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Static Turning Analysis of Vehicles Subject to Externally Applied Forces – A Moment Arm Ratio Formulation

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SUMMARY

A formulation for representing the static turning response of a two-axle vehicle due to applied external or control forces is expressed in terms of a simple ratio of two distances along the vehicle longitudinal axis. The two distances coincide with points on the vehicle at which externally applied/control forces and their reactive inertial forces act with respect to the vehicle neutral steer point. The resulting formulation is equivalent to the rotational equilibrium equation written with respect to the neutral steer point. The method allows a simple "visual analysis" of the steady turning process by showing how key forces and associated moment arms can change with respect to one another due to vehicle modifications or different operating conditions, thereby affecting the static turning response of the vehicle.

1. INTRODUCTION

The intent of this paper is to present a simplified formulation of the static turning response of a two-axle, pneumatic-tired vehicle subject to arbitrary lateral external forces. The laterally applied forces may have a variety of sources, such as, the intentional control force introduced from a steered wheel, or, an external disturbance force produced by the aerodynamic shape of the vehicle's body. The basic approach separates forces acting on the vehicle mass into two categories. The first category of forces is termed here ***control*** forces and comprises any external force associated with steering or disturbing the vehicle. The second category of forces is termed ***motion*** forces and comprises only those tire force components associated with sideslip and yawing motion of the vehicle – absent of any steering. Inertial forces acting on the vehicle mass are also included in this latter category. The conventional analysis that follows helps to identify these force terms. Linear relationships are assumed so that forces may be superimposed. The net result is that the static turning motion of a vehicle can be described simply in terms of a ratio of two distances along the vehicle longitudinal axis. The two distances are those associated with points on the vehicle axis at which the control force(s) and motion forces act with respect to the neutral steer point and mass center of the vehicle.

Consequently, knowledge of how the neutral steer point, the mass center, or location of the control forces may change due to modifications to a vehicle, allows a simple visualization of how the static turning response of the vehicle

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will likewise be affected. The development and discussion which follows will arrive at a "formula" which encapsulates this concept. Three examples of applying its use to different vehicle control problems completes the paper.

2. MATHEMATICAL ANALYSIS

The linear equations governing the static turning motion of a two-axle vehicle equipped with pneumatic tires are well known [1, 2, 3, 4, 5, 6]. However, for purposes of notation and subsequent referral, the basic lateral force and yaw moment equilibrium equations written with respect to the vehicle mass center are presented in equations (1) and (2) respectively, using Figure I as a reference diagram.

$$F_{yf} + F_{yr} + F_c - mUr = 0 \tag{1}$$

$$F_c d + F_{yf} a - F_{yr} b = 0 \tag{2}$$

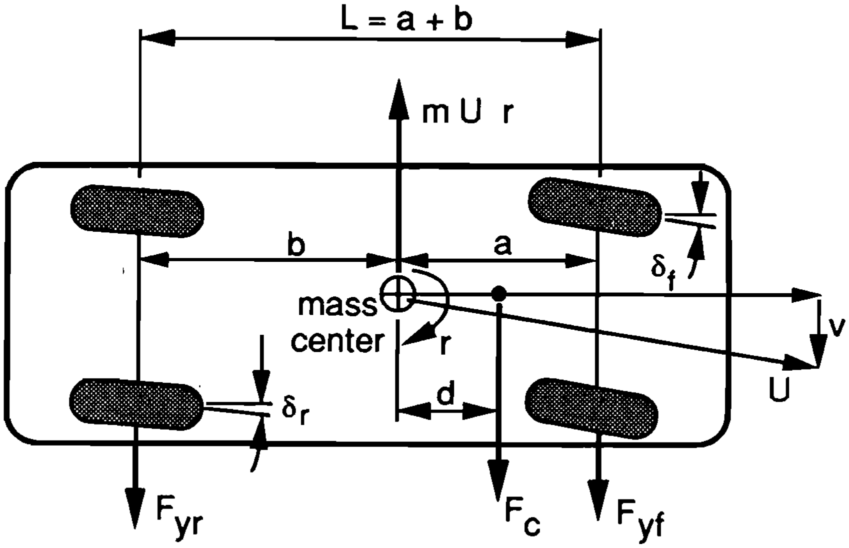


Figure I. Diagram of Forces Acting on Vehicle.

where,

- F_{yf} is the total lateral tire force at the front axle
- F_{yr} is the total lateral tire force at the rear axle
- F_c is an arbitrary lateral force (aerodynamic or otherwise)
- m is the vehicle mass
- U is the vehicle speed of travel

r is the vehicle yaw rate
d is a distance forward of the mass center which locates the applied lateral force F_c
a is the distance of the front axle forward of the mass center
b is the distance of the rear axle behind the mass center
L is the vehicle wheelbase = a + b

The linearized lateral tire forces are given by,

$$F_{yf} = -C_{af}[(v + a\,r)/U - \delta_f] \tag{3}$$

$$F_{yr} = -C_{ar}[(v - b\,r)/U - \delta_r] \tag{4}$$

where,

C_{af} is the front tire cornering stiffness (both tires)
C_{ar} is the rear tire cornering stiffness (both tires)
v is the vehicle sideslip velocity at the mass center
r is the vehicle yaw rate
δ_f is the steer angle of the front tires

and,

δ_r is the steer angle of the rear tires

Substituting equations (3)-(4) into equations (1)-(2) and rearranging, results in an equivalent set of two simultaneous equations for lateral force and yaw equilibrium involving sideslip velocity, v, and yaw rate, r:

$$F_c + C_{af}\delta_f + C_{ar}\delta_r = [(C_{af} + C_{ar})/U]v + [(C_{af}a - C_{ar}b)/U + mU]r \tag{5}$$

$$dF_c + aC_{af}\delta_f - bC_{ar}\delta_r = [(aC_{af} - bC_{ar})/U]v + [(C_{af}a^2 + C_{ar}b^2)/U]r \tag{6}$$

The force terms appearing on the left hand side of equation (5) are referred to here as *control* forces and include the arbitrary applied force F_c as well as those components of the total tire forces resulting from steering of the wheels. The force terms appearing on the right hand side of equation (5) are referred to here as *motion* forces and include the inertial force acting on the mass center as well as those components of the tire forces attributable only to sideslip and yaw motion (tire forces of the non-steered wheel).

By combining the left hand side control forces into one total force, F_T, acting at a distance, e, ahead of the mass center, the equations (5)-(6) can be simplified to:

$$F_T = [(C_{af} + C_{ar})/U]v + [(C_{af}a - C_{ar}b)/U + mU]r \tag{7}$$

$$eF_T = [(aC_{af} - bC_{ar})/U]v + [(C_{af}a^2 + C_{ar}b^2)/U]r \tag{8}$$

where, F_T and e are given by,

$$F_T = F_c + C_{af}\delta_f + C_{ar}\delta_r \tag{9}$$

and,

$$e = (dF_c + aC_{af}\delta_f - bC_{ar}\delta_r)/(F_c + C_{af}\delta_f + C_{ar}\delta_r) \tag{10}$$

Solution of equations (7)-(8) results in the following expression for the ratio of yaw rate to the total control force, F_T:

$$r/F_T = [(bC_{ar} - aC_{af}) + e(C_{af} + C_{ar})]/ [(a + b)^2 C_{af}C_{ar}/U - (aC_{af} - bC_{ar})\,mU] \tag{11}$$

By noting that the distance, c, from the neutral steer point forward to the mass center is given by the expression,

$$c = (bC_{ar} - aC_{af})/(C_{af} + C_{ar}) \tag{12}$$

and following its substitution into equation (11) and rearrangement of terms, the following simple expression is obtained which relates the vehicle yaw rate to the total control force:

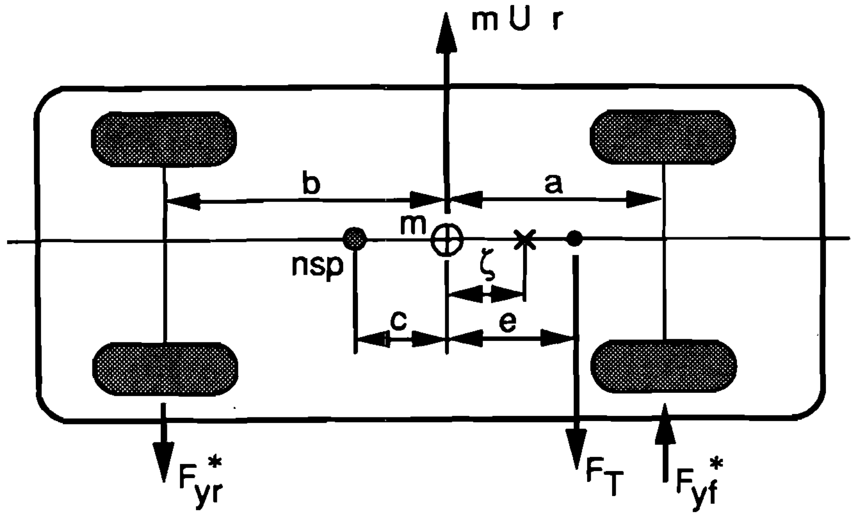
$$\boxed{r/F_T = [mU]^{-1}(c + e)/(c + \zeta)} \tag{13a}$$

or, in terms of the steady-state lateral acceleration, $a_y = Ur$,

$$\boxed{a_y/F_T = m^{-1}(c + e)/(c + \zeta)} \tag{13b}$$

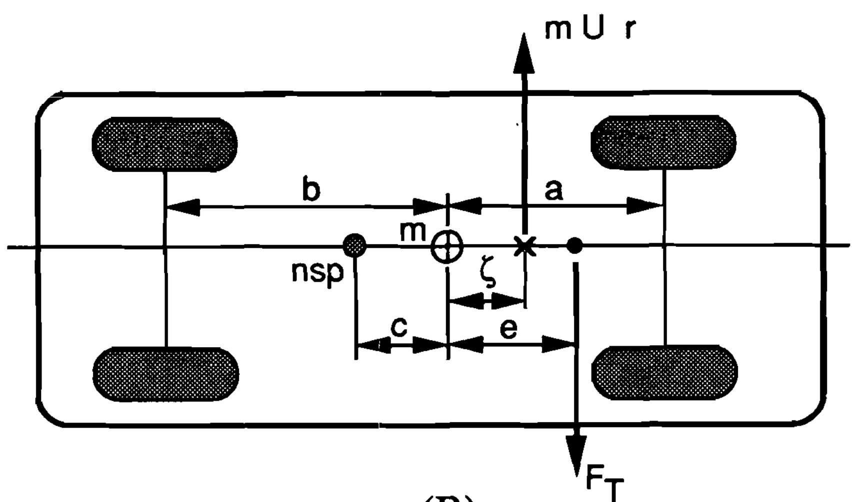
Figure 2 supplements equation (13) by illustrating the quantities involved. Equation (13) can be viewed as the yaw moment equilibrium equation written with respect to the neutral steer point. Since moment terms involving front and rear tire force components that depend upon sideslip velocity cancel one another about the neutral steer point (by definition), no coupling terms involving sideslip velocity at the neutral steer point appear in this form of the moment equilibrium equation. Accordingly, only those forces that affect the rotational equilibrium of the vehicle about the neutral steer point appear in Figure 2.

In diagram (A) of Figure 2, the distance (c +e) is the moment arm of the applied lateral control force about the neutral steer point. The distance c is the moment arm of the inertial force, ***mUr***, acting about the neutral steer point at the mass center. The distance ζ is equal to the resisting moment of the equal and opposite <u>non-steered</u> tire force components, F^*_{yf} and F^*_{yr}, generated by a pure rotation of the vehicle about the neutral steer point, normalized by the inertial force ***mUr*** (i.e., ζ = tire yaw damping moment about ***nsp/mUr***). Diagram (A) can be simplified by not including the tire damping forces, F^*_{yf} and F^*_{yr}, provided that the inertial force, ***mUr***, is translated forward to the point located by ζ,



(A)

or, equivalently:



(B)

Figure 2. Diagram of Forces and Moment Arms. (A) Intertial Force Acting at the Mass Center; (B) Intertial Force Located at Distance ζ Ahead of the Mass Center.

thereby retaining the tire yaw damping moment, $\zeta \cdot mUr$. See diagram (B) of Figure 2.

Under this latter formulation, if the vehicle is viewed as a simple lever, having a fulcrum or pivot point located at the neutral steer point (nsp), the steady turning process is analogous to a basic mechanical statics problem. The applied control force, F_T, acting at a distance (c + e), produces a force of reaction, mUr, acting at a counter-balancing moment arm distance $(c + \zeta)$. Subsequent examples will utilize diagram (B) of Figure 2 to illustrate the moment arm formulation in several applications.

The moment arms c and e depend upon the tire stiffnesses, mass distribution, and relative magnitudes and location of any steered wheel or additional external forces. The moment arm, ζ, is similarly dependent, but is also a function of the speed of the vehicle and is given by the expression,

$$\zeta = (a + b)^2 C_{af} C_{ar} / [(C_{af} + C_{ar}) m U^2] \tag{14}$$

As seen, the moment arm, ζ, which is proportional to the tire yaw damping moment, varies as $1/U^2$ — changing in magnitude from infinity to zero with increasing vehicle speed. Consequently, changes in vehicle speed strongly influence the moment arm, ζ, but have no effect on the moment arms c or e. For oversteer vehicles ($c < 0$ and the nsp lies ahead of the mass center), the classical condition for directional stability [1, 3] occurs in the diagram of Figure 2 when the moment arm ζ equals the moment arm-c: The forward speed at which this occurs is of course the critical speed, $U_{cr} = (-L/K)^{1/2}$, (where K is the understeer gradient) obtained when $\zeta = -c$. In contrast, for understeer vehicles ($c > 0$ and the nsp lies behind the mass center), the characteristic speed [6], $U_{ch} = (L/K)^{1/2}$, occurs when $\zeta = c$.

3. APPLICATIONS

Diagram (B) of Figure 2 and equation (13) can be used to visualize and predict changes in the turning response of vehicles when modifications (planned or otherwise) are introduced that cause the moment arms c, e, and ζ to become altered. The examples that follow help to demonstrate the graphical utility of the moment arm formulation for explaining different vehicle turning responses deriving from changes in vehicle understeer, speed, or applied forces.

Example *1:* Aerodynamic Crosswind Response with Fixed Steering.

This example addresses the static turning response of a vehicle subject to a constant crosswind and associated lateral aerodynamic force, F_a. The aerodynamic force, F_a, acts at the center of pressure located a distance, d, ahead of the center of mass. No steering is assumed to occur. See Figure 3. (Vehicle dynamicists and aerodynamicists have used this type of test as one measure of a

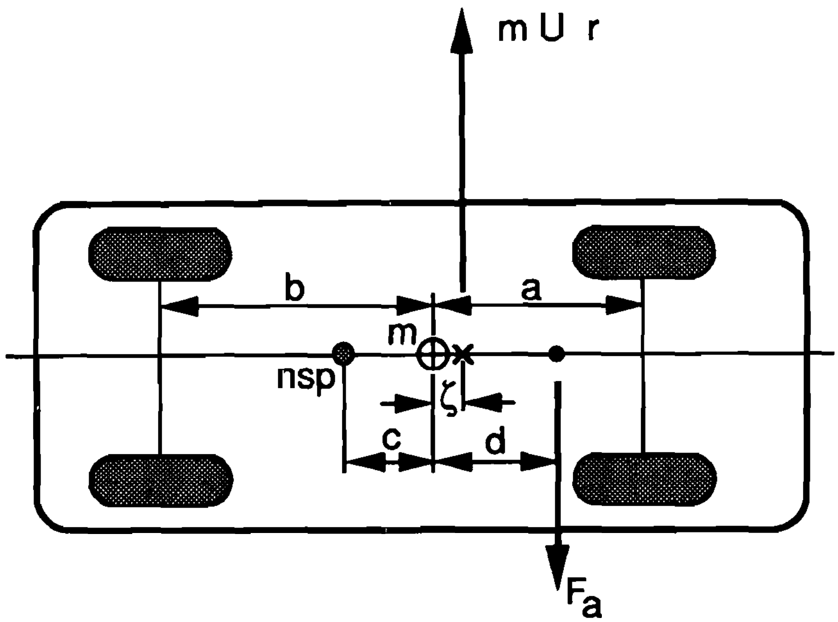


Figure 3. High speed — Aerodynamic Force Example.

vehicle's passive crosswind sensitivity [7, 8].) Utilizing equations (9) and (10), the total lateral control force, F_T, in this case becomes $F_T = F_a$, since 6, = $\delta_f = 0$. Likewise, the moment arm distance, e, in equation (10) is simply the distance from the mass center to the center of pressure (i.e., $e = d$). Equation (13a), relating the static yaw rate response to the applied aerodynamic force, then becomes, following these substitutions,

$$r/F_a = [mU]^{-1}(c+d)/(c+\zeta) \tag{15}$$

At very high speeds, where aerodynamic forces have a sizeable influence, the moment arm ζ is normally quite small ($\zeta < L/10$, where $L = a + b$, the vehicle wheelbase). For most understeer vehicles, the moment arm c ("static margin"$\bullet L$) will be in the range $L/10$ to $L/5$. For typical passenger cars, the distance to the center of pressure will be in the vicinity of $L/6$ to $L/3$. This then implies a diagram like that shown in Figure *3* with c, d, and ζ having the approximate relative lengths shown.

This diagram can be used to see how a baseline vehicle having these nominal characteristics, will respond to vehicle modifications that introduce a change in understeer. For example, if the rear tire cornering stiffness is reduced so as to introduce less understeer, the neutral steer point nsp would move forward, thereby shortening the moment arm c to a value c — A. Simultaneously, the moment arm, ζ, would be reduced slightly to a value $\zeta - \epsilon$. Equation (15) and

Figure 3 would then suggest that this influence would increase the baseline yaw rate since the ratio $(c - \Delta + d)/(c - \Delta + \zeta - \epsilon)$ would clearly increase (for $\zeta < d$).

In contrast to this, if the same baseline vehicle is operated in a similar or larger crosswind at slower speeds, two primary effects would occur. First, the aerodynamic slip angle would increase, producing a rearward shift in the center of pressure (due to typical non-linearities in passenger car aerodynamic properties) and because of the decreased speed, the moment arm ζ would increase. This would then result in a new diagram as shown in Figure 4 with moment arm, d, locating the center of aerodynamic pressure, with a length less than ζ. This new set of operating conditions results in a reduced passive wind sensitivity (for the same magnitude of crosswind force) because of: (1) the reduced moment arm d, and, (2) the increased tire damping moment arm ζ produced by the lower speed condition.

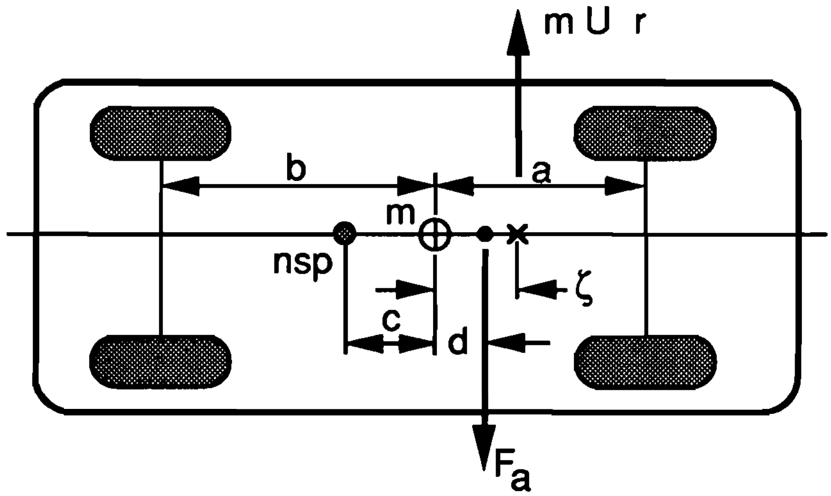


Figure 4. Lower Speed — Aerodynamic Force Example.

However, under these new baseline conditions, a decrease in understeer for the vehicle, similar to that noted above, can have an opposite effect and produce a <u>decrease</u> in the vehicle yaw rate. This is because the ratio of moment arms seen in equation (15) and Figure 4 is altered oppositely from that seen in Figure 3. This ratio, as before, is given by $(c - \Delta + d)/(c - \Delta + \zeta - \epsilon)$. Under conditions in which d is considerably less than ζ, the decrease in c to a value $c - \Delta$ will produce a reduced moment arm ratio, thereby resulting in a decreased yaw rate compared to its baseline response. (When d is in the vicinity of ζ the picture is less obvious because ϵ also reduces the denominator.)

Finally, for this latter example, even if the speed is not reduced, but the center of pressure is moved rearward through body aerodynamic styling changes alone, the passive wind sensitivity will still be reduced because of the shortened moment arm d. Obviously, as d moves rearward toward the neutral steer point, the passive wind sensitivity, as measured by the static yaw rate response to an aerodynamic side force, approaches zero (i.e., the moment arm d in equation (15) approaches a value of $-c$.

Example 2: Turning Response due to a Superelevated Road and Gravitational Side Force.

For non-steered vehicles subject to a gravitational side force from a laterally inclined road surface (assuming superelevation levels $\ll 1$), the application of equation (13) is similar to that discussed above for aerodynamic forces. However, in this case, the applied force is acting at the mass center and therefore the moment arm distance, e, in equation (13) and Figure 2 is now zero. Consequently, the static yaw rate response due to a constant gravitational side force, $F_g = W \bullet E$, is given by,

$$r/F_g = [mU]^{-1}(c)/(c+\zeta) \tag{16}$$

where, W is the weight of the vehicle and E is the superelevation of the road surface. Further, the ratio of centrifugal lateral acceleration (in g's) produced by an amount of road superelevation, E, is given by the simple ratio, $c/(c+\zeta)$. Thus, at low speeds where ζ is large, the resulting centrifugal acceleration induced by a road surface cross-slope is nil. At very high speeds where ζ is very small, the induced centrifugal acceleration (in g's) approaches the amount of roadway superelevation, E. For a neutral steer vehicle ($c=0$), no centrifugal acceleration or path curvature response is induced from the gravitational side force.

Example 3: Control Forces from Two-Wheel and Four-Wheel Steering Vehicles.

Control forces attributable to steered wheels can also be introduced and analyzed in a similar manner using this formulation. The control force terms $C_{\alpha f}\delta_f$ and $C_{\alpha r}\delta_r$, appearing on the left hand side of equations (5) and (6), are a result of steering the front and/or rear wheels. In the absence of any external aerodynamic or gravitational forces, the total control force, F_T, given in equation (9) for the front-wheel steering case, is seen to be,

$$F_T = F_{2ws} = C_{\alpha f}\delta_f \tag{17}$$

and the fixed location of this force, e, given by equation (10), becomes after the aforementioned substitutions:

$$e = a \tag{18}$$

These equations simply state that for the conventional front-wheel-only steered vehicle, the control force, $C_{\alpha f}\delta_f$, acts at the obvious front axle location and is located at the distance, a, forward of the mass center. See Figure 5.

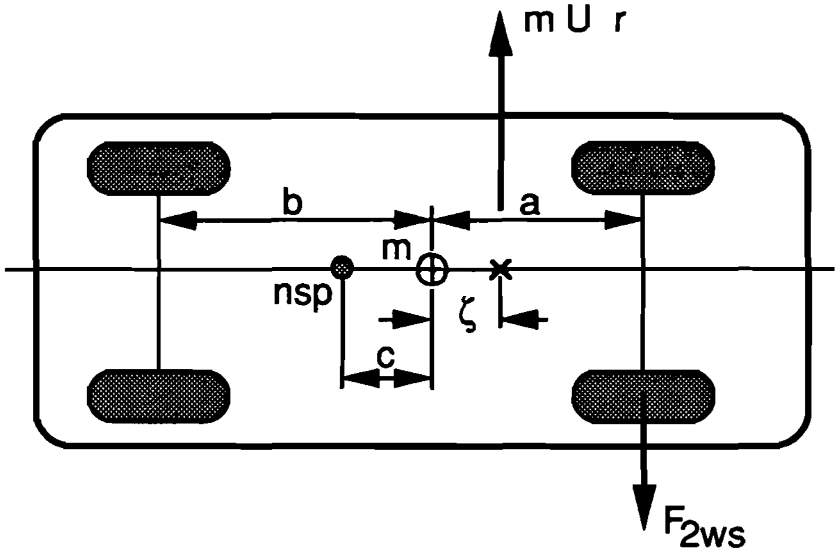


Figure 5. Conventional Front-Wheel Steer Vehicle.

The relationship between yaw rate and total lateral control force given by the basic relationship in equation (13), then becomes for the front-wheel steer vehicle:

$$r/F_{2ws} = [mU]^{-1}(c+a)/(c+\zeta) \tag{19}$$

If this is expressed in terms of the front wheel steer angle, the yaw rate to steer angle gain is given by,

$$r/\delta_f = C_{\alpha f}(c+a)/[(c+\zeta)mU] \tag{20}$$

The same formulation can also be extended to four-wheel steering vehicles. Consider a vehicle having four-wheel steering defined by the simple relationship, $\delta_r = k\delta_f$, so that the rear wheels are steered in proportion to the front wheels by an amount k. If $k>0$, the wheels are steered in the same direction. Such control strategies are often exercised at mid-range and higher speeds of travel. Under such schemes in actual practice, k usually takes on a value of about 0.3 [9]. If no additional lateral forces, F_c, are present, the total control force given by equation (9) is seen to be:

$$F_T = F_{4ws} = C_{\alpha f}\delta_f + C_{\alpha r}\delta_r = (C_{\alpha f} + kC_{\alpha r})\delta_f \tag{21}$$

and the fixed location of this force, e, given by equation (10), becomes after the substitution of the above front/rear steering relationship:

$$e = (aC_{\alpha f} - kbC_{\alpha r})/(C_{\alpha f} + kC_{\alpha r}) \tag{22}$$

indicating that the steering control force has now moved considerably rearward from the front axle.

The relationship between yaw rate and total lateral control force, given by the basic relationship in equation (13), then becomes for the four-wheel steer example:

$$r/F_{4ws} = [mU]^{-1}(c+e)/(c+\zeta) \tag{23}$$

with the moment arm, e, determined by equation (22). At highway speeds, the relative moment arm lengths for this four-wheel steering example are provided by the diagram of Figure 6.

In this figure, ζ is approximately $L/10$ because of the elevated speed. The moment arm e is assumed to be about $L/6$ (assuming $C_{\alpha f}$ = C,,, a/b = 0.6, and k = 0.3). Lastly, c is approximately $L/5$, reflecting a modest level of understeer.

For cases in which the front-rear steer relationship is more complicated than a simple gain factor, k, and which may change with vehicle speed and operating conditions, the location, e, of the control force, F_{4ws}, is no longer fixed and can move up and down the longitudinal axis of the vehicle. (Note that this is not the case with the conventional front-only steering vehicle in which e is fixed at the front axle position, e = a, as noted earlier.)

A good example of this control force "leveraging" is provided in most four-wheel steering systems when counter-steering of the rear wheels ($k<0$) occurs at low speeds to improve maneuverability in tight quarters. If equation (22) is re-examined with a negative value of k in mind, it is clear that the location of the four wheel steering control force, given by e, is extended out to a distance well beyond the front axle. For example, if k = −0.3 [9, 10], and assuming equal front and rear tire cornering stiffnesses, the length of e becomes,

$$e = (a+0.3b)/0.7 \tag{24}$$

Consequently, one primary advantage that four-wheel steering vehicles enjoy, by comparison to conventional front-wheel steered vehicles, is the great leeway available to them in locating the applied control force (for steering purposes) through intelligent alteration of its associated moment arm. Most four-wheel steering systems which alter the front-rear steering ratio with speed are simultaneously adjusting the effective location, e, of the steering control force, F_{4ws} and its magnitude. The product of e and F_{4ws} is the corresponding steering control moment. Conventional front-wheel steering vehicles maintain a constant gain relationship between the steering control force and the steering control input ($F_{2ws}=C_{\alpha f}\delta_f$) and a constant gain relationship between the steering control moment and the steering input (steering moment = $aC_{\alpha f}\delta_f$). However, most four-wheel steering systems alter these force/moment relationships by changing k as a function of speed. Ordinarily, such four-wheel steering systems produce larger steering forces and smaller steering moments at high speeds (compared to a comparable front-wheel steering vehicle), and, smaller steering forces and larger steering moments at low speeds. This can be seen from equations *(21)* and *(22)* and noting that their product, the steering control moment, is given by $(aC_{af} - kbC_{ar})\delta_f$, with k typically ranging from —0.3 to +0.3 as speed increases.

An interesting case develops when the rear/front steering ratio gain, k, is chosen in such a manner as to maintain e = ζ in Figure 6. That is, the location, e,

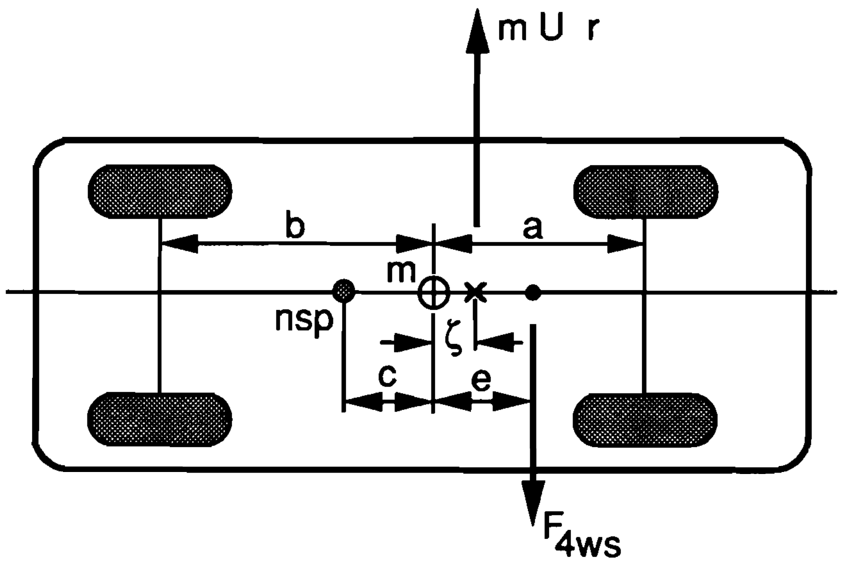


Figure 6. High Speed — Four-Wheel Steer Example.

of the control force varies with forward speed and identically to ζ. Examination of equation *(23)* shows that this produces a control system that has a constant relationship between steering control force, F_{4ws}, and steady-state lateral acceleration, regardless of speed, since mUr/F_{4ws} is unity under this scheme. By setting e equal to ζ, using the relationships provided by equations (14) and (22), the resulting speed-dependent expression for k becomes:

$$k = [-L^2 C_{af}^2 C_{ar} + mU^2(C_{af} + C_{ar})aC_{af}] / [L^2 C_{af} C_{ar}^2 + mU^2(C_{af} + C_{ar})bC_{ar}] \quad (25)$$

As the forward speed, U, approaches zero, the expression for k approaches $-C_{af}/C_{ar}$, resulting in low speed counter-steering. At very high speeds, k approaches aC_{af}/bC_{ar}, and produces front/rear steering of the same polarity. Interestingly, this concept is very similar to the one proposed by Sano et al. [9], in which their system is designed to produce a zero steady-state sideslip condition at the mass center for all speeds. The expression given by equation (25) results in a similar zero sideslip condition at low speed, but at the neutral steer point instead of the mass center.

4. CONCLUSIONS

The above moment arm formulation of the steady turning response of a two-axle vehicle offers a simple means for visualizing and predicting how the key forces acting on a vehicle, and their location, will affect the turning motion. Under this formulation, the vehicle can be viewed as a simple lever, with a pivot point at the neutral steer point, about which the inertial motion force and the applied external control forces are in rotational equilibrium. The inertial force can be viewed as a "force of reaction" to the applied control force within the context of this statics analogy, but which also describes the steady turning response of the vehicle. The method has direct application to a variety of vehicle studies such as: the interaction between vehicle handling and aerodynamic properties, gravitational force effects on vehicle turning induced by road surface geometry, and multi-wheel steering systems.

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