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Static, Isotropic Spacetime in New General Relativity

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New general relativity predicts a static, isotropic spacetime quite different from the Schwarzschild spacetime of general relativity, unless one of the two unknown parameters, ε, is exactly zero. In this paper we study its structure as a Riemannian spacetime, paying special attentions to motions of test particles therein.

Recently Kawai and Toma[1] have discussed singularities of the static, isotropic spacetime in new general relativity, a gravitational theory based on absolute (or tele-) parallelism.[2),*)] Here we study structures of the spacetime as probed by spin-1/2 particles and photons.

The static, isotropic spacetime of new general relativity is described by the parallel vector fields $\boldsymbol{b}=\{b_k{}^\mu(x)\}$ with

$$b_{(0)}{}^0=1/\sqrt{A(r)}\,,\qquad b_{(0)}{}^\alpha=b_{(a)}{}^0=0\,,$$

$$b_{(a)}{}^\alpha=\delta_a{}^\alpha/\sqrt{B(r)} \tag{1}$$

with indices α and a running over 1~3, where $A(r)$ and $B(r)$ are

$$A(r)=(1-GM/ar)^a(1+GM/br)^{-b}\,, \tag{2a}$$

$$B(r)=(1-GM/ar)^{2-a}(1+GM/br)^{2+b}\,. \tag{2b}$$

Here r is the radial coordinate and M denotes the mass of the central gravitating body. The two constants, a and b, are defined by

$$a=2\{\sqrt{(1-\varepsilon)(1-4\varepsilon)}-2\varepsilon\}/(1-5\varepsilon)=2+\varepsilon+0(\varepsilon^2)\,, \tag{3a}$$

$$b=2\{\sqrt{(1-\varepsilon)(1-4\varepsilon)}+2\varepsilon\}/(1-5\varepsilon)=2+9\varepsilon+0(\varepsilon^2) \tag{3b}$$

in terms of ε, which is one of the two unknown parameters (denoted by ε and c_3) of the theory.[2),**)] Comparison with solar-system experiments severely restricts the possible value of ε:

$$\varepsilon=-0.004\pm0.004\,. \tag{4}$$

So we shall assume henceforth that $|\varepsilon|\ll1$ even if it is nonvanishing.

The invariant distance ds^2 is expressed by

*) We use the same notations and conventions as in Ref. 2). The only exception is that the constants, a and b, of Eqs. (3a) and (3b) are denoted therein by p and q, respectively.

**) The parameter c_3 does not appear in isotropic spacetimes.

$$ds^2 = b^k{}_\mu b_{k\nu} dx^\mu dx^\nu$$
$$= -A(r)dt^2 + B(r)\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\}. \tag{5}$$

So we are using the isotropic coordinates. If ε is vanishing, then Eqs. (3a) and (3b) give $a=b=2$, and therefore the metric reduces to the Schwarzschild metric of general relativity written in the isotropic coordinates.

In a spherically symmetric spacetime with parallel vector fields of diagonal form like (1), the axial-vector part of the torsion tensor identically vanishes, and so the Dirac equation is locally Lorentz invariant: Namely, the Dirac equation does not change its form under local Lorentz transformations of parallel vector fields,

$$b_k{}^\mu(x) \to b'_k{}^\mu(x) = A_k{}^j(x) b_j{}^\mu(x)$$

accompanied with the transformation of Dirac spinor fields, $\psi(x) \to \psi'(x) = U(A)\psi(x)$. The transformed fields $b'_k{}^\mu$ are not parallel vector fields in general: Nevertheless, these fields do form a tetrad of the metric, and are sufficient for describing spin-1/2 particles and fields.

The Maxwell equation of new general relativity is of the same form as in general relativity, being completely described by the metric tensor alone. Therefore, as long as we explore spacetime structure by using spin-1/2 particles and photons as probes, the parallel vector fields cannot be determined uniquely, and ambiguity of making local Lorentz transformations is left unremoved.

These considerations imply that for an isotropic spacetime the underlying Weizenboeckian structure, on which new general relativity is based, is not observable by studying motions of test particles. Instead, it is the Riemannian structure that is explored by test particles: So we shall next investigate properties of the static, isotropic spacetime, regarding it as a Riemannian spacetime.

If the parameter ε is exactly zero, the metric (5) is the Schwarzschild metric as was noted earlier, and its structure is quite well-known.[3] In particular, when it is written in the isotropic coordinates, its form is invariant under the transformation, $r \to \bar{r} = (GM/2)^2/r$, and it covers only those regions of spacetime which are denoted by I and III in the Kruskal coordinates. These two regions represent two asymptotically flat spacetimes outside the event horizon located at $r = GM/2$.

If the parameter ε takes a small but non-zero value, $1 \gg |\varepsilon| \neq 0$, the constants, a and b, are both very close but unequal to 2: If $\varepsilon > 0$ (or $\varepsilon < 0$), then $a > 2$ and $b > 2$ (or $a < 2$ and $b < 2$) as is seen from Eqs. (3a) and (3b). Thus, since a (and hence $2-a$) are not integer, the metric functions, $A(r)$ and $B(r)$, become complex for $r < GM/a$. This represents that the spacetime is physically meaningful only outside the sphere of radius $r = GM/a$.

The singularity property of the metric (5) for $\varepsilon \neq 0$ is seen from the behavior of the Riemann-Christoffel curvature tensor near $r = GM/a$. It has been shown that the scalar curvature $R(\{\})$ and the quadratic invariants, $R(\{\})_{\mu\nu\rho\sigma} R(\{\})^{\mu\nu\rho\sigma}$ and $R(\{\})_{\mu\nu} R(\{\})^{\mu\nu}$, become infinite there.[1] Thus, the spherical shell of $r = GM/a$ is a singularity of spacetime, if the parameter ε takes a nonzero value, no matter how small its value may be. As will be seen shortly, this is a point singularity when $\varepsilon < 0$.

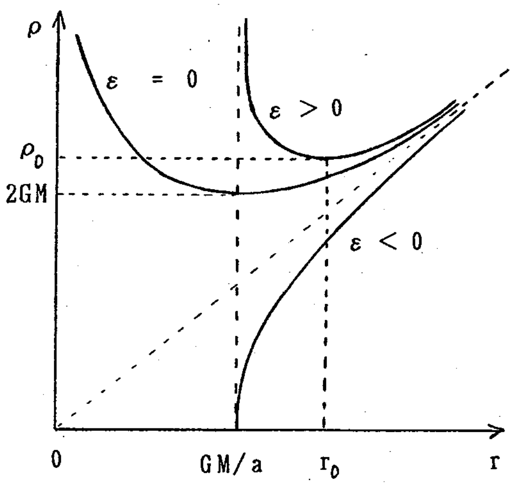

Fig. 1. The qualitative behaviors of ρ as a function of r.

In order to gain more insights about the metric (5), we shall rewrite it in the Schwarzschild coordinates:

$$ds^2 = -E(\rho)dt^2 + F(\rho)d\rho^2 + \rho^2\{d\theta^2 + \sin\theta^2 d\phi^2\}, \qquad (6)$$

where the radial coordinate ρ is defined by

$$\rho = r\sqrt{B(r)} = r(1-GM/ar)^{1-a/2} \times (1+GM/br)^{1+b/2}, \qquad (7)$$

and the metric functions $E(\rho)$ and $F(\rho)$ are given by

$$E(\rho) = A(r)$$

and

$$F(\rho) = B(r)(dr/d\rho)^2. \qquad (8)$$

The behavior of $\rho(r)$ is quite different depending on the value of ε. So we shall discuss the three cases, $\varepsilon=0$, $\varepsilon>0$ and $\varepsilon<0$, separately (see Fig. 1 for the qualitative behavior of ρ as a function of r):

(i) If $\varepsilon=0$ (i.e., $a=b=2$), ρ is well-behaved over the whole region, $0<r<\infty$, and it takes the minimum, $\rho=2GM$, at $r=GM/2$. The metric functions are given by $E(\rho)=(1-2GM/\rho)$ and $F(\rho)=1/(1-2GM/\rho)$, and therefore they are defined also for $0<\rho<2GM$, i.e., the region inside the event horizon.

(ii) If $1\gg\varepsilon>0$ (i.e., $a>2$ and $b>2$), then ρ is defined only over the region $r>GM/a$: For large r, ρ is nearly equal to r, and as r decreases it becomes minimum, $\rho_0=r_0\sqrt{B(r_0)}$, at $r=r_0\equiv(1/2)(1+\sqrt{\varepsilon})GM$, and then increases to infinity as r approaches GM/a. In contrast with the Schwarzschild metric, however, the radial, Schwarzschild coordinate cannot be extended to $\rho<\rho_0$, since both the metric functions $E(\rho)$ and $F(\rho)$ have complex, non-real values there. The region $\rho>\rho_0$ corresponding to $r>r_0$ represents an asymptotically flat spacetime: By contrast, another region $\rho>\rho_0$ for $r_0>r>GM/a$ is not asymptotically flat and instead, the boundary with infinitely large area at $\rho=\infty$ (i.e., at $r=GM/a$) is a singularity of spacetime, as has been noted above.

(iii) If $\varepsilon<0$ and $|\varepsilon|\ll1$ (i.e., $a<2$ and $b<2$), then ρ is zero for $r=GM/a$, and monotonically increases as r becomes larger. The sphere with radius $r=GM/a$ is therefore the point singularity located at the center.

The radial distance from the singularity at $r=GM/a$ to any other point r is finite for any value of $\varepsilon\neq0$, however: In fact, for any r larger than GM/a, we have

$$\int_{GM/a}^{r}\sqrt{B(r)}\,dr = \int_{GM/a}^{r}(1-GM/ar)^{1-a/2}(1+GM/br)^{1+b/2}dr < \infty \qquad (9)$$

since the integral converges at both ends.

Further information about the nature of the singularity can be obtained by considering a radially traveling photon in the metric (5). The world line is determined by the condition $ds^2=0$, and so we have in the isotropic coordinates

$$dt/dr = \pm\sqrt{B(r)/A(r)} = \pm(1-GM/ar)^{1-a}(1+GM/br)^{1+b}\,, \tag{10}$$

where the plus and the minus signs are for outgoing and ingoing photons, respectively. For a photon starting from r_1 at $t=0$, the coordinate time $t(r_1, r)$ for arriving at r is given by

$$t(r_1, r) = \pm\int_{r_1}^{r}(1-GM/ar)^{1-a}(1+GM/br)^{1+b}dr\,. \tag{11}$$

It follows from this expression that an outgoing photon emitted at an arbitrary point outside the sphere of radius $r=GM/a$ can escape to infinity. This represents that *the singularity at $r=GM/a$ for $\varepsilon\neq 0$ is naked*, not being covered with event horizon.

As an ingoing photon approaches the sphere $r=GM/a$, its coordinate time changes like

$$t(r_1, r) \underset{r\to GM/a}{\sim} \begin{cases} \text{a finite function of } r_1 & \text{for } \varepsilon<0\,, \\ -4GM\ln(1-GM/2r) & \text{for } \varepsilon=0\,, \\ \text{const}\times(1-GM/ar)^{2-a} & \text{for } \varepsilon>0\,. \end{cases} \tag{12}$$

If $\varepsilon<0$, the photon collides with the singularity in a finite coordinate time. When $\varepsilon>0$, on the other hand, it will take infinite coordinate time for a photon to collide with the singularity.

The physical time interval $\tau(r_1, r)$ for a photon to travel from r_1 to r, however, is not equal to the coordinate time interval itself, but should be defined by the following manner. Imagine a chain of fixed observers between r_1 and r, and suppose that each observer measures the time interval $d\tau$ for the photon to go to the next one. Adding up all these time intervals gives the physical time interval $\tau(r_1, r)$:

$$\begin{aligned}\tau(r_1, r) &= \int_{r_1}^{r} d\tau = \int_{r_1}^{r}(d\tau/dt)(dt/dr)dr \\ &= \pm\int_{r_1}^{r}(1-GM/ar)^{1-a/2}(1+GM/br)^{1+b/2}dr\,,\end{aligned} \tag{13}$$

where the plus and minus signs correspond to outgoing and ingoing photons, respectively, and $d\tau=\sqrt{A(r)}(dt/dr)dr$ means the time interval for the photon to go from r to $r+dr$, as measured by the observer fixed at r. Since we are assuming that $|\varepsilon|\ll 1$, the value of a is very close to 2, and the integral converges at $r=GM/a$. Thus, an ingoing photon does indeed collide with the singularity at $r=GM/a$ in a finite physical time interval.

The radial motion of a photon can be described most simply by using the advanced and retarded time parameters,

$$v = t + r^*\,,\quad w = t - r^*\,, \tag{14}$$

where r^* is defined by

$$r^* = \int_{r_2}^{r}\sqrt{B(r)/A(r)}dr = \int_{r_2}^{r}(1-GM/ar)^{1-a}(1+GM/br)^{1+b}dr \tag{15}$$

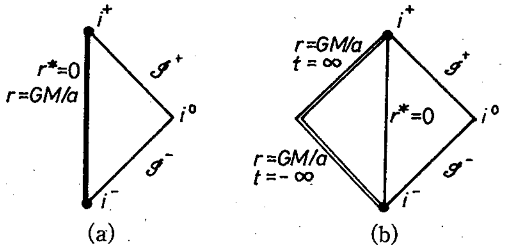

Fig. 2. The Penrose diagrams; (a) for $\varepsilon<0$ and (b) for $\varepsilon>0$.

with r_2 being a constant. If $\varepsilon<0$, we choose as $r_2=GM/a$, since the integral of (15) converges in this case. When $\varepsilon>0$, we choose as $r_2=r_0\{=(1/2)(1+\sqrt{\varepsilon})GM>GM/a\}$. The range of the variable r^* is then

$$\infty>r^*\geqq 0 \qquad \text{for} \quad \varepsilon<0,$$
$$\infty>r^*>-\infty \qquad \text{for} \quad \varepsilon>0. \qquad (16)$$

The singularity is located at $r^*=0$ for $\varepsilon<0$ and at $r^*=-\infty$ for $\varepsilon>0$, respectively. An ingoing (outgoing) photon travels along v=const (w=const) in the v-w plane.

Now introduce the new coordinates p and q by

$$v=\tan p, \qquad (\pi/2>p>-\pi/2)$$
$$w=\tan q. \qquad (\pi/2>q>-\pi/2) \qquad (17)$$

If $\varepsilon<0$, the range of p and q is further restricted by $p-q\geqq 0$, because $v-w=2r^*\geqq 0$ in this case.

We can then construct the Penrose diagram for the spacetime with the metric (5).[4] It describes causal structure of the spacetime in a simple manner (see Fig. 2). In this diagram null geodesics are at $\pm 45^\circ$ to the vertical axis, $r^*=0$. If $\varepsilon<0$, the singularity at $r=GM/a$ is timelike: The Penrose diagram is similar to the one for the Reissner-Nordstroem spacetime with $e^2>m^2$. Due to the timelike character of the singularity, there are no Cauchy surfaces. If $\varepsilon>0$, the singularity is null: There are Cauchy surfaces, over which all the past-directed timelike or null lines cross before hitting the singularity. In both cases with $\varepsilon\neq 0$, unlike in the Schwarzschild spacetime, timelike and null curves can always escape to infinity.

In summary, the static, isotropic spacetime predicted by new general relativity is quite different from the Schwarzschild spacetime of general relativity, unless the parameter ε is exactly zero. In particular, the spacetime has a naked singularity not covered with event horizon. When $\varepsilon>0$, the singular region is a spherical shell with infinitely large area. If $\varepsilon<0$, on the other hand, the singularity is point-like and is located at the center.

Finally, we mention an unsolved problem of proving the Birkhoff theorem in new general relativity. This will be indispensable when one applies new general relativity to gravitational collapse of stellar matters.

1) T. Kawai and N. Toma, Prog. Theor. Phys. **83** (1990), 1.
2) K. Hayashi and T. Shirafuji, Phys. Rev. **D19** (1979), 3524. Earlier references are quoted therein.
3) See, for example, C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), chap. 31.
4) For the Penrose diagram, see, for example, S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, Cambridge, 1973), chap. 5.