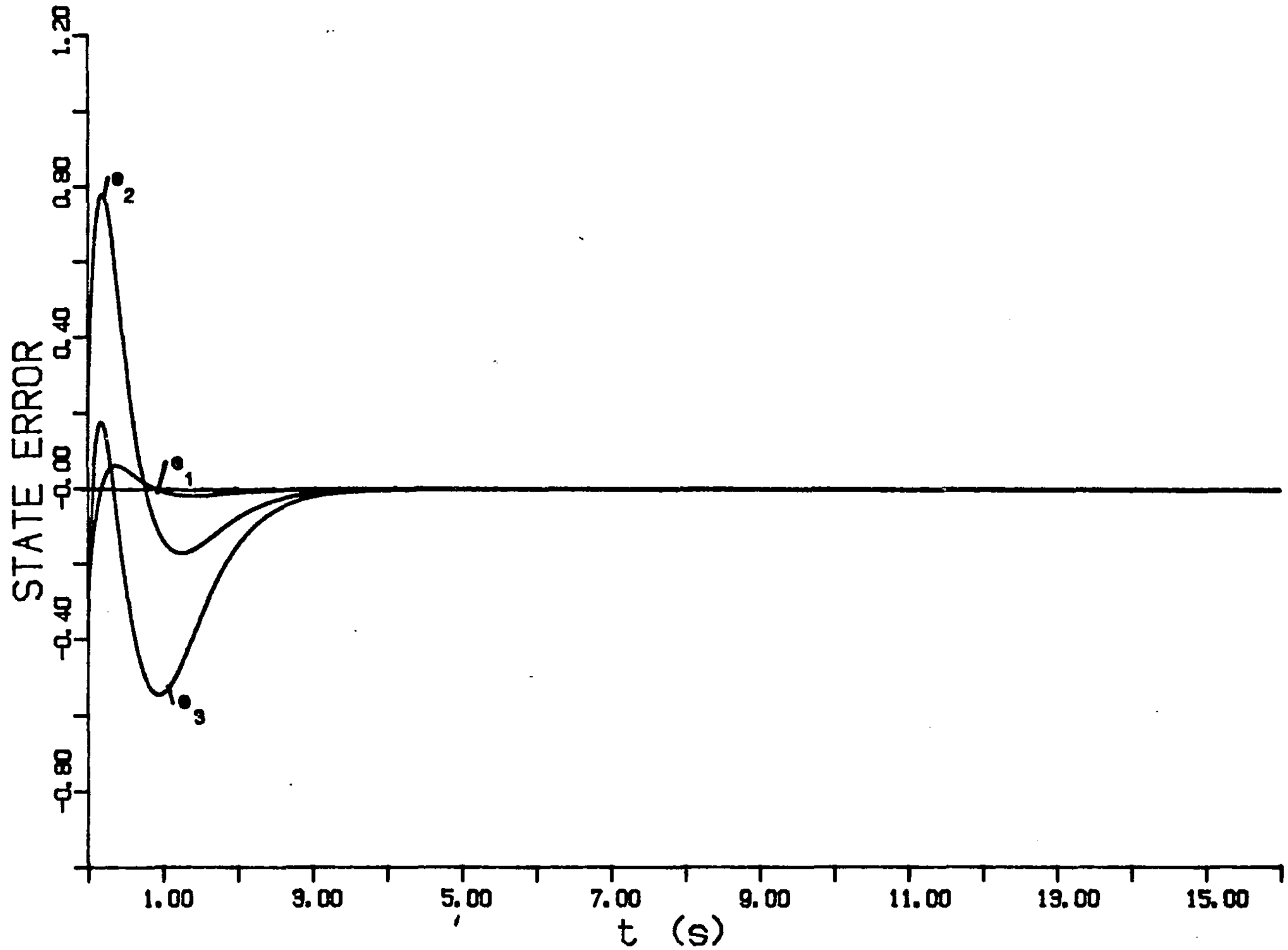

Figure 12: State error for nonlinear system, observer poles at −3

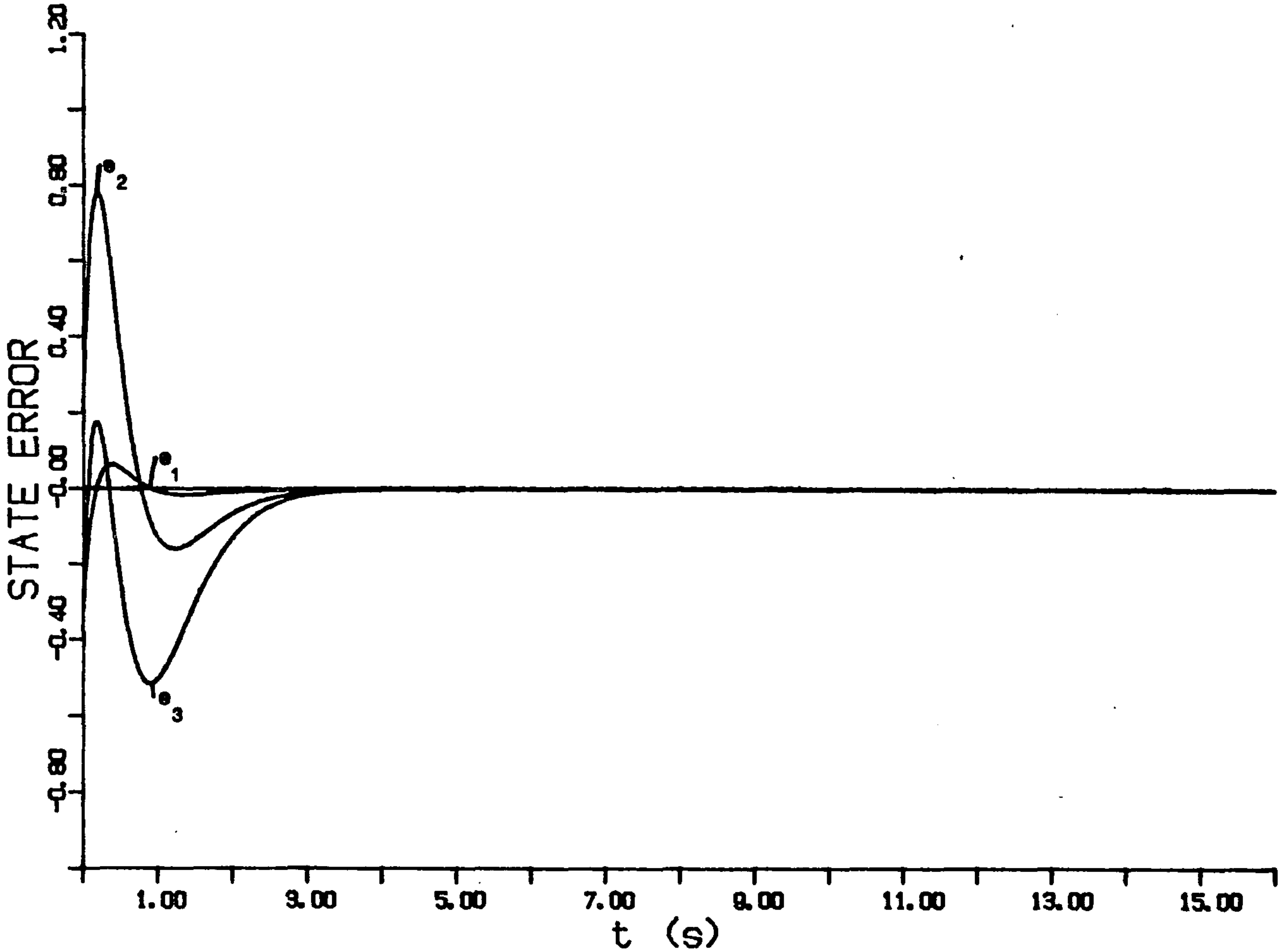

Figure 13: State error for linear system, observer poles at -3

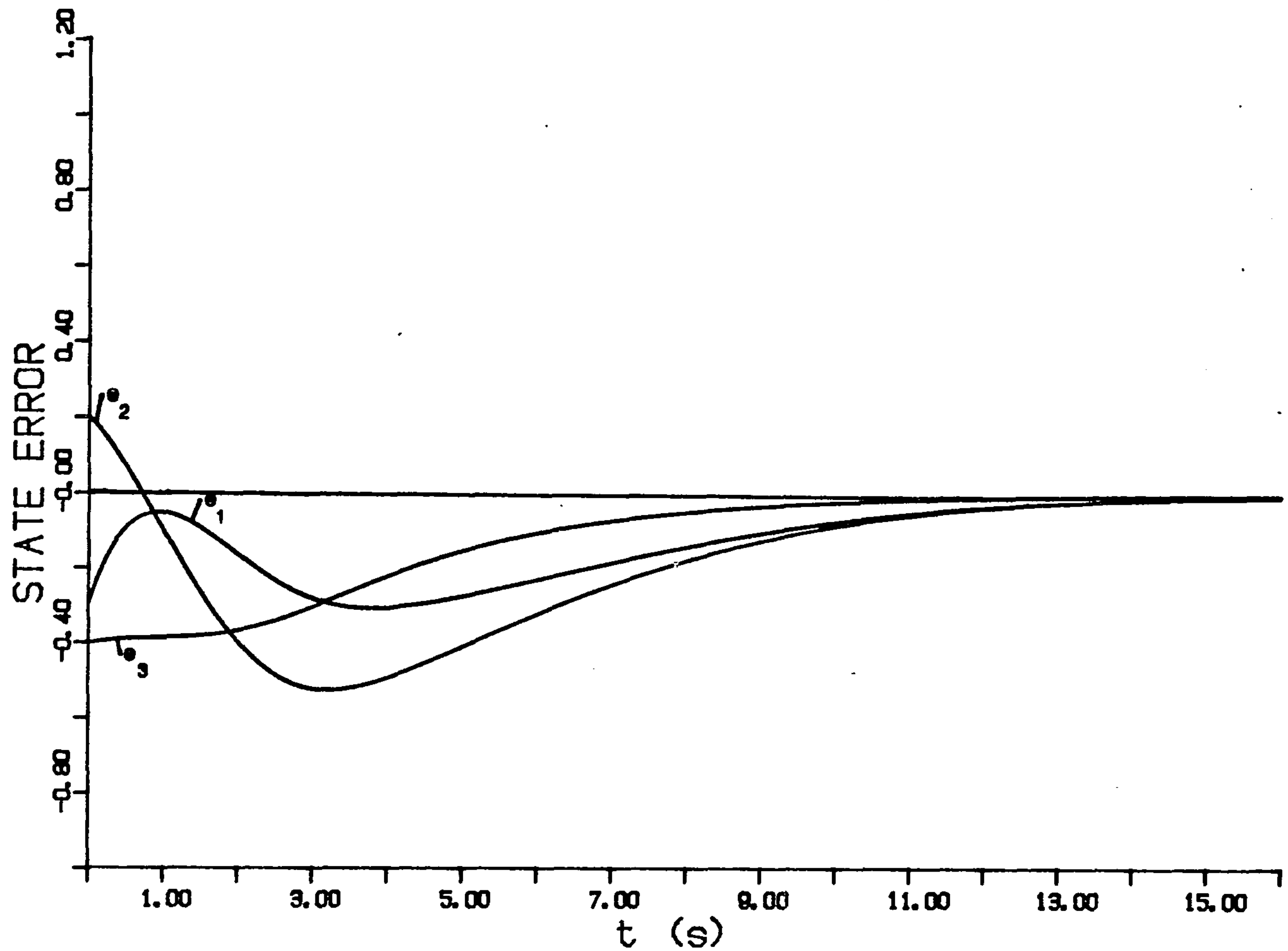

Figure 14: State error for nonlinear system, observer poles at −0.5

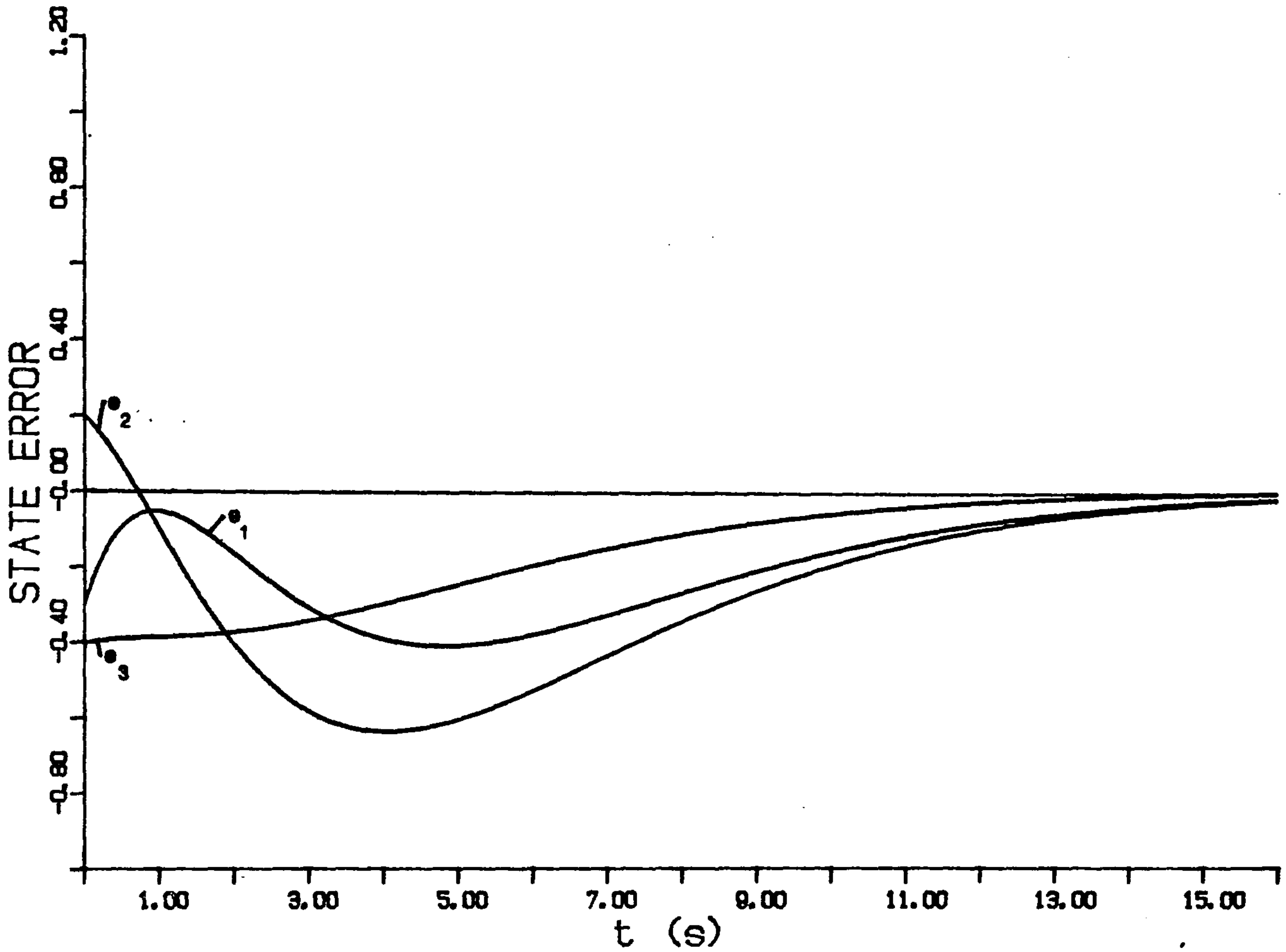

Figure 15: State error for linear system, observer poles at −0.5

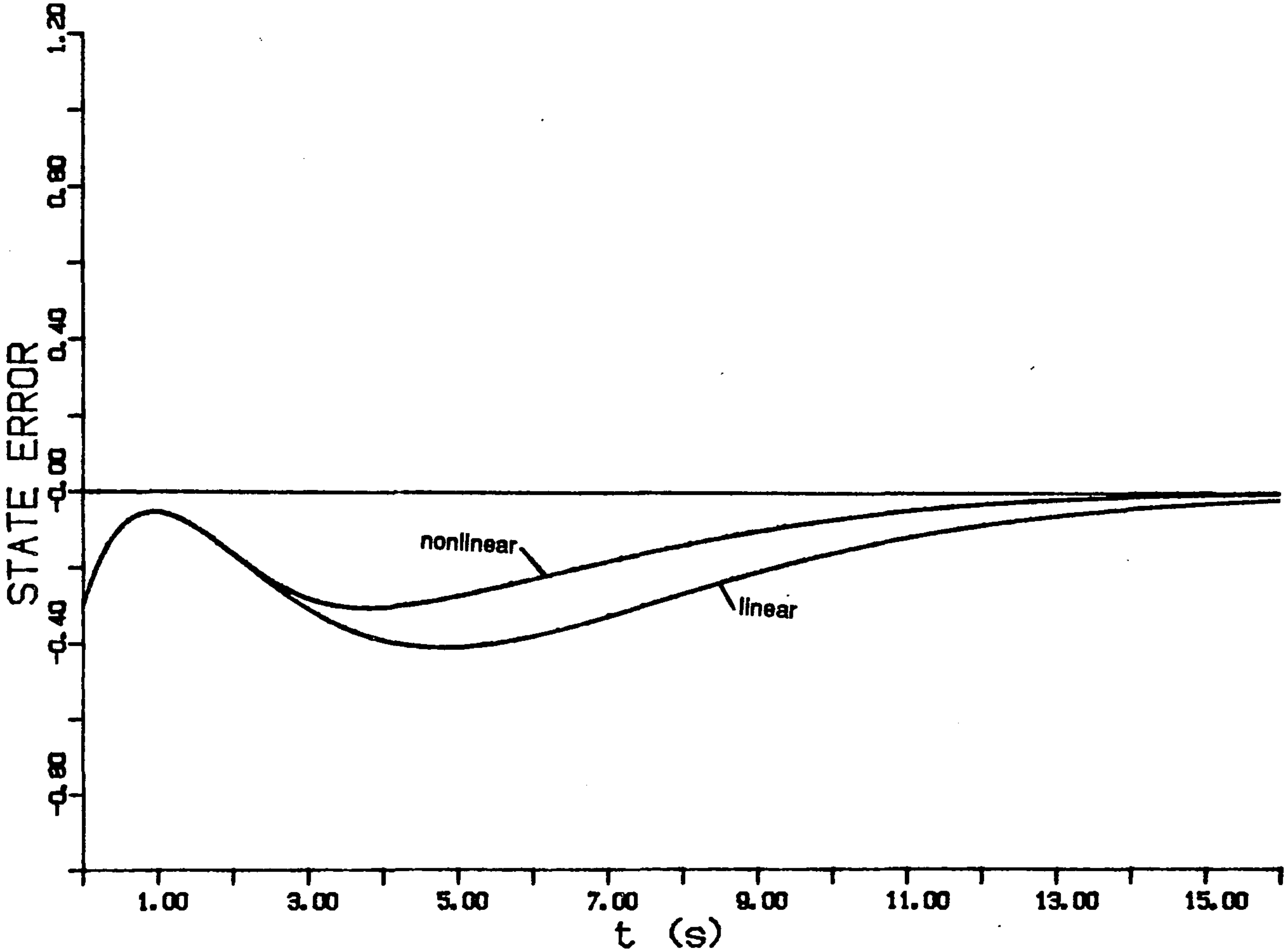

Figure 16: State error (e_1) for the nonlinear and the linear system, observer poles at -0.5

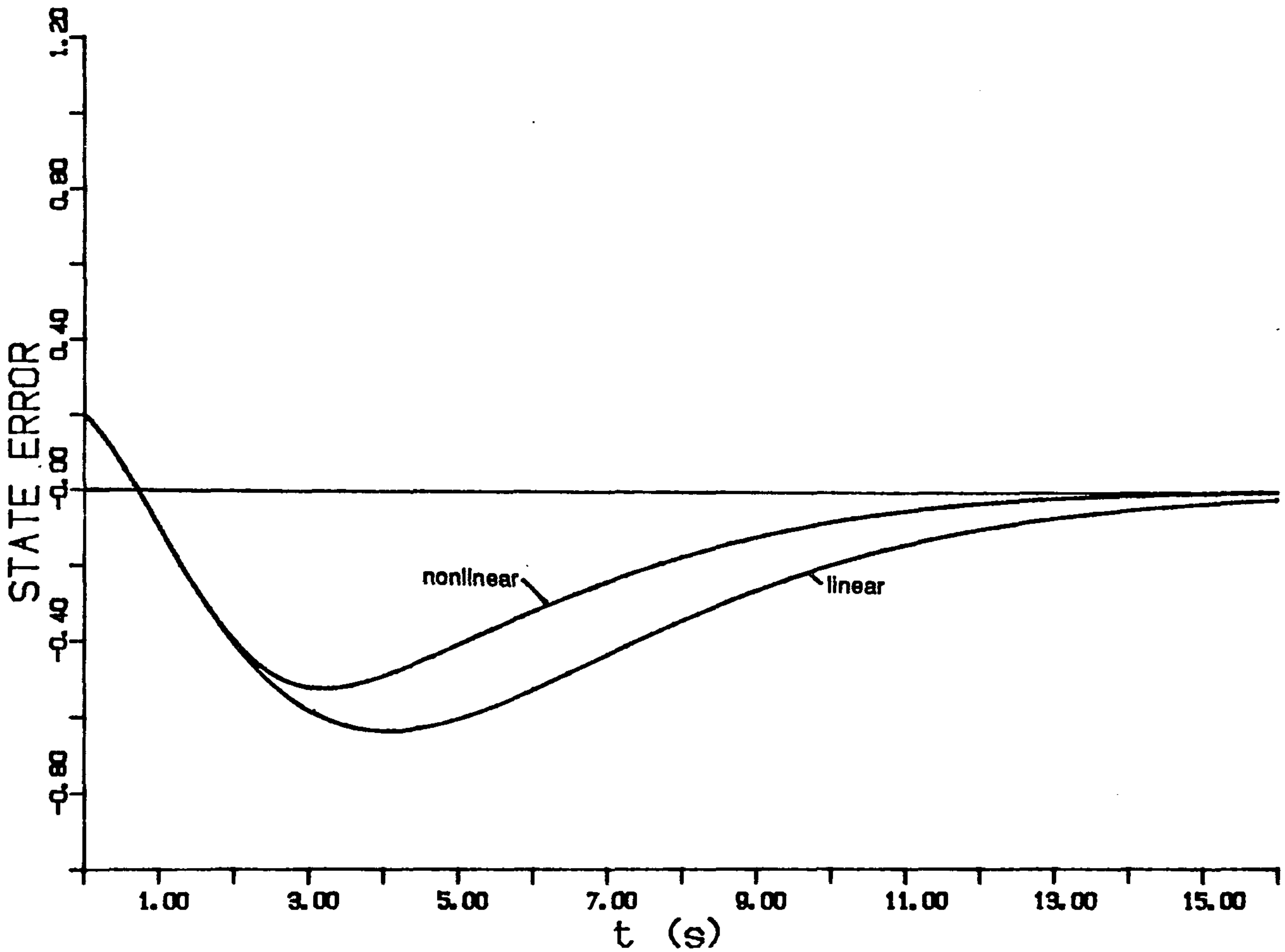

Figure 17: State error (e_2) for the nonlinear and the linear system, observer poles at -0.5

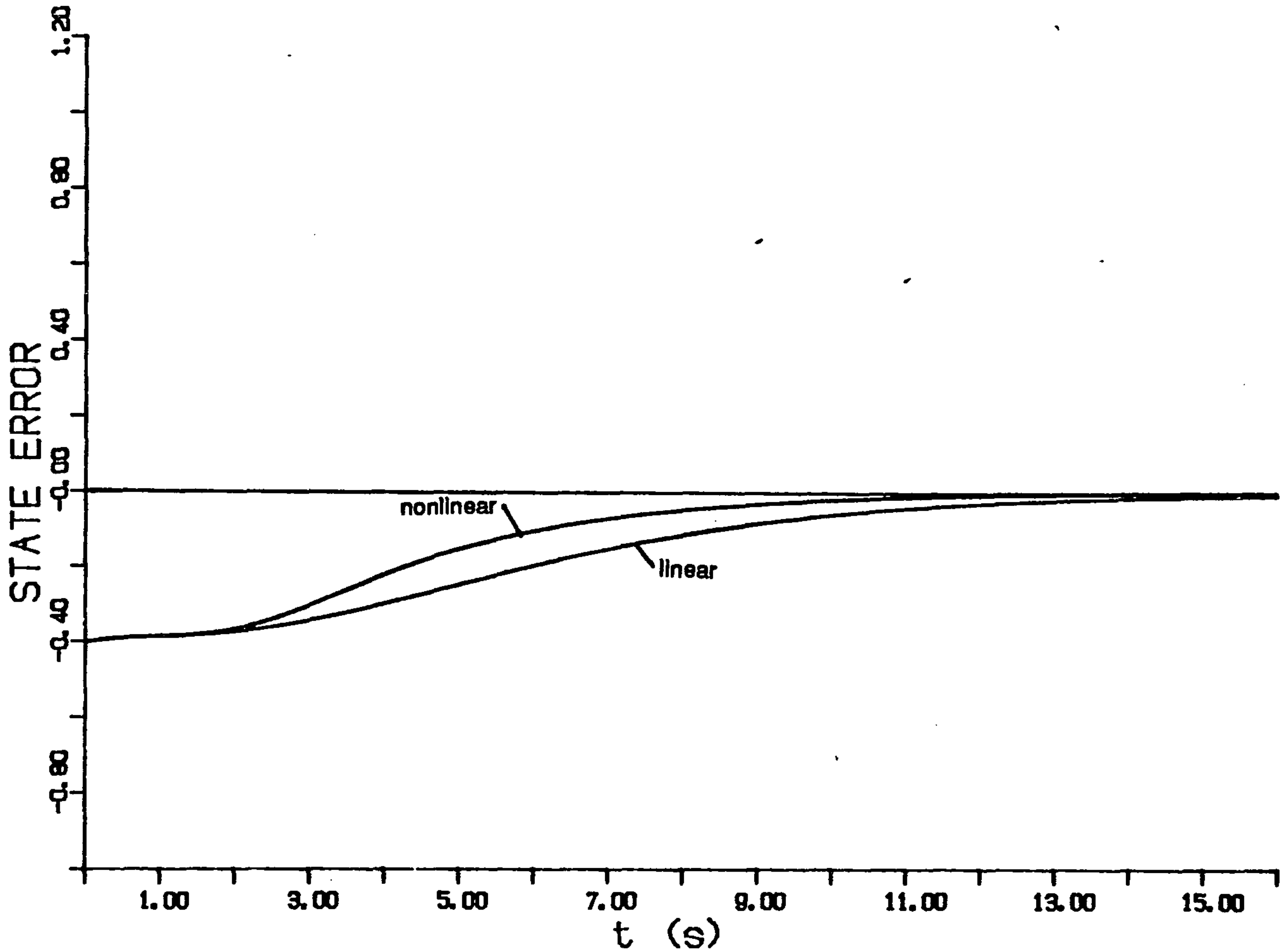

Figure 18: State error (e_3) for the nonlinear and the linear system, observer poles at -0.5

REFERENCES

[1] D'Azzo J.J., Houpis C.H., *Linear Control System Analysis and Design*, 2nd Ed., McGraw-Hill, 1981

[2] Mayr O., *The Origins of Feedback Control*, The M.I.T. Press, 1970

[3] Dorf R.C., *Modern Control Systems*, Addison-Wesley Publ. Comp., 1967

[4] Ogata K., *Modern Control Engineering*, Prentice-Hall, Inc., 1970

[5] Brogan W.L., *Modern Control Theory*, Quantum Publishers, Inc., 1974

[6] Astrom K.J., Wittenmark B., *Computer Controlled Systems Theory and Design*, Prentice-Hall Information and System Sciences Series, Ed. Kailath T., Prentice-Hall, Inc., 1984

[7] Athans M., Falb P.L., *Optimal Control*, McGraw-Hill Book Company, 1966

[8] Astrom K.J., "Theory and applications of adaptive control—a survey," *Automatica*, vol. 19, no. 5, pp. 471-486, 1983

[9] loannou P.A., Kokotovic P.V., *Adaptive Systems with Reduced Models*, Springer-Verlag, 1983

[10] Trinks W., *Governors and the Governing of Prime Movers*, Van Nostrand, Princeton, N.J., 1919

[11] Minorsky N., "Directional stability and automatically steered bodies," *J. Am. Soc. Nav. Eng.*, vol. 34, p. 280, 1922

[12] Nyquist H., "Regeneration theory," *Bell System Tech. J.*, vol. 11, pp. 126-47, 1932

[13] Black H.S., "Stabilized feedback amplifiers," *Bell Syst. Tech. J.*, 1934

[14] Hazen, "Theory of servomechanisms," *Journal of the Franklin Institute*, 1935

[15] *Theory of Servomechanism*, Eds.: James H.M., Nichols N.B., Phillips R.S., Boston Technical Publishers, Inc., 1964

[16] Bode H.W., *Network Analysis and Feedback Amplifier Design*, Van Nostrand, Princeton, N.J., 1945

[17] Hall A.C., "Application of circuit theory to the design of servomechanisms," *J. Franklin Inst.*, 1946

[18] Harris H., "The frequency response of automatic control systems," *Trans. AIEE*, vol. 65, pp. 539-546, 1946

[19] Evans W.R., "Graphical analysis of control systems," *Trans. AIEE*, vol. 67, pt. II, pp. 547-551, 1948

[20] Lefschetz S., *Differential Equations: Geometric Theory*, 2nd Ed., Pure and Applied Mathematics, vol. IV, Interscience Publishers, John Wiley and Sons, New York, 1963.

[21] Pontryagin L.S., *Ordinary Differential Equations*, Addison-Wesley Publishing Company, Inc., 1962

[22] Bellman R., *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc., 1953

[23] Kailath T., *Linear Systems*, Prentice-Hall, Inc., 1980

[24] Kalman R.E., "On the general theory of control systems," in *Proceedings of the First International Congress on Automatic Control*, Butterworth's, pp. 481-493, London 1960

[25] Gilbert E., "Controllability and observability in multivariable control systems," *SIAM J. Control*, vol. 1, pp. 128-151, 1963

[26] Kalman R.E., "Mathematical description of linear systems," *SIAM J. Control*, vol. 1, pp. 152-192, 1963

[27] Popov V.M., "On a new problem of stability for control systems," *Autom. Remote Control*, vol. 24, no. 1, pp. 1-23, 1963

[28] Kalman R.E., Falb P., Arbib M.S., *Topics in Mathematical System Theory*, McGraw-Hill, New York, 1969

[29] Rissanen J., "Control system synthesis by analogue computer based on the 'Generalized linear feedback' concept," in *Proceedings of the Symposium on Analog Computation Applied to the Study of Chemical Processes*, International Seminar, Brussels, pp. 1-13, Nov. 21-23, 1960, Presses Academiques Europeennes, Bruxelles, 1961

[30] Popov V.M., "Hyperstability and optimality of automatic systems with several control functions," *Rev. Roum. Sci. Tech. Ser. Electrotech. Energ.*, vol. 9, pp. 629-690, 1964

[31] Popov V.M., *Hyperstability of Control Systems*, Springer-Verlag New York, Inc., New York, 1973

[32] Rosenbrock H.H., "Distinctive problems of process control," *Chem. Eng. Prog.*, vol. 58, pp. 43-50, Sept. 1962

[33] Astrom K.J., *Introduction to Stochastic Control Theory*, Academic Press, New York, 1970

[34] Luenberger D.G., "Observing the state of a linear system," *IEEE Trans. Mil. Electron.*, MIL-8, pp. 74-80, 1964

[35] Bellman R., *Dynamic Programming*, Princeton, N.J., Princeton University Press, 1957

[36] Krasovskii N.N., "On the theory of optimum regulation," *Automation and Remote Control*, vol. 18, pp. 1005-1016, 1958

[37] Krasovskii N.N., "On the theory of optimum control," *Appl. Math. Mech.*, vol. 23, pp. 899-919, 1959

[38] LaSalle J.P., "Time optimal control systems," *Proc. Natl. Acad. Sci. U.S.*, vol. 45, pp. 573-577, 1959

[39] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mischenko E.F., *The Mathematical Theory of Optimal Processes*, John Wiley, New York, 1962

[40] Bellman R.E., Glicksberg I., Gross O.A., *Some aspects of the mathematical theory of control processes*, Report R-313, Santa Monica, Calif., The RAND Corporation, 1958

[41] Kalman R.E., "Contributions to the theory of optimal control," *Boletin de la Sociedad Matematica Mexicana*, vol. 5, pp. 102-119, 1960

[42] Kalman R.E., "A new approach to linear filtering and prediction problems," *J. Basic Eng.*, pp. 35-46, 1960

[43] Kalman R.E., Bucy R.S., "New results in linear filtering and prediction theory," *J. Basic Eng.*, pp. 95-108, 1961

[44] Whitaker H.P., Yamron J., Kezer A., *Design of model reference adaptive control systems for aircraft*, Report R-164, Instrumentation Laboratory, MIT, Cambridge, MA, 1958

[45] Gregory P.C., Ed., Proc. *Self Adaptive Flight Control System Symposium*, WADC Report No. 59-49, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, 1959

[46] Margolis M., Leondes C.T., "A parameter tracking servo for adaptive control systems," *IEEE Trans. on Autom. Contr.*, vol. AC-4, no. 2, 1959

[47] Osburn P.V., Whitaker H.P., Kezer A., "New developments in the design of adaptive control systems," *Inst. Aeronautical Sciences*, Paper 61-39, 1961

[48] Feldbaum A., *Optimal Control Systems*, Academic Press, New York, 1965

[49] Tsypkin Y.Z., *Foundations of the Theory of Learning Systems*, Academic Press, New York, 1973

[50] Landau I.D., "A survey of model reference adaptive techniques: theory and applications," *Automatica*, vol. 10, pp. 353-379, 1974

[51] Astrom K.J., Wittenmark B., "On self-tuning regulators," *Automatica*, vol. 9, pp. 185-199, 1973

[52] Landau I.D., *Adaptive Control - The Model Reference Approach*, Marcel Dekker, New York, 1979

[53] Narendra K.S., Monopoli R.V., Eds., *Applications of Adaptive Control*, Academic Press, New York, 1980

[54] Brockett R.W., "Nonlinear systems and differential geometry," *Proc. IEEE*, vol. 64, no. 1, pp. 61-72, Jan. 1976

[55] Vidyasager M., "New directions of research in nonlinear system theory," *Proc. IEEE*, vol. 74, no. 8, pp. 1060-1091, Aug. 1986

[56] Hermann R., "On the accessibility problem in control theory," in *Int. Symp. on Nonlinear Differential Equations and Nonlinear Mechanics*, New York: Academic Press, pp. 325-332, 1963

[57] Haynes G.W., Hermes H., "Nonlinear controllability via Lie theory," *SIAM J. Contr.*, vol. 8, pp. 450-460, 1970

[58] Brockett R.W., "System theory on group manifolds and coset spaces," *SIAM J. Contr.*, vol. 10, pp. 265-284, 1972

[59] Lobry C., "Controllbilite des sytemes non linearires," *SIAM J. Contr.*, vol. 8, pp. 573-605, 1970

[60] Lobry C., *Quelques aspects qualitatifs de la theorie de la commande*, These de doctorat d'etat, Universite de Grenoble, Grenoble, France, 1972

[61] Sussman H.J., Jurdjevic V.J., "Controllbility of nonlinear systems," *J. Differential Equations*, vol. 12, pp. 95-116, 1972

[62] Krener A.J., *A generalization of the Pontryagin maximal principle and the bang-bang principle*, Ph.D. dissertation, University of California, Berkeley, CA, 1971

[63] Krener A.J., "A generalization of Chow's theorem and the bang-bang theorem to nonlinear control problems," *SIAM J. Contr.*, vol. 12, pp. 43-52, 1974

[64] Hermann R., Krener A.J., "Nonlinear controllability and observability," *IEEE Trans. Autom. Contr.*, vol. AC-22, no. 5, pp. 728-740, 1977

[65] Gauthier J.P., Bornard G., "Observability for any u(t) of a class of nonlinear systems," *IEEE Trans. Autom. Contr.*, vol. AC-26, no. 4, pp. 922-926, 1981

[66] Zeitz M., "Controllability canonical (phase-variable) form for non-linear time-variable systems," *Int. J. Contr.*, vol. 37, no. 6, pp. 1449-1457, 1983

[67] Kou S.R., Elliott D.L., Tarn T.J., "Exponential observers for nonlinear dynamic systems," *Inf. Contr.*, vol. 29, pp. 204-216, 1975

[68] Sommer R., "Control design for multivariable non-linear time-varying systems," *Int. J. Contr.*, vol. 31, no. 5, pp. 883-891, 1980

[69] Bestle D., Zeitz M., "Canonical form observer design for non-linear time-variable systems," *Int. J. Contr.*, vol. 38, no. 2, pp. 419-431, 1983

[70] Krener A.J., Isidori A., "Linearization by output injection and nonlinear observers," *Syst. Contr. Lett.*, vol. 3, pp. 47-52, 1983

[71] Krener A.J., Respondek W., "Nonlinear observers with linearizable error dynamics," *SIAM J. Contr. Optim.*, vol. 23, no. 2, pp. 197-216, 1985

[72] Davison E.J., Kunze E.G., "Some sufficient conditions for the global and local controllability of nonlinear time-varying systems," *SIAM J. Control*, vol. 8, no. 4, pp. 489-497, Nov. 1970

[73] Takasaki G.M., Fenton R.E., "On the identification of vehicle longitudinal dynamics," *IEEE Trans. Autom. Contr.*, vol. AC-22, no. 4, pp. 610-615, Aug. 1977.

[74] Cormier W.H., *Experimental Studies in Vehicle Lateral Control*, Master of Science Thesis, Department of Electrical Engineering, The Ohio State University, Columbus, Ohio, Dec. 1978

[75] Fenton R.E., et.al., "On the steering of automated vehicles: Theory and experiment," *IEEE Trans. Autom. Contr.*, vol. AC-21, no. 3, pp. 306-315, June 1976

[76] Whalley R., "Ship motion control," *6th Ship Control System Symposium*, Ottawa, 1981

[77] Roberts G.N., *Warship Model Validation*, RNEC Control Engineering Department Report, no. 123, March 1986

[78] Friedland B., *Control System Design. An Introduction to State-Space Methods*, McGraw-Hill Book Company, Inc., 1986

[79] Etkin B., *Dynamics of Flight. Stability and Control*, John Wiley and Sons, Inc., 1963

[80] Kalman R.E., Bertram J.E., "Control system analysis and design via the 'Second Method' of Lyapunov. I. Continuous-time systems," *Trans. ASME Jounal of Basic Engineering*, vol. 82D, pp. 371-393, June 1960

[81] Chen C.T., *Linear System Theory and Design*, Holt, Rinehart and Winston, 1984

[82] Sussman H.J., "Single-input observability of continuous-time systems," *Math. Systems Theory*, 12, pp. 371-393, 1979.

[83] Bronshtein I.N., Semandyayer K.A., *Handbook of Mathematics*, 3rd Ed., Verlag Harri Deutch, Van Nostrand Reinhold Company, 1985.

[84] Davison E.J., Silverman L.M., Varaiya P.P., "Controllability of a class of nonlinear time-variable systems," *IEEE Trans. Autom. Contr.*, pp. 791-792, Dec. 1967.

[85] Zak S.H., MacCarley C.A. "State-feedback control of non-linear systems," *Int. J. Control*, vol. 43, no. 5, pp. 1497-1514, 1986.

[86] Su R., "On the linear equivalents of nonlinear systems," *Systems and Control Letters*, vol. 2, no. 1, pp. 48-52, July 1982.

[87] Hunt L.R., Su R., Meyer G., "Global transformations of nonlinear systems," *IEEE Trans. Autom. Contr.*, vol. AC-28, no. 1, pp. 24-31, Jan. 1983.

[88] Farlow S.J., *Partial Differential Equations for Scientists and Engineers*, Wiley, 1982.

[89] Zachmanoglou E.C., Thoe D.W., *Introduction to Partial Differential Equations with Applications*, The Williams and Wilkins Company, 1976.

[90] John F., *Partial Differential Equations*, 4th Ed., Springer-Verlag, 1982.

[91] Kamke E., *Differentialgleichungen Lösungsmethoden und Lösungen*, Akademische Verlagsgesellschaft, 1965.

[92] Moon P., Spencer D.E., *Partial Differential Equations*, D.C. Heath, Lexington, Massachusetts, 1969.

[93] Forsyth A.R., *Theory of Differential Equations*, vol. 1, Dover Publications, 1890.

[94] Ford L.R., *Differential Equations*, McGraw-Hill Book Company, 1933.

[95] The Matlab Group, *MACSYMA Reference Manual*, Version 8, Lab. Comput. Sci., M.I.T., Nov. 1975.

[96] Courant R., Hilbert D., *Methods of Mathematical Physics*, vol. II (Partial Differential Equations by R. Courant), Interscience Publishers, 1966.

[97] Takasaki G.M., Fenton R.E., "On the identification of vehicle longitudinal dynamics," *IEEE Trans. Autom. Contr.*, vol. AC-22, no. 4, Aug. 1977.

[98] Hauksdottir A.S., Fenton R.E., "On the design of a vehicle longitudinal controller," *IEEE Trans. Veh. Tech.*, vol. VT-34, no. 4, pp. 182-187, Nov. 1985.

[99] Fenton R.E., Chu P.M., "On vehicle automatic longitudinal control," *Transportation Science*, vol. 11, no. 1, Feb. 1977.