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Stationary Analysis of a Fluid Queue Driven by Some Countable State Space Markov Chain

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Abstract Motivated by queueing systems playing a key role in the performance evaluation of telecommunication networks, we analyze in this paper the stationary behavior of a fluid queue, when the instantaneous input rate is driven by a continuous-time Markov chain with finite or infinite state space. In the case of an infinite state space and for particular classes of Markov chains with a countable state space, such as quasi birth and death processes or Markov chains of the G/M/1 type, we develop an algorithm to compute the stationary probability distribution function of the buffer level in the fluid queue. This algorithm relies on simple recurrence relations satisfied by key characteristics of an auxiliary queueing system with normalized input rates.

Keywords fluid queues · Markov chains · uniformization · stationary regime · numerical algorithm

AMS 2000 Subject Classification Primary 60K25 · Secondary 60J27

1 Introduction

In the performance evaluation of packet telecommunication networks, fluid flow approximations prove very useful to analyze complex systems. Among fluid flow models, fluid queues with Markov modulated input rates play a key role in the recent developments of both queueing theory and performance evaluation of packet

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networks (see for instance Boxma and Dumas, 1998). The first studies of such queueing systems can be dated back to the works by Kosten (1984) and Anick et al. (1982), who analyzed in the early 1980's fluid models in connection with statistical multiplexing of several identical exponential on-off input sources in a buffer.

The above studies mainly focused on the analysis of the stationary regime and have given rise to a series of theoretical developments. For instance, Mitra (1987) and (1988) generalizes this model by considering multiple types of exponential on–off inputs and outputs. Stern and Elwalid (1991) consider such models for separable Markov modulated rate processes which lead to a solution of the equilibrium equations expressed as a sum of terms in Kronecker product form. Igelnik et al. (1995) derive a new approach, based on the use of interpolating polynomials, for the computation of the buffer overflow probability. Using the Wiener–Hopf factorization of finite Markov chains, Rogers (1994) shows that the distribution of the buffer level has a matrix exponential form, and Rogers and Shi (1994) explore algorithmic issues of that factorization. Ramaswami (1996) and da Silva Soares and Latouche (2002) respectively exhibit and exploit the similarity between stationary fluid queues in a finite Markovian environment and quasi birth and death processes. Ahn and Ramaswami (2003) establish a direct connection by stochastic coupling between fluid queues and quasi birth and death processes. Following the work of Sericola (1998) and that of Nabli and Sericola (1996), Nabli (2004) obtained an algorithm to compute the stationary distribution of a fluid queue driven by a finite Markov chain.

Most of the above cited studies have been carried out for finite modulating Markov chains. The analysis of a fluid queue driven by infinite state space Markov chains has also been addressed in many research papers. For instance, when the driving process is the $M/M/1$ queue, Virtamo and Norros (1994) solve the associated infinite differential system by studying the continuous spectrum of a key matrix. Adan and Resing (1996) consider the background process as an alternating renewal process, corresponding to the successive idle and busy periods of the $M/M/1$ queue. By renewal theory arguments, the fluid level distribution is given in terms of integral of Bessel functions. They also obtain the expression of Virtamo and Norros via an integral representation of Bessel functions. Barbot and Sericola (2002) obtain an analytic expression for the joint stationary distribution of the buffer level and the state of the $M/M/1$ queue. This expression is obtained by writing down the solution in terms of a matrix exponential and then by using generating functions that are explicitly inverted.

The Markov chain describing the number of customers in the $M/M/1$ queue is a specific birth and death process. Queueing systems with more general modulating infinite Markov chain have been studied by several authors. For instance, van Dorn and Scheinhardt (1997) studied a fluid queue fed by an infinite general birth and death process using spectral theory. In Sericola and Tuffin (1999), the authors consider a fluid queue driven by a general Markovian queue with the hypothesis that only one state has a negative drift. By using the differential system, the fluid level distribution is obtained in terms of a series, which coefficients are computed by means of recurrence relations. This study is extended to the finite buffer case in Sericola (2001).

Besides the study of the stationary regime of fluid queues driven by finite or infinite Markov chains, the transient analysis of such queues has been studied by using Laplace transform by Kobayashi and Ren (1992) and Ren and Kobayashi (1995) for exponential on–off sources. These studies have been extended to the Markov modulated input rate model by Tanaka et al. (1995). Sericola (1998) has obtained a transient solution based on simple recurrence relations, which are particularly interesting for their numerical properties. More recently, Ahn and Ramaswami (2004) use an approach based on an approximation of the fluid model by the amounts of work in a sequence of Markov modulated queues of the quasi birth and death type. When the driving Markov chain has an infinite state space, the transient analysis is more complicated. Sericola et al. (2005) consider the case of the $M/M/1$ queue by using recurrence relations and Laplace transforms.

With respect to all above cited studies, the contribution of the present paper is to propose to a general algorithm for computing the stationary distribution and related characteristics of an infinite buffer fluid queue driven by a Markov chain with a countable state space, while controlling the error made in the computations. The only assumption underlying the proposed algorithm is that some associated Markov chain can be uniformized. This means that the corresponding infinitesimal generator must have a finite norm. This restrictive assumption is needed because the solution we propose here is based on the uniformization technique (Ross, 1983). By using simple recurrence relations, we obtain in the case of infinite state spaces and for particular classes of countable Markov chains, such as quasi birth and death processes or Markov chains of the G/M/1 type, algorithms to compute the stationary distribution of the buffer level in the fluid queue. Clearly, the recurrence relations can also be used to compute the stationary distribution of the buffer level in the fluid queue, when the state space of the driving Markov chain is finite.

The paper is organized as follows. In Section 2, we present the model, the notation and the system of partial differential equations satisfied by the joint probability distribution function of the buffer level and of the state of the driving process. Section 3 is devoted to the resolution of that system of partial differential equations; the solution is obtained in terms of a series. We first construct an equivalent fluid queue with unit net input rates, which allows us to obtain simpler recurrence relations satisfied by the coefficient of the series. In Section 4, we give details of some algorithmic aspects of the buffer level distribution computation for general driving processes and we give, in Section 5, the precise algorithms for particular classes of driving processes.

2 Model Description

We consider an infinite capacity fluid queue driven by a continuous-time Markov chain $\{Z_t,\ t \geq 0\}$ on the countable state space S with infinitesimal generator $A = (a_{i,j})$. The Markov chain $\{Z_t\}$ is supposed to be stationary and ergodic. We denote by β_j the output rate from state j given by $\beta_j = -a_{j,j}$ and by π its stationary probability distribution that satisfies $\pi A = 0$. The stationary distribution π has been studied in several papers for various special types of countable Markov chains such as birth and death processes, quasi birth and death processes or Markov chains of the G/M/1 type. We refer the reader to Latouche and Ramaswami (1999), Meini (1998) and the references therein for the main results on this subject.

The fluid level in the queue at time t is denoted by X_t and the net input rate, defined as the difference between the input and the service rate, of the fluid queue at time t is denoted by r_{Z_t}. It is well-known, see for instance Tanaka (1995), that the joint distribution $F_j(t,x) = \mathbb{P}\{Z_t = j, X_t \leq x\}$ of the fluid level and of the state of the Markov chain $\{Z_t\}$ at time t satisfies the partial differential equation

$$\frac{\partial F_j(t,x)}{\partial t} + r_j \frac{\partial F_j(t,x)}{\partial x} = \sum_{i \in S} F_i(t,x) a_{i,j}. \tag{1}$$

When $X_0 = 0$, the function $F_j(t,x)$ has, on top of its usual jump at point $x = 0$, a jump at point $x = r_j t$ for $t > 0$. This jump corresponds to the case, when the Markov chain $\{Z_t\}$ starts and remains during the whole interval $[0, t)$ in states having the same net input rate and is carefully detailed in Sericola (1998).

We assume that the net input rates r_j are such that $E(r_{Z_t}) < 0$, that is, such that

$$\sum_{j \in S} r_j \pi_j < 0, \tag{2}$$

which means that the queue is stable and that a stationary regime exists for the system.

The stationary state of the Markov chain and of the fluid level are respectively denoted by Z and X. The joint distribution $F_j(x) = \mathbb{P}\{Z = j, X \leq x\}$ thus satisfies, for every $j \in S$ and $x \geq 0$,

$$F_j(x) = \lim_{t \longrightarrow \infty} F_j(t,x),$$

and

$$r_j \frac{dF_j(x)}{dx} = \sum_{i \in S} F_i(x) a_{i,j}.$$

The initial condition is given by $F_j(0) = 0$ for every j such that $r_j > 0$, since the queue cannot be empty if there is a positive drift. In addition, for every $j \in S$, we have by the law of total probability (Ross, 1983)

$$\lim_{x \longrightarrow \infty} F_j(x) = \mathbb{P}\{Z = j\} = \pi_j.$$

The differential equation can be written in matrix notation as

$$\frac{dF(x)}{dx} R = F(x) A, \tag{3}$$

where $F(x)$ is the infinite row vector, whose jth component is equal to $F_j(x)$, and where R is the diagonal matrix, whose jth diagonal element is r_j.

We suppose without any loss of generality that, as shown in da Silva Soares and Latouche (2002) or Sericola (2001), for every $j \in S$, we have $r_j \neq 0$, so that the matrix R is invertible. We moreover assume that the states of the Markov chain $\{Z_t, t \geq 0\}$ are ordered in such a way that r_j is an increasing function of j. We decompose the state space S with respect to the sign of the net input rates r_j, so that $S_- = \{j \in S \mid r_j < 0\}$ and $S_+ = \{j \in S \mid r_j > 0\}$.

In the following, we denote by $|M|$ the matrix obtained from matrix M by taking the absolute values of its entries.

3 Solution to the Basic Matrix Equation

3.1 Notation and Preliminary Results

In this paper, we are interested in computing the quantity $\mathbb{P}(X > x)$ for $x \geq 0$ and in designing an algorithm for obtaining the numerical value of this quantity with a predetermined round-off error ε. It turns out that instead of considering the original system, it is more convenient in the computations to introduce an auxiliary system. It is easy to check that the matrix $T = |R|^{-1}A$ is an infinitesimal generator. Define

$$\lambda = \sup\{\beta_j/|r_j| \ ; \ \ j \in S\} \tag{4}$$

and

$$r = \inf\{|r_j| \ ; \ \ j \in S\}. \tag{5}$$

In the following, we make the two following assumptions:

(H_1) The Markov chain with generator T can be uniformized, i.e., $\lambda < \infty$.
(H_2) The infimum of the absolute values of the net input rates is positive, i.e., $r > 0$.

Let ξ the stationary probability distribution corresponding to the matrix T, that is, the row vector such that $\xi T = 0$ and $\xi \mathbb{1} = 1$, where $\mathbb{1}$ is the infinite column vector with all entries equal to 1. It is easy to check that $\xi = \pi|R|/(\pi|R|\mathbb{1})$ or equivalently that $\pi = \xi|R|^{-1}/(\xi|R|^{-1}\mathbb{1})$.

We denote by $\mathbb{I}$ the diagonal matrix with entries equal to -1 for the states of S_- and equal to 1 for the states of S_+, that is

$$\mathbb{I} = \begin{pmatrix} -I_- & 0 \\ 0 & I_+ \end{pmatrix},$$

where I_- and I_+ are the identity matrices corresponding to the subsets S_- and S_+, respectively.

As suggested in da Silva Soares and Latouche (2002), we rewrite Eq. (3) as

$$\frac{dF(x)}{dx}|R|\mathbb{I} = F(x)|R|T. \tag{6}$$

This amounts to introducing an auxiliary fluid queue with net input rates given by the matrix $\mathbb{I}$ and driven by the Markov chain with infinitesimal generator T. This latter queue is stable and we denote by $\overline{X}$ and $\overline{Z}$ the stationary level in this queue and the stationary state of the driving Markov chain, respectively. The row vector $G(x)$ of the joint distributions $G_j(x) = \mathbb{P}\{\overline{Z} = j, \overline{X} \leq x\}$ satisfies

$$\frac{dG(x)}{dx}\mathbb{I} = G(x)T \text{ with } G_j(0) = 0 \text{ for } j \in S_+ \text{ and } \lim_{x \longrightarrow \infty} G(x) = \xi.$$

We thus have

$$F(x) = \frac{G(x)|R|^{-1}}{\xi|R|^{-1}\mathbb{1}} \text{ or equivalently } G(x) = \frac{F(x)|R|}{\pi|R|\mathbb{1}}.$$

The transient distribution of the couple $(\overline{Z}_t, \overline{X}_t)$ is denoted by the row vector $G(t, x)$, the jth component of this vector is $G_j(t, x) = \mathbb{P}\{\overline{Z}_t = j, \overline{X}_t \leq x\}$. Similarly to Eq. (1), this transient distribution satisfies the following partial differential equation

$$\frac{\partial G(t, x)}{\partial t} + \frac{\partial G(t, x)}{\partial x}\mathbb{I} = G(t, x)T. \tag{7}$$

By taking as initial conditions $\overline{X}_0 = 0$ and $\mathbb{P}\{\overline{Z}_0 = j\} = \xi_j$, for every $j \in S$, it is shown in Sericola (1998) that for every $t > 0$ and $x \in [0, t)$, we have

$$G(t, x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{t}\right)^k \left(1 - \frac{x}{t}\right)^{n-k} b(n, k), \tag{8}$$

where the infinite row vector $b(n, k) = (b_j(n, k),\ j \in S)$ is defined as follows. For $j \in S_-$,

$$b_j(n, n) = \xi_j \text{ for } n \geq 0, \tag{9}$$

$$b_j(n, k) = \frac{1}{2} b_j(n, k+1) + \frac{1}{2} \sum_{i \in S} b_i(n-1, k) p_{i,j} \text{ for } k = n-1, \ldots, 0. \tag{10}$$

and for $j \in S_+$

$$b_j(n, 0) = 0 \text{ for } n \geq 0, \tag{11}$$

$$b_j(n, k) = \sum_{i \in S} b_i(n-1, k-1) p_{i,j} \text{ for } k = 1, \ldots, n. \tag{12}$$

where $P = (p_{i,j})$ is the transition probability matrix of the uniformized Markov chain with generator T with respect to the rate λ. This matrix is related to T by the relation $P = I + T/\lambda$, where I is the identity matrix with dimension given by the context and λ is defined by Eq. (4). The stationary probability distribution ξ satisfies $\xi P = \xi$.

In the next section, we show how to use the auxiliary queueing system with net input rates given by matrix $\mathbb{I}$ and driven by the Markov chain with infinitesimal generator T to compute the probability distribution function (PDF) of the buffer X.

3.2 Computation of the Stationary PDF of the Buffer Level

By using the auxiliary queue with net input rates given by matrix $\mathbb{I}$ and driven by the Markov chain with infinitesimal generator T, we have the following representation for the complementary probability distribution function $\mathbb{P}(X > x)$.

Proposition 1 *For fixed $k \geq 0$, the sequence $(b(n,k), n \geq k)$ converges to a limit $b(k)$ as $n \to \infty$ and the complementary probability distribution function of the stationary buffer content X is given for fixed N by*

$$\mathbb{P}\{X > x\} = \sum_{k=0}^{N} e^{-\lambda x} \frac{(\lambda x)^k}{k!} (1 - b(N,k)\mathbb{v}) + e(x; N), \tag{13}$$

where

$$e(x; N) = \sum_{k=0}^{N} e^{-\lambda x} \frac{(\lambda x)^k}{k!} (b(N,k) - b(k))\mathbb{v} + \sum_{k=N+1}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} (1 - b(k)\mathbb{v}) \tag{14}$$

and the column vector $\mathbb{v}$ is given by

$$\mathbb{v} = \frac{|R|^{-1}\mathbb{1}}{\xi |R|^{-1}\mathbb{1}} = (\pi |R| \mathbb{1}) |R|^{-1}\mathbb{1}. \tag{15}$$

To prove Proposition 1, we first establish a technical lemma describing the properties of the sequence of vectors $b(n,k)$. For the sake of simplicity, an inequality between vectors of the same dimension means that the inequality stands for each of their entry (i.e., the inequality holds component by component).

Lemma 1 *The row vectors $b(n,k)$, $n \geq 0$ and $0 \leq k \leq n$, satisfy:*

(a) *For $n \geq 0$ and $1 \leq k \leq n$, $0 \leq b(n, k-1) \leq b(n,k) \leq \xi$.*
(b) *For $n \geq 1$ and $1 \leq k \leq n$, $b(n-1, k-1) \leq b(n,k)$.*
(c) *For $n \geq 1$ and $0 \leq k \leq n-1$, $b(n,k) \leq b(n-1,k)$.*
(d) *For $n \geq 1$ and $0 \leq k \leq n-2$, $b(n,k) \leq b(n-1,k+1)$.*

Proof Inequality (a) has been proven in Sericola (1998), in which we consider the vectors $\xi - b(n,k)$ instead of the $b(n,k)$. Inequality (d) follows from (c) and (a) by writing $b(n,k) \leq b(n-1,k) \leq b(n-1,k+1)$. Inequalities (b) and (c) are proven in the Appendix. □

We now proceed to the proof of Proposition 1.

Proof of Proposition 1 From Lemma 1, we deduce that for fixed k, the sequence $(b(n,k), n \geq k)$ is non increasing and non negative, and then converges as n goes to infinity. We denote by $b(k) = (b_j(k),\ j \in S)$ the limit of $b(n,k)$ when n tends to infinity. In other words, for $j \in S$, we have

$$\lim_{n \longrightarrow \infty} b_j(n,k) = b_j(k).$$

Let us now consider relation (8). When t goes to infinity, we have for every $x \geq 0$

$$\begin{aligned}
G(x) &= \lim_{t\to\infty} \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{t}\right)^k \left(1-\frac{x}{t}\right)^{n-k} b(n,k) \\
&= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} \lim_{t\to\infty} \sum_{n=0}^{\infty} e^{-\lambda(t-x)} \frac{(\lambda(t-x))^n}{n!} b(n+k,k) \\
&= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} \lim_{n\longrightarrow\infty} b(n+k,k) \\
&= \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} b(k),
\end{aligned}$$

where the third equality follows from the following well-known result : If u_n is a sequence converging to u when n goes to infinity, then the limit of the series $\sum_{n=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^n}{n!} u_n$ when x goes to infinity is equal to u. By using the above equation, we have

$$\begin{aligned}
\mathbb{P}\{X > x\} &= 1 - F(x)\mathbb{1} \\
&= 1 - G(x)\mathbb{v} \\
&= 1 - \sum_{k=0}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} b(k)\mathbb{v} \\
&= 1 - \sum_{k=0}^{N} e^{-\lambda x} \frac{(\lambda x)^k}{k!} b(k)\mathbb{v} - \sum_{k=N+1}^{\infty} e^{-\lambda x} \frac{(\lambda x)^k}{k!} b(k)\mathbb{v} \\
&= \sum_{k=0}^{N} e^{-\lambda x} \frac{(\lambda x)^k}{k!} (1 - b(N,k)\mathbb{v}) + e(x;N),
\end{aligned}$$

where $e(x; N)$ is defined by Eq. (14) and the vector $\mathbb{v}$ is defined by Eq. (15). This establishes Eq. (13). □

From Eq. (13), we see that we can evaluate the value of $\mathbb{P}(X > x)$ by computing the finite series in the right hand side of this equation with an error given by the term $e(x; N)$. Hence, if we want to compute this quantity with a prescribed round-off error ε, we have to find an upper bound for $e(x; N)$ by choosing N and by controlling the values of $(b(N,k) - b(k))$ for $k = 0, \ldots, N$ and $(1 - b(k)\mathbb{v})$. The remainder of this section is devoted to exhibiting the properties of the vectors $b(n,k)$ and $b(k)$. These properties will be used in the next section to design an algorithm for computing $\mathbb{P}(X > x)$ for a series of values $x_1 < \ldots < x_M$.

Proposition 2 *The sequence of vectors $b(k)$ is converging and*

$$\lim_{k\longrightarrow\infty} b(k) = \xi. \tag{16}$$

Proof By taking limits when n goes to infinity in the recurrence relations satisfied by the $b(n,k)$ see Eqs. (10), (11) and (12), we obtain

$$\text{For } j \in S_- : b_j(k) = \frac{1}{2} b_j(k+1) + \frac{1}{2} \sum_{i \in S} b_i(k) p_{i,j}, \quad k \geq 0, \tag{17}$$

$$\text{For } j \in S_+ : b_j(0) = 0, \tag{18}$$

$$b_j(k) = \sum_{i \in S} b_i(k-1) p_{i,j}, \quad k \geq 1. \tag{19}$$

It is easy to check by taking the limit when n goes to infinity in inequality (a) of Lemma 1 that

$$0 \leq b_j(k-1) \leq b_j(k) \leq \xi_j.$$

This proves that, for every $j \in S$, the sequence $b_j(k)$ converges when k tends to infinity. Since the sequence $b(k)$ converges, we have

$$\lim_{k \longrightarrow \infty} b(k) = \lim_{x \longrightarrow \infty} G(x) = \xi. \qquad \square$$

Another property of the sequence of vectors $b(k)$ is obtained by writing the flow equation, which states that the mean stationary input rate is equal to the mean stationary output rate. We write $\xi = (\xi_-, \xi_+)$, where ξ_- and ξ_+ denote the sub-vectors obtained from ξ by considering the states of S_- and S_+, respectively.

Proposition 3 *The quantities $b_j(0)$, $j \in S$ are such that*

$$\sum_{j \in S_-} b_j(0) = \sum_{j \in S_-} \xi_j - \sum_{j \in S_+} \xi_j = 2\xi_- \mathbb{1} - 1, \tag{20}$$

which establishes

$$\mathbb{P}\{\overline{X} = 0\} = 2\xi_- \mathbb{1} - 1. \tag{21}$$

Proof First note that the stability condition (2) is equivalent to the condition

$$\sum_{j \in S_-} \xi_j - \sum_{j \in S_+} \xi_j > 0.$$

Let a_j and d_j be respectively the input rate and the service rate of the fluid queue when $\overline{Z} = j$. The flow equation then becomes

$$\begin{aligned}\sum_{j \in S} a_j \xi_j &= \sum_{j \in S} d_j \mathbb{P}\{\overline{Z} = j, \overline{X} > 0\} + \sum_{j \in S_-} a_j \mathbb{P}\{\overline{Z} = j, \overline{X} = 0\} \\ &= \sum_{j \in S} d_j [\xi_j - G_j(0)] + \sum_{j \in S_-} a_j G_j(0)\end{aligned}$$

$$= \sum_{j\in S} d_j\xi_j - \sum_{j\in S_-} d_j G_j(0) + \sum_{j\in S_-} a_j G_j(0)$$
$$= \sum_{j\in S} d_j\xi_j + \sum_{j\in S_-} (a_j - d_j) G_j(0),$$

which can be written as

$$\sum_{j\in S}(a_j - d_j)\xi_j = \sum_{j\in S_-}(a_j - d_j)G_j(0).$$

Since for every $j \in S$, $G_j(0) = b_j(0)$, and for $j \in S_-$, we have $a_j - d_j = -1$, and for $j \in S_+$, we have $a_j - d_j = 1$, we obtain the first equality of Eq. (20). The second one is trivial by using the normalizing condition $\xi_+\mathbb{1} + \xi_-\mathbb{1} = 1$. To get relation (21) we simply write

$$\mathbb{P}\{\overline{X} = 0\} = \sum_{j\in S} G_j(0) = \sum_{j\in S_-} G_j(0) = \sum_{j\in S_-} b_j(0),$$

and then we use relation (20). □

By using the above proposition and relations (17), (18) and (19), it can be easily shown that for every $k \geq 0$, we have

$$\sum_{j\in S_-} b_j(k) - \sum_{j\in S+} b_j(k) = 2\xi_-\mathbb{1} - 1.$$

As we did for ξ, we write $b(n,k) = (b_-(n,k), b_+(n,k))$ and $b(k) = (b_-(k), b_+(k))$ where $b_-(n,k), b_+(n,k), b_-(k)$ and $b_+(k)$ are the sub-vectors restricted to the states of S_- and S_+, respectively.

Proposition 4 *The vectors $b(n,k)$ satisfy the inequalities*

$$b(n-1,k)\mathbb{1} - b(n,k)\mathbb{1} \leq b_-(n,0)\mathbb{1} - 2\xi_-\mathbb{1} + 1, \; \textit{for } n \geq 1 \textit{ and } 0 \leq k \leq n-1 \tag{22}$$

and

$$\xi_-\mathbb{1} - b_-(n,0)\mathbb{1} \leq b_+(n,n)\mathbb{1} \leq \xi_+\mathbb{1}, \; \textit{for } n \geq 0. \tag{23}$$

Proof By summation over j in relation (10) at point $(n, k-1)$ and in relation (12) at point (n,k), we obtain

$$2\sum_{j\in S_-} b_j(n,k-1) = \sum_{j\in S_-} b_j(n,k) + \sum_{i\in S} b_i(n-1,k-1) \sum_{j\in S_-} p_{i,j},$$

and

$$\sum_{j\in S_+} b_j(n,k) = \sum_{i\in S} b_i(n-1,k-1) \sum_{j\in S_+} p_{i,j}.$$

By adding these two relations, we obtain

$$2\sum_{j\in S_-} b_j(n,k-1) + \sum_{j\in S_+} b_j(n,k) = \sum_{j\in S_-} b_j(n,k) + \sum_{j\in S} b_j(n-1,k-1),$$

which can be written as

$$b_-(n,k)\mathbb{1} - b_+(n,k)\mathbb{1} = b_-(n,k-1)\mathbb{1} - b_+(n,k-1)\mathbb{1} + b(n,k-1)\mathbb{1} - b(n-1,k-1)\mathbb{1}.$$

Defining $d_{n,k} = b_-(n,k)\mathbb{1} - b_+(n,k)\mathbb{1}$, we obtain

$$b(n-1,k)\mathbb{1} - b(n,k)\mathbb{1} = d_{n,k} - d_{n,k+1}.$$

This quantity is positive from inequality (c) of Lemma 1. Hence, the sequence $d_{n,k}$ is decreasing with respect to integer k. We then have, from inequality (a) of Lemma 1,

$$\begin{aligned} 0 \le b(n-1,k)\mathbb{1} - b(n,k)\mathbb{1} &= d_{n,k} - d_{n,k+1} \\ &\le d_{n,0} - d_{n,n} = b_-(n,0)\mathbb{1} - \xi_-\mathbb{1} + b_+(n,n)\mathbb{1} \\ &\le b_-(n,0)\mathbb{1} - \xi_-\mathbb{1} + \xi_+\mathbb{1} = b_-(n,0)\mathbb{1} - 2\xi_-\mathbb{1} + 1, \end{aligned}$$

which completes the proof of relation (22).

The right hand side of inequality (23) is the right hand side of inequality (a) of Lemma 1. In order to prove the first inequality of Eq. 23, we use the fact that the sequence $d_{n,k}$ is decreasing with respect to integer k by writing that $d_{n,n} \le d_{n,0}$, that is

$$\xi_-\mathbb{1} - b_+(n,n)\mathbb{1} \le b_-(n,0)\mathbb{1},$$

which is the desired inequality. □

Propositions 3 and 4 are used in the next section in order to devise a numerical algorithm for computing the quantity $\mathbb{P}(X > x)$ for different values $x_1 < \ldots < x_m$.

4 Algorithmic Aspects

From Proposition 1, we see that we can compute $\mathbb{P}\{X > x\}$ with a prescribed precision ε by using the series in the right hand side of Eq. (13) as soon as we can guarantee that the term $e(x;N)$ defined by Eq. (14) is less than ε. This requires to adequately choose N and to control the terms $(b(N,k) - b(k))$ and $(1 - b(k)\mathbb{V})$. For this purpose, we first need to compute the vectors $b(k)$. However, this task is quite intricate because these vectors appear as limits of sequences of vectors $(b(n,k), n \ge k)$ defined by recurrence relations. By assuming that these sequences are sufficiently smooth (typically that there is no step decrease), we approximate $b(k)$ by $b(n,k)$ as soon as $b(n,k) - b(n+1,k) \le \varepsilon$, which means that the sequence $(b(n,k))$ is numerically stable and close to the limit $b(k)$.

Denote by v the maximal entry of vector $\mathbb{v}$. We have

$$v = \sup\{\mathbb{v}_j \,;\; j \in S\} = \frac{\sup\{1/|r_j| \,;\; j \in S\}}{\xi|R|^{-1}\mathbb{1}} = \frac{1}{r\xi|R|^{-1}\mathbb{1}} = \frac{\pi|R|\mathbb{1}}{r},$$

where r is defined Eq. (5).

We then define the integer N as the smallest integer n such that $v(b_-(n,0) - b_-(0))\mathbb{1} \leq \varepsilon$. From Proposition 3, we have $b_-(0)\mathbb{1} = 2\xi_-\mathbb{1} - 1$, so that N is defined as

$$N = \inf\{n \geq 0 \mid v(b_-(n,0)\mathbb{1} - 2\xi_-\mathbb{1} + 1) \leq \varepsilon\}. \tag{24}$$

With this choice for N, we subsequently show that we have $(b(N,k) - b(N+1,k)) \leq \varepsilon$ and $(1 - b(N,k)\mathbb{v}) \leq \varepsilon$ so that we can expect that

$$(1 - b(k)\mathbb{v}) \leq \varepsilon \quad \text{and} \quad (b(N,k) - b(k)) \leq \varepsilon,$$

which in turn implies that $e(x; N) \leq \varepsilon$ for all $x \in \mathbb{R}$.

For $k = 0$, we have, by definition of N and v,

$$(b(N,0) - b(0))\mathbb{v} \leq v(b(N,0) - b(0))\mathbb{1} \leq \varepsilon.$$

For $k = 1, \ldots, N$, we have from Proposition 4 and by definition of N

$$\begin{aligned}(b(N,k) - b(N+1,k))\mathbb{v} &\leq v(b(N,k) - b(N+1,k))\mathbb{1} \\ &\leq v(b(N+1,0)\mathbb{1} - 2\xi_-\mathbb{1} + 1) \\ &\leq v(b(N,0)\mathbb{1} - 2\xi_-\mathbb{1} + 1) \\ &\leq \varepsilon,\end{aligned}$$

so we expect that $(b(N,k) - b(k))\mathbb{v} \leq \varepsilon$.

For $k \geq N+1$, from inequality (23) and since $b_-(n,n) = \xi_-$, we get

$$2\xi_-\mathbb{1} - b_-(n,0)\mathbb{1} \leq b(n,n)\mathbb{1} \leq 1,$$

that is

$$0 \leq v(1 - b(n,n)\mathbb{1}) \leq v(b_-(n,0)\mathbb{1} - 2\xi_-\mathbb{1} + 1).$$

By definition of N, we have for $k \geq N$,

$$0 \leq v(1 - b(k,k)\mathbb{1}) \leq \varepsilon,$$

and we obtain for $k \geq N$

$$1 - b(k,k)\mathbb{v} = (\xi - b(k,k))\mathbb{v} \leq v(\xi - b(k,k))\mathbb{1} = v(1 - b_-(k,k)\mathbb{1}) \leq \varepsilon.$$

That is why we expect that $(1 - b(k)\mathbb{v}) \leq \varepsilon$, for $k \geq N$.

Table 1 Algorithm for the computation of $\mathbb{P}\{X > x\}$.

input : $x_1 < \cdots < x_M, \varepsilon$
output : $\mathbb{P}\{X > x\}$ for $x = x_1, \ldots, x_M$
$\lambda = \sup\{\beta_j/|r_j| \; ; \; j \in S\}$
$T = |R|^{-1} A$
Compute the stationary distribution ξ
Compute the column vector $\mathbb{v}$ and $v = \sup\{\mathbb{v}_j \; ; \; j \in S\}$
$P = I + T/\lambda$
$b_+(0, 0) = 0$
$b_-(0, 0) = \xi_-$
$n = 0$
while $v(b_-(n, 0)\mathbb{1} - 2\xi_-\mathbb{1} + 1) > \varepsilon$ **do**
  n = n+1
  $b_+(n, 0) = 0$
  for $k = 1$ **to** n **do** compute $b_+(n, k)$ from Eq. (12) **endfor**
  $b_-(n, n) = \xi_-$
  for $k = n - 1$ **downto** 0 **do** compute $b_-(n, k)$ from Eq. (10) **endfor**
endwhile
N = n
for $m = 1$ **to** M **do** $\mathbb{P}\{X > x_m\} = \sum_{k=0}^{N} e^{-\lambda x_m} \frac{(\lambda x_m)^k}{k!} (1 - b(N, k)\mathbb{v})$ **endfor**

The above considerations lead us to expect that $e(x; N) \leq \varepsilon$. We tested this approximation by considering fluid queues fed Markov chains with only one negative effective input rate. For such queues we developed in Sericola and Tuffin (1999) an exact algorithm which gives for a prescribed precision ε the same results as those obtained with the method presented in this paper. This supports the assumption made above on the smoothness of the sequences of vectors $(b(n, k), n \geq k), k \geq 0$.

The algorithm to compute the distribution $\Pr\{X > x\}$ for the M values $x_1 < x_2 < \cdots < x_M$ of x is then given in Table 1.

Of course, this algorithm can be applied for every finite Markov chain. For infinite state space Markov chains, the computation of the stationary distribution ξ and of the infinite row vectors $b(n, k)$ can be obtained only for special structures of the driving Markov chain. We deal with these particular types of structures in the following section.

5 Special Cases

In this section, we give a few examples of Markov chains for which the algorithm described in Section 4 can be used for estimating the probability distribution function of the buffer content in the stationary regime of a fluid queue driven by a Markov chain with a countable state space.

5.1 Fluid Queues Fed by Quasi Birth and Death Processes

The fluid queue is driven by a stationary ergodic continuous-time quasi birth and death (QBD) process, non necessarily homogeneous, represented by the two-dimensional Markov chain $\{Z_t\}$ on the state space S given by

$$S = \{(\ell, j) : \ell \in \mathbb{N}, 1 \le j \le m_\ell\},$$

where the first component is called the level and the second one is called the phase. The infinitesimal generator A of the QBD process is a block-tridiagonal infinite matrix that can be written as

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} & 0 & & \\ A_{1,0} & A_{1,1} & A_{1,2} & 0 & \\ 0 & A_{2,1} & A_{2,2} & A_{2,3} & \ddots \\ & 0 & A_{3,2} & A_{3,3} & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where, for $\ell \ge 0$, the matrices $A_{\ell,\ell}$ are square matrices of dimension m_ℓ and the sizes of the other sub-matrices are determined accordingly. The computation of the stationary distribution π (or ξ) of the QBD process has been studied in several papers for various types of QBD processes. We refer the reader to Latouche and Ramaswami (1999) and the references therein for the main results on this subject.

As we did for matrix A, we write the matrix R and the vectors $b(n, k)$ and ξ by using blocks, that is

$$R = diag(R_0, R_1, \cdots, R_\ell, R_{\ell+1}, \cdots),$$

where R_ℓ is the diagonal matrix containing the entries $r_{(\ell, j)}$ for $j = 1, \ldots, m_\ell$,

$$b(n, k) = (b_0(n, k), b_1(n, k), \ldots, b_\ell(n, k), \ldots),$$

with, for $\ell \ge 0$,

$$b_\ell(n, k) = (b_{(\ell,1)}(n, k), \ldots, b_{(\ell,m_\ell)}(n, k)),$$

and

$$\xi = (\xi_0, \xi_1, \ldots, \xi_\ell, \ldots),$$

with, for $\ell \ge 0$,

$$\xi_\ell = (\xi_{(\ell,1)}, \ldots, \xi_{(\ell,m_\ell)}).$$

We assume that $r_{(\ell,j)} > 0$ when ℓ is greater than a fixed $\ell_0 > 0$ and that $r_{(\ell,j)} < 0$ for $0 \le \ell \le \ell_0$. We thus have $S_- = \{(\ell, j) \in S : \ell \le \ell_0\}$ and $S_+ = \{(\ell, j) \in S : \ell > \ell_0\}$. The transition probability matrix $P = I + T/\lambda$ can then be written as

$$P = \begin{pmatrix} P_{0,0} & P_{0,1} & 0 & & \\ P_{1,0} & P_{1,1} & P_{1,2} & 0 & \\ 0 & P_{2,1} & P_{2,2} & P_{2,3} & \ddots \\ & 0 & P_{3,2} & P_{3,3} & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\begin{aligned} P_{\ell,\ell+1} &= |R_\ell|^{-1} A_{\ell,\ell+1}/\lambda, \\ P_{\ell,\ell} &= I + |R_\ell|^{-1} A_{\ell,\ell}/\lambda, \\ P_{\ell,\ell-1} &= |R_\ell|^{-1} A_{\ell,\ell-1}/\lambda. \end{aligned}$$

The relations (9), (10), (11), (12) can be written for $0 \le \ell \le \ell_0$ as

$$b_\ell(n,n) = \xi_\ell \text{ for } n \ge 0 \tag{25}$$

$$\begin{aligned} b_\ell(n,k) = \frac{1}{2} b_\ell(n,k+1) &+ \frac{1}{2}\left[b_{\ell-1}(n-1,k)P_{\ell-1,\ell} + b_\ell(n-1,k)P_{\ell,\ell}\right] \\ &+ \frac{1}{2} b_{\ell+1}(n-1,k)P_{\ell+1,\ell} \text{ for } k = n-1,\ldots,0 \end{aligned} \tag{26}$$

and for $\ell \ge \ell_0 + 1$ as

$$b_\ell(n,0) = 0 \text{ for } n \ge 0 \tag{27}$$

$$\begin{aligned} b_\ell(n,k) = b_{\ell-1}(n-1,k-1)P_{\ell-1,\ell} &+ b_\ell(n-1,k-1)P_{\ell,\ell} \\ &+ b_{\ell+1}(n-1,k-1)P_{\ell+1,\ell} \text{ for } k = 1,\ldots,n, \end{aligned} \tag{28}$$

where we set $b_{-1}(n-1,k)P_{-1,0} = 0$. The computation of each $b_\ell(n,k)$ requires a finite number of operations but the number of such vectors is infinite. This difficulty is overcome by the following result.

Lemma 2 *For $n \ge 0$ and $0 \le k \le n$, we have*

$$b_\ell(n,k) = 0 \textit{ for } \ell \ge \ell_0 + k + 1.$$

Proof We proceed by recurrence over index k. For $k = 0$, we have from relation (11), $b_j(n,0) = 0$ for $j \in S_+$ which means that $b_\ell(n,0) = 0$ for $\ell \ge \ell_0 + 1$.

Suppose that for a fixed index $k \ge 1$, we have $b_\ell(n,k-1) = 0$ for $\ell \ge \ell_0 + k$, for every $n \ge k-1$. This means in particular that

$$\begin{aligned} b_{\ell-1}(n-1,k-1) &= 0 \ \text{ for } \ell \ge \ell_0 + k + 1 \\ b_\ell(n-1,k-1) &= 0 \ \text{ for } \ell \ge \ell_0 + k \\ b_{\ell+1}(n-1,k-1) &= 0 \ \text{ for } \ell \ge \ell_0 + k - 1. \end{aligned}$$

Thus, we obtain, from relation (28), that $b_\ell(n,k) = 0$ for $\ell \geq \ell_0 + k + 1$. □

The above result means also that we only need to compute relation (28) for $\ell = \ell_0 + 1$ to $\ell = \ell_0 + k$.

5.2 Fluid Queues Fed by G/M/1 Type Markov Chains

Markov chains of the G/M/1 type have an infinitesimal generator with the following Hessenberg structure

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} & 0 & 0 & \cdots \\ A_{1,0} & A_{1,1} & A_{1,2} & 0 & \cdots \\ A_{2,0} & A_{2,1} & A_{2,2} & A_{2,3} & \ddots \\ A_{3,0} & A_{3,1} & A_{3,2} & A_{3,3} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

on the same state space S as that used for QBDs. The computation of the stationary distribution of such Markov chains can be found for instance in Meini (1998).

The matrix P has clearly the same structure and thus the recurrence relations (9), (10), (11), (12) become for $0 \leq \ell \leq \ell_0$

$$b_\ell(n,n) = \xi_\ell \text{ for } n \geq 0, \tag{29}$$

$$b_\ell(n,k) = \frac{1}{2} b_\ell(n,k+1) + \frac{1}{2} \sum_{h=\max(0,\ell-1)}^{\infty} b_h(n-1,k) P_{h,\ell} \text{ for } k = n-1, \ldots, 0, \tag{30}$$

and for $\ell \geq \ell_0 + 1$

$$b_\ell(n,0) = 0 \text{ for } n \geq 0, \tag{31}$$

$$b_\ell(n,k) = \sum_{h=\ell-1}^{\infty} b_h(n-1,k-1) P_{h,\ell} \text{ for } k = 1, \ldots, n. \tag{32}$$

In this case, the computation of each $b_\ell(n,k)$ requires an infinite number of operations and the number of such vectors is infinite. These difficulties are overcome by using Lemma 2, which is still valid. The recurrence relations (29), (30), (31), (32) thus become for $0 \leq \ell \leq \ell_0$

$$b_\ell(n,n) = \xi_\ell \text{ for } n \geq 0, \tag{33}$$

$$b_\ell(n,k) = \frac{1}{2} b_\ell(n,k+1) + \frac{1}{2} \sum_{h=\max(0,\ell-1)}^{\ell_0+k} b_h(n-1,k) P_{h,\ell} \text{ for } k = n-1, \ldots, 0, \tag{34}$$

and for $\ell \geq \ell_0 + 1$

$$b_\ell(n, 0) = 0 \text{ for } n \geq 0 \tag{35}$$

$$b_\ell(n, k) = \sum_{h=\ell-1}^{\ell_0+k-1} b_h(n-1, k-1) P_{h,\ell} \text{ for } k = 1, \ldots, n. \tag{36}$$

This means that we only need to compute relation (36) for $\ell = \ell_0 + 1$ to $\ell = \ell_0 + k$.

5.3 Non-skip-free G/M/1 Type Markov Chains

Clearly, our method also applies for more general driving processes such as Markov chains of the non-skip-free G/M/1 type. The corresponding infinitesimal generator is an H-Hessenberg matrix, that is a matrix of the following form.

$$A = \begin{pmatrix} A_{0,0} & \cdots & A_{0,H-1} & A_{0,H} & 0 & 0 & \cdots \\ A_{1,0} & \cdots & A_{1,H-1} & A_{1,H} & A_{1,H+1} & 0 & \cdots \\ A_{2,0} & \cdots & A_{2,H-1} & A_{2,H} & A_{2,H+1} & A_{2,H+2} & \ddots \\ A_{3,0} & \cdots & A_{3,H-1} & A_{3,H} & A_{3,H+1} & A_{3,H+2} & \ddots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Appendix

Proof of Lemma 1

In this Appendix, we prove inequalities (b) and (c) appearing in Lemma 1.

Proof of inequality (*b*) The proof is by mathematical induction on the index n. We use inequality (a) of Lemma 1, i.e. $0 \leq b(n, k) \leq \xi$.

For $n = 1$, we have

$$\begin{aligned} b_j(1, 1) &\geq 0 = b_j(0, 0) \quad \text{for } j \in S_+ \\ b_j(1, 1) &= \xi_j = b_j(0, 0) \quad \text{for } j \in S_-. \end{aligned}$$

Suppose that for a fixed index $n \geq 1$, we have

$$b(n, k) \geq b(n-1, k-1) \text{ for } 1 \leq k \leq n. \tag{37}$$

We must show that for $1 \leq k \leq n+1$, we have $b(n+1, k) \geq b(n, k-1)$.

Let $j \in S^+$. By using the recurrence hypothesis (37), we obtain

$$b_j(n+1, k) - b_j(n, k-1) = \sum_{i \in S} (b_i(n, k-1) - b_i(n-1, k-2)) p_{i,j} \geq 0.$$

We proceed again by recurrence over index k. Let $j \in S^-$. For $k = n+1$, we have $b_j(n+1, n+1) = 0 = b_j(n, n)$. Suppose that for a fixed index $k \le n$, we have

$$b_j(n+1, k+1) \ge b_j(n, k). \tag{38}$$

By using the two recurrence hypotheses (37) and (38), we have

$$\begin{aligned} b_j(n+1, k) - b_j(n, k-1) &= \frac{1}{2}(b_j(n+1, k+1) - b_j(n, k)) \\ &\quad + \frac{1}{2}\sum_{i \in S}(b_j(n, k) - b_j(n-1, k-1))\, p_{i,j} \\ &\ge 0, \end{aligned}$$

which completes the proof. □

Proof of inequality (c) We follow exactly the same steps as in the proof of inequality (b).

The proof is by mathematical induction on the index n. We use inequality (a) of Lemma 1, i.e. $0 \le b\,(n, k) \le \xi$.

For $n = 1$, we have

$$\begin{aligned} &b_j(0, 0) \ge 0 = b_j(1, 0) \quad \text{for } j \in S_+ \\ &b_j(0, 0) = \xi_j \ge b_j(1, 0) \quad \text{for } j \in S_-. \end{aligned}$$

Suppose that for a fixed index $n \ge 1$, we have

$$b(n-1, k) \ge b(n, k) \text{ for } 0 \le k \le n-1. \tag{39}$$

We must show that for $0 \le k \le n$, we have $b(n, k) \ge b(n+1, k)$.

Let $j \in S^+$. By using the recurrence hypothesis (39), we have

$$b_j(n, k) - b_j(n+1, k) = \sum_{i \in S}(b_i(n-1, k-1) - b_i(n, k-1))\, p_{i,j} \ge 0.$$

We proceed again by recurrence over index k. Let $j \in S^-$. For $k = n$, we have $b_j(n, n) = \xi_j \ge b_j(n+1, n)$. Suppose that for a fixed index $k \le n-1$, we have

$$b_j(n, k+1) \ge b_j(n+1, k+1). \tag{40}$$

By using the two recurrence hypotheses (39) and (40), we obtain

$$\begin{aligned} b_j(n, k) - b_j(n+1, k) &= \frac{1}{2}(b_j(n, k+1) - b_j(n+1, k+1)) \\ &\quad + \frac{1}{2}\sum_{i \in S}(b_j(n-1, k) - b_j(n, k))\, p_{i,j} \\ &\ge 0, \end{aligned}$$

which completes the proof. □

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