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Static Deformation of a Multilayered Sphere by Internal Sources

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Summary

The Thomson–Haskell matrix device is used to solve the problem of the static deformation of a multilayered spherical Earth model by buried sources. The model consists of $p-1$ concentric spherical shells plus an inner core; each shell as well as the core being homogeneous, isotropic and perfectly elastic. The point source is represented as a discontinuity in the motion-stress vector across the spherical surface passing through the source. Explicit series expressions in terms of layer matrices are obtained for the displacements and stresses at any point in the medium for three sources: a vertical strike-slip fault, a vertical dip-slip fault and a centre of explosion. The singular case corresponding to the Legendre polynomial of the first degree ($n = 1$) has been discussed in detail.

1. Introduction

During the last few years, elasticity theory of dislocations has been developed and applied by several investigators, e.g. Steketee (1958), Chinnery (1961, 1963), Maruyama (1964) and Press (1965). Mansinha & Smylie (1967) computed the changes in the products of inertia of the Earth due to rearrangement of masses associated with major earthquakes and then calculated their contribution to the excitation of the Chandler wobble and the secular polar shift. As a mathematical model, they used vertical, rectangular, strike-slip and dip-slip faults in a uniform half-space. However, there is no justification for using a half-space model in a problem with intrinsic spherical geometry. Ben-Menahem & Singh (1968) made a significant contribution by obtaining explicit expressions for the displacements at the free surface of a homogeneous non-gravitating sphere due to internal dislocations of arbitrary orientation. The numerical results were reported by Ben-Menahem, Singh & Solomon (1969, 1970) and Singh & Ben-Menahem (1969). Ben-Menahem & Israel (1970) obtained the displacement field at any point within the sphere and then calculated the inertia changes due to a displacement dislocation and a centre of explosion in a uniform sphere. These authors find that the spherical model of the Earth yields higher inertia changes than the corresponding half-space model.

The aim of the present paper is to generalize the results of Ben-Menahem & Israel (1970) for the displacement field due to a point source in a homogeneous sphere. The homogeneous sphere has been replaced by a multilayered sphere consisting of $p-1$ concentric spherical shells plus a solid core; each shell and the core being homogeneous, isotropic and perfectly elastic. The point source is represented as a discontinuity in

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the displacement and stress components across the spherical surface passing through the source and the Thomson–Haskell matrix technique (Thomson 1950; Haskell 1953) is applied. Explicit series expressions for the displacements and stresses at any point within the sphere are obtained for a vertical strike-slip fault, a vertical dip-slip fault and a centre of explosion by using the motion-stress vector obtained in an earlier paper (Wason & Singh 1971).

The sister problem of the static deformation of a multilayered half-space by buried sources has been recently solved by Singh (1970). The static deformation of a sphere resulting from surface mass loads has been studied by Slichter & Caputo (1960), Caputo (1961), Longman (1962) and Takeuchi, Saito & Kobayashi (1962).

2. Formulation of the problem

Consider the Earth as made up of $p-1$ concentric spherical shells plus an inner core; each shell as well as the core being homogeneous, isotropic and perfectly elastic. Let the ith spherical shell be bounded by the radii r_{i-1}, $r_i (r_{i-1} < r_i;\ i = 1, 2, \ldots, p)$ with centre at the origin of a spherical polar co-ordinates system (r, θ, ϕ) and let the elastic constants of the shell be λ_i, μ_i and its density ρ_i (Fig. 1). Evidently, $r_0 = 0$ and $r_p = a$, where a is the radius of the Earth.

In the ith shell the displacement vector $\mathbf{u}_i$ satisfies the vector Naviér equation of statical elasticity

$$\mu_i \nabla^2 \mathbf{u}_i + (\lambda_i + \mu_i) \operatorname{grad} \operatorname{div} \mathbf{u}_i = 0, \tag{2.1}$$

and one may take (Singh 1970, equations (5) and (6))

$$\mathbf{u}_i(r) = \sum_{m,n} \mathbf{u}_i^{mn}(r), \tag{2.2}$$

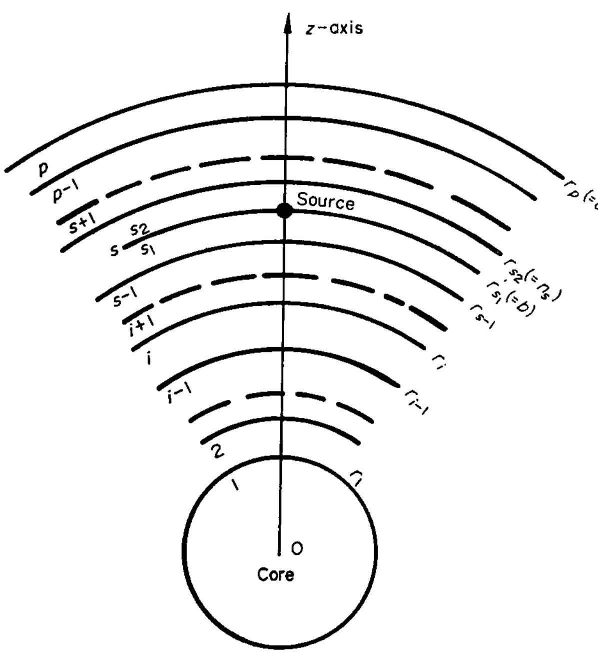

FIG. 1. Geometry of the layered sphere and the sources.

with

$$\mathbf{u}_i^{mn}(r) = A_{1,i}^{mn}\mathbf{N}_{m,n+1}^{-} + A_{2,i}^{mn}\mathbf{N}_{m,n-1}^{+} + B_{1,i}^{mn}\mathbf{F}_{m,n-1}^{-} + B_{2,i}^{mn}\mathbf{F}_{m,n+1}^{+} + C_{1,i}^{mn}\mathbf{M}_{m,n}^{-} + C_{2,i}^{mn}\mathbf{M}_{m,n}^{+}, \quad (2.3)$$

where **N**, **F** and **M** are the three independent vector solutions of equation (2.1). $A_{1,i}$, $A_{2,i}$, $B_{1,i}$, $B_{2,i}$, $C_{1,i}$ and $C_{2,i}$ are six arbitrary constants belonging to the ith shell, to be determined by the application of the boundary and source conditions.

We put

$$\left.\begin{aligned}
\mathbf{P}_{m,n} &= \mathbf{e}_r\, Y_{m,n}(\theta,\phi),\\
(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,n} &= \left[\mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right] Y_{m,n}(\theta,\phi),\\
(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,n} &= \left[\mathbf{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \mathbf{e}_\phi \frac{\partial}{\partial\theta}\right] \frac{\partial}{\partial\phi}\, Y_{m,n}(\theta,\phi),\\
Y_{m,n}(\theta,\phi) &= P_n^m(\cos\theta)(\alpha_m\cos m\phi + \beta_m \sin m\phi).
\end{aligned}\right\} \quad (2.4)$$

Here α_m, β_m are constants and shall be specified later at the time of introducing the source.

On expressing the vectors **N**, **F** and **M**, in terms of the mutually orthogonal surface vector harmonics $\mathbf{P}_{m,n}$, $\mathbf{B}_{m,n}$ and $\mathbf{C}_{m,n}$ (Ben-Menahem & Singh 1968, equations (2.13), (2.14), (2.16)), equation (2.3) reduces to the form (the superscript of the entities **u**, x, y, z, A_1, A_2, B_1, B_2, C_1 and C_2 is mn throughout, unless otherwise specified)

$$\mathbf{u}_i(r) = x_i(r)\,\mathbf{P}_{m,n} + y_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,n} + z_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,n}, \quad (2.5)$$

where

$$x_i(r) = -(n+1)\,r^{-n-2}A_{1,i} + nr^{n-1}A_{2,i} + \frac{n[(n+1)\lambda_i + (n+3)\mu_i]}{2(\lambda_i+2\mu_i)}\,.\,r^{-n}B_{1,i} + \frac{(n+1)[n\lambda_i + (n-2)\mu_i]}{2(\lambda_i+2\mu_i)}\,r^{n+1}B_{2,i}, \quad (2.6)$$

$$y_i(r) = r^{-n-2}A_{1,i} + r^{n-1}A_{2,i} + \frac{(2-n)\lambda_i + (4-n)\mu_i}{2(\lambda_i+2\mu_i)}\,r^{-n}B_{1,i} + \frac{(n+3)\lambda_i + (n+5)\mu_i}{2(\lambda_i+2\mu_i)}\,r^{n+1}B_{2,i}, \quad (2.7)$$

$$z_i(r) = r^{-n-1}C_{1,i} + r^n C_{2,i}. \quad (2.8)$$

Using equation (2.4) of Ben-Menahem & Israel (1970), the stress vector across the surface r = constant corresponding to the displacement vector (2.5) may be expressed in the following form (the superscript of the quantities **T**, X, Y and Z is mn throughout, unless otherwise mentioned):

$$\mathbf{T}_i(r) = X_i(r)\,\mathbf{P}_{m,n} + Y_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,n} + Z_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,n}, \quad (2.9)$$

where

$$\frac{X_i(r)}{\mu_i} = 2(n+1)(n+2)\,r^{-n-3}A_{1,\,i}+2n(n-1)\,r^{n-2}A_{2,\,i}$$

$$-\frac{n[(n^2+3n-1)\,\lambda_i+n(n+3)\,\mu_i]}{\lambda_i+2\mu_i}\,r^{-n-1}B_{1,\,i}$$

$$+\frac{(n+1)[(n^2-n-3)\,\lambda_i+(n^2-n-2)\,\mu_i]}{\lambda_i+2\mu_i}\,r^n B_{2,\,i}, \tag{2.10}$$

$$\frac{Y_i(r)}{\mu_i} = -2(n+2)\,r^{-n-3}A_{1,\,i}+2(n-1)\,r^{n-2}A_{2,\,i}$$

$$+\frac{(n^2-1)\,\lambda_i+(n^2-2)\,\mu_i}{\lambda_i+2\mu_i}\,r^{-n-1}B_{1,\,i}$$

$$+\frac{n(n+2)\,\lambda_i+(n^2+2n-1)\,\mu_i}{\lambda_i+2\mu_i}\,r^n B_{2,\,i}, \tag{2.11}$$

$$\frac{Z_i(r)}{\mu_i} = -(n+2)\,r^{-n-2}C_{1,\,i}+(n-1)\,r^{n-1}C_{2,\,i}. \tag{2.12}$$

Equations (2.5) and (2.9) may be written as follows:

$$\left.\begin{aligned} \mathbf{u}_i &= {}_R\mathbf{u}_i+{}_L\mathbf{u}_i,\\ \mathbf{T}_i &= {}_R\mathbf{T}_i+{}_L\mathbf{T}_i, \end{aligned}\right\} \tag{2.13}$$

where

$$\left.\begin{aligned} {}_R\mathbf{u}_i &= x_i\,\mathbf{P}_{m,\,n}+y_i(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,\,n},\\ {}_R\mathbf{T}_i &= X_i\,\mathbf{P}_{m,\,n}+Y_i(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,\,n}, \end{aligned}\right\} \tag{2.14}$$

and

$$\left.\begin{aligned} {}_L\mathbf{u}_i &= z_i(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,\,n},\\ {}_L\mathbf{T}_i &= Z_i(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,\,n}. \end{aligned}\right\} \tag{2.15}$$

From equations (2.6) to (2.8), (2.10) to (2.12), it is obvious that the original problem splits into two independent problems represented by equations (2.14) and (2.15), referred to as the *R*- and the *L*-problem, respectively. The reason is that we can satisfy the equations of motion and boundary conditions (as will be seen in the following section) by the *R*-problem and the *L*-problem independently. One may solve these two problems separately and then use equation (2.13) to get the complete solution; the two solutions may be called as the spheroidal and toroidal response, respectively, of a multilayered sphere to static sources. The counterpart in the dynamic case is the separation of Rayleigh and Love wave problems and spheroidal and toroidal oscillations.

3. Formal solution

(a) *Solution of the R-problem*

We define the *column* matrices J_i and K_i (the superscript of the matrices J and K is *mn* throughout, unless otherwise mentioned) by

$$[J_i(r)] = [x_i(r), y_i(r), X_i(r), Y_i(r)], \tag{3.1}$$

$$[K_i] = [A_{1,\,i}, A_{2,\,i}, B_{1,\,i}, B_{2,\,i}]. \tag{3.2}$$

From equations (2.6), (2.7), (2.10), (2.11), (3.1) and (3.2), we have

$$J_i(r) = M_i^n(r)\, K_i, \tag{3.3}$$

where the elements of the matrix $M_i^n(r)$ are (omitting the subscript i of λ_i and μ_i):

$$\left.\begin{aligned}
&(11) = -(n+1)\, r^{-n-2}, && (12) = nr^{n-1},\\
&(13) = Q_1\, n[(n+1)\,\lambda + (n+3)\,\mu]\, r^{-n}, && (14) = Q_1(n+1)[n\lambda + (n-2)\,\mu]\, r^{n+1},\\
&(21) = r^{-n-2}, && (22) = r^{n-1},\\
&(23) = Q_1[(2-n)\,\lambda + (4-n)\,\mu]\, r^{-n}, && (24) = Q_1[(n+3)\,\lambda + (n+5)\,\mu]\, r^{n+1},\\
&(31) = 2(n+1)(n+2)\,\mu r^{-n-3}, && (32) = 2n(n-1)\,\mu r^{n-2},\\
&(33) = -2Q_1\, n[(n^2+3n-1)\,\lambda && (34) = 2Q_1(n+1)[(n^2-n-3)\,\lambda\\
&\qquad\qquad + n(n+3)\,\mu]\,\mu r^{-n-1}, && \qquad\qquad + (n^2-n-2)\,\mu]\,\mu r^n,\\
&(41) = -2(n+2)\,\mu r^{-n-3}, && (42) = 2(n-1)\,\mu r^{n-2},\\
&(43) = 2Q_1[(n^2-1)\,\lambda && (44) = 2Q_1[n(n+2)\,\lambda\\
&\qquad\qquad + (n^2-2)\,\mu]\,\mu r^{-n-1}, && \qquad\qquad + (n^2+2n-1)\,\mu]\,\mu r^n,
\end{aligned}\right\} \tag{3.4}$$

where

$$Q_1 = 1/[2(\lambda+2\mu)]. \tag{3.5}$$

Equation (3.3) yields

$$K_i = [M_i^n(r)]^{-1}\, J_i(r), \tag{3.6}$$

where the matrix $[M_i^n(r)]^{-1}$ is the inverse of the square matrix $M_i^n(r)$. The elements of $[M_i^n(r)]^{-1}$ are found to be (omitting the subscript i of λ_i and μ_i):

$$\left.\begin{aligned}
&(11) = 2Q_2[(n^2-n-3)\,\lambda && (12) = 2Q_2\, n[n(n+2)\,\lambda\\
&\qquad\qquad + (n^2-n-2)\,\mu]\, r^{n+2}, && \qquad\qquad + (n^2+2n-1)\,\mu]\, r^{n+2},\\
&(13) = -Q_2[n\lambda + (n-2)\,\mu]\,\mu^{-1}\, r^{n+3}, && (14) = -Q_2\, n[(n+3)\,\lambda\\
& && \qquad\qquad + (n+5)\,\mu]\,\mu^{-1}\, r^{n+3},\\
&(21) = 2Q_3[(n^2+3n-1)\,\lambda && (22) = -2Q_3(n+1)[(n^2-1)\,\lambda\\
&\qquad\qquad + n(n+3)\,\mu]\, r^{-n+1}, && \qquad\qquad + (n^2-2)\,\mu]\, r^{-n+1},\\
&(23) = Q_3[(n+1)\,\lambda && (24) = Q_3(n+1)[(2-n)\,\lambda\\
&\qquad\qquad + (n+3)\,\mu]\,\mu^{-1}\, r^{-n+2}, && \qquad\qquad + (4-n)\,\mu]\,\mu^{-1}\, r^{-n+2},\\
&(31) = 2(n-1)\,\acute{\mathbf{Q}}_3\, r^n, && (32) = 2(n^2-1)\,\acute{\mathbf{Q}}_3\, r^n,\\
&(33) = -\acute{\mathbf{Q}}_3\,\mu^{-1}\, r^{n+1}, && (34) = -(n+1)\,\acute{\mathbf{Q}}_3\,\mu^{-1}\, r^{n+1},\\
&(41) = -2(n+2)\,\acute{\mathbf{Q}}_2\, r^{-n-1}, && (42) = 2n(n+2)\,\acute{\mathbf{Q}}_2\, r^{-n-1},\\
&(43) = -\acute{\mathbf{Q}}_2\,\mu^{-1}\, r^{-n}, && (44) = n\acute{\mathbf{Q}}_2\,\mu^{-1}\, r^{-n},
\end{aligned}\right\} \tag{3.7}$$

where

$$\left.\begin{aligned}
&\acute{\mathbf{Q}}_2 = 1/[(2n+1)(2n+3)],\\
&\acute{\mathbf{Q}}_3 = 1/[(2n-1)(2n+1)],\\
&Q_2 = Q_1\,\acute{\mathbf{Q}}_2,\ Q_3 = Q_1\,\acute{\mathbf{Q}}_3.
\end{aligned}\right\} \tag{3.8}$$

From equations (3.3) and (3.6), the displacements and stresses at the top and the bottom of the ith shell are connected through the relation

$$J_i(r_i) = N_i^n J_i(r_{i-1}), \tag{3.9}$$

where the layer matrix N_i^n is given by

$$N_i^n = M_i^n(r_i)[M_i^n(r_{i-1})]^{-1}. \tag{3.10}$$

Continuity of the displacement and stress components at the interface $r = r_{i-1}$ yields

$$J_i(r_{i-1}) = J_{i-1}(r_{i-1}). \tag{3.11}$$

Hence equation (3.9) becomes

$$J_i(r_i) = N_i^n J_{i-1}(r_{i-1}). \tag{3.12}$$

Let a point source be situated at the point $r = b$, $\theta = 0$. Let the source layer be designated as layer s bounded by the radii r_{s-1}, r_s. We divide the source layer into two spherical shells s_1 and s_2 of identical properties. The shell s_1 is bounded by the radii $r = r_{s-1}$, $r_{s_1}(= b)$, and the shell s_2 by $r = r_{s_1}$, $r_{s_2}(= r_s)$ (Fig. 1). Due to the presence of the source the displacement and/or the stress vector may be discontinuous across the spherical surface $r = b$.

Let the matrix representation of the source be

$$J_{s_2}(r_{s_1}) - J_{s_1}(r_{s_1}) = D^{mn}. \tag{3.13}$$

For a specific source the source matrix D^{mn} is known.

From equation (3.9) and (3.11), we get by iteration

$$J_p(r_p) = N_p^n N_{p-1}^n \dots N_{s_2}^n J_{s_2}(r_{s_1}), \tag{3.14}$$

$$J_{s_1}(r_{s_1}) = N_{s_1}^n N_{s-1}^n \dots N_2^n J_1(r_1). \tag{3.15}$$

It may be shown (Haskell 1953) that for the source layer s

$$N_{s_2}^n N_{s_1}^n = N_s^n. \tag{3.16}$$

Equations (3.13)–(3.16) now yield

$$J_p(r_p) = U_p^n J_1(r_1) + V_p^n D^{mn}, \tag{3.17}$$

where

$$U_p^n = N_p^n N_{p-1}^n \dots N_2^n, \tag{3.18}$$

$$V_p^n = N_p^n N_{p-1}^n \dots N_{s_2}^n. \tag{3.19}$$

The boundary conditions: (i) that the surface tractions vanish at the surface $r = a$, and (ii) the displacements and stresses are bounded at the origin $r = 0$, give rise to the following equations

$$[J_p(a)] = [x_p(a), y_p(a), 0, 0], \tag{3.20}$$

$$[K_1] = [0, A_{2,1}, 0, B_{2,1}]. \tag{3.21}$$

Equations (3.3), (3.17), (3.20) and (3.21) result in

$$[x_p(a), y_p(a), 0, 0] = [E_p^n][0, A_{2,1}, 0, B_{2,1}] + [F_p^{mn}], \tag{3.22}$$

where

$$E_p^n = U_p^n M_1^n(r_1), \tag{3.23}$$

$$F_p^{mn} = V_p^n D^{mn}. \tag{3.24}$$

Equation (3.22) is equivalent to the following four equations:

$$\left.\begin{aligned} x_p(a) &= (E_p{}^n)_{12}\, A_{2,\,1} + (E_p{}^n)_{14}\, B_{2,\,1} + (F_p{}^{mn})_1, \\ y_p(a) &= (E_p{}^n)_{22}\, A_{2,\,1} + (E_p{}^n)_{24}\, B_{2,\,1} + (F_p{}^{mn})_2, \end{aligned}\right\} \tag{3.25}$$

$$\left.\begin{aligned} 0 &= (E_p{}^n)_{32}\, A_{2,\,1} + (E_p{}^n)_{34}\, B_{2,\,1} + (F_p{}^{mn})_3, \\ 0 &= (E_p{}^n)_{42}\, A_{2,\,1} + (E_p{}^n)_{44}\, B_{2,\,1} + (F_p{}^{mn})_4. \end{aligned}\right\} \tag{3.26}$$

The set (3.26) yields

$$\left.\begin{aligned} A_{2,\,1} &= [(E_p{}^n)_{34}\, (F_p{}^{mn})_4 - (E_p{}^n)_{44}\, (F_p{}^{mn})_3]/\Delta, \\ B_{2,\,1} &= [(E_p{}^n)_{42}\, (F_p{}^{mn})_3 - (E_p{}^n)_{32}\, [F_p{}^{mn})_4]/\Delta, \end{aligned}\right\} \tag{3.27}$$

where

$$\Delta = (E_p{}^n)_{32}\, (E_p{}^n)_{44} - (E_p{}^n)_{42}\, (E_p{}^n)_{34}. \tag{3.28}$$

It may easily be seen that at any point in the ith spherical shell

$$J_i(r) = M_i{}^n(r)[M_i{}^n(r_{i-1})]^{-1}\, J_i(r_{i-1}), \tag{3.29}$$

where

$$[J_i(r_{i-1})] = [E_{i-1}^n][0, A_{2,\,1}, 0, B_{2,\,1}] + (\Psi_{i-1}^{mn}], \tag{3.30}$$

with

$$\Psi_{i-1}^{mn} = \begin{Bmatrix} F_{i-1}^{mn}; & \text{for } i > s_2, \\ D^{mn}; & \text{for } i = s_2, \\ 0; & \text{for } i < s_2. \end{Bmatrix} \tag{3.31}$$

The matrices E_{i-1}^{mn} and F_{i-1}^{mn} are given by equations (3.23) and (3.24) respectively, with p replaced by $i-1$. Putting the values of $A_{2,\,1}$ and $B_{2,\,1}$ from equation (3.27) into equation (3.30), we obtain

$$\begin{aligned} \left(J_i(r_{i-1})\right)_q = \{ & [(E_{i-1}^n)_{q4}\, (E_p{}^n)_{42} - (E_{i-1}^n)_{q2}\, (E_p{}^n)_{44}](F_p{}^{mn})_3 \\ & + [(E_{i-1}^n)_{q2}\, (E_p{}^n)_{34} - (E_{i-1}^n)_{q4}\, (E_p{}^n)_{32}](F_p{}^{mn})_4 \\ & + \Delta(\Psi_{i-1}^{mn})_q\}/\Delta, \ (q = 1, 2, 3, 4). \end{aligned} \tag{3.32}$$

(b) *Solution of the L-problem*

The treatment of this problem is exactly similar to that of the R-problem. Hence we give below only important results omitting the details of their derivation. Previous notation is retained and the subscript L is prefixed to the quantities for distinction from the corresponding quantities for the R-problem.

We define the *column* matrices

$$[{}_LJ_i(r)] = [z_i(r), Z_i(r)], \tag{3.33}$$

$$[{}_LK_i] = [C_{1,\,i}, C_{2,\,i}]. \tag{3.34}$$

At any point in the ith shell, we have

$${}_LJ_i(r) = {}_LM_i{}^n(r)\, {}_LK_i, \tag{3.35}$$

where the matrix ${}_LM_i{}^n(r)$ is given by

$$[{}_LM_i{}^n(r)] = \begin{bmatrix} r^{-n-1} & r^n \\ -(n+2)\, \mu_i\, r^{-n-2} & (n-1)\, \mu_i\, r^{n-1} \end{bmatrix}. \tag{3.36}$$

Further

$${}_LJ_i(r_i) = {}_LN_i{}^n\, {}_LJ_i(r_{i-1}), \tag{3.37}$$

where the layer matrix ${}_LN_i^n$ is given by

$$ {}_LN_i^n = {}_LM_i^n(r_i)[{}_LM_i^n(r_{i-1})]^{-1}, \tag{3.38}$$

with

$$[{}_LM_i^n(r)]^{-1} = \frac{1}{2n+1}\begin{bmatrix}(n-1)\,r^{n+1} & -\mu_i^{-1}\,r^{n+2}\\ (n+2)\,r^{-n} & \mu_i^{-1}\,r^{-n+1}\end{bmatrix}. \tag{3.39}$$

Using the continuity condition

$$ {}_LJ_i(r_{i-1}) = {}_LJ_{i-1}(r_{i-1}), \tag{3.40}$$

equation (3.37) becomes

$$ {}_LJ_i(r_i) = {}_LN_i^n\,{}_LJ_{i-1}(r_{i-1}). \tag{3.41}$$

The source is represented through the relation

$$ {}_LJ_{s_2}(r_{s_1}) - {}_LJ_{s_1}(r_{s_1}) = {}_LD^{mn}. \tag{3.42}$$

The counterpart of equation (3.22) in this case is

$$[z_p(a), 0] = [{}_LE_p^n][0, C_{2,1}] + [{}_LF_p^{mn}], \tag{3.43}$$

where

$$ {}_LE_p^n = {}_LU_p^n\,{}_LM_1^n(r_1), \tag{3.44}$$

$$ {}_LF_p^{mn} = {}_LV_p^n\,{}_LD^{mn}, \tag{3.45}$$

$$\left.\begin{aligned} {}_LU_p^n &= {}_LN_p^n\,{}_LN_{p-1}^n \cdots {}_LN_2^n,\\ {}_LV_p^n &= {}_LN_p^n\,{}_LN_{p-1}^n \cdots {}_LN_{s2}^n.\end{aligned}\right\} \tag{3.46}$$

Equation (3.43) is equivalent to the following two equations:

$$z_p(a) = ({}_LE_p^n)_{12}\,C_{2,1} + ({}_LF_p^{mn})_1, \tag{3.47}$$

$$0 = ({}_LE_p^n)_{22}\,C_{2,1} + ({}_LF_p^{mn})_2. \tag{3.48}$$

The last equation gives

$$C_{2,1} = -({}_LF_p^{mn})_2/({}_LE_p^n)_{22}. \tag{3.49}$$

Corresponding to equation (3.29), we now have

$$ {}_LJ_i(r) = {}_LM_i^n(r)[{}_LM_i^n(r_{i-1})]^{-1}\,{}_LJ_i(r_{i-1}), \tag{3.50}$$

where

$$[{}_LJ_i(r_{i-1})] = [{}_LE_{i-1}^n][0, C_{2,1}] + [{}_L\Psi_{i-1}^{mn}], \tag{3.51}$$

with

$$ {}_L\Psi_{i-1}^{mn} = \begin{Bmatrix} {}_LF_{i-1}^{mn}; & \text{for } i > s_2,\\ {}_LD^{mn}; & \text{for } i = s_2,\\ 0; & \text{for } i < s_2.\end{Bmatrix} \tag{3.52}$$

The matrices ${}_LE_{i-1}^n$ and ${}_LF_{i-1}^{mn}$ are given by equations (3.44) and (3.45) respectively, with p replaced by $i-1$. Equations (3.49) and (3.51) give

$$\left({}_LJ_i(r_{i-1})\right)_q = [({}_LE_p^n)_{22}\,({}_L\Psi_{i-1}^{mn})_q - ({}_LE_{i-1}^n)_{q2}\,({}_LF_p^{mn})_2]/({}_LE_p^n)_{22}, \quad (q = 1, 2). \tag{3.53}$$

For every n the constants $A_{2,1}$, $B_{2,1}$ and $C_{2,1}$ must satisfy equations (3.26) and (3.48). For $n = 0$, we may take $A_{2,i} = B_{1,i} = C_{1,i} = C_{2,i} = 0$. For $n = 1$, however, there arises some difficulty in evaluating the constants. It will be shown in a later section that for $n = 1$, the two equations of the set (3.26) become identical and equation (3.48) is an identity. In order to determine the constants $A_{2,1}$, $B_{2,1}$ and $C_{2,1}$, we need extra conditions consistent with the physics of the problem. These have been discussed in Section 5.

4. Specification of the source

Equations (3.29) and (3.50) are general expressions for the displacements and stresses at any point in the ith layer induced by an arbitrary point source for which the source matrices D^{mn} and ${}_LD^{mn}$ are known. Let the point source be a displacement dislocation. The six elementary displacement dislocation sources (kl), in the notation of Steketee (1958), are: $(kl) = (11)$, (22), (33), (23), (31) and (12). For $k = l$ the corresponding force system at the focus is a combination of a centre of compression and an additional double force without moment in the k-direction. For $k \neq l$, it is a combination of two coplaner, mutually perpendicular double forces with moments in opposite directions. The source (12) is pertinent to a vertical strike-slip fault and (23) to a vertical dip-slip fault.

Wason & Singh (1971, Section 4) obtained the source matrices D^{mn} and ${}_LD^{mn}$ for the six elementary displacement dislocation sources (kl). For easy reference, we reproduce below the non-zero components of these matrices. In these expressions γ stands for $U_0\, ds/(24\pi b^2)$, where U_0 is the amount of the constant dislocation over the fault area dS.

$$\left.\begin{aligned}
(D^{0,n})_1 &= \frac{2(2n+1)\gamma}{\lambda_s+2\mu_s}\left[\mu_s+(3\lambda_s+2\mu_s)\,\frac{\delta_{kl}}{\alpha_0}\right],\\
(D^{0,n})_3 &= -2(D^{0,n})_4 = \frac{2(2n+1)(3\lambda_s+2\mu_s)\,\mu_s\gamma}{(\lambda_s+2\mu_s)\,b}\left[1-\frac{4\delta_{kl}}{\alpha_0}\right],\\
(D^{1,n})_2 &= -({}_LD^{1,n})_1 = Q,\\
(D^{2,n})_4 &= -2({}_LD^{2,n})_2 = -2\mu_s Q/b,
\end{aligned}\right\} \tag{4.1}$$

where

$$Q = \frac{3(2n+1)\gamma}{n(n+1)}, \tag{4.2}$$

δ_{kl} is the Kronecker delta and the values of the non-zero constants α_m, β_m appearing in equation (2.4) for different kinds of sources are (Wason & Singh 1971, equation (2.10)):

$$\left.\begin{aligned}
(kl) &= (11);\ \alpha_0 = -2,\ \alpha_2 = 1.\\
(kl) &= (22);\ \alpha_0 = -2,\ \alpha_2 = -1.\\
(kl) &= (33);\ \alpha_0 = 4.\\
(kl) &= (23);\ \beta_1 = 2.\\
(kl) &= (31);\ \alpha_1 = 2.\\
(kl) &= (12);\ \beta_2 = 1.
\end{aligned}\right\} \tag{4.3}$$

Similarly, in the case of a centre of explosion, the non-zero components of the source matrices are (Wason & Singh 1971, Section 5):

$$\left.\begin{aligned}(D^{0,n})_1 &= -(2n+1)\,A_0/b^2,\\ (D^{0,n})_3 &= 4(2n+1)\,A_0\,\mu_s/b^3,\\ (D^{0,n})_4 &= -2(2n+1)\,A_0\,\mu_s/b^3.\end{aligned}\right\} \tag{4.4}$$

where A_0 depends upon the strength of the source and is of dimensions L^3. In this case the sole non-zero constant $\alpha_0 = 1$.

We now proceed to calculate explicit expressions for the displacements and stresses at any point in the layered medium, for three kinds of sources.

(I) vertical strike-slip fault ($m = 2$),

(II) vertical dip-slip fault ($m = 1$),

(III) centre of explosion ($m = 0$).

We obtain

$$\mathbf{u}_i^m(r) = \sum_{n=m}^{\infty} [x_i(r)\,\mathbf{P}_{m,n} + y_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,n} + z_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,n}], \tag{4.5}$$

$$\mathbf{T}_i^m(r) = \sum_{n=m}^{\infty} [X_i(r)\,\mathbf{P}_{m,n} + Y_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{B}_{m,n} + Z_i(r)(n(n+1))^{\frac{1}{2}}\,\mathbf{C}_{m,n}]. \tag{4.6}$$

The radial functions $x_i(r)$, $X_i(r)$, etc. occurring in these expressions are given by equations (3.29), (3.32), (3.50) and (3.53). The source terms of equations (3.32) and (3.53) are known completely by equations (3.31), (3.52), (4.1), (4.2), (4.4) and the following relations:

Case I

$$\left.\begin{aligned}(F_k^{2,n})_q &= -2\mu_s\,b^{-1}\,Q(V_k^n)_{q4}, && (q = 1, 2, 3, 4)\\ ({}_LF_k^{2,n})_q &= \mu_s\,b^{-1}\,Q({}_LV_k^n)_{q2}, && (q = 1, 2)\end{aligned}\right\} \tag{4.7}$$

Case II

$$\left.\begin{aligned}(F_k^{1,n})_q &= Q(V_k^n)_{q2}, && (q = 1, 2, 3, 4)\\ ({}_LF_k^{1,n})_q &= -Q({}_LV_k^n)_{q1}, && (q = 1, 2)\end{aligned}\right\} \tag{4.8}$$

Case III

$$(F_k^{0,n})_q = -(2n+1)\,A_0\,b^{-2}[(V_k^n)_{q1} - 4\mu_s\,b^{-1}(V_k^n)_{q3} + 2\mu_s\,b^{-1}(V_k^n)_{q4}],$$
$$(q = 1, 2, 3, 4). \tag{4.9}$$

Equations (4.7)–(4.9) are obtained from equations (3.24), (3.45), (4.1) and (4.4), and hold good for every n. However, equations (3.32) and (3.53) do not hold for the case $n = 1$. This singular case is discussed in the next section.

5. The singular case n = 1

From equations (3.36), (3.38) and (3.39), we note that for $n = 1$ the layer matrix ${}_LN_i^n$ is of the form

$$[{}_LN_i^1] = \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}, \; (b_1, b_2, b_3 \text{ are non-zero}) \tag{5.1}$$

Similarly, from equations (3.4), (3.7) and (3.10), we note that the elements of the layer matrix N_i^n for $n = 1$ exhibit the following properties:

$$\left.\begin{array}{ll} \text{(i)} & N_{31}+N_{32} = N_{41}+N_{42} = 0, \\ \text{(ii)} & N_{11}+N_{12} = N_{21}+N_{22}, \\ \text{(iii)} & N_{31}+2N_{41} = N_{32}+2N_{42} = 0, \\ \text{(iv)} & (N_{34}-2N_{33})+2(N_{44}-2N_{43}) = 0. \end{array}\right\} \quad (5.2)$$

The product of two matrices of the form (5.1) has the same form. Also the product matrix of two matrices having properties expressed by equation (5.2) has the same properties.

We shall now prove the assertion made in the concluding paragraph of Section 3. First consider the set (3.26). For $n = 1$, the two equations in the set (3.26) may be written as follows:

$$\begin{aligned}\big((U_p{}^1)_{q1}+(U_p{}^1)_{q2}\big) A_{2;\,1}^{m;\,1}+\frac{r_1}{\lambda_1+2\mu_1}&[(U_p{}^1)_{q1}(\lambda_1-\mu_1)r_1 \\ +(U_p{}^1)_{q2}(2\lambda_1+3\mu_1)r_1&+\big((U_p{}^1)_{q4}-2(U_p{}^1)_{q3}\big)(3\lambda_1+2\mu_1)\mu_1]B_{2;\,1}^{m;\,1} \\ +(V_p{}^1)_{q1}(D^{m,\,1})_1+(V_p{}^1)_{q2}&(D^{m,\,1})_2+(V_p{}^1)_{q3}(D^{m,\,1})_3+(V_p{}^1)_{q4}(D^{m,\,1})_4 \\ &= 0, \quad (q = 3, 4). \quad (5.3)\end{aligned}$$

From equations (3.18), (3.19), (4.1), (4.4) and (5.2), it can be seen that in the set (5.3) the first equation ($q = 3$) is a constant multiple of the second equation ($q = 4$), i.e. the two equations in the set (3.26) are identical.

Equation (3.48) for $n = 1$ reads

$$({}_LE_p{}^1)_{22}\,C_{2;\,1}^{m;\,1}+({}_LF_p{}^{m,\,1})_2 = 0. \quad (5.4)$$

From equations (3.36), (3.46) and (5.1), we have

$$({}_LE_p{}^1)_{22} = 0. \quad (5.5)$$

Similarly, equations (3.45), (3.46), (4.1), (4.4) and (5.1) yield

$$({}_LF_p{}^{m,\,1})_2 = 0. \quad (5.6)$$

Thus equation (5.4) is satisfied for all $C_{2;\,1}^{m;\,1}$. Hence the case $n = 1$ is singular (Ben-Menahem & Singh 1968, p. 439; Caputo 1961, p. 1480). The reason is that when $n = 1$, a part of the displacement field for the L-problem represents a rigid rotation and a part of the displacement field for the R-problem represents a rigid translation. Consequently, the corresponding stresses are zero and we cannot find the constants $A_{2;\,1}^{m;\,1}$, $B_{2;\,1}^{m;\,1}$ and $C_{2;\,1}^{m;\,1}$ from the usual boundary conditions. Therefore, to find the solution corresponding to $n = 1$, i.e. to determine the constants $A_{2;\,1}^{m;\,1}$, $B_{2;\,1}^{m;\,1}$ and $C_{2;\,1}^{m;\,1}$, we invoke the principles of the conservation of the angular momentum and the mass-centre (Ben-Menahem & Singh 1968, p. 443), i.e. we apply the following two conditions:

(i) the angular momentum of the sphere about its centre remains zero;

(ii) the centre of mass of the sphere is not displaced.

These two conditions imply that

$$\iiint r\,\mathbf{e}_r\times\mathbf{u}\,d\tau = 0, \quad (5.7)$$

and

$$\iiint \mathbf{u}\,d\tau = 0, \quad (5.8)$$

respectively. Here $d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$ and the integration is over the volume bounded by the sphere.

The only part of the displacement that gives any angular momentum for a sphere is the **M** vector with $n = 1$ (Jeffreys 1970, p. 474, para. 3). Similarly, only **N** and **F** vectors at $n = 1$ give non-zero contributions to the integral in equation (5.8). Thus, to find the values of the integrals in equations (5.7) and (5.8) we need consider only the displacement vector at $n = 1$.

The displacement vector at any point in the ith shell corresponding to $n = 1$ is given by

$$\mathbf{u}_i^{m,1}(r) = x_i^{m,1}(r)\,\mathbf{P}_{m,1} + y_i^{m,1}(r)\sqrt{2}\mathbf{B}_{m,1} + z_i^{m,1}(r)\sqrt{2}\mathbf{C}_{m,1}, \tag{5.9}$$

where the radial functions $x_i^{m,1}(r)$, $y_i^{m,1}(r)$ and $z_i^{m,1}(r)$, can be found from equations (3.29) and (3.50).

Equations (3.29), (3.30), (3.50), (3.51), (5.7) to (5.9) yield, after a lengthy but straightforward analysis,

$$Q_4 C_{2;1}^{m;1} + Q_5 = 0, \tag{5.10}$$

$$Q_6 A_{2;1}^{m;1} + Q_7 B_{2;1}^{m;1} + Q_8 = 0, \tag{5.11}$$

where

$$\left.\begin{aligned} Q_4 &= \frac{r_1^5}{5} + \sum_{i=2}^{p}\left[({}_Le_i)_{12}\left(\frac{r_i^2 - r_{i-1}^2}{2}\right) + ({}_Le_i)_{22}\left(\frac{r_i^5 - r_{i-1}^5}{5}\right)\right],\\ Q_5 &= \sum_{i=s_2}^{p}\left[({}_L\Psi_i^m)_1\left(\frac{r_i^2 - r_{i-1}^2}{2}\right) + ({}_L\Psi_i^m)_2\left(\frac{r_i^5 - r_{i-1}^5}{5}\right)\right], \end{aligned}\right\} \tag{5.12}$$

$$\left.\begin{aligned} Q_6 &= r_1^3 + \sum_{i=2}^{p}\left[(e_i)_{22}(r_i^3 - r_{i-1}^3) + \frac{2\lambda_i + 5\mu_i}{2(\lambda_i + 2\mu_i)}(e_i)_{32}(r_i^2 - r_{i-1}^2)\right.\\ &\qquad \left. + \frac{\lambda_i + \mu_i}{\lambda_i + 2\mu_i}(e_i)_{42}(r_i^5 - r_{i-1}^5)\right],\\ Q_7 &= \frac{(\lambda_1 + \mu_1) r_1^5}{\lambda_1 + 2\mu_1} + \sum_{i=2}^{p}\left[(e_i)_{24}(r_i^3 - r_{i-1}^3) + \frac{2\lambda_i + 5\mu_i}{2(\lambda_i + 2\mu_i)}(e_i)_{34}\right.\\ &\qquad \left. \times (r_i^2 - r_{i-1}^2) + \frac{\lambda_i + \mu_i}{\lambda_i + 2\mu_i}(e_i)_{44}(r_i^5 - r_{i-1}^5)\right],\\ Q_8 &= \sum_{i=s_2}^{p}\left[(\Psi_i^m)_2(r_i^3 - r_{i-1}^3) + \frac{2\lambda_i + 5\mu_i}{2(\lambda_i + 2\mu_i)}(\Psi_i^m)_3(r_i^2 - r_{i-1}^2)\right.\\ &\qquad \left. + \frac{\lambda_i + \mu_i}{\lambda_i + 2\mu_i}(\Psi_i^m)_4(r_i^5 - r_{i-1}^5)\right]. \end{aligned}\right\} \tag{5.13}$$

with

$$\left.\begin{aligned} e_i &= [M_i^1(r_{i-1})]^{-1} E_{i-1}^1,\\ {}_Le_i &= [{}_LM_i^1(r_{i-1})]^{-1}\, {}_LE_{i=1}^1,\\ \Psi_i^m &= [M_i^1(r_{i-1})]^{-1}\, \Psi_{i-1}^{m;1},\\ {}_L\Psi_i^m &= [{}_LM_i^1(r_{i-1})]^{-1}\, {}_L\Psi_{i-1}^{m;1}. \end{aligned}\right\} \tag{5.14}$$

The matrices on the R.H.S. of the above equation are obtained from the respective parent matrices by putting $n = 1$.

From equations (3.26), (5.10) and (5.11), we get

$$\left.\begin{aligned} A_{2;\,1}^{m;\,1} &= [Q_7(F_p^{m,\,1})_3 - Q_8(E_p^{\,1})_{34}]/\Delta_1, \\ B_{2;\,1}^{m;\,1} &= [Q_8(E_p^{\,1})_{32} - Q_6(F_p^{m,\,1})_3]/\Delta_1, \end{aligned}\right\} \tag{5.15}$$

$$C_{2;\,1}^{m;\,1} = -Q_5/Q_4, \tag{5.16}$$

where

$$\Delta_1 = Q_6(E_p^{\,1})_{34} - Q_7(E_p^{\,1})_{32}. \tag{5.17}$$

Putting the above values of $A_{2;\,1}^{m;\,1}$, $B_{2;\,1}^{m;\,1}$ and $C_{2;\,1}^{m;\,1}$ in equations (3.30), (3.51), we obtain

$$\begin{aligned} \big(J_i^{m,\,1}(r_{i-1})\big)_q = \{&[Q_7(F_p^{m,\,1})_3 - Q_8(E_p^{\,1})_{34}](E_{i-1}^1)_{q2} \\ &+ [Q_8(E_p^{\,1})_{32} - Q_6(F_p^{m,\,1})_3](E_{i-1}^1)_{q4} \\ &+ (\Psi_{i-1}^{m;\,1})_q\,\Delta_1\}/\Delta_1, \quad (q = 1, 2, 3, 4) \end{aligned} \tag{5.18}$$

$$\big({}_LJ_i^{m,\,1}(r_{i-1})\big)_q = [Q_4({}_L\Psi_{i-1}^{m;\,1})_q - Q_5({}_LE_{i-1}^1)_{q2}]/Q_4, \quad (q = 1, 2). \tag{5.19}$$

6. Discussion

Recently, observational evidence has been presented in support of the hypothesis that earthquakes may excite the Chandler wobble and produce the observed polar shift. Mansinha & Smylie (1967) developed a half-space theory while Ben-Menahem & Israel (1970) applied spherical theory taking Earth as a homogeneous sphere, to investigate theoretically the effect of earthquakes on the rotation of the Earth. However, for more accurate determination of the extent to which earthquakes are able to maintain Chandler wobble, additional observations and theoretical study of more realistic Earth models is needed. In the present paper, a multilayered spherical Earth model is taken and the displacement field at any point in the medium induced by a displacement dislocation or a centre of compression is obtained. The results obtained in the present investigation are being used to calculate the changes in the Inertia Tensor, due to a displacement dislocation and a centre of explosion in a multilayered sphere, and its effect on the Chandler wobble. The paper will be submitted for publication to the Geophysical Journal in due course.

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