States and Operators in the Spacetime Algebra

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Abstract

The spacetime algebra (STA) is the natural, representation-free language for Dirac's theory of the electron. Conventional Pauli, Dirac, Weyl and Majorana spinors are replaced by spacetime multivectors, and the quantum σ- and γ-matrices are replaced by two-sided multivector operations. The STA is defined over the reals, and the role of the scalar unit imaginary of quantum mechanics is played by a fixed spacetime bivector. The extension to multiparticle systems involves a separate copy of the STA for each particle, and it is shown that the standard unit imaginary induces correlations between these particle spaces. In the STA, spinors and operators can be manipulated without introducing any matrix representation or coordinate system. Furthermore, the formalism provides simple expressions for the spinor bilinear covariants which dispense with the need for the Fierz identities. A reduction to $2+1$ dimensions is given, and applications beyond the Dirac theory are discussed.

1 Introduction

In this paper we present a new, direct method of translation between conventional matrix-based approaches to spinors in 3 and 4 dimensions [1, 2], and the spacetime algebra (STA) formalism of Hestenes [3, 4, 5, 6]. This method quickly yields the Dirac equation and the spinor bilinear covariants in Hestenes' STA form. With spinors and quantum matrix operators expressed in the real STA, all algebraic manipulations can be performed without ever introducing a matrix representation. The result is a very powerful language for expressing and analysing the Dirac equation, which provides many new insights into the geometric substructure of the Dirac theory [6, 7].

We study the Pauli matrix algebra in Section 2, and demonstrate how quantum spin states are formulated in terms of the real geometric algebra of space (which is a subalgebra of the full STA). An extension to multiparticle systems is introduced, in which separate (commuting) copies of the STA are taken for each particle. These copies produce multiparticle states having more degrees of freedom than in conventional quantum mechanics. Imposition of the complex structure removes this extra freedom and shows how the scalar unit imaginary of quantum mechanics induces correlations between particle spaces by locking their phases together.

In Section 3 the Dirac algebra is studied using the full, relativistic STA. The STA form of the Dirac equation is derived and a table of Dirac spinor bilinear covariants in STA form is presented. In Section 4 a similar approach is presented for the Weyl representation, and it is shown how the 2-spinor calculus of Penrose & Rindler [8] can be reformulated to great advantage. We close with a discussion of other representations and dimensions; an appendix gives the STA translation procedure for the Majorana representation.

We work throughout with the spacetime algebra [3, 9], which is the geometric algebra of flat, Minkowski spacetime, and is generated by an orthonormal frame of vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$. The $\{\gamma_\mu\}$ satisfy the Dirac algebra

$$\gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = g_{\mu\nu} = \mathrm{diag}(+\,-\,-\,-), \tag{1.1}$$

but are to be considered as four independent unit vectors. The full STA is 16-dimensional, and is spanned by the basis

$$1, \qquad \{\gamma_\mu\} \qquad \{\sigma_k,\, i\sigma_k\}, \qquad \{i\gamma_\mu\}, \qquad i. \tag{1.2}$$

Here $i \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ is the unit pseudoscalar. It squares to -1, and anticommutes with vectors and trivectors. The spacetime bivectors $\{\sigma_k\}$, $k = 1\ldots3$ are defined by

$$\sigma_k \equiv \gamma_k\gamma_0, \tag{1.3}$$

and represent an orthonormal frame of vectors in the space relative to the γ_0 direction. The $\{\sigma_k\}$ generate the Pauli algebra of space, so that relative vectors $a_k\sigma_k$ are viewed as spacetime bivectors. To distinguish these from spacetime vectors we write the former in bold type — $\boldsymbol{a} = a_k\sigma_k$. The meaning of the σ_k is unambiguous so these are left in normal type.

We use the symbol $\tilde{}$ to denote the operation of (relativistic) reversion, which reverses the order of vectors in any given product. Angled brackets $\langle A\rangle_r$ are used to denote the projection of the grade-r part of A, with the scalar part written as $\langle A\rangle$. Other notations and conventions are detailed in Paper I [9] of this series.

2 Pauli Spinors

In this section we study spinors in the Pauli algebra of space. This is a useful preliminary to the discussion of relativistic spinors and the Dirac algebra. The algebra of space is generated by three orthonormal (relative) vectors $\{\sigma_k\}$, and is spanned by

$$1, \qquad \{\sigma_k\}, \qquad \{i\sigma_k\}, \qquad i. \tag{2.1}$$

Since $\sigma_1\sigma_2\sigma_3 = \gamma_0\gamma_1\gamma_3\gamma_3$, space and spacetime both share the same pseudoscalar i.

When working non-relativistically, it is necessary to distinguish between relative vectors $\{\sigma_k\}$ and relative bivectors $\{i\sigma_k\}$, both of which are bivectors in the full STA (1.2). Accordingly, we define two new operations in the STA,

$$\begin{aligned} A^\dagger &= \gamma_0\tilde{A}\gamma_0, \\ \bar{A} &= \gamma_0 A\gamma_0, \end{aligned} \tag{2.2}$$

which are the reversion and parity operations in the Pauli algebra. The presence of γ_0 in these definitions demonstrates that these operators are frame-dependent.

The Pauli operator algebra [1] is generated by the 2×2 matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

These operators act on 2-component complex spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2.4}$$

where ψ_1 and ψ_2 are complex numbers. Throughout, we adopt the convention that standard quantum operators appear with carets, and quantum states are written as kets and bras. The unit scalar imaginary of conventional quantum mechanics is written as j, to distinguish it from the geometric pseudoscalar i.

The column Pauli spinor $|\psi\rangle$ is placed in one-to-one correspondence with the even multivector ψ (which satisfies $\psi = \psi^\dagger$) through the identification [10]

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k i\sigma_k. \tag{2.5}$$

The action of the four quantum operators $\{\hat{\sigma}_k, j\}$ translates as

$$\begin{aligned} \hat{\sigma}_k|\psi\rangle &\leftrightarrow \sigma_k\psi\sigma_3 \qquad (k = 1, 2, 3) \\ j|\psi\rangle &\leftrightarrow \psi i\sigma_3. \end{aligned} \tag{2.6}$$

It is routine to verify these; for example

$$\hat{\sigma}_1|\psi\rangle = \begin{pmatrix} -a^2 + ja^1 \\ a^0 + ja^3 \end{pmatrix} \quad \leftrightarrow \quad \begin{matrix} -a^2 + a^1 i\sigma_3 \\ -a^0 i\sigma_2 + a^3 i\sigma_1 \end{matrix} = \sigma_1 \left(a^0 + a^k i\sigma_k\right) \sigma_3. \tag{2.7}$$

Finally, complex conjugation translates to

$$|\psi\rangle^* \quad \leftrightarrow \quad \sigma_2\psi\sigma_2. \tag{2.8}$$

The Pauli equation (in natural units),

$$j\partial_t|\psi\rangle = \frac{1}{2m}\left((-j\boldsymbol{\nabla} - e\boldsymbol{A})^2 - e\hat{\sigma}_k B^k\right)|\psi\rangle + eV|\psi\rangle, \tag{2.9}$$

can now be written (in the Coulomb gauge) as [7, 11]

$$\partial_t\psi i\sigma_3 = \frac{1}{2m}(-\boldsymbol{\nabla}^2\psi + 2e\boldsymbol{A}\cdot\boldsymbol{\nabla}\psi i\sigma_3 + e^2\boldsymbol{A}^2\psi) - \frac{e}{2m}\boldsymbol{B}\psi\sigma_3 + eV\psi, \tag{2.10}$$

where $\boldsymbol{B}$ is the magnetic field vector $B^k\sigma_k$. This translation achieves two important goals: the scalar unit imaginary is eliminated, and all terms (both operators and states) are real-space multivectors. Removal of the distinction between states and operators is an important conceptual simplification, arising naturally through the use of the STA.

In order to translate the spinor inner product $\langle\psi|\phi\rangle$ we need only consider its real part. This is given by

$$\Re\langle\psi|\phi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\phi\rangle, \tag{2.11}$$

so that, for example,

$$\begin{aligned} \langle\psi|\psi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\psi\rangle &= \langle(a^0 - ia^k\sigma_k)(a^0 + ia^k\sigma_k)\rangle \\ &= (a^0)^2 + a^k a^k. \end{aligned} \tag{2.12}$$

It follows that the full inner product becomes

$$\langle\psi|\phi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi i\sigma_3\rangle i\sigma_3. \tag{2.13}$$

The right hand side projects out the $\{1, i\sigma_3\}$ components from the geometric product $\psi^\dagger\phi$. We write this projection as $\langle A\rangle_S$, and observe that, for Pauli-even multivectors,

$$\langle A\rangle_S = \tfrac{1}{2}(A - i\sigma_3 A i\sigma_3). \tag{2.14}$$

As an example of (2.13), consider the expectation value

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\sigma_k\psi\sigma_3\rangle - \langle\psi^\dagger\sigma_k\psi i\rangle i\sigma_3 = \sigma_k\cdot\langle\psi\sigma_3\psi^\dagger\rangle_1, \tag{2.15}$$

which gives the mean value of spin measurements in the k direction. The STA form indicates that this is the component of the spin vector $\psi\sigma_3\psi^\dagger$ in the σ_k direction, so that $S = \psi\sigma_3\psi^\dagger$ is the coordinate-free form of this vector. The even multivector ψ can be decomposed into $\psi = \rho^{1/2}R$ where $RR^\dagger = 1$, so the spin vector is therefore

$$S = \rho R\sigma_3 R^\dagger. \tag{2.16}$$

This demonstrates that the 2-sided construction of the expectation value (2.15) is an instruction to rotate and dilate the fixed σ_3 axis into the spin direction. The original states of quantum mechanics therefore become operators in the STA, acting on vectors. This explains why spinors transform single-sidedly under active rotations of fields in space. If the vector S is rotated to a new vector $R'S\tilde{R}'$, then the corresponding spinor must transform to $R'\psi$. In this way, the fixed $\{\sigma_k\}$ frame is shielded from rotations, and there is no conflict with rotational invariance. We use the term 'spinor' to denote any object which transforms single-sidedly under a rotor R [9].

We can now consider the status of the fixed $\{\sigma_k\}$ frame which, as is clear from (2.10), only occurs explicitly on the right hand side of the spinor ψ. This is analogous to rigid-body dynamics, in which a rotating frame $\{e_k\}$, aligned with the principal axes of the rotating body, can be related to a fixed laboratory frame $\{\sigma_k\}$ by

$$e_k = R\sigma_k R^\dagger. \tag{2.17}$$

The dynamics is now contained in the rotor R, which can also be viewed as a normalised spinor. The e_k are unaffected by the choice of laboratory frame and, under an active rotation, the body is rotated about its centre of mass, whilst the laboratory frame is fixed. Such a rotation has $e_k \mapsto R' e_k R'^\dagger$, which is enforced by $R \mapsto R'R$, and (as with the Pauli theory) the fixed frame is shielded from rotations. Gauge invariance can now be interpreted as the requirement that the physics is unaffected by the position of the σ_1 and σ_2 axes in the $i\sigma_3$ plane. In terms of rigid-body dynamics, this means that the body behaves as a symmetric top. The analogy between rigid-body dynamics and the STA form of the Pauli theory is therefore strong [4], and we shall see in Section 3 that the analogy extends to Dirac theory.

An arbitrary operator $\hat{M}|\psi\rangle$ is replaced by a multilinear function $M(\psi)$ in the STA. Since ψ is now a 4-component multivector, the space of operators M is 16-dimensional (this is the dimension of the group Gl(4,R)), and is easily large enough to encompass the 8-dimensional Pauli operator algebra (which forms the group Gl(2,C)). The subset of multilinear functions which represent Pauli operators is defined by the requirement that M respect the complex structure:

$$\begin{aligned} j\hat{M}(j|\psi\rangle) &= -\hat{M}|\psi\rangle \\ \Rightarrow M(\psi i\sigma_3)i\sigma_3 &= -M(\psi). \end{aligned} \tag{2.18}$$

The set of M satisfying (2.18) is 8-dimensional, as required.

The Hermitian adjoint is defined by

$$\langle\psi|\hat{M}\phi\rangle = \langle\hat{M}^\dagger\psi|\phi\rangle, \tag{2.19}$$

which translates into two equations in the STA, one each for the real and imaginary parts. The real equation is

$$\langle\psi^\dagger M(\phi)\rangle = \langle M_{HA}^\dagger(\psi)\phi\rangle, \tag{2.20}$$

and the imaginary equation simply restates (2.18). The subscript of M_{HA} labels the STA operator representation of the Pauli (Hermitian) adjoint. In geometric algebra the adjoint to a linear function is defined via

$$\langle\bar{M}(\psi)\phi\rangle = \langle\psi M(\phi)\rangle, \tag{2.21}$$

where we have employed the overbar notation of Hestenes & Sobczyk [12]. It is always clear from the context when the overbar is being employed to denote the geometric adjoint or parity (2.2). The Pauli adjoint is now given by the combination of a reversion, the geometric adjoint, and a second reversion:

$$M_{HA}(\psi) = \bar{M}^\dagger(\psi^\dagger). \tag{2.22}$$

For example, if $M(\psi) = A\psi B$, then $M_{HA}(\psi) = A^\dagger\psi B^\dagger$. Since the STA action of the $\hat{\sigma}_k$ operators takes ψ into $\sigma_k\psi\sigma_3$, it follows that these operators are, properly, Hermitian. The attractive feature of (2.22) is that the Pauli operator algebra can now be fully integrated into the wider subject of multilinear function theory — the study of linear functions within geometric algebra [12].

2.1 2-Particle Pauli States

In quantum theory, 2-particle states are assembled from direct products of single-particle states. To represent these states in the STA, we must consider direct products of two copies of the STA itself. For present purposes, however, we need only take copies of the even (Pauli) subalgebra of the full STA. The direct product is usually written as $\otimes$, but we will drop this symbol and use superscripts to label the single-particle algebra from which any particular element is derived. Thus we write $\sigma_1 \otimes \sigma_1$ as $\sigma_1^1\sigma_1^2$, where the $\sigma_1^1\sigma_1^2$ product is commutative and associative. Wherever possible, we will further abbreviate $i^1\sigma_1^1$ to $i\sigma_1^1$ *et cetera*, and will write the unit element of either space simply as 1.

The full 2-particle Pauli algebra is $8 \times 8 = 64$ dimensional, and the spinor subalgebra is $4 \times 4 = 16$ dimensional — which is twice the dimension of the direct product of two 2-component complex spinors. This is because we have not taken the complex structure of the spinors into account. While the role of

j is played in the two single-particle spaces by right multiplication by $i\sigma_3^1$ and $i\sigma_3^2$ respectively, standard quantum mechanics does not distinguish between these operations. Therefore we must include a projection operator to ensure that right multiplication by $i\sigma_3^1$ or $i\sigma_3^2$ reduces to the same operation. This projector is

$$E^{12} = \tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2), \tag{2.23}$$

so that

$$E^{12} i\sigma_3^1 = E^{12} i\sigma_3^2 = J^{12}, \tag{2.24}$$

where

$$J^{12} = \tfrac{1}{2}(i\sigma_3^1 + i\sigma_3^2). \tag{2.25}$$

The (multiparticle) STA representation of a 2-particle Pauli spinor is therefore $\psi^1\phi^2 E^{12}$, where ψ^1 and ϕ^2 are spinors (even multivectors) in their own spaces. A complete basis for 2-particle spin states is

$$\begin{aligned}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad &\leftrightarrow \quad E^{12} \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad &\leftrightarrow \quad -i\sigma_2^1 E^{12} \\
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad &\leftrightarrow \quad -i\sigma_2^2 E^{12} \\
\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad &\leftrightarrow \quad i\sigma_2^1 i\sigma_2^2 E^{12}
\end{aligned} \tag{2.26}$$

and the complex structure is provided by

$$j|\psi\rangle \quad \leftrightarrow \quad \psi J^{12}. \tag{2.27}$$

This procedure extends to higher multiplicities, where all that is required is to find the ‘quantum correlator’ E satisfying

$$E i\sigma_3^j = E i\sigma_3^k = J \quad \text{for all } j,\, k. \tag{2.28}$$

The beauty of this approach is that all the operations defined for the single-particle STA extend naturally to the multiparticle algebra; for example the spinor inner product generalises to

$$\langle \psi^\dagger \phi \rangle_S = \langle \psi^\dagger \phi E \rangle E - \langle \psi^\dagger \phi J \rangle J. \tag{2.29}$$

A full development of multiparticle spacetime algebra, including generalisation to relativistic states, will be presented in a forthcoming paper.

We conclude this section with an illustration of the insights revealed by our approach. The 2-particle singlet state $|\epsilon\rangle$ is defined by

$$|\epsilon\rangle = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right\}, \tag{2.30}$$

which translates into

$$\epsilon = \tfrac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2). \tag{2.31}$$

This is normalised such that $\langle\epsilon^\dagger\epsilon\rangle_S = E$. It can be confirmed that ϵ satisfies

$$i\sigma_k^1\epsilon = -i\sigma_k^2\epsilon, \tag{2.32}$$

which we can use to provide a novel demonstration of the rotational invariance of ϵ. Under a joint rotation of 2-particle space, a spinor ψ transforms to $R^1R^2\psi$, where R^1 and R^2 are copies of the same rotor but in different spaces. It follows from (2.32) that, under a rotation, ϵ transforms as

$$\epsilon \mapsto R^1R^2\epsilon = R^1\tilde{R}^1\epsilon = \epsilon, \tag{2.33}$$

so that ϵ is a genuine 2-particle scalar.

3 Dirac Spinors

In this section we extend the procedure of Section 2 to show how Dirac spinors can be understood in terms of the geometry of *real* spacetime. This reveals the geometrical role of spinors in the Dirac theory, following the approach of Hestenes [4, 5]. Furthermore, this formulation is *representation-free*, highlighting the intrinsic content of the Dirac theory.

The subject of relativistic spinors has, of course, been much discussed in the literature; see for example the books by Penrose & Rindler [8, 13] and Benn & Tucker [14], or the list of references provided by Figueiredo *et al.* [15] These treatments invariably define spinors over the complex field, obscuring much of the associated spacetime geometry. It is through the employment of the *real* STA that this geometry is revealed.

We begin with the γ-matrices in the standard Dirac-Pauli representation [1],

$$\hat{\gamma}_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}. \tag{3.1}$$

A Dirac column spinor $|\psi\rangle$ is placed in one-to-one correspondence with an 8-component even element of the STA (1.2) via [10, 16]

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \\ -b^3 + jb^0 \\ -b^1 - jb^2 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k i\sigma_k + i(b^0 + b^k i\sigma_k). \tag{3.2}$$

The action of the operators $\{\hat{\gamma}_\mu, \hat{\gamma}_5, j\}$ (where $\hat{\gamma}_5 = \hat{\gamma}^5 = -j\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$) translates as

$$\begin{aligned} \hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma_\mu\psi\gamma_0 \quad (\mu = 0, \ldots, 3) \\ j|\psi\rangle &\leftrightarrow \psi\, i\sigma_3 \\ \hat{\gamma}_5|\psi\rangle &\leftrightarrow \psi\sigma_3, \end{aligned} \tag{3.3}$$

which are verified by simple computation; for example

$$\hat{\gamma}_5|\psi\rangle = \begin{pmatrix} -b^3 + jb^0 \\ -b^1 - jb^2 \\ a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \quad \leftrightarrow \quad \begin{matrix} -b^3 + b^0\sigma_3 + b^1 i\sigma_2 - b^2 i\sigma_1 \\ +a^0\sigma_3 + a^3 i - a^2\sigma_1 + a^1\sigma_2 \end{matrix} = \psi\sigma_3. \tag{3.4}$$

Complex conjugation in this representation translates as

$$|\psi\rangle^* \quad \leftrightarrow \quad -\gamma_2\psi\gamma_2. \tag{3.5}$$

As a simple application of (3.2) and (3.3), the Dirac equation itself,

$$\hat{\gamma}^\mu(j\partial_\mu - eA_\mu)|\psi\rangle = m|\psi\rangle, \tag{3.6}$$

becomes, upon postmultiplying by γ_0,

$$\nabla\psi i\sigma_3 - eA\psi = m\psi\gamma_0, \tag{3.7}$$

which is the form first discovered by Hestenes [3]. This translation is direct and unambiguous, leading to an equation which is not only coordinate-free (since the vectors $\nabla = \gamma^\mu \partial_\mu$ and $A = \gamma^\mu A_\mu$ no longer refer to any frame) but is representation-free as well! In manipulating (3.7) one needs only the algebraic rules for multiplying spacetime multivectors, and the equation can be solved completely without ever having to introduce a matrix representation. Equation (3.7) therefore expresses the intrinsic geometric content of the Dirac equation.

It is worth commenting on our choice of the $(+---)$ metric. We can translate our results to the opposite metric $(-+++)$ by introducing the operators

$$\hat{e}_\mu = j\hat{\gamma}_\mu, \tag{3.8}$$

so that the $\{\hat{e}_\mu\}$ generate the Dirac algebra with opposite signature. The corresponding vectors $\{e_\mu\}$ generate a 16-dimensional algebra whose even part is isomorphic to the even subalgebra of the STA. With σ_k now defined as $e_0 e_k$, we employ the same identification (3.2), and the action of the $\{\hat{e}_\mu\}$ translates into

$$\hat{e}_\mu|\psi\rangle \quad \leftrightarrow \quad -e_\mu \psi i e_3, \tag{3.9}$$

with j and $\hat{e}_5$ translating in the same way as (3.3). The Dirac equation in this signature is

$$\hat{e}^\mu(\partial_\mu + jeA_\mu)|\psi\rangle = m|\psi\rangle, \tag{3.10}$$

which, using (3.9) and defining $\nabla_e = e^\mu \partial_\mu$, becomes

$$\nabla_e \psi i\sigma_3 - eA\psi = m\psi e_0. \tag{3.11}$$

This is identical to (3.7), the difference in sign of $\nabla_e^2 = -\nabla^2$ being handled by $e_0^2 = -1$. It is usually argued that the signature of the metric is irrelevant in relativistic quantum mechanics, because superpositions of states are considered over the complex field. Here we see an alternative explanation, which does not require complex numbers. Although the full Clifford algebras of opposite signature are different, their even subalgebras (the spinors) are isomorphic, so that any operation on a spinor performed in one algebra has a counterpart in the second algebra. We can now continue to work in the STA with signature $(+---)$, with the understanding that it is always possible to translate between metrics. Physically, it would be very surprising if the sign of the signature ever made a difference.

In order to discuss the spinor inner product, it is necessary to distinguish between the two types of adjoint, Hermitian and Dirac. We write these as

$$\begin{array}{lll} \langle\bar{\psi}| & - & \text{Dirac adjoint} \\ \langle\psi| & - & \text{Hermitian adjoint,} \end{array} \tag{3.12}$$

which translate as follows,

$$\begin{array}{lll} \langle\bar{\psi}| & \leftrightarrow & \tilde{\psi} \\ \langle\psi| & \leftrightarrow & \psi^\dagger = \gamma_0\tilde{\psi}\gamma_0. \end{array} \tag{3.13}$$

This makes it clear that the Dirac adjoint is the natural frame-invariant choice. The inner product translates in the same manner as (2.13), so that

$$\langle\bar{\psi}|\phi\rangle \quad \leftrightarrow \quad \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi i\sigma_3\rangle i\sigma_3 = \langle\tilde{\psi}\phi\rangle_S, \tag{3.14}$$

which is also easily verified by direct calculation. In a companion paper [17] we shall work with the STA form of the Lagrangian for the Dirac equation, which to illustrate (3.14) we give here:

$$\mathcal{L} = \langle\bar{\psi}|(\hat{\gamma}_\mu(j\partial^\mu - eA^\mu) - m)|\psi\rangle \quad \leftrightarrow \quad \langle\nabla\psi i\gamma_3\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle. \tag{3.15}$$

By utilising (3.14) the STA forms of the Dirac spinor bilinear covariants [2] are readily found; for example

$$\langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle \quad \leftrightarrow \quad \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle - \langle\tilde{\psi}\gamma_\mu\psi i\gamma_3\rangle i\sigma_3 = \gamma_\mu\cdot\langle\psi\gamma_0\tilde{\psi}\rangle_1 \tag{3.16}$$

identifies the vector $\psi\gamma_0\tilde{\psi}$ as the coordinate-free representation of the Dirac current. Since $\psi\tilde{\psi}$ contains only scalar and pseudoscalar terms, we can define $\rho e^{i\beta} = \psi\tilde{\psi}$ and, assuming $\rho \neq 0$, ψ can be written as $\rho^{1/2}e^{i\beta/2}R$. The 'rotor' R satisfies $R\tilde{R} = 1$ and generates Lorentz transformations. The full set of bilinear covariants [18] can now be written as

$$\begin{array}{crcl} \text{Scalar} & \langle\bar{\psi}|\psi\rangle & \leftrightarrow & \langle\psi\tilde{\psi}\rangle = \rho\cos\beta \\ \text{Vector} & \langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle & \leftrightarrow & \psi\gamma_0\tilde{\psi} = \rho v \\ \text{Bivector} & \langle\bar{\psi}|j\hat{\gamma}_{\mu\nu}|\psi\rangle & \leftrightarrow & \psi i\sigma_3\tilde{\psi} = \rho e^{i\beta}S \\ \text{Pseudovector} & \langle\bar{\psi}|\hat{\gamma}_\mu\hat{\gamma}_5|\psi\rangle & \leftrightarrow & \psi\gamma_3\tilde{\psi} = \rho s \\ \text{Pseudoscalar} & \langle\bar{\psi}|j\hat{\gamma}_5|\psi\rangle & \leftrightarrow & \langle\psi\tilde{\psi}i\rangle = -\rho\sin\beta, \end{array} \tag{3.17}$$

where

$$\begin{aligned} v &= R\gamma_0\tilde{R} \\ s &= R\gamma_3\tilde{R} \end{aligned} \tag{3.18}$$

and

$$S = isv. \tag{3.19}$$

These are summarised neatly by the equation

$$\psi(1+\gamma_0)(1+i\gamma_3)\tilde{\psi} = \rho\cos\beta + \rho v + \rho e^{i\beta}S + i\rho s + i\rho\sin\beta. \tag{3.20}$$

The full Dirac spinor ψ contains (in the rotor R) an instruction to carry out a rotation of the fixed $\{\gamma_\mu\}$ frame into the frame of observables. The single-sided transformation law for the spinor ψ is thus interpreted and, as in the Pauli case, the $\{\gamma_\mu\}$ frame is shielded from active rotations of spacetime observables, so that its presence does not compromise Lorentz invariance. The analogy with rigid-body dynamics, discussed in Section 2, therefore extends immediately to the relativistic theory, where γ_0 is now interpreted as the fixed time-like vector which determines the laboratory frame.

Once the spinor bilinear covariants are written in STA form (3.17) they can be manipulated far more easily than in conventional treatments. The Fierz identities, which relate the various observables (3.17), are simple to derive [18] because there is no longer any need for complicated index manipulations [19, 20]. Furthermore, reconstituting ψ from the observables (up to a gauge transformation) is now a routine exercise, carried out by writing

$$\begin{aligned} \langle\psi\rangle_S &= \tfrac{1}{4}\left(\psi + \bar{\psi} + \sigma_3(\psi + \bar{\psi})\sigma_3\right) \\ &= \tfrac{1}{4}(\psi + \gamma_0\psi\gamma_0 + \sigma_3\psi\sigma_3 + \gamma_3\psi\gamma_3), \end{aligned} \tag{3.21}$$

so that

$$\psi\tilde{\langle\psi\rangle}_S = \tfrac{1}{4}\rho(e^{i\beta} + v\gamma_0 - e^{i\beta}Si\sigma_3 + s\gamma_3). \tag{3.22}$$

The right-hand side of (3.22) can be found directly from the observables, and the left-hand side gives ψ to within a complex multiple. Defining

$$Z = \tfrac{1}{4}\rho(e^{i\beta} + v\gamma_0 - e^{i\beta}Si\sigma_3 + s\gamma_3) \tag{3.23}$$

we find that, up to an arbitrary phase factor,

$$\psi = (\rho e^{i\beta})^{1/2}Z(Z\tilde{Z})^{-1/2}. \tag{3.24}$$

The identification of the rotor R suggests that the vector $R\gamma_0\tilde{R}$ can be interpreted as a 4-velocity field. This idea is examined critically in a further paper [7], in which integral curves (streamlines) are plotted following this 4-velocity. In conventional formulations of the Dirac equation, a velocity operator is found whose eigenvalues are plus or minus the speed of light. It is argued that an accurate determination of the velocity, by measuring the position of a particle at two different times, leads to a large uncertainty in the momentum, which then favours higher velocities. Even in the context of conventional Dirac theory, this argument has a number of problems [21]. Applied to the STA velocity, the argument contains the further flaw that it assumes an incorrect relation between velocity and momentum. The status of velocity in the conventional operator approach to Dirac theory remains problematic, whereas the STA formalism provides an unambiguous answer.

An arbitrary Dirac operator $\hat{M}|\psi\rangle$ is written as a multilinear function $M(\psi)$, which acts linearly on the entire even subalgebra of the STA. The 64 real dimensions of this space of linear operators are reduced to 32 by the constraint (2.18)

$$M(\psi i\sigma_3) = M(\psi)i\sigma_3 \tag{3.25}$$

and, proceeding as at (2.22), the formula for the Dirac adjoint is

$$M_{DA}(\psi) = \tilde{\bar{M}}(\tilde{\psi}). \tag{3.26}$$

Self-adjoint Dirac operators satisfy $\tilde{M}(\psi) = \bar{M}(\tilde{\psi})$, and clearly include the $\hat{\gamma}_\mu$. The Hermitian adjoint, M_{HA}, is derived in the same way:

$$M_{HA}(\psi) = \bar{M}^\dagger(\psi^\dagger), \tag{3.27}$$

in agreement with the non-relativistic equation (2.22).

We now examine two important operator classes: projection and symmetry operators. The particle/antiparticle projection operators translate into

$$\frac{1}{2m}(m \mp \hat{\gamma}_\mu p^\mu)|\psi\rangle \quad \leftrightarrow \quad \frac{1}{2m}(m\psi \mp p\psi\gamma_0), \tag{3.28}$$

and the spin-projection operators become

$$\tfrac{1}{2}(1 \pm \hat{\gamma}_\mu s^\mu \hat{\gamma}_5)|\psi\rangle \quad \leftrightarrow \quad \tfrac{1}{2}(\psi \pm s\psi\gamma_3). \tag{3.29}$$

Provided that $p\cdot s = 0$, the spin and particle projection operators commute.

The three discrete symmetries C, P and T translate equally simply (following the convention of Bjorken & Drell [1]):

$$\begin{array}{lcl} \hat{P}|\psi\rangle & \leftrightarrow & \gamma_0\psi(\bar{x})\gamma_0 \\ \hat{C}|\psi\rangle & \leftrightarrow & \psi\sigma_1 \\ \hat{T}|\psi\rangle & \leftrightarrow & i\gamma_0\psi(-\bar{x})\gamma_1, \end{array} \tag{3.30}$$

where $\bar{x} = \gamma_0 x \gamma_0$ is (minus) a reflection in the time-like γ_0 axis.

We stress that this analysis is not limited to the Dirac equation, but is applicable to any type of equation involving Dirac spinors. For example, the equation for a zero-charge particle with an anomalous magnetic moment [22],

$$j\left(\hat{\gamma}_\mu \partial^\mu + \hat{\gamma}_{\mu\nu}\kappa F^{\mu\nu}\right)|\psi\rangle = m|\psi\rangle, \tag{3.31}$$

becomes

$$\nabla\psi i\sigma_3 + \kappa F\psi i\gamma_3 = m\psi\gamma_0, \tag{3.32}$$

and provides a model for a relativistic oscillator [23] when used with the quadratic potential $A = \frac{1}{2}m\omega x\gamma_0 x$. The STA formalism can also be employed in the study of nonlinear spinor equations [24, 25] and semi-classical models. An example of the latter is the Lagrangian of Barut & Zanghi [26],

$$L = j\langle\dot{\bar{\psi}}|\psi\rangle + p^\mu(\dot{x}_\mu - \langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle) + eA^\mu\langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle, \tag{3.33}$$

which translates into

$$L = \langle\dot{\psi}i\sigma_3\tilde{\psi} + p\cdot(\dot{x} - \psi\gamma_0\tilde{\psi}) + eA\psi\gamma_0\tilde{\psi}\rangle. \tag{3.34}$$

This form of the Lagrangian has been used by one of us [16] to study its predictions for the behaviour of an electron in a magnetic field. The Lagrangian (3.34) is also considered in Paper III [17] of this series, where its symmetries are examined.

4 The Weyl Representation and 2-Spinor Calculus

In this section we develop a translation for the Weyl representation, after the manner of Section 3. This is the starting point for a reformulation of the 2-spinor calculus of Penrose & Rindler [8].

The Weyl (chiral) representation is defined by [2]

$$\hat{\gamma}_0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \tag{4.1}$$

and is obtained from the Dirac-Pauli representation (3.1) via the unitary matrix

$$\hat{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix}. \tag{4.2}$$

A spinor in this representation is written as

$$|\psi\rangle = \begin{pmatrix} |\kappa\rangle \\ |\bar{\omega}\rangle \end{pmatrix}, \tag{4.3}$$

where $|\kappa\rangle$ and $|\bar{\omega}\rangle$ are 2-component spinors. This is translated as

$$|\psi\rangle = \begin{pmatrix} |\kappa\rangle \\ |\bar{\omega}\rangle \end{pmatrix} \quad \leftrightarrow \quad \psi = \kappa\tfrac{1}{2}(1+\sigma_3) - \omega\tfrac{1}{2}(1-\sigma_3), \tag{4.4}$$

where κ and ω are the Pauli-even equivalents of the the 2-component complex spinors $|\kappa\rangle$, $|\bar{\omega}\rangle$, defined by equation (2.5). The action of the $\hat{\gamma}_\mu$ translates in the same way as for Dirac spinors (3.3), as we see from the relations

$$\hat{\gamma}_0|\psi\rangle = \begin{pmatrix} -|\bar{\omega}\rangle \\ -|\kappa\rangle \end{pmatrix} \quad \leftrightarrow \quad -\omega\tfrac{1}{2}(1+\sigma_3) + \kappa\tfrac{1}{2}(1-\sigma_3) = \gamma_0\psi\gamma_0 \tag{4.5}$$

and

$$\hat{\gamma}_k|\psi\rangle = \begin{pmatrix} -\hat{\sigma}_k|\bar{\omega}\rangle \\ \hat{\sigma}_k|\kappa\rangle \end{pmatrix} \quad \leftrightarrow \quad -\sigma_k\omega\sigma_3\tfrac{1}{2}(1+\sigma_3) - \sigma_k\kappa\sigma_3\tfrac{1}{2}(1-\sigma_3) = \gamma_k\psi\gamma_0. \tag{4.6}$$

We have used (2.6) and the fact that γ_0 commutes with all Pauli-even elements. The action of j and $\hat{\gamma}_5$ also translate as in (3.3), so that the chiral projection operators $\frac{1}{2}(1 \pm \hat{\gamma}_5)$ result in

$$\begin{aligned} \tfrac{1}{2}(1+\hat{\gamma}_5)|\psi\rangle &\leftrightarrow \psi\tfrac{1}{2}(1+\sigma_3) = \kappa\tfrac{1}{2}(1+\sigma_3) \\ \tfrac{1}{2}(1-\hat{\gamma}_5)|\psi\rangle &\leftrightarrow \psi\tfrac{1}{2}(1-\sigma_3) = -\omega\tfrac{1}{2}(1-\sigma_3). \end{aligned} \tag{4.7}$$

Our fundamental translation for a relativistic 2-spinor is consequently

$$|\kappa\rangle \quad \leftrightarrow \quad \kappa\tfrac{1}{2}(1+\sigma_3). \tag{4.8}$$

This differs from the Pauli spinor translation (2.5), and is closer to the usual 'minimal left ideal' definition of a spinor [14]. The true significance of (4.8) lies not in the algebraic structure of the Pauli algebra, however, but in the fact that the projector $\frac{1}{2}(1+\sigma_3)$ adds a scalar to a vector, and is a 3-space expression of a *null* vector. The reason for identifying the projector with a null vector will become apparent when we consider the vector current.

Under a Lorentz transformation the spinor ψ transforms to $R\psi$, so that the STA 2-spinors transform into

$$\begin{aligned} \kappa'\tfrac{1}{2}(1+\sigma_3) &= R\kappa\tfrac{1}{2}(1+\sigma_3) \\ \omega'\tfrac{1}{2}(1-\sigma_3) &= R\omega\tfrac{1}{2}(1-\sigma_3), \end{aligned} \tag{4.9}$$

and in the STA both spinors therefore have the *same* transformation law. If we examine how the corresponding column spinors transform, we find that

$$\begin{aligned} R\kappa\tfrac{1}{2}(1+\sigma_3) &= R_+\kappa\tfrac{1}{2}(1+\sigma_3) + R_-\kappa\sigma_3\tfrac{1}{2}(1+\sigma_3) \\ R\omega\tfrac{1}{2}(1-\sigma_3) &= R_+\omega\tfrac{1}{2}(1-\sigma_3) - R_-\omega\sigma_3\tfrac{1}{2}(1-\sigma_3) \end{aligned} \tag{4.10}$$

where $R_+ = \frac{1}{2}(R+R^\dagger)$ and $R_- = \frac{1}{2}(R-R^\dagger)$. The column spinors now have different transformation laws: the $|\kappa\rangle$ transforms simply enough to $\hat{R}|\kappa\rangle$, but the $|\bar{\omega}\rangle$ transforms under the operator equivalent of

$$\begin{aligned} R - R_- &= \gamma_0 R \gamma_0 \\ &= (\gamma_0 \tilde{R} \gamma_0)\tilde{} \\ &= (R^{-1})^\dagger, \end{aligned} \tag{4.11}$$

so that $|\bar{\omega}'\rangle = (\hat{R}^{-1})^\dagger|\bar{\omega}\rangle$. This splitting of Lorentz transformations into two distinct operations is an unattractive feature of the 2-spinor formalism. The problem is that in carrying out relativistic manipulations in the Pauli algebra, the 2-spinor formalism has already singled out a preferred time-like axis (the γ_0 axis). This breaks up expressions in a way that disguises their frame-independent meaning. By working in the even subalgebra of the STA this problem can be avoided, since recourse is always available to the full STA.

The transformation (4.9) shows why it is necessary to supplement the Pauli-even spinors κ, ω, with the projectors $\frac{1}{2}(1 \pm \sigma_3)$. An arbitrary rotor R would, in general, map a Pauli-even element into the full 8-dimensional Pauli algebra. The presence of a factor of $\frac{1}{2}(1 \pm \sigma_3)$ on the right, on the other hand, projects out a 4-dimensional space which is closed under the action of a rotor R, and therefore forms an invariant subspace.

We are now in a position to consider the 2-spinor calculus approach to relativistic quantum mechanics [8, 27]. The basic idea is to treat Weyl spinors as 2-component complex vectors, using standard vector-space notions. Thus the spinor $|\kappa\rangle$ is written as κ^A, and a skew metric tensor ϵ_{AB} is introduced whose components are defined to be those of the matrix $i\hat{\sigma}_2$. We can therefore write

$$\begin{aligned} \kappa^A &\leftrightarrow \kappa\tfrac{1}{2}(1+\sigma_3) \\ \kappa_A &\leftrightarrow i\sigma_2\kappa\tfrac{1}{2}(1+\sigma_3), \end{aligned} \tag{4.12}$$

although there is rarely any need to distinguish κ^A and κ_A, since the translation into the STA eliminates all reference to components.

In (4.3) there are two different types of spinor, which transform differently under Lorentz rotations. To distinguish these 'modules', the second type $|\bar{\omega}\rangle$ are usually given primed indices. In terms of components the two modules are related by $\bar{\kappa}_{A'} = (\kappa_A)^*$. The Pauli-even equivalent of $(\kappa_A)^*$ is $\sigma_2(i\sigma_2\kappa)\sigma_2 = \kappa i\sigma_2$, bringing us to the next aspect of our translation

$$\bar{\kappa}_{A'} \quad \leftrightarrow \quad -\kappa i\sigma_2\tfrac{1}{2}(1-\sigma_3), \tag{4.13}$$

where κ is the Pauli-even equivalent of κ^A. We have differed slightly from our previous definition [10] in order to maintain consistency with the conventions adopted in section 2. As before, κ^A and $\bar{\kappa}_{A'}$ transform differently under Lorentz rotations:

$$\begin{aligned} \kappa^A &\mapsto (R)^A{}_B\kappa^B \\ \bar{\kappa}_{A'} &\mapsto (R^{-1\dagger})_{A'}{}^{B'}\bar{\kappa}_{B'}, \end{aligned} \tag{4.14}$$

but these are merely separate manifestations of the same underlying transformation $\kappa \mapsto R\kappa$.

The 2-spinor inner products are written as $\kappa^A\omega_A$ and $\bar{\kappa}^{A'}\bar{\omega}_{A'}$. In order to translate these, we first expand

$$\begin{aligned} \kappa^A\omega_A &= \kappa^0\omega^1 - \kappa^1\omega^0 \\ &= (-a^0b^2 - a^3b^1 + a^2b^0 + a^1b^3) + j(-a^3b^2 + a^0b^1 - a^1b^0 + a^2b^3), \end{aligned} \tag{4.15}$$

where we have used the identification (2.5). It can be verified that (4.15) translates to

$$\kappa^A\omega_A \quad \leftrightarrow \quad \langle i\sigma_2\tilde{\kappa}\omega\rangle - i\sigma_3\langle i\sigma_1\tilde{\kappa}\omega\rangle = \langle i\sigma_2\tilde{\kappa}\omega\rangle_S, \tag{4.16}$$

where again $\langle\ \rangle_S$ denotes the projection of $\{1, i\sigma_3\}$ components. The 2-spinor inner product is antisymmetric, $\kappa^A\omega_A = -\omega^A\kappa_A$, which is seen from (4.16) as follows:

$$\begin{aligned}\langle i\sigma_2\tilde{\kappa}\omega\rangle_S &= -\langle\tilde{\omega}\kappa i\sigma_2\rangle_S \\ &= -\langle i\sigma_2\tilde{\omega}\kappa\rangle_S.\end{aligned} \tag{4.17}$$

Since $\bar{\kappa}^{A'}\bar{\omega}_{A'} = (\kappa^A\omega_A)^*$, we see similarly that

$$\bar{\kappa}^{A'}\bar{\omega}_{A'} \quad \leftrightarrow \quad \langle\tilde{\kappa}\omega i\sigma_2\rangle_S = \langle i\sigma_2(-\kappa i\sigma_2)\tilde{\ }(-\omega i\sigma_2)\rangle_S. \tag{4.18}$$

A particularly instructive way to express the 2-spinor inner product is to construct the full Dirac spinor

$$\psi = \kappa\tfrac{1}{2}(1+\sigma_3) - \omega i\sigma_2\tfrac{1}{2}(1-\sigma_3), \tag{4.19}$$

and observe that

$$\begin{aligned}\psi\tilde{\psi} &= \kappa i\sigma_2\tfrac{1}{2}(1-\sigma_3)\tilde{\omega} - \omega\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\kappa} \\ &= -\langle i\sigma_2\tilde{\kappa}\omega\rangle + i\langle i\sigma_1\tilde{\kappa}\omega\rangle.\end{aligned} \tag{4.20}$$

Apart from an irrelevant minus sign this has the same form as (4.16), except that the projection is now onto the $\{1, i\}$ components. The advantage of the form (4.20) is that Lorentz invariance is manifest, since $R\psi\tilde{\psi}\tilde{R} = \psi\tilde{\psi}$. Furthermore, (4.20) demonstrates that a 'normalised spin frame' in the sense of Penrose & Rindler [8] is equivalent to a Lorentz transformation. Equation (4.20) also defines the zero-component of the Cartan map [28] from $\mathcal{C}^2 \times \mathcal{C}^2$ to $\mathcal{C}^4$, which arises naturally in the context of the 2-particle STA.

In order to treat the Dirac equation, we need first the 2-spinor equivalents of the γ- matrices. These are the $\sigma_\mu^{AA'}$ and $\sigma^\mu_{A'A}$ operators, which are the 2-spinor forms of the 'vector' operators $\hat{\sigma}_\mu = (1, \hat{\sigma}_i)$ and $\hat{\bar{\sigma}}_\mu = (1, -\hat{\sigma}_i)$. A full derivation of the translation of these operators can be carried out in the multiparticle STA, but it is obvious from (4.5) and (4.6) that this must be:

$$\begin{aligned}\sigma_\mu^{AA'}\bar{\kappa}_{A'} &\leftrightarrow -\gamma_\mu\kappa i\sigma_2\tfrac{1}{2}(1-\sigma_3)\gamma_0 \\ \sigma^\mu_{A'A}\kappa^A &\leftrightarrow \gamma^\mu\kappa\tfrac{1}{2}(1+\sigma_3)\gamma_0.\end{aligned} \tag{4.21}$$

If the γ_0 is (anti)commuted through to the left hand side, it changes the sign of the projector. This agrees with the behaviour of the $\sigma_\mu^{AA'}$ and $\sigma^\mu_{A'A}$ operators, which interchange modules. We now take the 2-spinor form of the Dirac equation,

$$\begin{aligned} j\sigma_\mu^{AA'}\partial^\mu\bar{\omega}_{A'} &= m\kappa^A \\ j\sigma^\mu_{A'A}\partial_\mu\kappa^A &= m\bar{\omega}_{A'}, \end{aligned} \tag{4.22}$$

and translate these into the pair of equations:

$$\begin{aligned} -\nabla\omega i\sigma_2\tfrac{1}{2}(1-\sigma_3)\gamma_0 i\sigma_3 &= m\kappa\tfrac{1}{2}(1+\sigma_3), \\ \nabla\kappa\tfrac{1}{2}(1+\sigma_3)\gamma_0 i\sigma_3 &= -m\omega i\sigma_2\tfrac{1}{2}(1-\sigma_3). \end{aligned} \tag{4.23}$$

With ψ defined as in (4.19), these combine into the single equation

$$\nabla\psi i\sigma_3\gamma_0 = m\psi, \tag{4.24}$$

which is Pauli-even. Postmultiplying by γ_0 recovers equation (3.7), demonstrating the true representation-independence of this equation.

Of the pair of equations (4.22), Penrose & Rindler [8] write '*an advantage of the 2-spinor description is that the γ-matrices disappear completely – and complicated γ-matrix identities simply evaporate!*'. While this is true, the comment applies even more strongly to the equation (4.24), in which complicated 2-spinor identities have also been eliminated!

The translation (4.21), together with the inner product (4.20), enables us to write the 2-spinor vector current as

$$\begin{aligned} \kappa_A\sigma_\mu^{AA'}\bar{\kappa}_{A'} \quad\leftrightarrow\quad & \gamma_\mu\gamma_0\kappa i\sigma_2\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\kappa} + \text{reverse} \\ & = -\gamma_\mu\cdot(\kappa(\gamma_0+\gamma_3)\tilde{\kappa}). \end{aligned} \tag{4.25}$$

This identifies the null vector $-\kappa(\gamma_0+\gamma_3)\tilde{\kappa}$ as the relativistic current, justifying our earlier comment that the idempotent $\frac{1}{2}(1+\sigma_3)$ should be viewed as a (Pauli) null vector rather than as an algebraic projector. If we define the spinor

$$\psi = \kappa\tfrac{1}{2}(1+\sigma_3), \tag{4.26}$$

the null vector is given (up to a factor of one-half) by $\psi\gamma_0\tilde{\psi}$. In this way, the observables in the 2-spinor approach are unified with those from the translation of the Dirac-Pauli representation.

We conclude this section with two examples of how the STA formulation considerably simplifies 2-spinor equations. The first is the relativistic spinor equations discussed by Marx [29]:

$$\begin{aligned} j\partial_\mu \sigma^\mu_{A'A}\kappa^A &= -m\bar{\kappa}_{A'}, \\ j\partial^\mu \sigma_\mu^{AA'}\bar{\omega}_{A'} &= m\omega^A. \end{aligned} \tag{4.27}$$

These translate into the pair of equations

$$\begin{aligned} \nabla\gamma_0\kappa\tfrac{1}{2}(1-\sigma_3)i\sigma_3 &= m\kappa i\sigma_2\tfrac{1}{2}(1-\sigma_3), \\ -\nabla\gamma_0\omega i\sigma_2\tfrac{1}{2}(1+\sigma_3)i\sigma_3 &= m\omega\tfrac{1}{2}(1+\sigma_3) \end{aligned} \tag{4.28}$$

which, on defining the full spinor ψ as in (4.19), can be written as

$$\begin{aligned} \nabla\psi\tfrac{1}{2}(1+\sigma_3)i\sigma_3 &= m\psi\tfrac{1}{2}(1+\sigma_3)i\sigma_2\gamma_0, \\ \nabla\psi\tfrac{1}{2}(1-\sigma_3)i\sigma_3 &= m\psi\tfrac{1}{2}(1-\sigma_3)i\sigma_2\gamma_0 \end{aligned} \tag{4.29}$$

and combined into the single equation

$$\nabla\psi i\sigma_1 = -m\psi\gamma_0, \tag{4.30}$$

which is much easier to interpret and solve. Equation (4.30) is not gauge-invariant, since $\psi i\sigma_3$ is not a solution of (4.30) if ψ is. However, the only difference between (4.30) and (3.7) is the replacement of the right $i\sigma_3$ with an $i\sigma_1$, so that any solution of the free-field Dirac equation (3.7) is transformed into a solution of (4.30) by right-multiplying by $e^{-i\sigma_2\pi/4}$. The transformation $\psi \mapsto \psi e^{-i\sigma_2\pi/4}$ is natural in the STA, where it may be viewed as an electroweak gauge transformation, but it is much harder to express in the standard formalism because it mixes $|\psi\rangle$ and $|\psi\rangle^*$.

Our second application of the STA approach to 2-spinors is the Lagrangian introduced by Plyuschay [30, 31], which provides a classical model for a massless, spinning particle. The Lagrangian is

$$L = -\tfrac{1}{2}\dot{x}^\mu\sigma_\mu^{AA'}\bar{\kappa}_{A'}\kappa_A + k_a R_a - s\rho, \tag{4.31}$$

where the k_a are a set of three Lagrangian multipliers,

$$\begin{aligned} R_1 &= \kappa^A\dot{\kappa}_A + \bar{\kappa}^{A'}\dot{\bar{\kappa}}_{A'} \\ R_2 &= j(\kappa^A\dot{\kappa}_A - \bar{\kappa}^{A'}\dot{\bar{\kappa}}_{A'}) \\ R_3 &= e^\mu\sigma_\mu^{AA'}(\bar{\kappa}_{A'}\dot{\kappa}_A + \dot{\bar{\kappa}}_{A'}\kappa_A), \end{aligned} \tag{4.32}$$

e^μ and s are constants, and

$$\rho = \frac{1}{E} e^\mu \sigma_\mu^{AA'} j (\bar{\kappa}_{A'} \dot{\kappa}_A - \dot{\bar{\kappa}}_{A'} \kappa_A) \tag{4.33}$$

$$E = e^\mu \sigma_\mu^{AA'} \bar{\kappa}_{A'} \kappa_A. \tag{4.34}$$

On translation into the STA, we find

$$\begin{array}{rcl} R_1 & \leftrightarrow & 2\langle \dot{\kappa} i\sigma_2 \tilde{\kappa} \rangle \\ R_2 & \leftrightarrow & 2\langle \dot{\kappa} i\sigma_1 \tilde{\kappa} \rangle \\ R_3 & \leftrightarrow & -2\langle e\dot{\kappa}(\gamma_0 + \gamma_3)\tilde{\kappa} \rangle, \end{array} \tag{4.35}$$

so that R_1 and R_2 are the $i\sigma_1$ and $i\sigma_2$ components of the bivector $\dot{\kappa} i\sigma_3 \tilde{\kappa}$. After variation of the Lagrangian, e is set equal to γ_0, whereupon R_3 becomes $2\langle \dot{\kappa}\tilde{\kappa} \rangle$, which is no longer Lorentz-invariant. This is the reason for introducing the arbitrary vector e. The STA approach provides a more attractive way to circumvent this problem: replace κ by an arbitrary Dirac spinor ψ, and then impose the constraint that ψ be Pauli-even after variation. The result is the same, but the STA Lagrangian is considerably more compact:

$$L = -2s \frac{\langle \dot{\psi} i\sigma_3 \tilde{\psi} \rangle}{\langle \psi \tilde{\psi} \rangle} + \tfrac{1}{2} \langle \dot{x} \psi (\gamma_0 + \gamma_3) \tilde{\psi} \rangle + \langle \dot{\psi} i\sigma_3 B \tilde{\psi} \rangle. \tag{4.36}$$

Here the three scalar Lagrangian multipliers have been replaced by a single space-like bivector. The Lagrangian (4.36) is now manifestly Lorentz-invariant, and can be manipulated using the approach developed in Paper III [17] of this series.

5 Further Developments

In this section we consider two final topics. The first is the more general matrix representation theory of the Dirac algebra, and the second is the Dirac equation in $2+1$ dimensions [32], which is important in the study of anyonic systems (these are systems with fractional statistics).

5.1 Changes of Representation

In the matrix theory, a change of representation is carried out by a 4×4 complex matrix $\hat{S}$, which defines new matrices

$$\hat{\gamma}'_\mu = \hat{S}\hat{\gamma}_\mu\hat{S}^{-1}, \tag{5.1}$$

with a corresponding spinor transformation $|\psi\rangle \mapsto \hat{S}|\psi\rangle$. For the Dirac equation, it is also a requirement that the transformed Hamiltonian be Hermitian, restricting (5.1) to a unitary transformation

$$\hat{\gamma}'_\mu = \hat{S}\hat{\gamma}_\mu\hat{S}^\dagger, \quad \hat{S}\hat{S}^\dagger = 1. \tag{5.2}$$

The STA approach proceeds instead by finding the analogue of the map (3.2) corresponding to any given representation. In Section 4 this was carried out for the Weyl matrices, and in Appendix A a construction is given for the Majorana representation. Finding the map is routine once the form of $\hat{S}$ which relates the required representation to the Dirac-Pauli matrices is determined. In constructing this map there is freedom to rescale the STA equivalent spinor, which in Section 4 was used to remove some factors of $\sqrt{2}$.

Once the map has been found, the action of j and the $\{\hat{\gamma}_\mu, \hat{\gamma}_5\}$ matrices translates in the same way as for the Dirac and Weyl representations (3.3), and the action of the C, P and T operators is given by equation (3.30). The STA form of these operators is independent of the matrix representation from which they were obtained, and the Dirac equation always takes on the same form (3.7), which is manifestly representation-free.

The Hermitian and Dirac adjoints always translates as in (3.13), although the separate transpose and complex conjugation operations retain some dependence on representation. For example, complex conjugation in the Dirac-Pauli and Weyl representations translates as (3.5),

$$|\psi\rangle^* \quad \leftrightarrow \quad -\gamma_2\psi\gamma_2, \tag{5.3}$$

whereas in the Majorana representation (Appendix A), we find that

$$|\psi\rangle^* \quad \leftrightarrow \quad \psi\sigma_2. \tag{5.4}$$

This clearly limits the usefulness of the complex conjugation operator. Equation (5.4) also demonstrates that, in the Majorana representation, complex conjugation coincides with charge conjugation (up to a conventional phase factor).

5.2 $2+1$ Dimensions

We can adapt the approach of Section 2 to describe the Dirac theory in $2+1$ dimensions. The result could be derived from the full Dirac equation (3.7) by ignoring one of the spatial directions, but it is useful to make contact with the standard matrix theory approach. A suitable representation is given by the 2×2 matrices [32]

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{\gamma}_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}. \tag{5.5}$$

A 2-component complex spinor is placed in one-to-one correspondence with an even element ψ via

$$|\psi\rangle = \begin{pmatrix} \alpha + ja^0 \\ -a^2 - ja^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = \alpha + ia^k\gamma_k, \tag{5.6}$$

where $i = \gamma_0\gamma_1\gamma_2$. The quantum matrix operators translate in the expected manner:

$$\begin{aligned} \hat{\gamma}_k|\psi\rangle &\leftrightarrow \gamma_k\psi\gamma_0 \\ j|\psi\rangle &\leftrightarrow \psi\gamma_1\gamma_2 = \psi i\gamma_0, \end{aligned} \tag{5.7}$$

and the Dirac equation in $2+1$ dimensions becomes simply

$$\nabla\psi i = m\psi, \tag{5.8}$$

or

$$\nabla\psi\gamma_1\gamma_2 = m\psi\gamma_0. \tag{5.9}$$

This does indeed have the same form as the full Dirac equation (3.7), and could have been found by projecting out the γ_3 direction. Equation (5.9) can be analysed with the techniques described in other papers of this series [7, 17].

6 Discussion and Conclusions

The spacetime algebra is the natural language for relativistic quantum mechanics, expressing the physics in a manifestly Lorentz-*in*variant way. All relations can be manipulated without ever introducing a matrix representation, greatly simplifying the algebra involved. Furthermore, every term in an expression is given a clear geometric meaning. The unit scalar imaginary of quantum mechanics is replaced by a fixed bivector in real space, and the 2-particle generalisation of this bivector results in a projection operator which correlates each particle's space. This strongly suggests that a scalar unit imaginary is unnecessary for quantum mechanics, a point frequently emphasised by Hestenes [6]. The real formulation also has clear conceptual benefits, especially at the multiparticle level. For example, the construction of 2-particle observables demonstrates that the quantum correlator locks particle phases together and describes non-local interactions between particles. This analysis will be given a relativistic extension in a forthcoming paper, in which the generalisation to curved spacetime will also be discussed.

It has been suggested [33] that the STA cannot contain as much structure as the complex Dirac algebra, since it is defined over the reals. We have demonstrated that this is not true; the STA not only incorporates the Dirac theory, it also extends it in ways that were previously unavailable. The geometry of real physical spacetime is richer than is often realised, and spacetime algebra is the natural language to exploit this richness.

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A Majorana Spinors

The Majorana representation [2] is defined by

$$\hat{\gamma}_0 = \begin{pmatrix} 0 & \hat{\sigma}_2 \\ \hat{\sigma}_2 & 0 \end{pmatrix} \qquad \hat{\gamma}_1 = \begin{pmatrix} -j\hat{\sigma}_3 & 0 \\ 0 & -j\hat{\sigma}_3 \end{pmatrix} \tag{A.1}$$

$$\hat{\gamma}_2 = \begin{pmatrix} 0 & \hat{\sigma}_2 \\ -\hat{\sigma}_2 & 0 \end{pmatrix} \qquad \hat{\gamma}_3 = \begin{pmatrix} j\hat{\sigma}_1 & 0 \\ 0 & j\hat{\sigma}_1 \end{pmatrix}. \tag{A.2}$$

Our translation for spinors in this representation is:

$$|\psi\rangle = \begin{pmatrix} a^0 + jb^0 \\ -a^2 + jb^2 \\ a^1 + jb^1 \\ -a^3 - jb^3 \end{pmatrix} \quad \leftrightarrow \quad \psi = \phi\tfrac{1}{2}(1+\sigma_2) + \omega i\sigma_2\tfrac{1}{2}(1-\sigma_2), \tag{A.3}$$

where

$$\begin{aligned} \phi &= a^0 + ia^k\sigma_k \\ \omega &= b^0 + ib^k\sigma_k. \end{aligned} \tag{A.4}$$

This ensures that the $\{\hat{\gamma}_\mu\}$ and j operators translate as for the Dirac and Weyl representations. Verification is a matter of computation. The simplest translation to verify is j, since

$$\psi i\sigma_3 = -\omega\tfrac{1}{2}(1+\sigma_2) + \phi i\sigma_3\tfrac{1}{2}(1-\sigma_2) \tag{A.5}$$

has the effect of switching the rows of the column spinor and introducing a minus sign, as required. The Dirac equation, the bilinear covariants, and the C, P and T operations all translate in the same manner as in Section 3, although complex conjugation (which is representation-dependent) translates differently:

$$|\psi\rangle^* \quad \leftrightarrow \quad \psi\sigma_2. \tag{A.6}$$

It follows that Majorana (real) spinors are represented in the STA as

$$|\psi\rangle_{maj} \quad \leftrightarrow \quad \psi\tfrac{1}{2}(1+\sigma_2), \tag{A.7}$$

and are therefore very similar to 2-spinors, but with a different choice of null vector for the Pauli projector. With charge conjugation defined as in (3.30) we see that, in the Majorana representation, charge conjugation and complex conjugation play the same role. Applications of this translation include the classical model of Hasiewicz *et al.* [34] and the geometric construction of Gibbons [35].