State space realization and inversion of 2-D systems

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State Space Realization and Inversion of

2-D Systems

by

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ABSTRACT

In this paper the state space realization results of [1] for causal
2-D systems are generalized to a much larger class of 2-D systems. We
introduce a generalized notion of a state space realization for which
the state can still be recursively evaluated. The results include a
realization method for a class of NSHP filters. In the second part
we introduce inverse 2-D systems with inherent delay. Some results
concerning existence of an inverse with inherent delay for a 2-D system
will be given. It will be shown that, in general, a causal 2-D system
cannot have a causal inverse (with inherent delay). Furthermore it
will be shown that a causal 2-D system always has an inverse with
inherent delay in the larger class of 2-D systems mentioned above.

1. INTRODUCTION

In this paper the results concerning state space realization of a causal 2-D
system (as described in [1]) will be generalized to a larger class of 2-D sys-
tems. These systems, which will be called weakly causal, are closely related
to the so called Non Symmetric Half Plane filters (NSHP filters). For the use
of NSHP filters see for instance [2]. The proposed method gives us a genera-
lized notion of a state space realization for which the state and therefore
the output, can still be evaluated in a recursive way. In the second part of
the paper inversion of 2-D systems is considered. Also inversion with inhe-
rent delay will be treated and there weakly causal systems arise in a natural
way. Inverse systems or inverse filters are tools for the restoration of de-
graded images, although there are many problems concerning the applicability.
For instance if the degradation is due to noise then certainly inverse fil-
tering is not advisable as a restoration method. However, if noise is not
important as a degradation source, then inverse filtering can give reasonable
good restorations in many situations. Many aspects of image restoration are
treated in [12] and [13]. These papers certainly provide more motivational
material to study inverse 2-D systems. The paper will be concerned with sca-
lar systems but the realization part can be modified in an obvious way to in-
clude the multivariable case. The state space model for a causal system will
be Roessers model. This model is equivalent (conceptually) to the model in
[1]. It is preferable for us in this paper because it contains less matrices.
Let us now describe a 2-D I/O systems ([1])

1.1.
$$y_{kh} = \sum_{i=0, j=0}^{\infty, \infty} F_{k-i, h-j} u_{ij} \quad k, h = 0, 1, 2, \ldots$$

where y_{kh}, $F_{m,n}$, u_{ij} are reals such that

1.2.
$$F_{m,n} = 0 \text{ for } m < 0 \text{ or } n < 0 .$$

Using a formal power series approach (or 2-D Z-transform) defined by

$$\hat{y}(s,z) = \sum_{k=0, h=0}^{\infty, \infty} y_{kh} s^{-h} z^{-k}$$

(1.1) becomes after transformation

$$\hat{y}(s,z) = \hat{F}(s,z) \hat{u}(s,z)$$

where

$$\hat{F}(s,z) = \sum_{m=0, n=0}^{\infty, \infty} F_{m,n} s^{-n} z^{-m} .$$

We will assume that $\hat{F}(s,z)$ is a real rational function in the variables
s and z. Thus $\hat{F}(s,z) = P(s,z)/Q(s,z)$ where $P(s,z)$ and $Q(s,z)$ are polynomials
in two variables (see also [1]). Now it is clear that $\hat{F}(s,z)$ can also be
seen as a rational function in z where the coefficients are polynomials
in s. We then have a condition, equivalent to (1.2), for a rational
$\hat{F}(s,z)$, which we will state without proof.

1.3 <u>Lemma</u>.

Suppose $\hat{F}(s,z) = P(s,z)/Q(s,z)$ is a rational function in s and z
such that

$$\hat{F}(s,z) = \sum_{(k,h)\in\mathbb{Z}^2} F_{kh} s^{-h} z^{-k}$$

then (1.2) is equivalent to

1° $\deg_z(Q) \geq \deg_z(P)$

2° the degree in s of the coefficients of the highest power
in z of Q is not less then than the degree in s of all
the other coefficients of P and Q. □

As in [1] a rational $\hat{F}(s,z)$ satisfying (1.2) will be called a causal
transfer function.

1.4 <u>Theorem</u> [1].

Every causal transfer function has a state space realization which
can be written in the form (Roesser)

1.5.a.
$$\begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

1.5.b.
$$y_{kh} = [C_1 \mid C_2] \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D u_{kh}$$

The vector $\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix}$ is called the (local) state. The matrices have appropriate

dimensions and the initial conditions are

$$x_{0,h} = 0, \quad a_{k,0} = 0 \qquad k,h = 0,1,2,\ldots.$$

Furthermore, to every state space realization (1.5) corresponds a causal
transfer function. $\square$

s and z can also be given an interpretation in terms of shifts in the
following way

$$z(x)_{kh} = x_{k+1,h} \; ; \; s(a)_{kh} = a_{k,h+1}$$

so (1.5.a) can be written as

$$\begin{bmatrix} z(x)_{kh} \\ s(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh} \qquad k,h = 0,1,2,\ldots.$$

In this paper the realization technique of [1] will be generalized
to a larger class of transfer functions and a generalized version
of (1.5) will be obtained. Furthermore it turns out that, investigating
inverse 2-D systems with inherent delay [3], [4], [5], one may not
hope for state space equations of type (1.5) and one has to admit
a more general class which we will call weakly causal systems.

2. WEAKLY CAUSAL 2-D SYSTEMS

Consider the 2-D I/O system

2.1
$$y_{kh} = \sum_{(i,j) \in J} F_{k-i, h-j} \, u_{ij} \qquad\qquad (k,h) \in J \subset \mathbb{Z}^2$$

The index set J will be specified later on.

As usual the double sequence $F = (F_{m,n})$, $(m,n) \in \mathbb{Z}^2$ is called the impulse response (also point spread function) ($\mathbb{Z}$ denotes the set of integers)

The support of F is the set

$$S_F = \{(m,n) \mid (m,n) \in \mathbb{Z}^2, \, F_{m,n} \neq 0\}$$

A cone C is a subset of $\mathbb{R}^2$ such that if $(x,y) \in C$ then $(\lambda x, \lambda y) \in C$ for all $\lambda \geq 0$. ($\mathbb{R}$ denotes the field of reals). The closed first quadrant of $\mathbb{R}^2$ will be denoted by Q_1.

2.2. Definition

The I/O system (2.1) will be called weakly causal if

$$S_F \subset C \, , \quad J \subset C$$

for some closed convex cone C satisfying

2.3
$$I : C \cap (-C) = \{0\}$$
$$II: Q_1 \subset C$$

From now on C will always denote a closed convex cone satisfying I and II. In the next it will be shown that under certain conditions we can construct a generalized local state space model. This state space model will be such that states and therefore outputs can recursively be computed. The main idea in this procedure is to "move" the impulse response of a weakly causal I/O system into the first quadrant in such a way that its structure, important for the proposed realization method, is kept unaffected. The obtained impulse response is then the point spread function of a causal 2-D system. For this system a state space realization can be constructed (see [1]) and can be given an interpretation as a generalized state space realization for the weakly causal I/O system. Therefore we will be interested in invertible mappings φ

$$\varphi: C \cap \mathbb{Z}^2 \to Q_1 \cap \mathbb{Z}^2$$

such that the origin is a fixed point ($\varphi(0,0) = (0,0)$).

This map φ will "move" the impulse response of the weakly causal I/O system into Q_1. Because we are primarily interested in a state space model for the weakly causal I/O system this map φ has to be one-one and onto. This can be achieved if we define such a φ for a somewhat larger set than $C \cap \mathbb{Z}^q$. A useful way to do this is to introduce the notion of causality cone.

2.4. Definition

A causality cone C_c is the intersection of two halfplanes $H_{p,r}$ and $H_{q,t}$ where

$$H_{p,r} = \{(x,y) \mid (x,y) \in \mathbb{R}^2, \; px + ry \geq 0\}$$

$$H_{q,t} = \{(x,y) \mid (x,y) \in \mathbb{R}^2, \; qx + ty \geq 0\}$$

where p, r, q, t are nonnegative integers satisfying $qr - pt = -1$.

Lemma

Every causality cone has the properties I and II (see (2.3)).

Proof

The proof is straightforward and will be deleted.   □

Remark

Every causality cone induces a partial order on $\mathbb{Z}^2$ in the same way as Q_1 (the causality cone for a causal system) does. (see [10]). The impulse response of a weakly causal I/O system is an example of a function with past-finite support (see[6]).

2.5. Lemma

Suppose that C is a closed convex cone satisfying I and II. Then there exists a causality cone C_c such that $C \subset C_c$.

Proof

$C \subset C'$ where C' is the intersection of two halfplanes $H_{p',r'}$ and $H_{q',t'}$ such that $q'r' - p't' < 0$ and p' and r' are coprime. Then there exists integers q_1 and t_1 such that $q_1 r' - p' t_1 = -1$ and thus $(q_1 + np')r' - p'(t_1 + nr') = -1$ for all $n \in \mathbb{Z}$. Because of $q'/t' < p'/r'$ we have for sufficiently large n_0 that $q'/t' < (q_1 + n_0 p')/(t_1 + n_0 r')$. Now take $p = p'$, $r = r'$, $q = q_1 + n_0 p'$, $t = t_1 + n_0 r'$ and $C_c = H_{p,r} \cap H_{q,t}$ is a causality cone satisfying $C \subset C_c$.   □

Remark

(2.5) is a lemma on existence of C_c. In fact C_c is not unique at all. We now have the following

2.6. Theorem

If C_c is a causality cone, then there exists a (one-one and onto) map φ such that

$$\varphi: C_c \cap \mathbb{Z}^2 \to Q_1 \cap \mathbb{Z}^2$$

$$\varphi(k_1 + k_2, h_1 + h_2) = \varphi(k_1, h_1) + \varphi(k_2, h_2) .$$

Proof.

Suppose $C_c = H_{p,r} \cap H_{q,t}$ then the map φ defined by

$$\varphi(k,h) = (pk + rh, qk + ht)$$

is a possible one. □

We will now take a formal power series point of view for (2.1) (or apply the 2-D Z-transform to (2.1)) although, strictly speaking, the series are not formal power series in the sense that only nonnegative powers of s^{-1} and z^{-1} occur. However in the following it will become clear that we may call the series expansions under consideration formal power series, because of the isomorphism result of theorem (2.7). Then we obtain

$$\hat{y}(s,z) = \hat{F}(s,z)\hat{u}(s,z) .$$

It follows that

2.7. Theorem

For any $C_c = H_{p,r} \cap H_{q,t}$, the set

$$\hat{S}_{p,r,q,t} = \{\hat{F}(s,z) \mid \text{there exist } F_{kh} \text{ such that}$$

$$\hat{F}(s,z) = \sum_{(k,h)\in\mathbb{Z}^2} F_{kh} s^{-h} z^{-k} \text{ and } S_F \subset C_c\}$$

is a ring with the usual addition and multiplication.
Furthermore $\hat{S}_{p,r,q,t}$ is isomorphic to $\hat{S}_{1,0,0,1}$.

Proof

Define the ring homomorphism

by
$$\Phi: \hat{S}_{p,r,q,t} \to \hat{S}_{1,0,0,1}$$

$$\Phi(\hat{F})(\alpha,\beta) = \sum_{m=0,n=0}^{\infty,\infty} F_{\varphi^{-1}(n,m)} \alpha^{-m}\beta^{-n}$$

where φ is the same as in (2.6). Now the proof is just a matter of verification that Φ is indeed a ring isomorphism. □

The above theorem gives us the possibility of transforming a weakly causal I/O system into a causal I/O system. Causality is one of the requirements for applying the realization method of [1]. Rationality of the formal power series, associated with the impulse response, is another requirement. The next theorem states that this feature is automatically conserved by a transformation as is applied in theorem (2.7).

2.8. Theorem

Let Φ be the ring isomorphism defined in the above theorem.
Then $\Phi(\hat{F})(\alpha,\beta)$ is rational iff $\hat{F}(s,z)$ is rational.

Proof

The equality

$$F_{\varphi^{-1}(n,m)} \alpha^{-m} \beta^{-n} = F_{kh} \alpha^{-qk-ht} \beta^{-pk-rh} = F_{kh}(\alpha^t\beta^r)^{-h} (\alpha^q\beta^p)^{-k}$$

implies

$$\Phi(\hat{F})(\alpha,\beta) = \hat{F}(\alpha^t\beta^r, \alpha^q\beta^p)$$

from which the result follows.   □

2.9. Definition

A rational $\hat{F}(s,z)$ corresponding to a weakly causal I/O system will be
called a weakly causal transfer function.

For a causal transfer function we have that the formal power series expansion
is unique. For weakly causal transfer functions we can state a somewhat modi-
fied result. It is not quite unexpected that uniqueness of a series expansion
is closely related to the causality cone associated with the I/O system under
consideration. This result is stated in the next theorem.

2.10. Theorem

Suppose $\hat{F}(s,z)$ is a weakly causal transfer function and that

$$\hat{F}(s,z) = \sum_{(k,h)\in\mathbb{Z}^2} F_{kh} s^{-h} z^{-k} = \sum_{(k,h)\in\mathbb{Z}^2} G_{kh} s^{-h} z^{-k}$$

Furthermore, suppose that $S_F \subset C_c$, $S_G \subset C_c$ where C_c is a causality
cone. Then $F_{kh} = G_{kh}$ for all k,h.

Proof.

Let Φ be defined as in (2.7). Now $\Phi(\hat{F})(\alpha,\beta)$ is a causal transfer
function. The formal power series expansion for $\Phi(\hat{F})(\alpha,\beta)$ is unique
(see [9 ch I]).
This proves the theorem.   □
Notice that a weakly causal transfer function may have more than one
formal power series representation with support in a (different by
(2.10)) causality cone.

Example

$$\frac{1}{s-z} = \frac{1}{s} \sum_{k=0}^{\infty} \left(\frac{z}{s}\right)^k \text{ with } C_c = \{(x,y) \mid y \geq 0, \; x \geq -y\}$$

$$\frac{1}{s-z} = -\frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{s}{z}\right)^k \text{ with } \bar{C}_c = \{(x,y) \mid x \geq 0, \; y \geq -x\}$$

2.11. Lemma

The isomorphism Φ, as defined in (2.7), can also be described by
the substitution

$$s = \alpha^t \beta^r \quad , \quad z = \alpha^q \beta^p$$

with inverse

$$\alpha = s^p z^{-r} \quad , \quad \beta = s^{-q} z^t$$

Proof

This follows from the proof of theorem (2.8).   □

The fact that the inverse of the substitution in lemma (2.11) can also be gi-
ven in the above form, with integer exponents, is due to the condition
$qr - pt = -1$ which in fact is the reason for the one-one character of Φ.

Next we will derive a state space realization for a weakly causal transfer
function. For this purpose we transform this weakly causal transfer
function into a causal one. Then we construct a state space realization
for this causal transfer function like is done in [1]. Now the ring
isomorphism Φ can also be defined for the obtained state space realization.
This is done by means of lemma (2.11). We will now describe the
procedure in more detail.

Suppose $\hat{F}(s,z)$ is a weakly causal transfer function and let $\hat{F}(s,z) \in \hat{S}_{p,r,q,t}$.
Suppose $T(\alpha,\beta) = \hat{F}(\alpha^t \beta^r, \alpha^q \beta^p)$.
Then $T(\alpha,\beta)$ is a causal transfer function. Now, by Theorem (1.4)
$T(\alpha,\beta)$ has a state space realization with dynamical equations

2.12.a
$$\begin{bmatrix} \beta(x)_{kh} \\ \alpha(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

2.12.b
$$y_{kh} = [C_1 \mid C_2] \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D u_{kh}$$

and

$$T(\alpha,\beta) = D + [C_1 \mid C_2]\begin{bmatrix} \beta I-A_1 & -A_2 \\ -A_3 & \alpha I-A_4 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Since $\beta = s^{-q}z^t$, $\alpha = s^p z^{-r}$ equation (2.12.a) can be written as

$$\begin{bmatrix} s^{-q}z^t(x)_{kh} \\ s^p z^{-r}(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u_{kh}$$

or

$$\begin{bmatrix} z^t(x)_{kh} \\ s^p(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 s^q & A_2 s^q \\ A_3 z^r & A_4 z^r \end{bmatrix}\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 s^q \\ B_2 z^r \end{bmatrix}u_{kh}$$
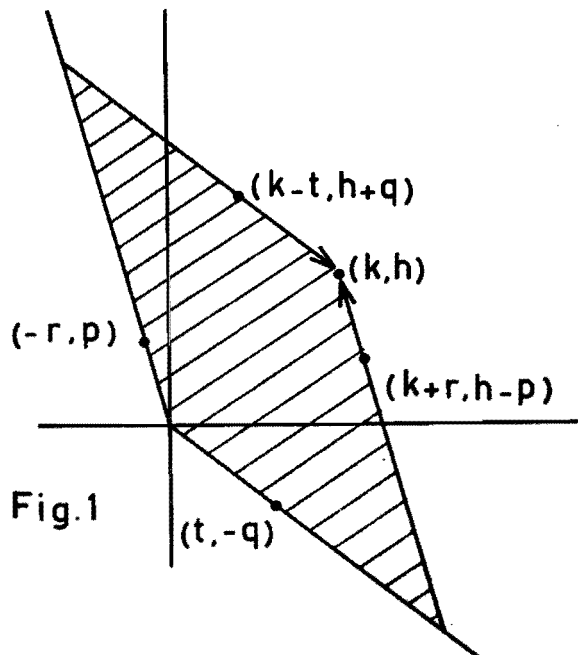
i.e.

2.13
$$x_{k+t,h} = A_1 x_{k,h+q} + A_2 a_{k,h+q} + B_1 u_{k,h+q}$$
$$a_{k,h+p} = A_3 x_{k+r,h} + A_4 a_{k+r,h} + B_2 u_{k+r,h}$$

$rh + pk \geq 0$
$th + qk \geq 0$

Instead of the initial condition for (1.5) we now have the following

$$x_{-rm,pm} = 0 \qquad m = 0,1,2,\ldots$$
$$a_{tn,-qn} = 0 \qquad n = 0,1,2,\ldots$$

See also figure 1

Fig.1

Observe that, in order to compute the (local) state at (k,h), only the states in the shaded area have to be known. From this we can also see that the state can be evaluated in a recursive way and for every state a finite number of steps is necessary.

Although the model is not a first order model, as is the case for causal 2-D systems, the recursive character is maintained. This, of course, is one of the most important features of any state space model. An advantage of the model is that there is a link with causal state space models through the transformation in lemma (2.5). Therefore, properties of causal state space systems and causal I/O systems may carry over to the weakly causal case and in fact do. However we will not be concerned with these aspects. The use of the proposed model will become clear when we are dealing with inversion of 2-D systems.

Remark

Because there are many ring isomorphisms transforming a weakly causal I/O systems into a causal one we may not expect uniqueness results.

The derived realization technique can also be applied to a class of NSHP filters. (Non Symmetric Half Plane filters).

2.14. Definition

An NSHP filter is an I/O system with support in an NSHP, i.e. a subset of $\mathbb{Z}^2$ of the following kind

$$\{(k,h) \mid (k,h) \in \mathbb{Z}^2, \ k > 0 \text{ or } (k = 0 \text{ and } h \geq 0)\} \ .$$

For more details on NSHP filters we refer to [2]. Now, consider an NSHP filter with support in a set H_q where

$$H_q = \{(k,h) \mid (k,h) \in \mathbb{Z}^2, \ k \geq 0, \ h \geq -qk \text{ for some positive integer } q\}.$$

It is clear that H_q is in fact a causality cone so that the above method can be applied.

Remark

Al the above results can immediately be generalized to the multivariable case. We have chosen not to do so in this paper because in the sequel we will only be concerned with the scalar case.

Remark

By allowing transformations like $\alpha = s^{-1}$, $\beta = z^{-1}$ one can realize transfer functions having their support in a closed convex cone C containing another quadrant. C still has to satisfy $C \cap (-C) = \{0\}$.

3. INVERTIBILITY OF 2-D SYSTEMS

We will now be concerned with the invertibility of weakly causal 2-D
systems. To this end we consider the ring $\hat{S}_{p,r,q,t}$ and we have

3.1. Theorem

Suppose $\hat{F}(s,z) \in \hat{S}_{p,r,q,t}$. Then

1°. If $\hat{F}(s,z)$ has a weakly causal inverse $\hat{F}^{-1}(s,z)$ then
$\hat{F}^{-1}(s,z) \in \hat{S}_{p,r,q,t}$.

2°. $\hat{F}(s,z)$ has a weakly causal inverse iff $F_{0,0} \neq 0$

Proof

$\hat{S}_{p,r,q,t}$ is isomorphic to $\hat{S}_{1,0,0,1}$, the isomorphism being the above
defined Φ. Now $\Phi(\hat{F})(\alpha,\beta)$ is invertible iff $F_{0,0} \neq 0$ (see [8 ch VII]).
From this 1° and 2° follow immediately. □

Consider (2.12) and suppose $D = F_{00} \neq 0$. Then the inverse system is

$$
\begin{bmatrix} \beta(x)_{kh} \\ \alpha(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 - B_1 D^{-1} C_1 & A_2 - B_1 D^{-1} C_2 \\ A_3 - B_2 D^{-1} C_1 & A_4 - B_2 D^{-1} C_2 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 D^{-1} \\ B_2 D^{-1} \end{bmatrix} y_{kh}
$$

$$
u_{kh} = \begin{bmatrix} -D^{-1} C_1 & | & -D^{-1} C_2 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D^{-1} y_{kh}
$$

Theorem (3.1) gives a positive result only if $F_{0,0} \neq 0$. In that case a weak-
ly causal inverse exists. If this condition is not satisfied it is possible
to introduce a generalized notion of inverse system (compare [3], [4]). To
this end we will now consider inverses with inherent delay (short w.i.d.).

3.2. Definition

Suppose $\hat{F}(s,z)$ is a weakly causal transfer function. Then a weakly
causal transfer function $\hat{G}(s,z)$ is said to be an inverse with inherent
delay (M,N) if

1° There exists a causality cone C_c such that

$$S_F \subset C_c, \ S_G \subset C_c, \ (M,N) \in C_c$$

$2^o \ \hat{G}(s,z)\hat{F}(s,z) = \dfrac{1}{z^M s^N}$ □

Remark

2° is an immediate generalization of the 1-D counterpart, while 1° is necessary because otherwise the product $\hat{G}(s,z)\hat{F}(s,z)$ cannot be well defined. Compare also the last remark of this paper.

Now we have (for Φ and φ see (2.7), (2.6) respectively)

3.3. Theorem

Suppose $\hat{F}(s,z)$ is a weakly causal transfer function and $\Phi(\hat{F})(\alpha,\beta) = T(\alpha,\beta)$ is a causal transfer function. If $U(\alpha,\beta)$ is a weakly causal inverse of $T(\alpha,\beta)$ w.i.d. (M',N'), then $\Phi^{-1}(U)(s,z)$ is a weakly causal inverse of $\hat{F}(s,z)$ w.i.d. $\varphi^{-1}(M',N')$.

Proof

Suppose $U(\alpha,\beta)T(\alpha,\beta) = \dfrac{1}{\beta^{M'}\alpha^{N'}}$ then

$$\Phi^{-1}(U.T)(s,z) = \frac{1}{z^M s^N} \ \text{or} \ \Phi^{-1}(U)(s,z)\Phi^{-1}(T)(s,z) = \frac{1}{z^M s^N}$$

where $(M,N) = \varphi^{-1}(M',N')$. □

By the above theorem it is clear that we can restrict ourselves to causal transfer functions. Now, one might expect (as in the 1-D case) every causal transfer function to have a causal inverse w.i.d. However this is not the case. A condition for this to be true is as follows.

3.4. Theorem

Suppose $\hat{F}(s,z)$ is a causal transfer function, so that

$$\hat{F}(s,z) = \frac{p_0(s) + p_1(s)z + \ldots + p_m(s)z^m}{q_0(s) + q_1(s)z + \ldots + q_n(s)z^n}, p_m(s) \neq 0, \ q_n(s) \text{ is monic and} \neq 0,$$

where $n \geq m$ and $\deg_s(q_n(s) \geq \deg_s(q_i(s))$ $i = 0,\ldots,n-1$,

$$\deg_s(q_n(s)) \geq \deg_s(p_j(s)) \quad j = 0,\ldots,m.$$

Then $\hat{F}(s,z)$ has a causal inverse w.i.d. iff

$$\deg_s(p_m(s)) \geq \deg_s(p_j(s)) \quad j = 0,\ldots,m-1.$$

Proof

Let $M = n-m$ and $N = \deg_s(q_n(s)) - \deg_s(p_m(s))$. Then $\hat{F}(s,z) z^M s^N$ is invertible and the inverse is causal. Therefore $\hat{F}(s,z)$ has a causal inverse w.i.d. (M,N).

The "only if" part is also clear. □

Observe that for every (M_1,N_1) such that $M_1 \geq M$ and $N_1 \geq N$ there exists a causal transfer function which can serve as an inverse w.i.d. (M_1,N_1). Furthermore the inverse w.i.d. (M,N) is invertible without delay. By theorem (3.4) not every causal transfer function has a causal inverse w.i.d. but in the next we will show that every causal transfer function does have a weakly causal inverse w.i.d., which is invertible without delay. This again demonstrates that the concept of weakly causal I/O systems is useful and therefore also the state space model (2.13).

Suppose $\hat{F}(s,z)$ is a causal transfer function. Let $S_F \subset Q_1$ be its support and define $\text{conv}^+ S_F$ by

$$\text{conv}^+ S_F = \text{conv } S_F + Q_1$$

where $\text{conv } S_F$ denotes the convex hull of S_F (the intersection of all convex sets containing S_F).

Furthermore, let (M,N) be an extremal point of $\text{conv}^+ S_F$ (a point such that $\text{conv}^+ S_F \backslash (M,N)$ is still convex). Then it is clear that $(M,N) \in S_F$ (see also [11 ch VIII]).

Furthermore, $\hat{H}(s,z) = \hat{F}(s,z) z^M s^N$ is a weakly causal transfer function and $H_{0,0} \neq 0$. Hence, by theorem (3.1), $\hat{H}(s,z)$ has a weakly causal inverse. Therefore, $\hat{F}(s,z)$ has a weakly causal inverse w.i.d. (M,N), which itself is invertible, namely $\hat{H}^{-1}(s,z)$.

Summarizing we have

3.5. Theorem

Suppose $\hat{F}(s,z)$ is a causal transfer function. Let (M,N) be an extremal point of $\text{conv}^+ S_F$ then there exists an inverse w.i.d. (M,N) which itself is an invertible (without delay) weakly causal transfer function. □

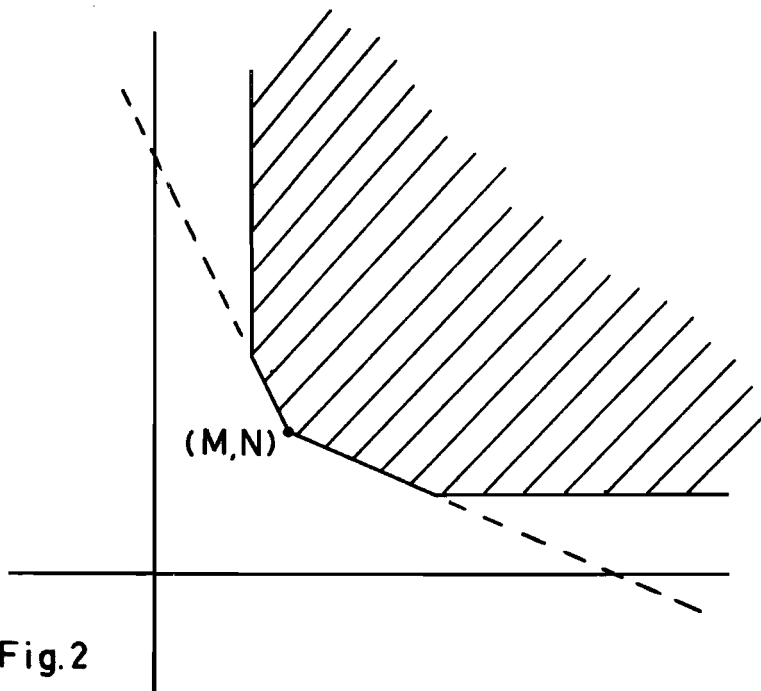Some geometrical insight is provided by figure 2.

Fig.2

The shaded area denotes $conv^+S_F$, the dotted lines correspond to a possible shifted cone giving rise to the required causality cone.

The idea in the above theorem is the following. Look for a point (M,N) in S_F such that there exists a causality cone (a closed convex cone satisfying (2.3) will also suffice because of lemma (2.5)) with the property that if we shift it in a way such that the origin (0,0) becomes (M,N) the support S_F is still contained in the shifted causality cone. Then (M,N) is a possible candidate for the inherent delay. Furthermore this holds for all the extremal points of $conv^+S_F$.

Remark

The extremal points of $conv^+S_F$ are the analogues of the minimal delay in the 1-D case because they give rise to inverse transfer functions which are invertible without delay.

In the 1-D case every delay larger then the minimal delay may serve as an inherent delay for some inverse (then this inverse is not invertible without delay). Because of the lack of a natural order in the 2-D case an inverse with minimal delay is not well defined. Therefore we will characterize all the possible delays corresponding to weakly causal inverses of some causal transfer function. The construction of possible inverses with inherent delay will be based on theorem (3.5) which is concerned with inverses w.i.d. which are invertible themselves. The following theorem enables us to construct more inverses w.i.d. whenever one inverse w.i.d. (which is itself invertible) based on an extremal point of $conv^+S_F$ is known.

3.6. Theorem

Suppose $\widehat{G}(s,z)$ is a weakly causal invertible transfer function. Let $S_G \subset C$ where C is a closed convex cone satisfying (2.3).
Let $(M,N) \in \mathbf{Z}^2 \setminus (-C)$. Then $\dfrac{\widehat{G}(s,z)}{z^M s^N}$ is a weakly causal transfer function.

Proof

If $\hat{G}(s,z)$ is invertible then $G_{0,0} \neq 0$. This means that for every $(M,N) \in \mathbb{Z}^2 \setminus (-C)$ there exists a causality cone such that if we shift it to (M,N) it still contains S_G. Therefore $\hat{G}(s,z)/z^M s^N$ is weakly causal. The second assertion follows from $G_{0,0} \neq 0$. $\qquad\qquad\square$

Now consider a causal transfer function $\hat{F}(s,z)$.

Let $\mathcal{M}$ and $\mathcal{N}$ denote the sets:

3.7
$$\mathcal{M} = \{M \mid (M,N) \in \text{conv}^+ S_F \text{ for some integer } N\}$$
$$\mathcal{N} = \{N \mid (M,N) \in \text{conv}^+ S_F \text{ for some integer } M\}$$

Let $\bar{M}, \bar{N}$ be defined by

3.8
$$\bar{M} = \min_{M \in \mathcal{M}} M \quad , \quad \bar{N} = \min_{N \in \mathcal{N}} N$$

We can now characterize the set of possible inherent delays corresponding to some causal transfer function.

3.9. Theorem

Suppose $\hat{F}(s,z)$ is a causal transfer function. Let $\bar{M}, \bar{N}$ be defined as in (3.8). Then we have

1°. If $M > \bar{M}$ or $N > \bar{N}$ there exists a weakly causal inverse $\hat{G}_{M,N}(s,z)$ w.i.d. $(\bar{M},N)$.

If $M \leq \bar{M}$ and $N \leq \bar{N}$ and $(M,N) \neq (\bar{M},\bar{N})$ there does not exist a weakly causal inverse w.i.d.

2°. $\hat{G}_{M,N}(s,z)$ is invertible (without delay) iff (M,N) is an extremal point of $\text{conv}^+ S_F$.

3°. $\hat{G}_{M,N}(s,z)$ is causal iff $(\bar{M},\bar{N}) \in \text{conv}^+ S_F$ and $M \geq \bar{M}$, $N \geq \bar{N}$.

Proof

Applying theorem (3.6) to $\hat{F}(s,z) z^M s^N$, for every extremal point (M,N) of $\text{conv}^+ S_F$, gives the proof of 1°. The proof of 2° follows from theorem (3.5). The proof of 3° follows from theorem (3.4). $\qquad\square$
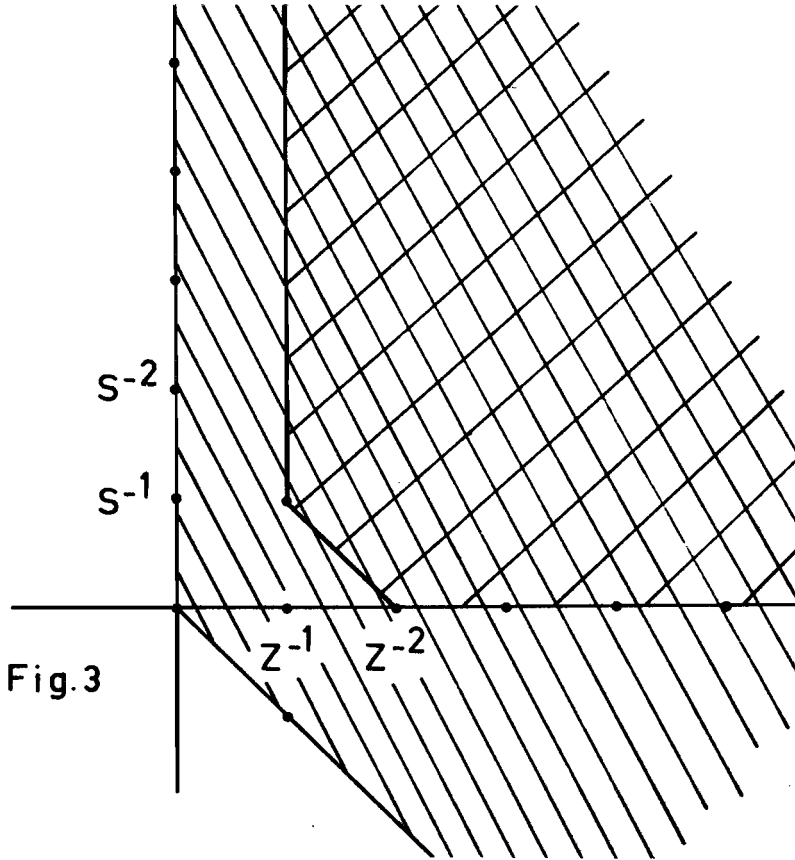
Example

Suppose, we have a causal I/0 system with transfer function
$\hat{F}(s,z) = \dfrac{s+z}{-sz+(s-1)z^2}$. Observe that $\hat{F}(s,z)$ does not satisfy the conditions of theorem (3.4) so that $\hat{F}(s,z)$ does not have a causal

inverse.

Now consider figure 3.

Fig. 3

Conv$^+$S$_F$ is the double shaded area, $(\bar{M}, \bar{N}) = (1,0)$. The extremal points of conv$^+$S$_F$ are $(2,0)$ and $(1,1)$. By theorem (3.9) there exists a weakly causal inverse w.i.d. $(0,1)$.

Indeed if we take $\hat{G}_{0,1}(s,z) = \dfrac{-sz+(s-1)z^2}{sz+s^2}$ then $\hat{G}_{0,1}(s,z) \cdot \hat{F}(s,z) = \dfrac{1}{s}$

and $\hat{G}_{0,1}(s,z)$ is weakly causal, for if we substitute (see lemma 2.11)

$$s = \alpha\beta^2 \ , \ z = \beta$$

Then $\hat{G}_{0,1}(\alpha\beta^2, \beta) = \dfrac{-\alpha\beta+\alpha\beta^2-1}{\alpha\beta+\alpha^2\beta^2}$ which is a causal transfer function.

By theorem (3.9), there exists a weakly causal inverse w.i.d. (1.1) which is invertible itself without delay. The weakly causal inverse w.i.d. $(1,1)$ is

$$\hat{G}_{1,1}(s,z) = \dfrac{-s+(s-1)z}{sz+s^2}$$

$\hat{G}_{1,1}(s,z)$ is weakly causal, for if we substitute

$$s = \alpha \ , \ z = \alpha\beta$$

then $\hat{G}_{1,1}(\alpha,\alpha\beta) = \dfrac{-1+(\alpha-1)\beta}{\alpha+\alpha\beta}$ which is causal and invertible. The shaded

area in fig. 3 denotes the causality cone of $\hat{G}_{1,1}(s,z)$.

Let $T(\alpha,\beta) = \hat{G}_{1,1}(\alpha,\alpha\beta)$. Now $T(\alpha,\beta)$ can be state space realized as

follows

$$\begin{bmatrix} \beta(x)_{kh} \\ \\ \alpha(a)_{kh} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{hk} \\ \\ a_{kh} \end{bmatrix} + \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} y_{kh}$$

$$\bar{u}_{kh} = \begin{bmatrix} -1 & | & -1 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + y_{kh}$$

and we obtain a state space realization of $\hat{G}_{1,1}(s,z)$ analogous to (2.13)

$$x_{k+1,h} = -x_{k,h+1} + y_{k,h+1} \qquad k=0,1,\dots$$

$$a_{k,h+1} = y_{kh} \qquad h = -k,-k+1,\dots$$

$$\bar{u}_{kh} = -x_{kh} - a_{kh} + y_{kh}$$

and $u_{kh} = \bar{u}_{k+1,h+1}$ because of the inherent delay

$$x_{0,h} = 0, \ a_{k,-k} = 0, \ h = 0,1,\dots \qquad k = 0,1,\dots$$

Observe that $S_F \subset Q_1 \subset S_{G_{1,1}}$. At first stage y_{kh} is defined for

$(k,h) \in Q_1$. We have to add zero's in the sense that $y_{kh} = 0$ for

$(k,h) \in S_{G_{1,1}} \setminus Q_1$.

Remark

In an expression like $\hat{y}(s,z) = \hat{F}(s,z)\hat{u}(s,z)$ the product is only
defined in the case where $\hat{y}(s,z)$, $\hat{F}(s,z)$, $\hat{u}(s,z)$ have their support
in the same causality cone (belong to the same ring).

An example of what may happen is the following

Suppose

$$\hat{F}(s,z) = \frac{1}{s-z} = \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \ldots$$

$$\hat{u}(s,z) = \frac{1}{s+z} = \frac{1}{z} - \frac{s}{z^2} + \frac{s^2}{z^3} - \ldots$$

Here the product of the two formal power series can not be well defined whereas $\frac{1}{s-z} \cdot \frac{1}{s+z} = \frac{1}{s^2-z^2}$.

ACKNOWLEDGMENT

4. CONCLUSIONS

In this paper we introduced a state space realization for the so called
weakly causal I/O systems (weakly causal transfer functions). It has
been shown that this considerably enlarges the class of realizable
transfer functions. Furthermore it was shown that a class of NSHP
filters can be state space realized using this method. In the latter part
we introduced inverse 2-D systems and a generalization of the concept
"inherent delay". We showed that the state space realization, obtained
in the first part, could be used for inverse systems with inherent
delay and that, in general, a causal 2-D system cannot have a causal
inverse (even with inherent delay). State space realization of a
multivariable weakly causal 2-D system can be handled in a completely
analogous way (see [1]). Although we described the 2-D case, the results
of this paper can also be derived for the multidimensional case.
The transformation of a weakly causal I/O system,to obtain a causal one,can
be seen as a unimodular transformation of an integer lattice (compare [14]).
This observation (made by one of the reviewers) may be useful if one is in-
terested in further generalization.

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