

Static and dynamic displacements of foundations on heterogeneous multilayered soils

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An analytical-numerical formulation is presented for dynamic and static analysis of strip foundations on an elastic isotropic medium consisting of heterogeneous layers. Each layer is characterized by an S-wave velocity that increases or decreases linearly with depth, a constant material density, a constant Poisson's ratio equal to $1/4$ and a constant linearly-hysteretic critical damping ratio. The solution, based on a transformation that uncouples the wave equations in closed-form, is 'exact' in that it properly accounts for the true boundary conditions at the layer interfaces and the surface. Results are presented for two characteristic soil profiles (half-space and stratum on rigid rock) in the form of normalized load-displacement ratios as functions of key dimensionless factors that influence the foundation behaviour during static and dynamic vertical, horizontal or moment loading. An interesting equivalence is established between a heterogeneous and a homogeneous halfspace, both having the same moduli at a depth equal to the foundation halfwidth (for translational motions) or to $1/2$ the foundation halfwidth (for rotation), i.e. for low frequency factors, the two media yield displacements of about the same average level, although the occurrence of resonance phenomena due to total wave reflection in the heterogeneous medium leads to fluctuations of the corresponding curves around the mean values.

Cet article présente une formulation analytique-numérique pour l'analyse dynamique et statique de fondations continues sur un milieu isotrop élastique formé de couches hétérogènes. Chaque couche est caractérisée par une vitesse d'onde en S qui augmente ou diminue linéairement en fonction de la profondeur, une densité de matière constante, un coefficient de Poisson constant égal à $1/4$ et un coefficient d'amortissement critique, linéairement hystérique, constant. La solution, basée sur une transformation qui découple les équations d'onde de forme fermée, est exacte en ce qu'elle tient correctement compte des conditions de limites réelles aux interfaces des couches et à la surface. Les résultats se rapportent à deux profils de sol caractéristiques (demi-espace et couche sur roche rigide) et sont présentés sous la forme de rapports charge-déplacement normalisés comme fonctions de facteurs clés sous dimension qui influencent le comportement des fondations lors de charges statiques et dynamiques verticales, horizontales ou soumises à des moments. Une équivalence intéressante est établie entre un demi-espace hétérogène et un demi-espace homogène, tous deux de même module à une profondeur égale à la demi-largeur de la fondation (pour des mouvements de translation) ou à la moitié de la demi-largeur de la fondation (pour rotation), c'est-à-dire que pour des facteurs basse fréquence, les deux milieux engendrent des déplacements d'un niveau moyen à peu près similaire, bien que les phénomènes de résonance qui se produisent par suite de la réflexion d'onde totale dans le milieu hétérogène conduisent à des fluctuations des courbes correspondantes de part et d'autre de valeurs moyennes.

The problem of estimating foundation settlements caused by static building loads has received considerable attention by civil engineers. The theory of elasticity has been widely applied for nearly a century (Boussinesq, 1885) to obtain solutions for a variety of idealized soil models (homogeneous halfspace, homogeneous stratum, etc.) and for a number of foundation configurations (circular, rectangular, strip, etc.). On the other hand, the response of foundations to dynamic loads was first studied analytically in the 1930's (Reissner, 1936; Barkan, 1938) in connection with the design of foundations for vibrating machinery. But it is the last two decades that have witnessed an unprecedented scientific interest in the dynamic problem, primarily in relation to the study of soil-structure interaction during earthquakes and the safety of off-shore platforms during wave-storms.

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NOTATION

\bar{a}	dimensionless frequency factor (= $\omega B/\beta_0$)	K	foundation stiffness (load-displacement ratio); $K = K' + iK''$
B	halfwidth of the foundation	\bar{K}	normalized foundation stiffness (equation (27)); $\bar{K} = \bar{K}' + i\bar{K}''$
b	rate of heterogeneity (unit m^{-1})	u, w	horizontal, vertical displacements
\tilde{b}	dimensionless rate of heterogeneity (= bB)	u_0, w_0	amplitude of foundation horizontal, vertical displacements
D	foundation compliance (load-displacement ratio); $D = D' + iD''$	α	P wave velocity in the soil; $\alpha = \alpha_0(1 + bz)$
\bar{D}	normalized foundation compliance; $\bar{D} = \bar{D}' + i\bar{D}''$	β	S wave velocity in the soil; $\beta = \beta_0(1 + bz)$
G	shear modulus; $G = G_0(1 + bz)^2$	α_0, β_0	velocities at the top of a layer ($z = 0$)
G_0	shear modulus at the top of a layer ($z = 0$)	Θ_0	amplitude of foundation rotation
H	layer thickness	λ	linear-hysteretic critical damping ratio
I_f, K_f	modified Bessel functions of order f , first and second kind, respectively	ν	Poisson's ratio
		ω	circular frequency (rad/s)

The majority of the existing solutions to both static and dynamic problems are based on the assumption of soil homogeneity. It is true that multi-layered elastic media have been studied extensively (e.g. Burmister, 1945, and Verstraeten, 1967, with the static problem; Luco, 1976, and Gazetas and Roesset, 1976, with the dynamic problem). Few solutions, however, are available to problems involving smooth variation of soil modulus within a particular soil layer. Such a variation is of interest not only with soils, whose stiffness depends on the effective overburden pressure and the degree of overconsolidation (which are both functions of depth within each layer), but also with rocks adjacent to excavations, for which the degree of induced loosening decreases with distance below the surface of excavation.

Stresses and displacements produced by vertical static uniform pressures in a medium whose moduli increase linearly with depth, i.e. $G = G_0 + mz$ where G_0, m are constants, have been analytically derived by Lekhitskii (1962) and by Gibson and his co-workers (e.g. Gibson, 1967; Brown and Gibson, 1972; Awojobi and Gibson, 1973, Gibson, 1974; Awojobi, 1974). It was discovered that for an incompressible medium, i.e. when Poisson's ratio $\nu = 1/2$, the stress distribution in the soil is hardly influenced by the 'rate' of heterogeneity and that for the particular case of zero surface modulus ($G_0 = 0$) this distribution is identical with the distribution in a homogeneous halfspace, regardless of the foundation geometry. The surface settlement, on the other hand, being sensitive to the assumed heterogeneity, becomes directly proportional to the applied normal pressure when $G_0 = 0$, independent of the size and shape of the loaded area. That is, such a medium behaves like a 'Winkler-Space' and not as a 'Homogeneous Half-Space'. This behaviour is not exactly true under drained conditions (i.e. when $\nu < 0.5$): the horizontal stresses are very sensitive to ν and the surface settlement under the load tends to infinity as $G_0 \rightarrow 0$. Nevertheless, the pattern of surface settlement is only moderately dependent on ν and approaches the settlement pattern of the 'Winkler-Space' for small values of G_0 . It is thereby concluded that the distribution of stresses under rigid foundations, on a heterogeneous medium are more uniform than the simple elastic theory (homogeneous halfspace) predicts.

Essentially the same conclusions were drawn by Carrier and Christian (1973) who employed the finite element method to study the settlement of a rigid circular foundation on soil having linear variation of modulus with depth (i.e. $G = G_0 + mz$). A different type of heterogeneity

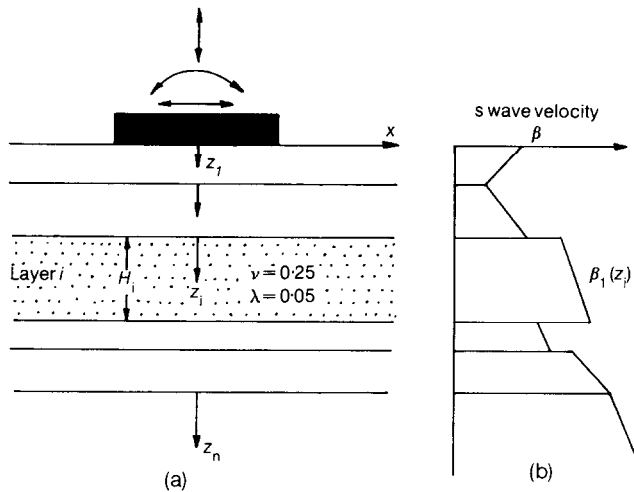


Fig. 1. Problem geometry, coordinate axes and shear-wave velocity profile

was studied by Holl (1949), Rostovtsen and Khramevskaya (1971) and Kassir (1972) who considered the modulus increasing as a power function of depth, i.e. $G = G_0 z^m$, and Chuaprasert and Kassir (1974) who studied the more general case of $G = G_0 [1 + (z/c)]^m$ whereby G_0 and c are constants and $m = (1 - 2\nu)/\nu$. (For $\nu = 1/3$, $m = 1$ and their solution corresponds to that of Awojobi and Gibson, 1973). They found that with increasing degree of heterogeneity (i.e. m), normal and shear stresses affect soil at greater depth, in agreement with intuition that expects stiffer material to attract larger stresses. The importance of this conclusion, however, is limited by the fact that Poisson's ratio also varies with m ($\nu = 1/3$ for $m = 1$; $\nu = 4/9$ for $m = 1/4$). Thus, it cannot be unveiled how much of the above effect is caused by the increase in m and how much by the decrease in ν , although both changes seem to have some participation.

Consideration of continuous¹ heterogeneity of the soil seems to be even more necessary when studying the dynamic response of foundations. Although the use of numerical (e.g. finite element) methods which simulate the continuous variation of the modulus by dividing the medium into a number of homogeneous layers of increasing stiffness, may be sufficient for static problems (Carrier *et al.*, 1973), it can lead to inaccurate results for high frequencies of vibration. The reason is a large degree of reflection of waves with small wavelength at surfaces of velocity discontinuity, not actually present in the soil. At the other side of the spectrum, waves of large wavelengths (low frequencies) will be artificially reflected at the lower boundary that the numerical solutions require.

Awojobi, in a series of publications (1972, 1973, 1974), studied particular aspects of the dynamic response of rigid circular or strip foundations on a medium obeying 'Gibson's' variation of modulus with depth. By approximately solving the exact dual integral equations governing the mixed boundary value problem for special limiting cases (e.g. static case, low or high frequency cases, etc.) he reached several conclusions of practical significance, some of which are addressed in a later section herein (presentation of results).

In this Paper a dynamic theory is presented to study the vertical, horizontal and rocking displacements of a rigid massless strip foundations resting on a multi-layered elastic isotropic halfspace. Each layer is characterized by a shear-wave velocity, β , that varies linearly with

¹ 'Continuous' is used here to distinguish from the 'layered' heterogeneity.

depth; a constant Poisson's ratio, $\nu = 0.25$; and a constant material density, ρ (Fig. 1). The variation of velocity with depth is expressed as

$$\beta = \beta_0(1 + bz) \quad (1)$$

where the rate of heterogeneity b can be any real number, positive or negative. For such a medium, the equations of motion are transformed into two uncoupled equations using a technique proposed by Gupta (1966) and are subsequently solved in closed-form. For a multi-layered halfspace loaded by a surface strip foundation a semi-analytical formulation is then developed similar in form with, although computationally more involved than, the method developed for homogeneous layers by the Author and Roesset (1976, 1979). By allowing the frequency of vibration to vanish, general solutions for the static deformations can also be obtained with the developed formulation.

Results of the method compare favourably with known analytical or numerical solutions and the limiting cases of a uniform stratum on rigid rock and a uniform halfspace are recovered as the rate of heterogeneity tends to zero. The key dimensionless factors that influence the foundation behaviour are identified and their importance is demonstrated through a series of parametric studies and through extensive comparisons with results pertinent to homogeneous multilayered soils.

FORMULATION OF THE PROBLEM

Stresses and displacements in a layer

For conditions of plane strain, appropriate for strip loading, the governing equations of motion of a heterogeneous elastic medium with S -wave velocity varying linearly with depth (equation (1)) and Poisson's ratio $\nu = 0.25$ are

$$G(z) \left[3 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial^2 w}{\partial x \partial z} + \frac{2b}{(1 + bz)^2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right] = \rho \frac{\partial^2 u}{\partial t^2} \quad (2a)$$

$$G(z) \left[3 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial z} + \frac{2b}{(1 + bz)^2} \left(\frac{\partial u}{\partial x} + 3 \frac{\partial w}{\partial z} \right) \right] = \rho \frac{\partial^2 w}{\partial t^2} \quad (2b)$$

in which $G(z) = G_0(1 + bz)^2$ is the shear modulus of the soil, with $G_0 = \rho\beta_0^2$; u and w are the horizontal and vertical displacements, respectively.

Gupta (1966) presented a method of uncoupling equations (2) in terms of pseudo-dilatational and pseudo-distortional wave potentials, Φ and Ψ' , defined by:

$$\{d\} = G\nabla(G^{-1}\Phi) + G^{-1}\nabla \times (G\Psi') \quad (3)$$

where $\{d\} = \{u, w\}^T$ is the displacement vector.

Substitution of (3) in (2) leads, after some straightforward but lengthy operations, to two uncoupled equations:

$$\frac{\partial^2 \Psi'}{\partial x^2} + \frac{\partial^2 \Psi'}{\partial z^2} = \frac{1}{[\beta(z)]^2} \frac{\partial^2 \Psi'}{\partial t^2} \quad (4a)$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{[\alpha(z)]^2} \frac{\partial^2 \Phi}{\partial t^2} \quad (4b)$$

where $\alpha(z) = \{(\lambda + 2G)/\rho\}^{1/2}$ is the P wave velocity. For $\nu = 0.25$, $\lambda = G$ and

$$\alpha(z) = 3\beta(z) = 3\beta_0(1 + bz).$$

By a logical generalization of the terminology used for waves in homogeneous media, Φ is identified with P waves and Ψ with S waves. It should be noted, though, that such P waves are not purely irrotational, nor are S waves purely equivoluminal, in heterogeneous media. This can be shown easily, by using equation (3) together with the definitions of dilation² and rotation³ of the elastic theory.

To obtain the general harmonic solution of, say, equation (4b), let

$$\Phi(x, z; t) = F(x)Z(z) \exp(i\omega t) \dots \dots \dots (5)$$

where ω = the wave frequency (rad/s).

Substituting (5) in (4b) leads to

$$\frac{1}{F(x)} \frac{d^2 F(x)}{dx^2} = -\frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} - \frac{\omega^2}{\alpha_0^2(1+bz)^2} \dots \dots \dots (6)$$

Since the left-side of (6) depends only on x and the right-side only on z , both sides must equal a constant value independent of x and z . Calling this constant $-k^2$ leads to:

$$F(x) = E' e^{-ikx} \dots \dots \dots (7)$$

(E' is integration constant)

and

$$\frac{d^2 Z}{dz^2} + \left(\frac{\omega^2}{\alpha_0^2(1+bz)^2} - k^2 \right) Z = 0 \dots \dots \dots (8)$$

Letting $Z = y(1+bz)^{\frac{1}{2}}$, and

$$s = \frac{1+bz}{b} k \dots \dots \dots (9)$$

transforms equation (8) into a modified Bessel equation:

$$\frac{d^2 y}{ds^2} + \frac{1}{s} \frac{dy}{ds} - \left[1 + \frac{f^2}{s^2} \right] y = 0 \dots \dots \dots (10)$$

with

$$f = \left(\frac{1}{4} - \frac{\omega^2}{\alpha_0^2 b^2} \right)^{\frac{1}{2}} \dots \dots \dots (10a)$$

Equation (10) has a general solution of the form

$$y = C_1 I_f(s) + C_2 K_f(s) \dots \dots \dots (12)$$

in which C_1, C_2 are constants of integration and I_f, K_f are the modified Bessel functions of order f , first and second kind, respectively (Watson, 1948).

Finally, combining (5), (7), (9) and (12) yields

$$\Phi(x, z; t) = (1+bz)^{\frac{1}{2}} \left[A' I_f \left(k \frac{1+bz}{b} \right) + B' K_f \left(k \frac{1+bz}{b} \right) \right] e^{i(\omega t - kx)} \dots \dots (13a)$$

A completely analogous procedure for (4a) yields

$$\Psi(x, z; t) = (1+bz)^{\frac{1}{2}} \left[A'' I_g \left(h \frac{1+bz}{b} \right) + B'' K_g \left(h \frac{1+bz}{b} \right) \right] e^{i(\omega t - hx)} \dots \dots (13b)$$

² dilation: $\Theta = \partial u/\partial x + \partial w/\partial z$.
³ rotation: $\Omega = (\partial u/\partial z - \partial w/\partial x)/2$.

with
$$g = \left(\frac{1}{4} - \frac{\omega^2}{\beta_0^2 b^2} \right)^{\frac{1}{2}} \dots \dots \dots (11b)$$

whereby A', B', A'', B'' are integration constants.

The two components of the displacement can now be determined from (3):

$$u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi'}{\partial z} - \frac{2b}{1+bz} \Psi' \dots \dots \dots (14a)$$

$$w = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi'}{\partial x} - \frac{2b}{1+bz} \Phi \dots \dots \dots (14b)$$

with Φ and Ψ' given by (13).

The stresses are subsequently obtained from (14) and the stress-displacement relations of the elastic theory, for $\nu = 0.25$:

$$\begin{aligned} \sigma_{zz} \equiv \sigma &= G(z) \left[3 \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} \right] \\ &= G(z) \left[\frac{\partial^2 \Phi}{\partial x^2} + 3 \frac{\partial^2 \Phi}{\partial x^2} + 6 \frac{b^2 \Phi}{(1+bz)^2} - \frac{6b}{1+bz} \frac{\partial \Phi}{\partial z} + 2 \frac{\partial^2 \Psi'}{\partial x \partial z} - \frac{2b}{1+bz} \frac{\partial \Psi'}{\partial x} \right] \dots \dots \dots (15a) \end{aligned}$$

$$\sigma_{xz} \equiv \tau = G(z) \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = G(z) \left[2 \frac{\partial^2 \Phi}{\partial x \partial z} - \frac{2b}{1+bz} \frac{\partial \Phi}{\partial x} + \frac{\partial^2 \Psi'}{\partial x^2} - \frac{\partial^2 \Psi'}{\partial z^2} - \frac{2b}{1+bz} \frac{\partial \Psi'}{\partial z} + \frac{2b^2 \Psi'}{(1+bz)^2} \right] (15b)$$

Performing the operations indicated by equations (14) and (15), with Φ and Ψ' given by (13), is a straightforward but quite tedious operation, mainly because the derivatives of a Bessel function are related to Bessel functions of higher order, through recurrence relations of the form (e.g. Watson, 1948)

$$\frac{d}{dy} [I_\nu(y)] = \frac{\nu}{y} I_\nu(y) + I_{\nu+1}(y)$$

The results for the stresses and displacements can be expressed in matrix form:

$$\{P\} = [X] \{E\} \dots \dots \dots (16)$$

in which $\{P\} = \{\sigma, \tau, u, w\}^T$, $\{E\} = \{A', A'', B', B''\}^T$. The elements X_{ij} of the transfer matrix $[X]$ are given in Appendix A.

Boundary conditions at layer interfaces

Adhering to the physical requirement of continuity of stresses (σ, τ) and displacements (u, w) at the 'rough' interface between two layers leads to $(n-1)$ sets of equations of the form

$$\{P_i(H_i)\} = \{P_{i+1}(0)\}, \quad i = 1, 2, \dots, n-1 \dots \dots \dots (17)$$

for the system of coordinate axes displayed in Fig. 1. Equation (17) can be written (using 16b) as

$$[B_i] \{E_i\} = [T_{i+1}] \{E_{i+1}\}, \quad i = 1, 2, \dots, n-1 \dots \dots \dots (17a)$$

where $B_i = X_i(H_i)$, $T_{i+1} = X_{i+1}(0)$.

Since (17b) holds for any value of x along the interface, the terms $\exp(-ikx)$ and $\exp(-ihx)$ should be identically equal. This leads to

$$k_i = h_i = k_{i+1} = h_{i+1} = \text{constant} \dots \dots \dots (17b)$$

which is an alternate expression of Snell's law of refraction. Because of (17b), the terms expressing the variation of $\{P\}$ in the x direction are omitted in (17a).

Elimination of the vector $\{E\}$ from (17a) yields

$$P_1(0) = [R]\{P_n(0)\} \dots \dots \dots (18a)$$

with

$$[R] = \prod_{i=1}^{n-1} ([T_i][B_i]^{-1}) \dots \dots \dots (18b)$$

For the n layer (i.e. the halfspace) the coefficients A' , B' must vanish, since $I_v(z) \rightarrow \infty$ as $z \rightarrow \infty$. Such a condition would lead to displacements increasing with depth, which is physically impossible with a surface wave source. Thus one can write

$$P_n(0) = [T_n] \begin{Bmatrix} 0 \\ 0 \\ A'' \\ B'' \end{Bmatrix} \dots \dots \dots (19)$$

and by combining (18a) and (19) to obtain

$$\{d\} = [Q_{22}][Q_{12}]^{-1}\{s\} \dots \dots \dots (20)$$

where

$$[Q] = [R][T_n] \dots \dots \dots (20a)$$

$$\{d\} = \{u_1(0), w_1(0)\}^T \quad \text{and} \quad \{s\} = \{\sigma_1(0), \tau_1(0)\}^T \dots \dots \dots (20b)$$

Equation (20) relates stresses and displacements at the top surface and can be used directly if the applied foundation stresses are known.

Displacement due to a uniform foundation pressure

This is a simplified version of the true problem. It corresponds to reality only if the foundation has zero stiffness and thus acts as a membrane, uniformly distributing the pressure applied on the foundation. Since many of the existing solutions were developed for this type of loading, its consideration here was thought appropriate.

The surface boundary conditions for a foundation of width $2B$ are

$$\sigma_1(x) = q_v e^{i\omega t} \quad \text{for } |x| \leq B \dots \dots \dots (21a)$$

$$\sigma_1(x) = 0 \quad \text{for } |x| > B \dots \dots \dots (21b)$$

$$\tau_1(x) = 0 \quad \text{for } -\infty < x < \infty \dots \dots \dots (21c)$$

if only a vertical pressure q_v is applied. In case of simultaneous application of uniform horizontal pressure q_h (21c) should be substituted by

$$\tau_1(x) = q_h e^{i\omega t} \quad \text{for } |x| \leq B \dots \dots \dots (21d)$$

$$\tau_1(x) = 0 \quad \text{for } |x| > B \dots \dots \dots (21e)$$

Equations (21a) to (21b) or (21d) to (21e) can be expressed through a single equation holding for all x , from $-\infty$ to $+\infty$, should the applied pressures, q_a ($a = v$ or h), be expanded in a series of periodic functions of the form

$$\tilde{q}_a(\xi) e^{i\xi x} \dots \dots \dots (22)$$

where $\tilde{q}_a(\xi)$ is the Fourier transform of $q_a(x)$

$$\tilde{q}_a(\xi) = \int_{-\infty}^{\infty} q_a(x) e^{-i\xi x} dx \dots \dots \dots (22a)$$

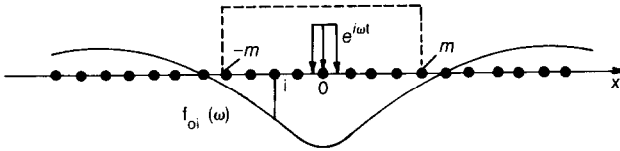


Fig. 2. Physical interpretation of $f_{0i}(\omega)$

Thus for each particular value of the parameter ξ , (20) yields

$$\{\tilde{d}(\xi)\} e^{-ikx} = [Q_{22}] [Q_{12}]^{-1} \{\tilde{q}(\xi)\} e^{i\xi x} \quad \dots \quad (23)$$

where

$$\{\tilde{q}(\xi)\} = \{\tilde{q}_v(\xi), q_h(\xi)\}^T \quad \dots \quad (23a)$$

By setting $k = -\xi$, the Fourier transform \tilde{d} of the displacement vector can be readily obtained from (23). The inverse Fourier transform of \tilde{d} yields the displacement vector

$$\{d(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\tilde{d}(\xi)\} e^{i\xi x} d\xi \quad \dots \quad (23b)$$

and, thus, the problem is solved. Notice, however, that analytical evaluation of (23b) appears impossible even for the simplest case of a halfspace. Instead, a discrete Fast Fourier transform algorithm has been implemented and the displacements are determined at a finite number of equidistant grid points on the surface. Details on the accuracy of the technique and the requirements for obtaining good and inexpensive solutions (minimum number of total grid points, number of grid points under the foundation, appropriate distance between the points, etc.) can be found in Gazetas and Roesset (1976).

Mixed boundary conditions for rigid foundations

The exact boundary conditions in this case are

$$\left. \begin{aligned} u_1(x) &= u_0 e^{i\omega t} \\ w_1(x) &= (w_0 + \Theta_0 x) e^{i\omega t} \end{aligned} \right\} \text{for } |x| \leq B \quad \dots \quad (24a)$$

$$\dots \quad (24b)$$

$$\left. \begin{aligned} \sigma_1(x) &= 0 \\ \tau_1(x) &= 0 \end{aligned} \right\} \text{for } |x| > B \quad \dots \quad (24c)$$

$$\dots \quad (24d)$$

in which u_0 and w_0 are the horizontal and vertical displacements of the centre of the foundation and Θ_0 is the angle of rotation of the foundation with respect to the horizontal (in rads).

Because of the mixed nature of conditions (22), they cannot be directly used with (20). Instead, the vertical and horizontal displacements $f_{0i}(\omega)$, of every point i on the surface due to a harmonically varying time with vertical or horizontal line traction of unit amplitude at the origin, i.e. at $x = 0$, are first determined (Fig. 2). The procedure is completely analogous with that described previously for a uniform pressure. Letting the foundation-soil interface be represented by $2m + 1$ grid points (Fig. 2) and noticing that as the applied traction moves to any other point the displacements at all points just shift by the same amount, leads to

$$\{d\} = [f] \{q\} \quad \dots \quad (25)$$

where $\{d\}$ and $\{q\}$ are the vectors of the displacements and stresses of all the points of the contact surface [of dimension $2(2m + 1)$] and $[f]$ is the flexibility matrix. It is then a simple problem of matrix algebra to relate displacement amplitudes u_0, w_0, Θ_0 with amplitudes of

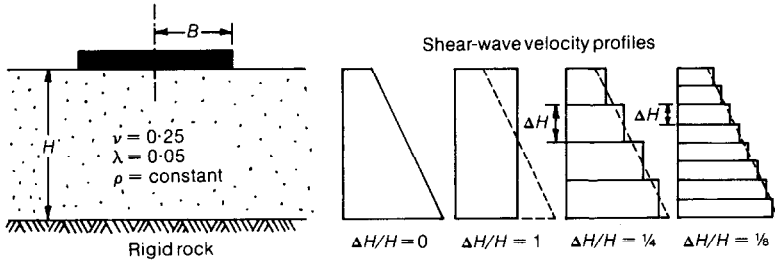


Fig. 3. Approximation of the heterogeneous layer ($\Delta H/H = 0$) with multi-layered homogeneous strata ($H/H = 1, 1/4, 1/8$)

external forces H_0, V_0, M_0 applied on the rigid foundation. Taking into account (24), (26) and the equilibrium of the massless foundation yields

$$\{d_0\} = [D]\{F_0\} \dots \dots \dots (26)$$

where $\{d_0\} = \{u_0, \Theta_0, w_0\}^T$, $\{F_0\} = \{H_0, M_0, V_0\}^T$ and $[D]$ is the foundation compliance matrix. The expression for obtaining $[D]$ is given in Appendix B. $[D]$ or its inverse, the stiffness matrix $[K]$, fully describe the response of the massless foundation to external applied loads, static or dynamic.

NUMERICAL RESULTS

This section presents characteristic numerical results from a series of extensive parametric analyses that have been conducted in order to: check the accuracy of the developed semi-analytical method against other numerical or analytical techniques; and to identify the basic variables of the problem and study their influence on the static and dynamic displacements of rigid strip foundations.

The results are displayed in the form of normalized foundation stiffnesses (i.e. load-displacements ratios) or compliances (i.e. displacement-load ratios) as functions of dimensionless groups of the key relevant parameters. Following the technique of dimensional analysis, as expressed by Riebuchinski's theorem (Gibson, 1974), one can state

$$\frac{K_v}{G_0} \text{ or } \frac{K_H}{G_0} \text{ or } \frac{K_R}{G_0 B^2} = f\left(\frac{H}{B}, bH, \frac{\omega B}{\beta_0}, \nu, \lambda\right) \dots \dots \dots (27)$$

whereby K_v, K_H and K_R are the foundation stiffnesses (per unit length) due to vertical, horizontal and moment loading; H is a characteristic length of the soil profile, herein taken as the depth of the soil deposit from surface to bedrock. Restricting Poisson's ratio to a single value $\nu = 0.25$ (for which the presented method is theoretically correct) and the hysteretic damping ratio to $\lambda = 0.05$, allows (27) to be expressed as:

$$\tilde{K} = f(H/B, \tilde{b}, \tilde{a}) \dots \dots \dots (27a)$$

in which: $\tilde{b} = bB$ (dimensionless rate of heterogeneity) and $\tilde{a} = \omega B/\beta_0$ (dimensionless frequency). Alternatively, for comparison with the results of a homogeneous stratum, K and ω are normalized with respect to the modulus $G_{H/2}$ at the middle of the stratum and (27a) is transformed to

$$\tilde{K} = f_2(H/B, \tilde{b}, \tilde{a}) \dots \dots \dots (28)$$

Static stiffnesses

To illustrate the accuracy of the method, the horizontal, rocking and vertical static stiffnesses of a rigid foundation on the surface of a heterogeneous soil stratum are first studied. The

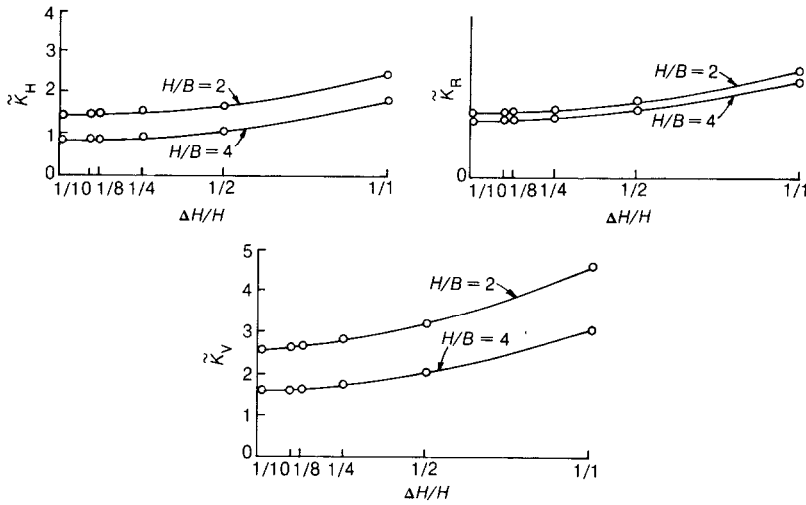


Fig. 4. Convergence of static stiffness to the ones of the heterogeneous stratum as $\Delta H/H \rightarrow 0$

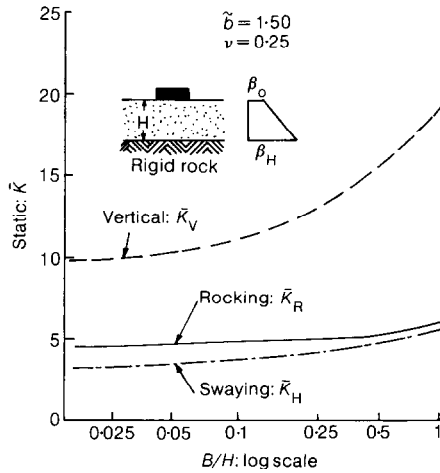


Fig. 5. Effect of layer depth on static stiffnesses

results are compared to the stiffnesses obtained by considering the stratum as consisting of a number of homogeneous layers, using available analytical (Gazetas and Roesset, 1976) or numerical (Chiang-Liang, 1974) techniques. Figure 3 explains how the stratum has been divided into an increasing number of layers of decreasing thickness, ΔH . Each layer has the same S wave velocity with the heterogeneous stratum at a depth corresponding to the middle of the particular layer. Figure 4 displays the dependence of the dimensionless stiffnesses \bar{K} on the relative thickness of the homogeneous layers $\Delta H/H$ for two typical values of the depth ratio ($H/B = 2$ and 4) and a single value of the dimensionless rate of heterogeneity ($\bar{b} = 1.5$).

For both H/B ratios, all three stiffnesses of the multi-layered profiles converge smoothly to the stiffnesses of the heterogeneous stratum, as the layer thickness ΔH tends to zero. In fact as soon as $\Delta H/H = 1/8$, the difference of the two types of profiles is negligible, if only with

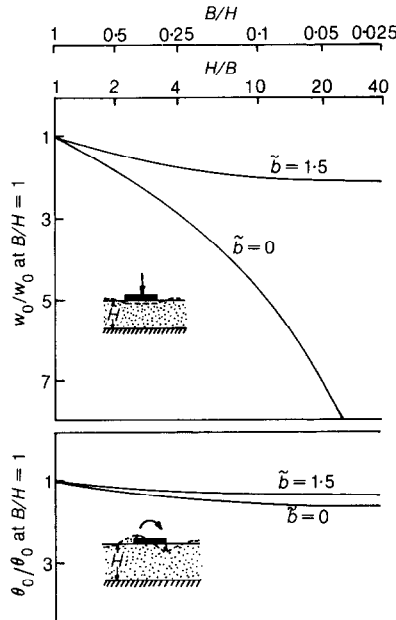


Fig. 6. Dependence of vertical settlement and of rotation on heterogeneity and depth ratio.

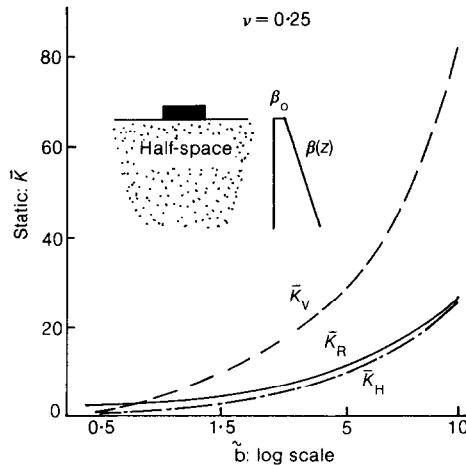


Fig. 7. Effect of the dimensionless rate of heterogeneity \tilde{b} on static stiffnesses

static displacements. The increase of the stiffnesses at larger $\Delta H/H$ values just reflects the higher modules of the corresponding profiles near the surface.

Notice also in Fig. 4 the relatively large sensitivity of vertical settlements to stratum depth. The deep strata ($H/B = 4$) experience settlements that are about 60% larger than the settlements of the corresponding shallow strata ($H/B = 2$) when $\tilde{b} = 1.5$. Horizontal displacements also increase significantly with increasing depth of the deposit (e.g. 40–50% for H/B increasing from 2 to 4) but rotation due to moment loading is practically independent of H/B . This phenomenon has already been established for homogeneous deposits loaded either by circular (Kausel, 1974)

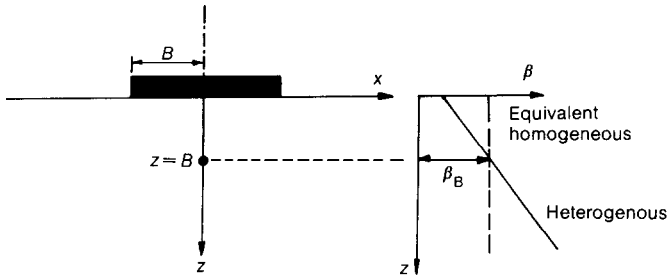


Fig. 8. Definition of equivalent homogeneous medium

or by strip (Gazetas *et al.*, 1979) foundations. It suggests that moment loading affects the soil at a much shallower depth compared to horizontal and, especially, to vertical loading.

This is further demonstrated in Fig. 5 which shows the dependence of the dimensionless load–displacement ratio \bar{K} on the depth ratio, B/H , for a heterogeneous deposit having $\tilde{b} = 1.5$. The rocking stiffness remains practically constant while stratum depth increases beyond a critical value of about $2B$ (i.e. $B/H = 0.5$). It is concluded that, due to moment surface loading, strains below a depth equal to foundation width are negligible for horizontal and vertical loading. The critical depths are approximately equal to five and seven times the foundation width, respectively.

A convenient way to reveal whether heterogeneity or other factors are responsible for confining the strain field to the above-mentioned limits is to compare the dependence of foundation displacements on stratum thickness, for a homogeneous ($\tilde{b} = 0$) and a heterogeneous ($\tilde{b} = 1.5$) deposit. Such a comparison is displayed in Fig. 6 for vertical and moment loading. It is evident that:

- (a) On a homogeneous deposit, as the layer thickness increases the vertical foundation settlement becomes progressively larger, tending to infinity as the stratum tends to a halfspace. In contrast, on heterogeneous soil the influence of layer depth on foundation settlement becomes smaller and smaller as H/B increases. Beyond approximately $H \simeq 14B$ the additional settlement is negligibly small. It consequently appears that heterogeneity is the primary cause of the reduced zone of influence of the surface loading.
- (b) For both the homogeneous and the heterogeneous soils, layer depth has an insignificant effect on the rotation of a foundation due to moment loading, except perhaps for very shallow deposits ($B/H > 1$). Therefore, heterogeneity has only a very small contribution to limiting the depth over which significant deformations take place. Instead, this is primarily due to the peculiarity of moment loading (zero resultant force) which produces stresses decaying exponentially with depth (Saint-Venant principle).

Notice that conclusion (a) qualitatively agrees with the finding of Awojobi (1974) that the settlement of a uniformly loaded strip footing on ‘Gibson’ soil is independent of the thickness of a stratum underlain by a rigid but smooth base.

The above conclusions were reached with a particular value of the dimensionless rate of heterogeneity, i.e. $\tilde{b} = 1.5$. It is useful to study the sensitivity of foundation displacements or stiffnesses to variations of \tilde{b} . To this end, Fig. 7 depicts the dependence on \tilde{b} of the normalized vertical, horizontal and rocking stiffnesses of a rigid foundation on a heterogeneous halfspace. As anticipated, increasing \tilde{b} leads to higher stiffnesses. Again the vertical stiffness shows the

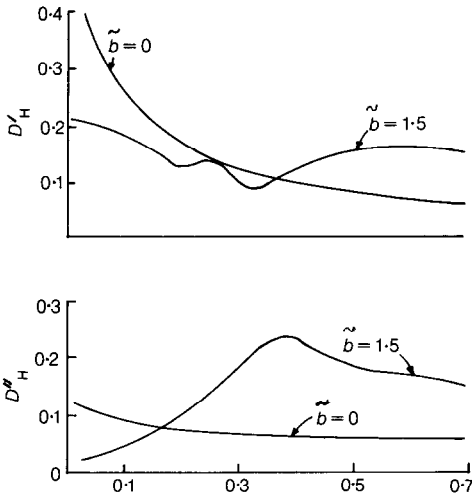


Fig. 9. Horizontal dynamic compliance for a heterogeneous ($\tilde{b} = 1.5$) and a homogeneous ($\tilde{b} = 0$) halfspace

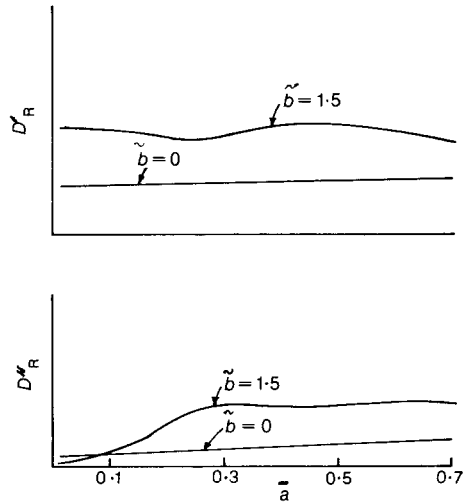


Fig. 10. Rocking dynamic compliances for a heterogeneous ($\tilde{b} = 1.5$) and a homogeneous ($\tilde{b} = 0$) halfspace

largest increase and the rocking the least. As $\tilde{b} \rightarrow 0$, i.e. as the medium tends to a homogeneous halfspace, the translational stiffnesses (K_V and K_H) tend to zero, in agreement with classical elastic theory, as discussed previously. Moment loading, on the other hand, yields a finite stiffness which, according to Muskhelishvili (1963), is given by

$$K_R = \frac{\pi}{2(1-\nu)} \left(1 + \left[\frac{\ln(3-4\nu)}{\pi} \right]^2 \right) \dots \dots \dots (29)$$

For $\nu = 0.25$, (29) yields $K_R \approx 2.2$. This value is in reasonable agreement with the portrayed value of $K_R \approx 2.4$ for $\tilde{b} < 0.5$.

The effect of \tilde{b} on the stiffnesses of a foundation resting on a stratum with a particular depth ratio H/B is of a similar nature.

Vibrations on a halfspace

One way to study the effects of heterogeneity on the dynamic response of a rigid foundation resting on a halfspace is to compare with the results for a homogeneous halfspace whose shear-wave velocity is the same as that of the heterogeneous space at a depth equal to the foundation halfwidth B (Fig. 8). Figures 9 to 11 show the horizontal, rocking and vertical displacement-load ratios (compliances), D as functions of the frequency factor, \bar{a} for the homogeneous and the heterogeneous halfspaces sketched in Fig. 8. It is reminded that D is a complex number and can be written in the form

$$D = D' + iD'' \quad (i = \sqrt{-1}) \dots \dots \dots (30)$$

with the real part D' representing the recoverable component of deformation while the imaginary part D'' expresses the dissipation of energy by waves propagating away from the foundation (radiation damping) and by hysteresis and friction in the soil (internal damping).

The main effects of heterogeneity on the foundation compliances, as can be seen in Figs 9 to 11 are as follows.

The recoverable parts of the displacements-load ratios (compliances) are not smoothly varying functions of \bar{a} , as in the case of a homogeneous halfspace, but exhibit peaks and

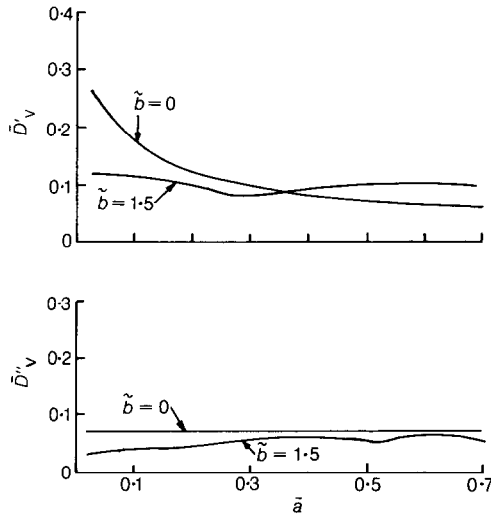


Fig. 11. Vertical dynamic compliances for a heterogeneous ($\tilde{b} = 1.5$) and a homogeneous ($\tilde{b} = 0$) halfspace valleys that are the result of resonance phenomena. It is noted that wave rays in heterogeneous media are not straight lines but curves $x(z)$ described by the differential equation

$$\frac{x''}{(1+(x')^2)^{\frac{3}{2}}} = p\beta' \quad \dots \quad (31)$$

where the prime (') indicates derivative with respect to z and p is the ray parameter $p = \sin \Theta_0/\beta_0$ in which Θ_0 is the angle of immergence of the ray into the ground (Fig. 12). With linear variation of velocity, equation (31) describes a family of circles for each value of p

$$\left[x - \frac{(1-p^2\beta_0^2)^{\frac{1}{2}}}{\beta_0 bp} \right]^2 + \left(z + \frac{1}{b} \right)^2 = \frac{1}{(\beta_0 bp)^2} \quad \dots \quad (32)$$

Thus total reflection of waves is in fact possible and does not require a velocity discontinuity. The reflecting waves are responsible for more severe foundation response, as is schematically illustrated in Fig. 12. This type of resonance phenomena are reminiscent of the dynamic behaviour of foundations on a homogeneous soil stratum (e.g. Kausel, 1974; Luco, 1976; Gazetas *et al.*, 1976, 1979). In the latter case, however, total reflection occurs only at specific frequency ranges (around the natural frequencies of the stratum) and leads to more pronounced peaks.

For frequencies close to zero, the imaginary parts \bar{D}'' exhibit values smaller than those for a homogeneous halfspace, since total reflection phenomena, as described above, restrict the amount of energy that is carried away, contrary to what is happening in a homogeneous halfspace. As the frequency factor \bar{a} increases, Rayleigh waves appear, carrying laterally part of the input energy and causing \bar{D}'' to increase more than a homogeneous halfspace theory would predict. The phenomenon is more distinguishable in the case of lateral vibrations, at least for the frequency range shown.

For the translational modes of vibration, the two media lead to very similar average values of the real part of the compliance functions over the frequency range portrayed. For example average $\bar{D}_H \simeq 0.15$; average $\bar{D}_V \simeq 0.12$. However, the homogeneous halfspace (having velocity $\beta = \beta_B$) is significantly stiffer against rocking vibrations than the heterogeneous halfspace. It can easily be confirmed that the two media would yield comparable displacement levels if they

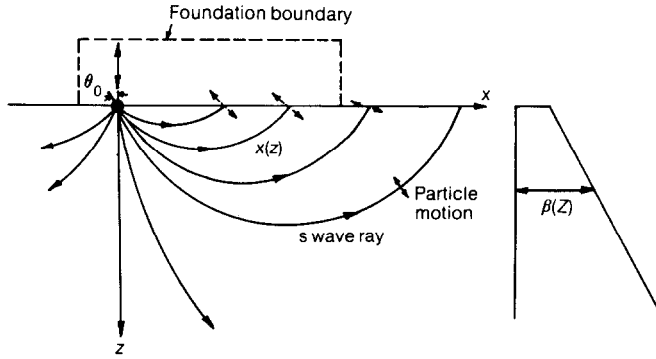


Fig. 12. Total reflection of wave rays in a heterogeneous medium increases the amplitude of foundation vibrations

had the same S wave velocity at a depth equal to $\frac{1}{2}B$. This further confirms the concept of a 'shallow pressure bulb' for the rocking motion, addressed previously with regard to static loading.

It can, therefore, be concluded that the 'effective' depth of the soil medium is approximately equal to the foundation halfwidth B for the translational modes of vibration and equal to $\frac{1}{2}B$ for the rotational mode.

It is worth noting that Awojobi (1972) showed that the resonant frequency factors (\bar{a}_m) of heavy⁴ circular foundation on a 'Gibson' halfspace are nearly the same with those predicted by the 'equivalent'⁵ homogeneous halfspace, in case of vertical vibrations. The same equivalence seems to apply in this case. Indeed, if \tilde{M} is the mass ratio defined by

$$\tilde{M} = \frac{M}{\rho B^2} \dots \dots \dots (33)$$

where M = total foundation mass per unit length, the dimensionless amplitude of vertical motion is obtained by

$$\bar{w}_0 = \left[\frac{\bar{D}_v'^2 + \bar{D}_v''^2}{(1 - \tilde{M}\bar{a}^2 \bar{D}_v')^2 + (\tilde{M}\bar{a}^2 \bar{D}_v'')^2} \right]^{\frac{1}{2}} \dots \dots \dots (34)$$

(Richard *et al.*, 1970). Approximating by straight horizontal lines the compliance functions in the frequency range $0 < \bar{a} < 0.7$, i.e. assuming $\bar{D}_v' \simeq 0.12$ and $\bar{D}_v'' \simeq 0.05$, leads to maximum displacement at resonant factor(s), \bar{a}_m , such that

$$(1 - 0.12\tilde{M}\bar{a}_m^2)^2 + (0.05\tilde{M}\bar{a}_m^2)^2 = \text{minimum} \dots \dots \dots (35)$$

from which

$$\bar{a}_m \simeq \sqrt{\frac{7.1}{\tilde{M}}} \dots \dots \dots (36)$$

Thus, as long as $\tilde{M} > 15$, \bar{a}_m would lie in the interval (0, 0.7) for which the 'equivalent' homogeneous halfspace exhibits very similar values of \bar{D}_v' and \bar{D}_v'' . Therefore equation (26) applies approximately for this medium as well.

Vibrations on a soil stratum on rigid rock

Figures 13(a) and (b) compare the real parts of the foundation stiffnesses as functions of the

⁴ That is, with mass ratios M , larger than about 20.

⁵ That is, whose shear modulus is the same as that of Gibson soil at a depth equal to the foundation radius.

dimensionless frequency, for a heterogeneous ($b = 1.5$) and a homogeneous ($\tilde{b} = 0$) stratum that have the same modulus as a depth equal to B , for two values of the depth ratio ($H/B = 2$ and $H/B = 4$). The conclusions can be summarized as follows.

Although resonance phenomena due to total wave reflection occur in both media, significant differences are observed between the two sets of stiffness functions. They can be attributed to two factors: the frequency selectivity of the homogeneous stratum; only specific frequencies lead to resonance and thus the corresponding stiffness functions exhibit sharper peaks and steeper valleys than those of the heterogeneous stratum; and the reflection of waves before they reach the rock surface in the heterogeneous medium; this leads to higher resonant frequencies, as if the 'effective' thickness of the heterogeneous layer is reduced. A similar phenomenon has been observed during vibrations of a heterogeneous deposit triggered by vertically propagating S waves (Gazetas, 1979).

CONCLUDING REMARKS

An analytical-numerical method has been presented to study the static and dynamic behaviour of strip foundations on the surface of an elastic isotropic medium consisting of heterogeneous layers. As schematically illustrated in Fig. 1(b) any soil profile consisting of horizontal layers with an arbitrary variation of S wave velocity with depth, can be readily handled with this theory. Also foundations that are infinitely stiff or infinitely flexible ('rigid' or 'uniform' applied pressures) and have rough or smooth contact surfaces ('complete' or 'relaxed' boundary) can be treated equally well. Results, however, have been presented only for the most interesting case of rigid, rough foundations.

For the two characteristic types of soil profiles studied in this Paper, namely the halfspace and the stratum-on-rigid-base, it has been shown that the main factors influencing the normalized displacements of massless foundations are

- (a) the dimensionless rate of heterogeneity, $\tilde{b} = bB$;
- (b) the stratum-depth to foundation-halfwidth ratio, H/B , and
- (c) the frequency factor $\tilde{a} = \omega B/\beta_0$.

The influence of each of the above factors has been demonstrated through a number of parametric plots and through extensive comparisons with pertinent results for homogeneous multilayered soils.

The results of these studies suggest an interesting 'equivalence' between a heterogeneous and a homogeneous medium that have the same modulus at a depth equal to the foundation halfwidth (for vertical and horizontal loading) or equal to $1/2$ the foundation halfwidth (for moment loading); i.e. for low frequency factors the two media yield displacements or rotations of about the same average level, although the fluctuations around the mean values can be quite different for the two media, because of differences in the occurrence of resonance phenomena.

The importance of two other factors, namely the Poisson's ratio ν and the critical damping ratio λ , has not been investigated in the Paper. One can approximately estimate the effects of their variation from relevant studies with other types of heterogeneous (e.g. Gibson, 1974) or layered homogeneous soils (e.g. Luco, 1976). The effect of high frequency factors also remains to be investigated.

ACKNOWLEDGEMENT

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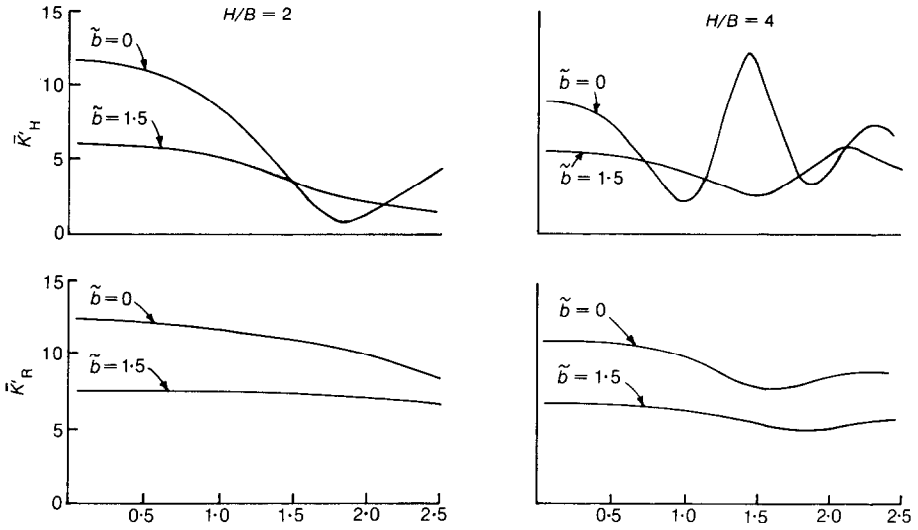


Fig. 13. Swaying and rocking dynamic stiffness functions (homogeneous or heterogeneous layer on rock: left hand side, $H/B = 2$; right hand side, $H/B = 4$). Only real parts shown

APPENDIX A: THE ELEMENTS OF THE TRANSFER MATRIX (X) IN EQUATION (16)

Calling: $a = 1 + bz$ and $I_v(y) = I_v$, where $y = ha/b$ and $k = h$ (equation (17b)), simplifies X_{ij} to

$$X_{11}/G = n_1 I_f + n_2 I_{f+1} + n_3 I_{f+2}$$

$$X_{12}/G = i(n_4 I_g - 2n_3/3I_{g+1})$$

$$X_{13}/G = n_1 K_f - n_2 K_{f+1} + n_3 I_{f+2}$$

$$X_{14}/G = i(n_4 K_g + 2n_3/3K_{g+1})$$

with

$$n_1 = 3b^2 a^{-1/2}(0.75 - 2f + f^2) - h^2 a^{\frac{1}{2}}$$

$$n_2 = 6bha^{-1/2} f$$

$$n_3 = 3h^2 a^{\frac{1}{2}}$$

$$n_4 = hba^{-1/2} - 2h^2 a^{\frac{1}{2}}$$

$$X_{21}/G = i(n_5 I_f - 2n_3/3I_{f+1})$$

$$X_{22}/G = n_6 I_g - n_7 I_{g+1} - n_3/3I_{g+2}$$

$$X_{23}/G = i(n_5 K_f + 2n_3/3K_{f+1})$$

$$X_{24}/G = n_6 K_g + n_7 K_{g+1} - n_3/3K_{g+2}$$

with

$$n_5 = bha^{-1/2}(1 - 2f)$$

$$n_6 = b^2 a^{-1/2}(1.25 - 2g - g^2) - h^2 a^{\frac{1}{2}}$$

$$n_7 = 2bha^{-1/2}(g + 2)$$

$$X_{31} = -in_8 I_f$$

$$X_{32} = -n_9 I_g - n_8 I_{g+1}$$

$$X_{33} = -in_8 K_f$$

$$X_{34} = -n_9 K_g + n_8 I_{g+1}$$

with

$$\begin{aligned}
 n_8 &= ha^4 \\
 n_9 &= ba^{-1}(2.5+g) \\
 X_{41} &= n_{10} I_f + n_8 I_{f+1} \\
 X_{42} &= -in_8 I_g \\
 X_{43} &= n_{10} K_f - n_8 K_{f+1} \\
 X_{44} &= -in_8 K_g \\
 n_{10} &= ba^{-1}(1.5+f)
 \end{aligned}$$

with

APPENDIX B: DETERMINATION OF THE COMPLIANCE MATRIX

where

$$\begin{aligned}
 \{d_0\} &= [D] \{F\} \\
 \{d_0\} &= \begin{Bmatrix} u_0 \\ \theta_0 \\ w_0 \end{Bmatrix} \text{ the displacement matrix} \\
 \{F_0\} &= \begin{Bmatrix} H_0 \\ M_0 \\ V_0 \end{Bmatrix} \text{ the load matrix}
 \end{aligned}$$

The compliance matrix is obtained as follows: $[D] = [K]^{-1}$
where

$$[K] = [L]^T [f] [L]^{-1}$$

with $[f]$ given in (26), (26a) and

$$\begin{aligned}
 [L] &= [[l_1] \dots [l_i] \dots [l_{2m+1}]] \\
 [l_i] &= \begin{bmatrix} 1 & 0 \\ 0 & X_i \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

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Dynamic displacements of offshore structures due to low frequency sinusoidal loading

I. M. SMITH* and F. MOLENKAMP†

Two of the most important aspects of foundation design for cyclic loading conditions are the assessment of material degradation due to cycling (the fatigue problem) and of dynamic amplification of the foundation response (the resonance problem). The first of these seems hardly amenable to calculation at all, except in the form of back-analysis of laboratory element and model test data or of field observations. The second, on the other hand, seems ill-suited to laboratory investigation and may therefore be estimated by calculation. For this purpose a simple bilinear soil model is presented and evaluated in comparison with a linearized equivalent. The performance of the model in relation to fatigue predictions is also explored.

L'évaluation de la dégradation des matériaux due aux charges cycliques (problème de la fatigue), d'une part, et l'évaluation de l'implification dynamique de la réponse des fondations (problème de la résonance), d'autre part, constituent deux des aspects les plus importants de la réalisation de fondations soumis à des charges cycliques. Dans le premier de ces cas, toute forme de calcul semble pratiquement impossible, mise à part l'analyse à rebours de données d'essais de modèles et éléments en laboratoire ou d'observations in situ. Le second cas, par contre, se prête mal aux recherches en laboratoire et peut donc être évalué par calcul. C'est dans ce but qu'est présenté un modèle de sol bilinéaire simple, lequel est évalué et comparé à un équivalent linéarisé. La performance du modèle par rapport aux prévisions de fatigue fait également l'objet d'une étude.

INTRODUCTION

In recent years, dynamic analysis of structure–foundation systems has assumed considerable importance in two major fields of civil engineering practice, namely in the design of structures to withstand earthquakes and/or ocean waves. Massive power stations have had to be sited in seismically active regions while oil and gas have had to be extracted from deep sea locations using fixed steel or concrete production structures. Some of these sites are also subject to seismic disturbance.

The design problems are characterized by two main difficulties which are not encountered under 'standard' civil engineering conditions. Firstly, the loading imposed on structures by earthquakes or waves is alternating in character, and is applied in the form of many (hundreds at least) cycles. It is well known that all civil engineering materials degrade under such cyclic loading; metals, concrete, plastics all tend to become softer and weaker, at least for slowly applied cycles. However, materials with internal viscosity exhibit particularly complex behaviour in that they apparently grow stiffer and stronger with increased rate of loading, so that their overall behaviour can be an amalgamation of the two contrary trends. This is the case for 'clays' and in addition the behaviour of all soils is complicated because the material degradation leads to porewater pressure rises which themselves can accelerate the degradation process. The foregoing could be termed the 'fatigue problem'.

The second difficulty could be termed the 'resonance problem', and depends on the frequency of the cyclic disturbing forces. Over a certain range of forcing frequencies relative to the natural frequency of the structure–foundation system, inertial amplification of the response

Discussion on this Paper closes 1 September, 1980. For further details see inside back cover.

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