Static universe and cosmic field.

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A Bruno Finzi nel suo 70mo compleanno

Summary. - *It is found that in the presence of the cosmic field, which singles out a time-like direction, a static closed homogeneous isotropic universe of the Einstein type is possible, provided matter is present. On the other hand, a universe of the de Sitter type is not permitted by the field equations.*

Recently [1] it was pointed out that, if one tries to set up a model of the universe which is homogeneous and isotropic and which oscillates without passing through a singular state, one is led to the existence of a scalar field, referred to as a « cosmic field », which is a special case of the C-field of HOYLE and NARLIKAR [2]. The gradient of this cosmic field singles out a time-like direction in space-time. The presence of such a field requires the introduction of terms in the field equations which are somewhat analogous to the cosmological Λ-term which was introduced by EINSTEIN [3] when he was looking for a closed static universe. The field equations can be written in the form

$$R_{\mu}^{\nu} - \frac{1}{2}\delta_{\mu}^{\nu}R_{\alpha}^{\alpha} + \Lambda\delta_{\mu}^{\nu} + \chi_{,\alpha}\chi^{,\alpha}\delta_{\mu}^{\nu} - 2\chi_{,\mu}\chi^{,\nu} = -8\pi T_{\mu}^{\nu}, \tag{1}$$

where χ is a function describing the cosmic field and satisfying the equation

$$\chi^{;\alpha}{}_{;\alpha} = 0. \tag{2}$$

Λ is the cosmological constant, and the other symbols have their usual meanings.

A question which arises is: what is the nature of the solution of the field equations for the case of a homogeneous, isotropic, static universe?

Let us consider the form of the line element in the static, spherically symmetric case. With a suitable choice of coordinates, it can be written

$$ds^2 = -e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + e^{\nu}dt^2, \tag{3}$$

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where $\lambda = \lambda(r)$, $\nu = \nu(r)$. If the three-dimensional space is homogeneous and isotropic (i.e., a space of constant curvature), one has

$$e^{\lambda} = \frac{1}{1 - \frac{r^2}{R^2}}, \tag{4}$$

where $\frac{1}{R^2}$ can be taken to be positive, negative, or zero, according to the sign of the curvature of the space. The case $\frac{1}{R^2} > 0$ corresponds to a closed universe.

If we assume that the matter (and radiation) in the universe can be described by a constant density ρ and a constant pressure p, the matter being at rest in our frame of reference, then we can write for the energy-momentum density tensor (with x^1, x^2, x^3, $x^4 = r$, θ, φ, t),

$$T_1^1 = T_2^2 = T_3^3 = -p, \qquad T_4^4 = \rho, \qquad T_\mu^\nu = 0 \;\; (\mu \neq \nu). \tag{5}$$

For the line element of Eq. (3), the solution of Eq. (2) which satisfies the condition of isotropy is simply

$$\chi = \alpha t + \beta \qquad (\alpha,\ \beta = \text{const.}), \tag{6}$$

so that

$$\chi_{\cdot 4} = \alpha, \qquad \chi_{\cdot k} = 0 \qquad (k = 1,\ 2,\ 3). \tag{6a}$$

The field equations (1) give three relations:

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda + \alpha^2 e^{-\nu} = 8\pi p, \tag{7}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right) + \Lambda + \alpha^2 e^{-\nu} = 8\pi p, \tag{7a}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda + \alpha^2 e^{-\nu} = 8\pi\rho, \tag{7b}$$

corresponding to the cases, $(\mu, \nu) = (1, 1)$, $(2, 2)$, $(4, 4)$. Here primes denote derivatives with respect to r.

Let us first take $\nu = 0$. Then the solution of Eq. (7) is given by Eq. (4) with

$$\frac{1}{R^2} = \alpha^2 + \Lambda - 8\pi p. \tag{8}$$

One that finds Eq. (7a) is then also satisfied. If one substitutes (4) into Eq. (7b), one gets the relation

$$\frac{3}{R^2} - \Lambda + \alpha^2 = 8\pi\rho. \tag{8a}$$

From Eqs. (8) and (8a) one gets

$$\alpha^2 = 4\pi(\rho + p) - \frac{1}{R^2}, \tag{9}$$

$$\Lambda = 4\pi(p - \rho) + \frac{2}{R^2}. \tag{9a}$$

If we consider the case of an EINSTEIN universe, characterized by $\nu = 0$ and λ given by Eq. (4) with $\frac{1}{R^2} > 0$, then we see from Eq. (9) that one must have

$$\alpha^2 < 4\pi(\rho + p), \tag{10}$$

so that the EINSTEIN universe cannot be empty.

For the sake of simplicity, let us take $\Lambda = 0$. Then one gets

$$\frac{1}{R^2} = 2\pi(\rho - p), \tag{11}$$

$$\alpha^2 = 2\pi(\rho + 3p). \tag{11a}$$

We see that one can then obtain a static EINSTEIN universe for any values of ρ and p, provided $\rho > p$ (which is a condition that must hold on the basis of physical considerations).

If we now let Λ be different from zero and consider the case of an empty universe, with $\rho = p = 0$, we see from Eq. (9) that

$$\alpha^2 = -\frac{1}{R^2}, \tag{12}$$

so that $\frac{1}{R^2} < 0$, and we do not have a closed universe.

DE SITTER [4] found a solution of the EINSTEIN field equations, corresponding to a closed, empty universe, for which λ is given by Eq. (4), and ν is given by

$$e^\nu = 1 - \frac{r^2}{R^2}, \tag{13}$$

with

$$\frac{1}{R^2} = \frac{\Lambda}{3}, \tag{14}$$

so that one can take $\frac{1}{R^2} > 0$.

In the presence of the cosmic field, the situation is quite different. If we substitute (4) into (7b), we get

$$\alpha^2 e^{-\nu} = 8\pi\rho - \frac{3}{R^2} + \Lambda, \tag{15}$$

so that, with $\alpha \neq 0$, ν is constant, and Eq. (13) cannot be satisfied. If ν is constant, it can be taken to vanish, without loss of generality, and we are then back to the case discussed previously.

We see then that, with the cosmic field present, a de SITTER universe is not possible. The only closed static universe is of the EINSTEIN type and, as we have seen, this is possible only if matter is present.

REFERENCES

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