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STATIONARY DYNAMICAL SYSTEMS

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Abstract. Following works of Furstenberg and Nevo and Zimmer we present an outline of a theory of stationary (or m-stationary) dynamical systems for a general acting group G equipped with a probability measure m. Our purpose is two-fold: First to suggest a more abstract line of development, including a simple structure theory. Second, to point out some interesting applications; one of these is a Szemerédi type theorem for $SL(2,\mathbb{R})$.

Contents

Introduction

Classical ergodic theory was developed for the group of real numbers $\mathbb{R}$ and the group of integers $\mathbb{Z}$. Later generalizations to $\mathbb{R}^d$ and $\mathbb{Z}^d$ actions evolved and more recently the theory has been vastly extended to handle more general concrete and abstract amenable groups. There however the theory finds a natural boundary, since by definition it deals with measure preserving actions on measurable or compact spaces, and these need not exist for a non-amenable group. Of course semi-simple Lie groups or non-commutative free groups admit many interesting measure preserving actions, but for many other natural actions of these groups no invariant measure exists.

Following works of Furstenberg (e.g. [7], [8], [9], [11]) and Nevo and Zimmer (e.g. [23], [24], [25]), we present here an outline of a theory of stationary (or m-stationary) dynamical systems for a general acting group G equipped with a probability measure

[1]A preliminary version of this work has been in circulation as a preprint for several years now but for technical reasons was not previously submitted for publication.

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m. By definition such a system comprises a compact metric space X on which G acts by homeomorphisms and a probability measure μ on X which is m stationary; i.e. it satisfies the convolution equation $m * \mu = \mu$. The immediate advantage of stationary systems over measure preserving ones is the fact that, given a compact G-space X, an m-stationary measure always exists and often it is also quasi-invariant.

The aforementioned works, as well as e.g. [19] and the more recent works [2] and [3], amply demonstrate the potential of this new kind of theory and our purpose here is two-fold. First to suggest a more abstract line of development, including a simple structure theory, and second, to point out some interesting applications.

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1. Stationary dynamical systems

Definitions: Let G be a locally compact second countable topological group, m an **admissible** probability measure on G. I.e. with the following two properties: (i) For some $k \geq 1$ the convolution power μ^{*k} is absolutely continuous with respect to Haar measure. (ii) the smallest closed subgroup containing supp (m) is all of G. Let $(X, \mathcal{B})$ be a standard Borel space and let G act on it in a measurable way. A probability measure μ on X is called m-**stationary**, or just **stationary** when m is understood, if $m * \mu = \mu$. As shown by Nevo and Zimmer, every m-stationary probability measure μ on a G-space X is quasi-invariant; i.e. for every $g \in G$, μ and $g\mu$ have the same null sets.

Given a stationary measure μ the quintuple $\mathcal{X} = (X, \mathcal{B}, G, m, \mu)$ is called an m-**dynamical system**, or just an m-**system**. (Usually we omit the σ-algebra $\mathcal{B}$ from the notation of an m-system, and often also the group G and the measure m). An m-system $\mathcal{X}$ is called **measure preserving** if the stationary measure is in fact G-invariant. With no loss of generality we may assume that the Borel space X is a compact metric space and that the action of G on X is by homeomorphisms. For a compact metric space X, the space of probability Borel measures on X with the weak* topology will be denoted by $M(X)$; it is a compact convex metric space. When G acts on X by homeomorphisms the closed convex subset of $M(X)$ consisting of m-stationary measures will be denoted by $M_m(X)$. By the Markov-Kakutani fixed point theorem $M_m(X)$ is non-empty. We say that the m-system (X, μ) is **ergodic** if μ is an extreme point of $M_m(X)$ and that it is **uniquely ergodic** if $M_m(X) = \{\mu\}$. It is easy to see that when μ is ergodic every G-invariant measurable subset of X has μ measure 0 or 1. Unless we say otherwise we will assume that an m-system is ergodic.

When $\mathcal{X} = (X, \mathcal{B}, G, m, \mu)$ and $\mathcal{Y} = (Y, \mathcal{A}, G, m, \nu)$ are two m-dynamical systems, a measurable map $\pi : X \to Y$ which intertwines the G-actions and satisfies $\pi_*(\mu) = \nu$ is called a **homomorphism** of m-stationary systems. We then say that $\mathcal{Y}$ is a **factor** of $\mathcal{X}$, or that $\mathcal{X}$ is an **extension** of $\mathcal{Y}$.

Let $\Omega = G^{\mathbb{N}}$ and let $P = m^{\mathbb{N}} = m \times m \times m \dots$ be the product measure on Ω, so that (Ω, P) is a probability space. We let $\xi_n : \Omega \to G$, denote the projection onto the n-th coordinate, $n = 1, 2, \dots$. We refer to the stochastic process $(\Omega, P, \{\eta_n\}_{n \in \mathbb{N}})$, where $\eta_n = \xi_1 \xi_2 \cdots \xi_n$ as the m-**random walk on** G.

A real valued function $f(g)$ for which $\int f(gg')\,dm(g') = f(g)$ for every $g \in G$ is called **harmonic**. For a harmonic f we have

$$\begin{aligned} &E(f(g\xi_1\xi_2\cdots\xi_n\xi_{n+1}|\xi_1\xi_2\cdots\xi_n) \\ &= \int f(g\xi_1\xi_2\cdots\xi_n g')\,dm(g') \\ &= f(g\xi_1\xi_2\cdots\xi_n), \end{aligned}$$

so that the sequence $f(g\xi_1\xi_2\cdots\xi_n)$ forms a **martingale**.

For $F \in C(X)$ let $f(g) = \int F(gx)\,d\mu(x)$, then the equation $m * \mu = \mu$ shows that f is harmonic. It is shown (e.g.) in [8] how these facts combined with the martingale convergence theorem lead to the following:

1.1. **Theorem.** *The limits*

$$\lim_{n\to\infty} \eta_n\mu = \lim_{n\to\infty} \xi_1\xi_2\cdots\xi_n\mu = \mu_\omega, \tag{1.1}$$

exist for P almost all $\omega \in \Omega$.

The measures μ_ω are the **conditional measures** of the m-system $\mathcal{X}$. We let Ω_0 denote the subset of Ω where the limit (1.1) exists. The fact that μ is m-stationary can be expressed as:

$$\int \xi_1(\omega)\mu dP(\omega) = m * \mu = \mu.$$

By induction we have

$$\int \xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega)\mu dP(\omega) = \mu,$$

and passing to the limit we also have the **barycenter equation**:

$$\int \mu_\omega dP(\omega) = \mu. \tag{1.2}$$

There is a natural "action" of G on Ω defined as follows. For $\omega = (g_1, g_2, g_3, \dots) \in \Omega$ and $g \in G$, $g\omega \in \Omega$ is given by $g\omega = (g, g_1, g_2, g_3, \dots)$. (This is not an action in the usual sense; e.g. $g^{-1}(g\omega) \neq \omega$.) It is easy to see that for every $g \in G$ and $\omega \in \Omega_0$, $\mu_{g\omega} = g\mu_\omega$, so that Ω_0 is G-invariant. The map $\zeta : \Omega \to M(X)$ given P a.s. by $\omega \mapsto \mu_\omega = \lim_n \xi_1\xi_2\cdots\xi_n\mu$, sends the measure P onto a probability measure, $\zeta_*P = P^* \in M(M(X))$; i.e. P^* is the distribution of the $M(X)$-valued random variable $\zeta(\omega) = \mu_\omega$. Clearly for each $k \geq 1$, the random variable $\zeta_k = \lim_{n\to\infty} \xi_k\xi_{k+1}\cdots\xi_{k+n}\mu$ has the same distribution P^* as $\zeta(\omega)$. We also have $\zeta_k = \xi_k\zeta_{k+1}$. The functions $\{\zeta_k\}$ therefore satisfy:

(a) ζ_k is a function of $\xi_k, \xi_{k+1}, \dots$
(b) all the ζ_k have the same distribution,
(c) ξ_k is independent of $\zeta_{k+1}, \zeta_{k+2}, \dots$
(d) $\zeta_k = \xi_k\zeta_{k+1}$.

In other words, the $M(X)$-valued stochastic process $\{\zeta_k\}$ is an m**-process** in the sense of definition 3.1 of [8] and it follows that the measure P^* is m-stationary (condition (d)) and that $\Pi(\mathcal{X}) = (M(X), G, m, P^*)$ is an m-system [1].

[1] The "barycenter" equation (1.2) is what makes the "quasifactor" $\Pi(\mathcal{X})$ meaningful in the general measure theoretical setup, where X is just a standard Borel space; see e.g. [16]

Definitions: We call the m-system $\mathcal{X} = (X, G, m, \mu)$, m**-proximal** (or a "boundary" in the terminology of [8]) if P a.s. the conditional measures $\mu_\omega \in M(X)$ are point masses. Clearly a factor of a proximal system is proximal as well. Let $\pi : (X, G, m, \mu) \to (Y, G, m, \nu)$ be a homomorphism of m-dynamical systems. We say that π is a **measure preserving homomorphism** (or extension) if for every $g \in G$ we have $g\mu_y = \mu_{gy}$ for ν almost all y. Here the probability measures $\mu_y \in M(X)$ are those given by the disintegration $\mu = \int \mu_y d\nu(y)$. It is easy to see that when π is a measure preserving extension then also (with obvious notations), P a.s. $g(\mu_\omega)_y = (\mu_\omega)_{gy}$ for ν almost all y . Clearly, when $\mathcal{Y}$ is the trivial system, the extension π is measure preserving iff the system $\mathcal{X}$ is measure preserving. We say that π is an m**-proximal homomorphism** (or extension) if P a.s. the extension $\pi : (X, \mu_\omega) \to (Y, \nu_\omega)$ is a.s. 1-1, where ν_ω are the conditional measures for the system $\mathcal{Y}$. Clearly, when $\mathcal{Y}$ is the trivial system, the extension π is m-proximal iff the system $\mathcal{X}$ is m-proximal. When there is no room for confusion we sometimes say proximal rather than m-proximal.

Proposition 3.2 of [8] can now be formulated as:

1.2. **Proposition.** *For every m-dynamical system $\mathcal{X}$ the system $\Pi(\mathcal{X}) = (M(X), P^*)$ is m-proximal. It is a trivial, one point system, iff $\mathcal{X}$ is a measure preserving system.*

Given the group G and the probability measure m, there exists a unique universal m-proximal system $(\Pi(G, m), \eta)$ called the **Poisson boundary** of the pair (G, m). Thus every m-proximal system (X, μ) is a factor of the system $(\Pi(G, m), \eta)$.

Given an m-system (X, μ) let

$$h_m(X, \mu) = -\int_G \int_X \log\big(\frac{dg\mu}{d\mu}(x)\big) d\mu(x) dm(g),$$

or

$$h_m(X, \mu) = -\sum m(g) \int_X \log\big(\frac{dg\mu}{d\mu}(x)\big) d\mu(x),$$

when G is discrete. This nonnegative number is the m-**entropy** of the m-system (X, μ). We have the following theorem (see [6], [24]).

1.3. **Theorem.** (1) *The m-system (X, μ) is measure preserving iff $h_m(X, \mu) = 0$.*
(2) *More generally, an extension of m-systems $\pi : (X, \mu) \to (Y, \nu)$ is a measure preserving extension iff $h_m(X, \mu) = h_m(X, \nu)$.*
(3) *An m-proximal system (X, μ) is isomorphic to the Poisson system $(\Pi(G, m), \eta)$ iff*

$$h_m(X, \mu) = h_m(\Pi(G, m), \eta).$$

Typically the conditional measures μ_ω are singular to the measure μ. In fact we have the following statement.

1.4. **Theorem.** *Let $\mathcal{X} = (X, G, \mu)$ be an m-system with the property that a.s. the conditional measures μ_ω are absolutely continuous with respect to μ $(\mu_\omega << \mu)$. Then μ is G-invariant; i.e. $\mathcal{X}$ is measure preserving.*

Proof. We consider the usual unitary representation of G on $H = L_2(X, \mu)$ given by

$$U_g f(x) = f(g^{-1}x) u(g^{-1}, x), \qquad \text{with} \qquad u(g, x) = \sqrt{\frac{dg^{-1}\mu}{d\mu}}.$$

For $\omega \in \Omega_0$ let $f_\omega = \frac{d\mu_\omega}{d\mu} \in L_1(\mu)$ denote the Radon-Nikodym derivative of μ_ω w.r.t. μ, and put $h_\omega = \sqrt{f_\omega}$. Then for $\omega \in \Omega_0$, $g \in G$ and $f \in L_2(X,\mu)$, denoting $v(g,x) = \frac{dg^{-1}\mu}{d\mu}$, we get

$$\begin{aligned}\int f(x)dg\mu_\omega(x) &= \int f(gx)f_\omega(x)d\mu(x)\\ &= \int f(x)f_\omega(g^{-1}x)dg\mu(x)\\ &= \int f(x)f_\omega(g^{-1}x)v(g^{-1},x)d\mu(x).\end{aligned}$$

Hence $f_{g\omega} = (f_\omega \circ g^{-1}) \cdot v(g^{-1},\cdot)$ and

$$h_{g\omega} = (h_\omega \circ g^{-1}) \cdot u(g^{-1},\cdot) = U_g h_\omega.$$

It is now easy to see that the map $\mu_\omega \mapsto h_\omega$ from Ω_0 into the unit ball B of $H = L_2(X,\mu)$, is a Borel isomorphism which intertwines the G-action on Ω_0 with the unitary action of G on B. If we let Y be the weak closure of the set of functions $\{h_\omega : \omega \in \Omega_0\}$ in B, we get a compact G-space (Y,G) by restricting the unitary representation $g \mapsto U_g$ to Y. Such a G-space is WAP and our theorem follows from theorem 7.4 in section 7 below, which asserts that every m-stationary measure on Y is G-invariant. (For the definition and basic properties of **weakly almost periodic** (WAP) G-systems we refer e.g. to [16, Chapter 1].) □

2. Examples

1. Let $G = SL(2,\mathbb{R})$ and let m be any absolutely continuous right and left K invariant probability measure on G such that supp(m) generates G as a semigroup. G acts on the compact space X of rays emanating from the origin in $\mathbb{R}^2$—which is homeomorphic to the unite circle in $\mathbb{R}^2$. Normalized Lebesgue measure μ is the unique m-stationary measure on X. G acts as well on the space $Y = \mathbb{P}^1$ of lines in $\mathbb{R}^2$ through the origin (the projective line) and the natural map $\pi : X \to Y$, that sends a ray in X to the unique line that contains it in Y, is a 2 to 1 homomorphism of m-systems, where we take $\nu = \pi(\mu)$. It is easy to see that (Y,ν) is m-proximal and that π is a measure preserving extension. It can be shown that (Y,ν) is the unique m-proximal system so that in particular (Y,ν) is the Poisson boundary $\Pi(G,m)$.

2. ([7]) Let G be a connected semisimple Lie group with finite center and no compact factors. Let $G = KNA$ be an Iwasawa decomposition, $S = AN$ and $P = MAN$, the corresponding minimal parabolic subgroup. Set $X = G/S$, $Y = G/P$ and let m be an admissible probability measure on G. More specifically we assume that m is absolutely continuous with respect to Haar measure, right and left K-invariant, and supp(μ) generates G as a semigroup. Then

(1) There exists on Y a unique m-stationary measure ν (which is the unique K-invariant probability measure on Y) such that the m-system (Y,ν) is m-proximal. In fact (Y,ν) is the Poisson boundary $\Pi(G,m)$ and the collection of m-proximal systems coincides with the collection of homogeneous spaces G/Q with Q a parabolic subgroup of G.

(2) For any m-stationary measure μ on X the natural projection $(X,\mu) \overset{\pi}{\to} (Y,\nu)$ is a measure preserving extension.

3. ([23]) Let G be a connected semisimple Lie group with finite center, no compact factors, and $\mathbb{R}$-rank $(G) \geq 2$. Let m be an admissible probability measure on G and let (X,G) be a compact metric G-space. Let P be a minimal parabolic subgroup of G and λ a P-invariant probability on X. Let ν_0 be the unique m-stationary probability measure on G/P. Let $\tilde{\nu}_0$ be any probability measure on G which projects onto ν_0 under the natural projection of G onto G/P, and put $\mu = \tilde{\nu}_0 * \lambda$ (it follows from [7], that (X,μ) is an m-system, and moreover that any m-stationary measure on X is of this form). Suppose further that the measure preserving P-action (X,λ) is mixing. Then there exists a parabolic subgroup $Q \subset G$, a Q-space Y, and a Q-invariant probability measure η on Y such that the m-system (X,μ) is isomorphic to the "induced" m-system $Y \underset{Q}{\times} G/Q = ((Y \times G)/Q, \tilde{\eta})$, where $\tilde{\eta}$ is an m-stationary measure. In particular (X,μ) is a measure preserving extension of an m-proximal system G/Q, and μ is G-invariant iff $Q = G$.

In the following examples let G be the free group on two generators, $G = F_2 = \langle a, b \rangle$, and $m = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}})$.

4. (See [8]) Let Z be the space of right infinite reduced words on the letters $\{a, a^{-1}, b, b^{-1}\}$. G acts on Z by concatenation on the left and reduction. Let η be the probability measure on Z given by

$$\eta(C(\epsilon_1, \dots, \epsilon_n)) = \frac{1}{4 \cdot 3^{n-1}},$$

where for $\epsilon_j \in \{a, a^{-1}, b, b^{-1}\}$, $C(\epsilon_1, \dots, \epsilon_n) = \{z \in Z : z_j = \epsilon_j,\ j = 1, \dots, n\}$. The measure η is m-stationary and the m-system $\mathcal{Z} = (Z,\eta)$ is m-proximal. In fact $\mathcal{Z}$ is the Poisson boundary $\Pi(F_2, m)$.

5. Let $Y = \{0,1\}$, $\nu = \frac{1}{2}(\delta_0 + \delta_1)$, and the action be defined by $a\epsilon = \bar{\epsilon}$, $b\epsilon = \bar{\epsilon}$ for $\epsilon \in \{0,1\}$, where $\bar{0} = 1$ and $\bar{1} = 0$. $\mathcal{Y} = (Y,\nu)$ is a measure preserving system.

6. Let $X = Y \times Z$, $\mu = \nu \times \eta$, where Y, Z, ν, η are as above, and let the action of G on X be defined as follows:

$$\begin{aligned} a(\epsilon, z) &= (\bar{\epsilon}, a_\epsilon z), \qquad & a^{-1}(\epsilon, z) &= (\bar{\epsilon}, a_{\bar{\epsilon}}^{-1} z), \\ b(\epsilon, z) &= (\bar{\epsilon}, b_\epsilon z), \qquad & b^{-1}(\epsilon, z) &= (\bar{\epsilon}, b_{\bar{\epsilon}}^{-1} z), \end{aligned}$$

where for $g \in G$ we let $g_0 = e$ and $g_1 = g$. Finally let $\pi : X \to Y$ be the projection on the first coordinate. One can check that $m * \mu = \mu$ so that $\mathcal{X}$ is an m-system, and that the extension π is a relatively proximal extension. We claim that the following system is a description of $\Pi(\mathcal{X}) = (M, P^*)$. Let $M = \{\langle(\epsilon, z), (\bar{\epsilon}, z')\rangle : \epsilon \in \{0,1\},\ z, z' \in Z\}$, here $\langle\cdot,\cdot\rangle$ denotes the *unordered* pair. The measure P^* is given by

$$P^*\big((\{\epsilon\} \times A) \times (\{\bar{\epsilon}\} \times B) \cup (\{\bar{\epsilon}\} \times B) \times (\{\epsilon\} \times A)\big) = \eta(A)\eta(B),$$

for $A, B \subset Z$ and $\epsilon \in \{0,1\}$. It is not hard to see that, although the m-system $\mathcal{X}$ is not measure preserving, it admits no nontrivial m-proximal factor.

7. A small variation on example 6 gives an example of a similar nature, with the conditional measures μ_ω being continuous. Take Y to be the diadic adding machine $Y = \{0,1\}^{\mathbb{N}} = \{\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \dots) : \epsilon_i \in \{0,1\}\}$, let $X = Y \times Z$, and define the action of F_2 on X by:

$$a(\epsilon, z) = (\epsilon + \mathbf{1}, a_\epsilon z), \qquad a^{-1}(\epsilon, z) = (\epsilon + \mathbf{1}, a_\epsilon^{-1} z),$$
$$b(\epsilon, z) = (\epsilon + \mathbf{1}, b_\epsilon z), \qquad b^{-1}(\epsilon, z) = (\epsilon + \mathbf{1}, b_{\epsilon+\mathbf{1}}^{-1} z),$$

where $\mathbf{1} = (1, 0, 0, \dots)$ and $a_\epsilon = e$ when $\epsilon_1 = 0$, $a_\epsilon = a$ when $\epsilon_1 = 1$, and b_ϵ is defined similarly.

8. Let G be the closed subgroup of the Lie group $GL(4, \mathbb{R})$ consisting of all 4×4 matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

with $A, B \in GL(2, \mathbb{R})$. We let G act on the subspace X of the projective space $\mathbb{P}^3$ consisting of the disjoint union of the two one dimensional projective spaces $\mathbb{P}^1$, which are naturally embedded in $\mathbb{P}^3$, the quotient space of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Call these two copies X_1 and X_2 respectively. There is a natural projection from (X, G) onto the two-point G-system $(Y, G) = (\{X_1, X_2\}, G)$. Let m be an admissible probability on G and μ an m-stationary measure on X. Then it is easy to see that the m-system (X, μ) is an m-proximal extension of the (measure preserving) two-point system Y. Moreover the m-system (X, μ) has no nontrivial m-proximal factor. If we let $Z \subset M(X)$ be the collection of measures of the form:

$$Z = \{\frac{1}{2}(\delta_{x_1} + \delta_{x_2}) : x_i \in X_i,\ i = 1, 2\},$$

then one can check that the elements of Z are the conditional measures μ_ω of the m-system (X, μ). It follows that $(M(X), P^*)$ is isomorphic as an m-system to the symmetric product $\mathbb{P}^1 \times \mathbb{P}^1 / \{\mathrm{id}, \mathrm{flip}\}$.

9. ([24]) Let $G = SL(2, \mathbb{R})$ and fix an admissible K-invariant measure m on G. In [24, Theorem 3.1] Nevo and Zimmer construct a co-compact lattice $\Gamma < G = SL(2, \mathbb{R})$, a Γ-space Z and an m-stationary measure η on the induced G-space $X = G/\Gamma \underset{\Gamma}{\times} Z$, with the property that $0 < h_\eta(X) < h_\nu(Y)$, where $Y = \Pi(G, m)$ and ν is the unique m-stationary probability measure on Y (see example **1** above).

Claim: The m-system (X, η, G) admits no nontrivial m-proximal factors.

Proof. There is a unique m-proximal G-system, namely the Poisson boundary $(\Pi(G, m), \nu)$. Since the entropy of the m-system $(G/\Gamma \underset{\Gamma}{\times} Z, \eta, G)$ is strictly lower than the entropy of $(\Pi(G, m), \nu)$, the former cannot admit the latter as a factor. □

3. Joinings

Definitions: Let $\mathcal{X}$ and $\mathcal{Y}$ be two m-systems. We say that a probability measure λ on $X \times Y$ is an m**-joining** of the measures μ and ν if it is m-stationary and its marginals are μ and ν respectively. In contrast to the situation in the class of measure preserving dynamical systems, the product measure $\mu \times \nu$ is usually not m-stationary and therefore not an m-joining. On the other hand we have the following natural construction. We let the probability measure $\lambda \in M(X \times Y)$ be defined by

$$\lambda = \mu \curlyvee \nu = \int \mu_\omega \times \nu_\omega dP(\omega).$$

The equation

$$g\lambda = \int \mu_{g\omega} \times \nu_{g\omega} dP(\omega),$$

for each $g \in G$, implies

$$\begin{aligned}\int g\lambda dm(g) &= \int \int g\mu_\omega \times g\nu_\omega dP(\omega) dm(g) \\ &= \int \int \mu_{g\omega} \times \nu_{g\omega} dP(\omega) dm(g) \\ &= \int \mu_\omega \times \nu_\omega dP(\omega) = \lambda;\end{aligned}$$

i.e. λ is m-stationary. We call the m-system $\mathcal{X} \curlyvee \mathcal{Y} = (X \times Y, \lambda)$, the m-**join** of the two m-systems $\mathcal{X}$ and $\mathcal{Y}$. We use the notation $\mathcal{X} \vee \mathcal{Y}$ to denote any joining of the systems $\mathcal{X}$ and $\mathcal{Y}$; e.g. when they are both factors of a third m-system $\mathcal{Z}$ then we usually mean $\mathcal{X} \vee \mathcal{Y}$ to be the factor of $\mathcal{Z}$ defined by the smallest σ-algebra containing $\mathcal{X}$ and $\mathcal{Y}$.

3.1. **Proposition.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be two m-systems,*

1. *if $\mathcal{X}$ is measure preserving then $\mu \curlyvee \nu = \mu \times \nu$;*
2. *if $\mathcal{X}$ is m-proximal then*

$$\mu \curlyvee \nu = \int \delta_{x_\omega} \times \nu_\omega dP(\omega)$$

is the unique m-joining of the two systems.

Proof. (1) Since the conditional measures for $\mathcal{X}$ satisfy $\mu_\omega = \mu$ a.s.,

$$\begin{aligned}\mu \curlyvee \nu &= \int \mu_\omega \times \nu_\omega dP(\omega) \\ &\int \mu \times \nu_\omega dP(\omega) \\ &= \mu \times \int \nu_\omega dP(\omega) \\ &= \mu \times \nu.\end{aligned}$$

(2) Let λ be any m-joining of μ and ν. Our assumption now is that the conditional measures of $\mathcal{X}$ are a.s. point masses δ_{x_ω}, whence the conditional measures $\lambda_\omega = \lim \xi_1 \xi_2 \cdots \xi_n \lambda$ have marginals δ_{x_ω} and ν_ω on X and Y respectively. This means $\lambda_\omega = \delta_{x_\omega} \times \nu_\omega$ and therefore

$$\lambda = \int \lambda_\omega dP(\omega) = \int \delta_{x_\omega} \times \nu_\omega dP(\omega) = \mu \curlyvee \nu.$$

∎

3.2. **Proposition.** (1) *The only endomorphism of a proximal system is the identity automorphism.*
(2) *For every m-system (X, μ) there is a unique maximal proximal factor.*

Proof. (1) Let $\alpha : X \to X$ be an endomorphism of the proximal system (X, μ). Consider the map $\phi : x \mapsto \theta_x = \frac{1}{2}(\delta_x + \delta_{\alpha(x)})$ of X into $M(X)$. This induces a quasifactor $(M(X), \lambda)$ where $\lambda = \phi_*(\mu)$. Now the conditional measures of the proximal system $(M(X), \lambda)$ are point masses of the form δ_{θ_x}. On the other hand applying the barycenter map b to the limits:

$$\xi_1(\omega) \cdots \xi_n(\omega)\lambda \to \delta_{\theta_{x(\omega)}},$$

we get

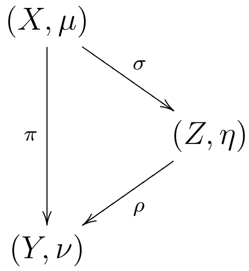
$$\xi_1(\omega) \cdots \xi_n(\omega)\mu \to \delta_{x(\omega)}.$$

Thus $b(\delta_{\theta_{x(\omega)}}) = \theta_{x(\omega)} = \delta_{x(\omega)}$ a.e.; i.e. $\alpha(x) = x$ a.e.

(2) It is easy to check that the join of all proximal factors of an m-system (X, μ) is a maximal proximal factor of (X, μ). ☐

4. A structure theorem for stationary systems

4.1. **Proposition.** *Let*



be a commutative diagram of m-systems
(1) *if π is a measure preserving extension then so are ρ and σ.*
(2) *if π is a proximal extension then so are ρ and σ.*

Proof. (1) Let

$$\mu = \int \mu_y d\nu(y) = \int \mu_z \, d\eta(z), \qquad \eta = \int \eta_y d\nu(y),$$

be the disintegrations of μ over Y and Z and of η over Y respectively. We assume that for all g and ν almost every y, $g\mu_y = \mu_{gy}$, hence

$$g\eta_y = g\sigma\mu_y = \sigma g\mu_y = \sigma\mu_{gy} = \eta_{gy},$$

so that ρ is a measure preserving extension. Now, since

$$\mu = \int \mu_y d\nu(y) = \int \mu_z \, d\eta(z) = \int \Big(\int \mu_z \, d\eta_y(z) \Big) d\nu(y),$$

the uniqueness of disintegration shows that

$$\mu_y = \int \mu_z \, d\eta_y(z).$$

Thus for $g \in G$ we have:

$$g\mu_y = g\Big(\int \mu_z \; d\eta_y(z)\Big) = \int g\mu_z d\eta_y(z),$$

and also

$$g\mu_y = \mu_{gy} = \int \mu_z \; d\eta_{gy}(z) = \int \mu_z \; dg\eta_y(z) = \int \mu_{gz} \; d\eta_y(z).$$

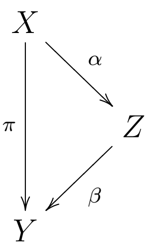
Again the uniqueness of disintegration yields $g\mu_z = \mu_{gz}$, so that also σ is a measure preserving extension.

(2) This is a straightforward consequence of the definition of m-proximal extension. □

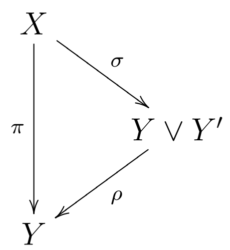
Let us call an m-system (X, μ) **standard** if there exists a homomorphism $\pi : (X, \mu) \to (Y, \nu)$ with (Y, ν) proximal and the homomorphism π a measure preserving extension. Note that with this terminology the results described in the examples **2** and **3** above can be stated as saying that the stationary systems (X, μ) described there are standard (of a very particular kind, namely measure preserving extensions of boundaries of the form G/Q with $Q \subset G$ a parabolic subgroup).

4.2. **Proposition.** (1) *The structure of a standard system as a measure preserving extension of a proximal system is unique.*

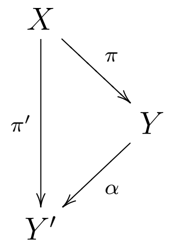
(2) *Let (X, μ) be a standard m-system: $\pi : (X, \mu) \to (Y, \nu)$ with (Y, ν) proximal and the homomorphism π a measure preserving extension. If $\alpha : (X, \mu) \to (Z, \eta)$ is a measure preserving homomorphism then there is a commutative diagram:*



Proof. (1) Let (X, μ) be a standard m-system: $\pi : (X, \mu) \to (Y, \nu)$ with (Y, ν) proximal and the homomorphism π a measure preserving extension. If $\pi' : (X, \mu) \to (Y', \nu')$ is another factor with (Y', ν') proximal, then the system $Y \vee Y'$ is also m-proximal and we have the diagram:

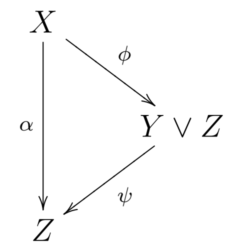


Now ρ is clearly a proximal extension and by proposition 4.1 it is also a measure preserving extension. Thus ρ is an isomorphism, so that Y' is a factor of Y. We now have the diagram:



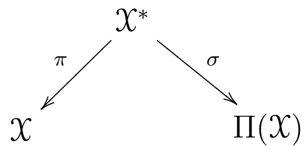
If π' is a measure preserving homomorphism then by proposition 4.1 so is α and being also a proximal homomorphism it is necessarily an isomorphism.

(2) Consider the diagram:



Since α is a measure preserving homomorphism so is ψ (proposition 4.1). On the other hand, since Y is proximal it follows that ψ is a proximal extension. Thus ψ is an isomorphism and we deduce that Y is a factor of Z as required. ☐

4.3. **Theorem** (A structure theorem for stationary systems). *Let $\mathcal{X} = (X, \mu)$ be an m-system, then there exist canonically defined m-systems $\mathcal{X}^* = (X^*, \mu^*)$, and $\Pi(\mathcal{X}) = (M, P^*)$, with $\mathcal{X}^*$ standard and $\Pi(\mathcal{X})$ m-proximal, and a diagram*



where π is an m-proximal extension, and σ is a measure preserving extension. Thus every m-system admits an m-proximal extension which is standard. The m-system $\mathcal{X}$ is measure preserving iff $\Pi(\mathcal{X})$ is trivial. The m-system $\mathcal{X}$ is m-proximal iff both π and σ are isomorphisms. We call $\mathcal{X}^*$ the **standard cover** *of $\mathcal{X}$.*

Proof. We let $\mathcal{X}^* = \mathcal{X} \curlyvee \Pi(\mathcal{X})$. Thus $X^* = X \times M(X)$ and the measure $\mu^* = \mu \curlyvee P^*$ is defined by the integral

$$\mu^* = \int \mu_\omega \times \delta_{\mu_\omega} dP(\omega). \tag{4.1}$$

The assertions of the theorem now follow from propositions 1.2 and 3.1, however for clarity and completeness we give below a more detailed proof. Denote by π and σ the projections on the first and second coordinates respectively. Clearly $\mathcal{X}^* = (X^*, \mu^*)$ is a joining of the systems (X, μ) and $(M(X), P^*)$ in the sense that $\pi(\mu^*) = \mu$, and $\sigma(\mu^*) = P^*$. We show next that μ^* is m-stationary. For $g \in G$ we have a.s.

$$g\mu_\omega = \mu_{g\omega}, \tag{4.2}$$

hence

$$g\mu^* = \int \mu_{g\omega} \times \delta_{\mu_{g\omega}} dP(\omega),$$

hence

$$\begin{aligned}\int_G g\mu^* dm(g) &= \int_G \int_\Omega \mu_{g\omega} \times \delta_{\mu_{g\omega}} dP(\omega) dm(g) \\ &= \int \mu_{\xi_1\omega} \times \delta_{\mu_{\xi_1\omega}} dP(\omega) \\ &= \int \mu_\omega \times \delta_{\mu_\omega} dP(\omega) = \mu^*.\end{aligned}$$

Now (4.1) gives the disintegration of μ^* with respect to P^*, i.e. w.r.t. σ, and (4.2) shows that σ is a measure preserving extension.

Next we mimic the proof of proposition 3 in [8] in order to show that the measures $\theta_\omega = \mu_\omega \times \delta_{\mu_\omega}$ are the conditional measures of the m-system (X^*, μ^*), i.e. we will show that a.s.

$$\lim \xi_1\xi_2 \cdots \xi_n \mu^* = \theta_\omega. \tag{4.3}$$

First observe that

$$\theta_\omega = \lim_{n\to\infty} \xi_1\xi_2 \cdots \xi_n(\mu \times \delta_\mu).$$

Write $\theta_1(\omega) := \theta_\omega$ and let

$$\theta_k = \lim_{l\to\infty} \xi_k\xi_{k+1} \cdots \xi_{k+l}(\mu \times \delta_\mu),$$

so that $\xi_1\xi_2 \cdots \xi_n\theta_{n+1} = \theta_1$. For a bounded continuous function f on X^* and a measure $\iota \in M(X^*)$ we write $f(\iota) = \int_{X^*} f(x^*)d\iota(x^*)$. Now for any such f we have

$$\begin{aligned}&\int_{X^*} f(\xi_1\xi_2 \cdots \xi_n x^*) d\mu^*(x^*) \\ =&\int_\Omega \int_X f(\xi_1\xi_2 \cdots \xi_n(x, \delta_{\omega'}) d\mu_{\omega'}(x) dP(\omega') \\ =&\int_\Omega f(\xi_1\xi_2 \cdots \xi_n(\mu_{\omega'} \times \delta_{\delta_{\omega'}}) dP(\omega') \\ =&E\big(f(\xi_1\xi_2 \cdots \xi_n(\theta_{n+1})|\xi_1\xi_2 \cdots \xi_n\big) \\ =&E\big(f(\theta_1)|\xi_1\xi_2 \cdots \xi_n\big) \to f(\theta_1) = f(\mu_\omega \times \delta_{\mu_\omega}),\end{aligned}$$

where the convergence in the last line follows from the martingale convergence theorem. Since clearly a.s. $\pi : (X \times M, \mu_\omega \times \delta_{\mu_\omega}) \to (X, \mu_\omega)$ is 1-1, we see that π is an m-proximal extension. This completes the proof of the theorem. □

4.4. **Theorem.** *If (X, μ) is an m-system with maximal entropy (i.e. $h_m(X, \mu) = h_m(\Pi(G, m))$); then (X, μ) is standard and it admits the Poisson boundary $\Pi(G, m)$ as its maximal proximal factor.*

Proof. Let $\pi : X^* \to X$ be the standard cover of (X, μ), so that in particular π is a proximal extension. Since π does not raise entropy it is a measure preserving extension (theorem 1.3). Thus π is an isomorphism and X is a standard system whose maximal proximal factor has maximal entropy. Again theorem 1.3 implies that this factor is isomorphic to $\Pi(G, m)$. $\square$

4.5. **Theorem.** *Let $\mathcal{X} = (X, \mu)$ be an m-system which admits a strict tower of proximal and measure preserving extensions*

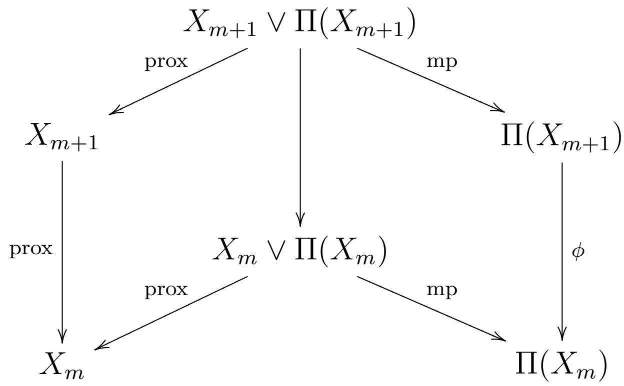
$$X \cdots \to X_{n+1} \to X_n \to \cdots \to X_2 \to X_1 \to X_0.$$

Assume that X_0 is the maximal proximal factor of $\mathcal{X}$, that the proximal and measure preserving maps alternate, and that each such map is maximal. Thus $X_{2n+1} \to X_{2n}$ is measure preserving and if $X \to Y \to X_{2n+1} \to X_{2n}$ is such that $Y \to X_{2n}$ is also measure preserving then $Y = X_{2n+1}$. Likewise $X_{2n+2} \to X_{2n+1}$ is proximal and if $X \to Y \to X_{2n+2} \to X_{2n+1}$ is such that $Y \to X_{2n+1}$ is also proximal then $Y = X_{2n+2}$. Let

$$\Pi(X) \cdots \to \Pi(X_{n+1}) \to \Pi(X_n) \cdots \to \Pi(X_2) \to \Pi(X_1) \to \Pi(X_0) = X_0 \tag{4.4}$$

be the corresponding sequence of homomorphisms. (Since for every n the map $\phi : X_{2n+1} \to X_{2n}$ is measure preserving, $\Pi(X_{2n+1}) \to \Pi(X_{2n})$ is an isomorphism.) If at any stage in this sequence we have that $\Pi(X_{2n+2}) \to \Pi(X_{2n+1})$ is an isomorphism, then $X = X_{2n+1}$.

Proof. For convenience we write $m = 2n + 1$. In the diagram



by assumption, ϕ is an isomorphism and therefore all the maps on the right of the central vertical arrow $X_{m+1} \vee \Pi(X_{m+1}) \to X_m \vee \Pi(X_m)$ are measure preserving maps. On the other hand all the arrows on the left of this arrow are proximal maps. We conclude that $X_{m+1} \vee \Pi(X_{m+1}) \to X_m \vee \Pi(X_m)$ is both measure preserving and proximal, hence an isomorphism. However this implies that also $X_{m+1} \to X_m$ is an isomorphism. Since we assumed that at each stage the extension is maximal we now realize that the whole tower above X_m collapses, i.e. $X = X_m = X_{2n+1}$. $\square$

4.6. **Corollary.** *For $G = SL(n, \mathbb{R})$ and K-invariant admissible m, every strict maximal tower is of height $\leq n$.*

Proof. As was shown in [7] the Poisson (G, m)-space $\Pi(G, m)$ is the flag manifold on $\mathbb{R}^n$. Since every proximal G-system is a factor of $\Pi(G, m)$, every sequence of the form (4.4) is defined by a nested sequence of parabolic subgroups, whence of length at most n. $\square$

Examples: 10. Applying the construction of the structure theorem to example **3.** in section 1, we obtain the following description for the m-system $(X^*, \mu^*) = (X \times M, \mu \curlyvee \nu)$. X^* can be taken as the subset of $X \times X$ consisting of all (*ordered*) pairs $((\epsilon, z), (\bar{\epsilon}, z'))$, $\epsilon \in \{0, 1\}$, $z, z' \in Z$, with the diagonal action $g((\epsilon, z), (\bar{\epsilon}, z')) = (g(\epsilon, z), g(\bar{\epsilon}, z'))$. The measure μ^* is then given by

$$\frac{1}{2}(\delta_0 + \delta_1) \times \eta \times \eta.$$

11. As we have seen (proposition 3.2), for every m-system (X, μ) there is a uniquely defined maximal proximal factor. This is not always the case with respect to measure preserving factors. We produce next an example of a product system $(X, \mu) = (Z \times Y, \eta \times \nu)$ where (Z, η) is m-proximal and (Y, ν) is measure preserving—so that (X, μ) is standard—with a factor (Y', ν') which is also measure preserving but such that the factor $Y \vee Y'$ of X is not measure preserving.

We let $G = F_2$, the free group on two generators a and b, $m = \frac{1}{4}(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}})$. Let Z be the Poisson boundary $\Pi(F_2, m)$ which we can take as the space of right infinite reduced words on the letters $\{a, a^{-1}, b, b^{-1}\}$ with the natural Markov measure η as in example (4) above. The system (Y, ν) will be the Bernoulli system $Y = \{0, 1\}^{F_2}$ with product measure $\nu = \{\frac{1}{2}, \frac{1}{2}\}^{F_2}$. Thus ν is an invariant measure under the natural action of F_2 on Y by translations. Clearly $\mu = \eta \times \nu = \eta \curlyvee \nu$ is m-stationary, so that (X, μ) is an m-system.

Next let A be the subset $\{z \in Z : a \text{ is the first letter of } z\}$, and let $\phi : Z \to Y$ be the continuous function defined by $(\phi(z))_g = 1_A(gz)$. We observe that the map $\Phi : X \to Y$ defined by $\Phi(z, y) = z + y \pmod 1$ is an equivariant continuous map. Let $Y' = \Phi(X)$ and $\nu' = \Phi_*(\mu)$. It is now easy to check that (Y', ν') is a measure preserving factor of the M-system (X, μ), which is isomorphic to the Bernoulli system (Y, ν). However it is also clear that the factor $Y \vee Y'$ of (X, μ) is a non-measure preserving m-system. In fact $Y \vee Y'$ admits the non-trivial proximal factor $Z' = \phi(Z)$.

4.7. **Remark.** For ergodic probability measure preserving transformations there is a more satisfactory structure theorem (due to Furstenberg [10], [12] and independently to Zimmer [27], [28]) according to which every such system is canonically presented as a weakly mixing extension of a measure-distal system (the latter is defined as a tower, possibly of infinite height, of compact extensions). In topological dynamics there is an analogous theorem for a minimal dynamical system (X, G) (see [4], [26]). However, as in our theorem 4.3, one is forced in this setting to first associate with X a proximal extension $X^* \to X$ so that only X^* has the required structure of a weakly mixing extension of a PI-system (where the latter is a tower of alternating proximal and isometric extensions). In [15] there is an example of a minimal dynamical system (X, T) which does not admit nontrivial factors that are either proximal or incontractible (this is the analogue of a measure preserving system in topological dynamics). We do not know how to construct a similar example for stationary systems. Such an example will show that in some sense one can not do better than what one gets in theorem 4.3.

4.8. **Problem.** Are there a group G, a probability measure m on G, and an ergodic m-stationary system $\mathcal{X} = (X, \mu, G)$ such that $\mathcal{X}$ does not admit nontrivial factors that are either proximal or measure preserving?

5. Nevo-Zimmer theorem in an abstract setup

Definitions:

(1) Notations as in theorem 4.3, we say that the quasifactor $\Pi(\mathcal{X}) = (M, P^*)$ is **mixingly embedded** in the m-system $\mathcal{X} = (X, \mu)$, if the measure preserving extension $\sigma : \mathcal{X}^* \to \Pi(\mathcal{X})$ is a mixing extension; i.e. if for every $f \in L_\infty(\mu^*)$ and every sequence $g_n \to \infty$ in G,
$$w^*\text{-}\lim g_n(f - E^{\Pi(\mathcal{X})} f) = 0,$$
where $E^{\Pi(\mathcal{X})} f$ is the conditional expectation of f with respect to the factor $\Pi(\mathcal{X})$.

(2) For an m-system (X, μ) and a subset V of $L_\infty(\mu)$, let
$$\mathcal{F}(V) = \{w^*\text{-}\lim g_n f : f \in V,\ g_n \to \infty\},$$
the set of all weak* limit points of sequences $g_n f$ where $f \in V$ and $g_n \to \infty$. Let $\mathcal{F}(V)$ be the smallest σ-algebra with respect to which all members of $\mathcal{F}(V)$ are measurable. Call the m-system (X, μ) **reconstructive** with respect to V if $\mathcal{F}(V)$ is the full σ-algebra of measurable sets on X.

5.1. **Theorem.** *Let $\mathcal{X} = (X, \mu)$ be an m-system such that*

(1) *The canonical m-proximal quasifactor $\Pi(\mathcal{X}) = (M, P^*)$ is mixingly embedded in $\mathcal{X}$.*

(2) *$\Pi(\mathcal{X})$ is a reconstructive m-system with respect to the subspace*
$$V = E^{\Pi(\mathcal{X})}(C(X)).$$

Then the m-proximal quasifactor $\Pi(\mathcal{X})$ is actually a factor of $\mathcal{X}$.

Proof. Consider an arbitrary continuous function f on X, $f \in C(X) \subset L_\infty(\mu) \subset L_\infty(X \times M(X), \mu^*)$, and the corresponding function $\tilde{f} \in L_\infty(P^*)$ on $M(X)$ defined by:
$$\tilde{f}(\mu_\omega) = \int_X f(x) d\mu_\omega = E^{\Pi(\mathcal{X})} f.$$
By assumption (1), for every sequence $g_n \to \infty$ in G for which $w^*\text{-}\lim g_n \tilde{f}$ exists, we have
$$\hat{f} = w^*\text{-}\lim g_n f = w^*\text{-}\lim g_n \tilde{f}, \tag{5.1}$$
hence $\hat{f}$ is in the w^*-closed subspace $L_\infty(\mu) \cap L_\infty(P^*)$.

On the other hand, by assumption (2), with the subspace $V = \{\tilde{f} : f \in C(X)\} = E^{\Pi(\mathcal{X})}(C(X))$, the smallest σ-algebra with respect to which all the functions:
$$\{\hat{f} = w^*\text{-}\lim g_n \tilde{f} : f \in C(X),\ g_n \to \infty\}$$
are measurable is the full σ-algebra of measurable sets on $\Pi(\mathcal{X})$. It thus follows that with respect to μ^*, $L_\infty(P^*) \subset L_\infty(\mu)$ and the proof is complete. □

5.2. **Corollary.** *Let $\mathcal{X} = (X, \mu)$ be an m-system such that the canonical m-proximal quasifactor $\Pi(\mathcal{X}) = (M, P^*)$ is mixingly embedded in $\mathcal{X}$. Then for every $f \in L_\infty(X, \mu)$, for a.e. ω*

$$w^*\text{-}\lim \xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega) f \equiv \tilde{f}(\mu_\omega).$$

Proof. In the proof of theorem 5.1 taking $g_n = \eta_n(\omega) = \xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega)$ we have, for every $h \in L_1(\mu^*)$ by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int_{X^*} \tilde{f}(\xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega)\mu_{\omega'}) h(x, \mu_{\omega'})\, d\mu^*(x, \mu_{\omega'})$$
$$= \tilde{f}(\mu_\omega) \int_{X^*} h(x, \mu_{\omega'})\, d\mu^*(x, \mu_{\omega'});$$

i.e. $w^*\text{-}\lim \xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega)\tilde{f} \equiv \tilde{f}(\mu_\omega)$, ω-a.s. In view of (5.1) we deduce that ω-a.s.

$$w^*\text{-}\lim \xi_1(\omega)\xi_2(\omega)\cdots\xi_n(\omega) f \equiv \tilde{f}(\mu_\omega),$$

□

Example: Let T and S be two discrete countable groups, m_S and m_T probability measures on S and T respectively such that the corresponding Poisson spaces $\Pi(S, m_S)$ and $\Pi(T, m_T)$ are nontrivial. We form the product group $G = T \times S$ and the product measure $m = m_T \times m_S$.

5.3. **Theorem.** *For $G = T \times S$ as above the Poisson spaces for the couples (G, m), (T, m_T) and (S, m_S) satisfy:*

$$\Pi(G, m) = \Pi(T, m_T) \times \Pi(S, m_S).$$

Proof. Clearly the systems $\Pi(T, m_T)$ and $\Pi(S, m_S)$ can be viewed as G m-systems and as such they are proximal. Thus these systems are factors of the m-system $\Pi(G, m)$. It is now easy to check that if η_T and η_S are the m-stationary measures on $\Pi(T, m_T)$ and $\Pi(S, m_S)$ respectively then the measure $\eta_T \curlyvee \eta_S = \eta_T \times \eta_S$. Whence $\Pi(T, m_T) \times \Pi(S, m_S)$ is a factor of the system $\Pi(G, m)$. Since the entropy of both systems is $h_m(\Pi(T, m_T), \eta_T) + h_m(\Pi(S, m_S), \eta_S)$ we can now apply theorem 1.3 to conclude that $\Pi(G, m) = \Pi(T, m_T) \times \Pi(S, m_S)$. □

5.4. **Remark.** Another proof of this fact follows directly from the characterization of the Poissson boundary of (G, m) as the space of ergodic components of the time shift in the path space of the random walk due to Kaimanovich and Vershik, [21].

5.5. **Lemma.** *Let $(X, \mathcal{B}, \mu), (Y, \mathcal{F}, \nu)$ be two probability spaces, $\mathcal{A}$ a sub-σ-algebra of $\mathcal{F}$ and $f \in L_\infty(X \times Y, \mu \times \nu)$. If for μ a.e. $x \in X$ the function $f_x(y) = f(x, y)$ is $\mathcal{A}$ measurable, then f is $\mathcal{B} \times \mathcal{A}$ measurable.*

5.6. **Theorem.** *Let (X, μ, G) be an m-system. If the canonical m-proximal quasifactor $\Pi(\mathcal{X}) = (M(X), P^*)$ is mixingly embedded in $\mathcal{X}$. Then $\Pi(\mathcal{X})$ is a factor of (X, μ).*

Proof. In view of theorem 5.1, all we have to show is that $\Pi(\mathcal{X})$ is a reconstructive m-system with respect to the subspace $V = E^{\Pi(\mathcal{X})}(C(X))$. For $f \in C(X)$ the function $\tilde{f}$ is $\Pi(\mathcal{X})$ measurable. Since $\Pi(\mathcal{X})$ is a factor of $\Pi(G)$, by theorem 5.3, lifting $\tilde{f}$ to $\Pi(G)$

we can write $\tilde{f}$ as a function of two variables $\tilde{f}(u,v)$, with $u \in \Pi(S)$ and $v \in \Pi(T)$. Now for almost every $v_0 \in \Pi(T)$ there exists a sequence $t_n^{v_0} \in T$ with $\lim t_n^{v_0} v = v_0$ for μ_T almost every $v \in \Pi(T)$. Thus, by (5.1), we see that

$$\tilde{f}^{v_0}(u) = w^*\text{-}\lim t_n^{v_0} f = w^*\text{-}\lim t_n^{v_0} \tilde{f}(u,v) = \tilde{f}(u,v_0)$$

is $\mathfrak{X}$ measurable. Similarly for almost every $u_0 \in \Pi(S)$ the function $\tilde{f}_{u_0}(v) = f(u_0,v)$ is $\mathfrak{X}$ measurable. By lemma 5.5, $\tilde{f}$ is $\mathfrak{X}$ measurable and since the subspace $V = \{\tilde{f} : f \in C(X)\}$ generates $C(\Pi(X))$ as an algebra, we conclude that $\Pi(X)$ is a reconstructive m-system with respect to V. □

6. A Szemerédi type theorem for $SL(2,\mathbb{R})$.

In this section G will denote the Lie group $SL(2,\mathbb{R})$ and we write $G = KAN$ for the standard Iwasawa decomposition of G; in particular K is the subgroup of 2 by 2 orthogonal matrices.

Recall that a **mean** on a topological group G is a positive linear functional ρ on $LUC(G)$ with $\rho(\mathbf{1}) = 1$. Here $LUC(G)$ denotes the commutative C^*-algebra of bounded, complex valued, left uniformly continuous functions on G. ($f : G \to \mathbb{C}$ is **left uniformly continuous** if for every $\epsilon > 0$ there exists a neighborhood V of the identity element $e \in G$ such that $\sup_{g\in G} |f(vg) - f(g)| < \epsilon$ for every $v \in V$.) The set of means on G forms a w^*-closed convex subset of $LUC(G)^*$ and we say that an element of this set is m**-stationary** if $m * \rho = \rho$. By the Markov-Kakutani fixed point theorem the set of m-stationary means is nonempty.

Let Z be the (compact Hausdorff) Gelfand space corresponding to the C^*-algebra $\mathfrak{L} = LUC(G)$. Recall that Z can be viewed as the space of non-zero continuous C^*-homomorphisms of the C^*-algebra $\mathfrak{L}$ into $\mathbb{C}$. In particular, for each $g \in G$ the evaluation map $z_g : F \mapsto F(g)$ is an element of Z. The fact that $\mathfrak{L}$ is G-invariant (i.e. for $f \in \mathfrak{L}$ and $g \in G$ also $f_g \in \mathfrak{L}$, where $f_g(h) = f(gh)$) implies that there is a naturally defined G-action on Z. We have $gz_e = z_g$ for every $g \in G$, and it follows directly that the G-orbit of the point z_e is dense in Z. Also by the construction of the Gelfand space we obtain a natural isomorphism of the commutative C^*-algebras $\mathfrak{L}$ and $C(Z)$. Let $\tilde{f}$ denote the element of $C(Z)$ which corresponds to f under this isomorphism. Now according to Riesz' representation theorem we identify $LUC(G)^*$ with the Banach space of complex regular Borel measures on Z. In this setting a mean on G is identified with a probability measure on Z. If L is a subset of G and ρ is a mean on G, we say that L is **charged by** ρ and write $\rho(L) > 0$ if

$$\mu_\rho(\mathrm{cls}\,\{z_g : g \in L\}) > 0.$$

Here μ_ρ is the probability measure on Z which corresponds to the mean ρ. It is easy to check that with respect to the natural G-action on Z the mean $m * \rho$ (defined by $m * \rho(f) = \int_G \rho(f_g)\, dm(g)$) corresponds to the measure $m * \mu_\rho$, so that ρ is m-stationary if and only if the measure μ_ρ is m-stationary.

6.1. **Theorem.** *Let m be an admissible probability measure on G and let ρ a K-invariant m-stationary mean on $G = SL(2,\mathbb{R})$. If $\rho(L) > 0$ for a subset $L \subset G$ then for every $\epsilon > 0$, $k \geq 1$ and a compact set $Q \subset G$, there exist a_0 and h in $G \setminus Q$ such that*

$$a_0, ha_0, \ldots, h^k a_0 \in L_\epsilon,$$

where $L_\epsilon = \{g \in G : d(g, L) < \epsilon\}$.

6.2. **Lemma.** *(A correspondence principle) Let* G *be a locally compact group. Given a nonempty subset* $L \subset G$*, there exists a compact metric* G*-space* X *and an open subset* $A \subset X$ *such that*

$$g_1^{-1}A \cap g_2^{-1}A \cap \cdots \cap g_k^{-1}A \neq \emptyset \Longrightarrow g_1 a, \dots, g_k a \in L_\epsilon,$$

for some $a \in G$*. If, moreover,* m *is a probability measure on* G *and* ρ *an* m*-stationary mean on* G *with* $\rho(L) > 0$*, then there exists an* m*-stationary probability measure* μ *on* X *with* $\mu(A) \geq \rho(L)$*.*

Proof. Let $f : G \to [0, 1]$ be a left uniformly continuous function such that $f(g) = 1$ for every $g \in L_{\epsilon/2}$ and $f(g) = 0$ for every $g \notin L_\epsilon$. Let $\mathcal{A}$ be the uniformly closed subalgebra of the algebra $LUC(G)$ of complex valued bounded left uniformly continuous functions on G generated by the orbit $\{f_g : g \in G\}$ of f, where $f_g(h) = f(gh)$. Let X be the (compact metric) Gelfand space corresponding to $\mathcal{A}$. The fact that $\mathcal{A}$ is G-invariant implies that there is a naturally defined G-action on X. Clearly the restriction map $\pi : Z \to X$, where with the above notation Z is the Gelfand space corresponding to $\mathcal{L}$, is a homomorphism of the corresponding dynamical systems. We denote $x_g = \pi(z_g)$, so that $gx_e = x_g$ and $\operatorname{cls}\{gx_e : g \in G\} = X$.

By the construction of the Gelfand space we obtain a natural isomorphism of the commutative C^*-algebras $\mathcal{A}$ and $C(X)$. Let $\hat{f}$ denote the element of $C(X)$ which corresponds to f under this isomorphism.

Clearly the restriction of ρ to $\mathcal{A}$ defines an m-stationary probability measure μ on X, so that the system (X, μ, G) is an m-system. In fact $\mu = \pi_*(\mu_\rho)$.

Let $A = \{x \in X : \hat{f}(x) > 1/2\}$ and consider the set $N(x_e, A) = \{g \in G : gx_e \in A\}$. We clearly have $\{g \in G : f(g) > 1/2\} = \{g \in G : gx_e \in A\}$. Note that indeed

$$\begin{aligned}
\mu(A) &= \int_X \mathbf{1}_A \, d\mu = \int_X \mathbf{1}_{\{\hat{f} > 1/2\}} \, d\mu \\
&\geq \int_X \mathbf{1}_{\{\hat{f} = 1\}} \, d\mu = \int_Z \mathbf{1}_{\{\tilde{f} = 1\}} \, d\mu_\rho \\
&\geq \rho(L_{\epsilon/2}) \geq \rho(L).
\end{aligned}$$

Now assume $g_1^{-1}A \cap g_2^{-1}A \cap \cdots \cap g_k^{-1}A \neq \emptyset$ and let x be a point in this intersection. Then $g_i x \in A$ for $i = 1, \dots, k$ and choosing $a \in G$ so that $d(ax_0, x)$ is sufficiently small, we also have $g_i a x_0 \in A$ for $i = 1, \dots, k$. Thus $f(g_i a) > 1/2$ and we conclude that $g_1 a, g_2 a, \dots, g_k a \in L_\epsilon$. □

Proof of the theorem. By the correspondence principle, lemma 6.2, we can associate with L an m-system (X, μ, G) and an open subset $A \subset X$ with $\mu(A) \geq \rho(L) > 0$ such that

$$\mu(g_1^{-1}A \cap g_2^{-1}A \cap \cdots \cap g_k^{-1}A) > 0 \Longrightarrow g_1 a, g_2 a, \dots, g_k a \in L_\epsilon, \tag{6.1}$$

for some $a \in G$.

Let

$$\begin{array}{ccc} & (X^*, \mu^*) & \\ \overset{\pi}{\swarrow} & & \overset{\sigma}{\searrow} \\ (X, \mu) & & \Pi(\mathfrak{X}) = (Y, \lambda), \end{array}$$

be the canonical standard cover of (X, μ), given by theorem 4.3, and set $A^* = \pi^{-1}(A), B = \sigma(A^*)$.

Let

$$\mu^* = \int_Y \mu_y \times \delta_y \, d\lambda(y)$$

be the decomposition of μ^* over (Y, λ). If we write $A^* = \bigcup\{A_y \times \{y\} : y \in B\}$ then, by reducing B if necessary, we can assume that $\mu_y(A_y) \geq \delta$ for some $\delta > 0$.

In the present situation $Y = \mathbb{P}^1$, the projective line, and $\sigma(\mu^*) = \lambda$ is Lebesgue measure. Let $y_0 \in B$ be a Lebesgue density point of B.

Claim: If g is a parabolic element of $SL(2, \mathbb{R})$ with fixed point y_0 then for every N

$$\lambda(B \cap gB \cap g^2B \cap \cdots \cap g^N B) > 0.$$

Proof of claim: Of course $y_0 = gy_0$ is a Lebesgue density point of each of the sets $g^j B$ and therefore we can choose an interval $J \subset Y$ such that

$$\frac{|J \cap g^j B|}{|J|} \geq (1 - \frac{1}{2N})|J| \quad j = 1, \ldots, N,$$

whence $\lambda(J \cap \bigcap_{j=1}^N g^j B) \geq \frac{1}{2}|J|$.

Now back to the proof of theorem 6.1.

Szemerédi's theorem yields, for every positive integer k and $\delta > 0$, a positive integer $N = N(k, \delta)$ such that every subset of $\{1, 2, \ldots, N\}$ of size δN contains an arithmetic progression of length k.

Take g as in the above claim, and such that $\{g^n : n = 1, 2, \ldots\} \cap Q = \emptyset$. Denoting $B_0 = B \cap gB \cap g^2B \cap \cdots \cap g^N B$, we have for λ almost every point $y \in B_0$,

$$\sum_{i=1}^N \int_X \mathbf{1}_{A_i}(x) \, d\mu_{g^i y}(x) \geq \delta N,$$

where $A_i = A_{g^i y}$. Thus for some $A_{i_j}, 1 \leq j \leq [\delta N]$ the intersection $\cap_{j=1}^{[\delta N]} A_{i_j} \neq \emptyset$. If we take $N = N(k, \delta)$ then we can find s and d such that

$$\{s, s+d, s+2d, \ldots, s+kd\} \subset \{i_1, i_2, \ldots, i_{[\delta N]}\},$$

and some $x^* \in X^*$ with

$$g^s x^*, g^{s+d} x^*, g^{s+2d} x^*, \ldots, g^{s+kd} x^*$$

all in A^*, hence, with $x = \pi(x^*)$,

$$g^s x, g^{s+d} x, g^{s+2d} x, \ldots, g^{s+kd} x$$

all in A.

Since there are only finitely many possible s and d we conclude that for some pair s, d,

$$\mu(g^s A \cap g^{s+d} A \cap \cdots \cap g^{s+kd} A) > 0. \tag{6.2}$$

In fact

$$\begin{aligned}
&\mu(g^s A \cap g^{s+d} A \cap \cdots \cap g^{s+kd} A)\\
&= \mu^*(g^s A^* \cap g^{s+d} A^* \cap \cdots \cap g^{s+kd} A^*)\\
&= \int_{B_0} \int_X \prod_{j=0}^{k} \mathbf{1}_{A_{g^{s+jd}y}}(x)\, dg^{s+jd}\mu_y(x)\, d\lambda(y)\\
&= \int_{B_0} \int_X \prod_{j=0}^{k} \mathbf{1}_{A_{g^{s+jd}y}}(x)\, d\mu_{g^{s+jd}y}(x)\, d\lambda(y).
\end{aligned}$$

Thus, if $\mu(g^s A \cap g^{s+d} A \cap \cdots \cap g^{s+kd} A) = 0$ then also

$$\int_X \prod_{j=0}^{k} \mathbf{1}_{A_{g^{s+jd}y}}(x)\, d\mu_{g^{s+jd}y}(x)\, d\lambda(y) = 0$$

for λ a.e. $y \in B_0$, contradicting our assumption on s and d.

From (6.2), by the correspondence principle (6.1), we can find $a \in G$ with

$$g^{-s}a, g^{-(s+d)}a, \dots, g^{-(s+kd)}a \in L_\epsilon.$$

Setting $a_0 = g^{-s}a$ and $h = g^{-d}$ we finally get

$$a_0, ha_0, \dots, h^k a_0 \in L_\epsilon.$$

□

6.3. **Remark.** Independently of our work T. Meyerovich applies in a recent work [22], similar ideas in order to obtain multiple and polynomial recurrence lifting theorems for infinite measure preserving systems.

7. WAP actions are stiff

A compact topological dynamical system (X, G) is called **weakly almost periodic** or **WAP**, if for every $f \in C(X)$, the set $\{f_g : g \in G\}$ is relatively compact in the weak topology on $C(X)$ (where $f_g \in C(X)$ is defined by $f_g(x) = f(gx)$.) Ellis and Nerurkar [5] showed that (X, G) is weakly almost periodic if and only if every element p in the enveloping semigroup E of the system (X, G) is a continuous map. (Recall that $E = E(X, G)$, the **enveloping semigroup** of the compact topological dynamical system (X, G) is, by definition, the closure of the set of maps $\{g : X \to X : g \in G\}$ in the compact product space X^X, where the semigroup structure is defined by composition of maps.) As shown in [5] the enveloping semigroup$E = E(X, G)$ of a WAP system contains a unique minimal left ideal I which is in fact a compact topological group. Consequently, in a topologically transitive WAP system there is a unique minimal subset and the action of G on this minimal set is equicontinuous.

A topological dynamical system (X, G) is called **stiff with respect to** m or m-**stiff** if every m-stationary measure on X is G-invariant, (see [13]). Our goal in this section is to show that WAP systems are stiff (theorem 7.4 below).

7.1. **Lemma.** *Let (X, G) be a WAP dynamical system. Every element $p \in E$ defines an element $p_* \in E(M(X), G)$ and the map $p \mapsto p_*$ is an isomorphism of $E = E(X, G)$ onto $E(M(X), G)$. In particular the dynamical system $(M(X), G)$ is also WAP.*

Proof. If $g_i \to p$ is a net of elements of G converging to $p \in E = E(X,G)$, then by Grothendieck's theorem, for every $f \in C(X)$, $f \circ g_i \to f \circ p$ weakly in $C(X)$. Therefore, we have for every $\nu \in M(X)$ and $f \in C(X)$:

$$g_i\nu(f) = \nu(f \circ g_i) \to \nu(f \circ p) := p_*\nu(f).$$

It is easy to see that $p \mapsto p_*$ is an isomorphism of flows, whence a semigroup isomorphism. Finally as G is dense in both enveloping semigroups, it follows that this isomorphism is onto. □

In the sequel we will identify the two enveloping semigroups and will write p for both p and p_*.

We recall the following theorem of R. Azencott ([1], theorem I.2, page 11).

7.2. **Theorem.** *Let (X,G) be a topological dynamical system with X a compact metric space. Let μ be a probability measure on X. The following properties are equivalent:*

1. *For every $x \in X$, the measure δ_x is a weak* limit point of the set $\{g\mu : g \in G\}$ in $M(X)$.*
2. *For every countable dense subset D of X there exists a Borel subset A of X with $\mu(A) = 1$ and with the property that for every $x \in D$ there exists a sequence $g_n \in G$ such that*

$$\lim g_n y = x \qquad \forall y \in A.$$

We call a measure μ satisfying the equivalent conditions of theorem 7.2 a **contractible** measure, we then call the dynamical system (X,μ,G) a **contractible system**. Note that a contractible system is necessarily topologically transitive and moreover every point of the subset A belongs to the dense G_δ subset X_{tr} of the transitive points of X. Also note that every m-proximal system (X,μ,G), with $X = \mathrm{supp}\,(\mu)$, is contractible.

7.3. **Lemma.** *Let (X,G) be a WAP system. Let μ be an m-stationary probability measure on X with $X = \mathrm{supp}\,(\mu)$ such that the m-system (X,μ) is m-proximal. Then (X,G) is the trivial one point system.*

Proof. Let X_{tr} be the dense, G-invariant, G_δ subset of X consisting of all points with dense G-orbit. Let D and A be the subsets of X given by theorem 7.2 (2). Since D can be any countable dense subset of X, we can assume that $D \subset X_{tr}$. Fix a point $x_0 \in D$ and let $g_n \in G$ satisfy $\lim g_n y = x_0$ for every $y \in A$. We can assume that the limit $p = \lim g_n$ exists in the enveloping semigroup $E = E(X,G)$, and then $\lim g_n y = py = x_0$ for every $y \in A$. Since p is a continuous map and since clearly A is a dense subset of X, it follows that $px = x_0$ for every $x \in X$. The elements of the left ideal $I = Ep$ are in 1-1 correspondence with the points of X (with $q_x \in I$ defined by $q_x y = x, \forall y \in X$) and it follows that I is the unique minimal left ideal in E. It is now clear that (X,G) is a minimal proximal system. However in a WAP system the group action on the unique minimal subset is equicontinuous and we conclude that X consists of a single point. □

7.4. **Theorem.** *Let G be a locally compact second countable topological group, m a probability measure on G with the property that the smallest closed subgroup containing* $\mathrm{supp}\,(m)$ *is all of G. Then every WAP dynamical system (X, G) is m-stiff.*

Proof. Let $E = E(X, G)$ be the enveloping semigroup of the WAP system (X, G). Let μ be an m-stationary ergodic probability on X; we will show that $g\mu = \mu$ for every $g \in G$. As in section 1 we let

$$\lim_{n\to\infty} \eta_n \mu = \mu_\omega, \qquad \omega \in \Omega_0,$$

be the conditional measures of the m-system $\mathcal{X}$, and let $P^* \in M(M(X))$ be the distribution of the $M(X)$-valued random variable $\mu(\omega) = \mu_\omega$. Let now $Z = \mathrm{supp}\,(P^*) \subset M(X)$. Clearly Z is a closed G-invariant subset of $M(X)$, and by proposition 1.2 the m-dynamical system (Z, P^*, G) is m-proximal. By lemma 7.1 the dynamical system (Z, G) is WAP and therefore, by lemma 7.3, it is the trivial one point system. Since the barycenter of P^* is μ, we have $P^* = \delta_\mu$, and it follows that P^* as well as μ are G-invariant measures. □

8. The SAT property

The notion of SAT (strongly approximately transitive) dynamical systems was introduced by Jaworsky in [18], where he developed their theory for discrete groups. For these groups he shows that the stationary measure on the Poisson boundary is SAT. It was later used in a slightly stronger version (SAT*) by Kaimanovich [20] in order to study the horosphere foliation on a quotient of a CAT(-1) space by a discrete group of isometries G, using the SAT* property on the boundary of G.

Definitions Let G be a locally compact second countable topological group. We fix some right Haar measure $m = m_G$ and let e be the identity element of G.

8.1. **Definitions.**

1. A **Borel G-space** is a standard Borel space $(X, \mathcal{X}, G)$ with a Borel action $G \times X \to X$.
2. A **G-system** is a Borel G-space $(X, \mathcal{X}, G)$ equipped with a probability measure μ whose measure class is preserved by each element of G.
3. We say that a Borel probability measure μ on a Borel G-space is **strongly approximately transitive (SAT)** if the measure class of μ is preserved by each element of G and:

   For every $A \in \mathcal{X}$ with $\mu(A) > 0$ there is a sequence $g_n \in G$ such that $\lim_{n\to\infty} \mu(g_n A) = 1$.

   When μ is SAT we will say that the system $(X, \mathcal{X}, \mu, G)$ is SAT.
4. If Y is a compact metric space, G acts on Y via a continuous representation of G into Homeo (Y) and ν is a Borel probability measure whose measure class is preserved by each element of G, we will say that the dynamical system $(Y, \mathcal{B}(Y), \nu, G)$ is **topological** ($\mathcal{B}(Y)$ denotes the Borel field on Y).
5. Let $(X, \mathcal{X}, \mu, G)$ be a G-system. A topological G-system $(Y, \mathcal{B}(Y), \nu, G)$ is a **topological model** for $(X, \mathcal{X}, \mu, G)$ if $\mathrm{supp}\,(\nu) = Y$ and $(X, \mathcal{X}, \mu, G)$ and $(Y, \mathcal{B}(Y), \nu, G)$ are isomorphic as G measure boolean algebras; i.e. there is an equivariant isomorphism between the corresponding measure algebras.

(6) Recall that for a topological system (Y, G), we say that a probability measure ν on Y is **contractible** if for every $y \in Y$ there exists a sequence $g_n \in G$ such that, in the weak* topology, $\lim_{n\to\infty} g_n\nu = \delta_y$.
(7) Given a Borel system $(X, \mathcal{X}, G)$ we say that a probability measure μ on X is **absolutely contractible** if for each topological model (Y, ν, G) of $(X, \mathcal{X}, \mu, G)$ the measure ν on Y is contractible.

Our goal is to show that a measure μ is SAT iff it is absolutely contractible.

Contractible topological systems

We now use a second characterization of contractible measures ([1], theorem I.2, page 11).

8.2. **Theorem.** *Let (Y, G) be a topological dynamical system with Y a compact metric space. Let ν be a probability measure on Y. The following properties are equivalent:*

(1) *The measure ν is contractible; i.e. for every $y \in Y$, the measure δ_y is a weak* limit point of the set $\{g\nu : g \in G\}$ in $M(Y)$.*
(2) *The linear operator $P_\nu : C(Y) \to LUC(G)$ defined by*

$$P_\nu f(g) = \int_Y f(gy)\, d\nu(y), \tag{8.1}$$

is an isometry of the Banach space $C(Y)$ of continuous functions on Y into the Banach space $LUC(G)$ of bounded left uniformly continuous functions on G (with sup*-norm).*

We note that the operator P_ν can be extended to the larger Banach space $L^\infty(Y, \nu)$ using the same formula (8.1), and since for $f \in L^\infty(Y, \nu)$ and $g, h \in G$

$$\begin{aligned} |P_\nu f(g) - P_\nu f(h)| &= |\langle f, g\nu - h\nu\rangle| \\ &\le \|f\|_\infty \|g\nu - h\nu\|_{\text{total variation}} \\ &= \|f\|_\infty \|\nu - g^{-1}h\nu\|, \end{aligned}$$

we conclude that $P_\nu(L^\infty(Y, \nu)) \subset LUC(G)$.

G-continuous functions

Recall the following definition and representation theorem from [17].

8.3. **Definition.** Given a G-system $(X, \mathcal{X}, \mu, G)$, a function $f \in L^\infty(X, \mu)$ is called **G-continuous** if $f \circ g_n$ converges in norm to f in $L^\infty(X, \mu)$ whenever $g_n \to e$.

8.4. **Theorem.** *Let G be a Polish topological group. A boolean G system admits a topological model if and only if there exists a sequence of G-continuous functions that generates the σ-algebra (equivalently: separates points).*

We first remark that although in [17] the boolean system is assumed to be measure preserving the proof, in fact, goes through if one assumes only that the **measure class** is preserved. Next we note that the condition in the theorem of admitting a sequence of G-continuous functions that generates the σ-algebra, is always satisfied when the group G is, in addition, locally compact (see corollary 8.7 below). Thus, in this case, one recovers the classical result that ensures the existence of a topological model for every measure class preserving boolean action of a locally compact second countable group.

More importantly, we observe that an immediate corollary of the proof of theorem 2.2 in [17] is the following version of the theorem (still for a general Polish topological group G).

8.5. **Theorem.** *Let $(X, \mathcal{X}, \mu, G)$ be a a boolean system which satisfies the condition of theorem 8.4 and let f be a function in $L^\infty(X, \mu)$. Then there exists a topological model (Y, ν, G) such that the function $F \in L^\infty(Y, \nu)$ corresponding to f is in $C(Y)$ iff f is G-continuous.*

Thus when G is a locally compact second countable group, theorem 8.5 applies for every G system.

8.6. **Lemma.** *Let $(X, \mathcal{X}, \mu, G)$ be a G-system and $f \in L^\infty(X, \mu)$. Let $\psi : G \to \mathbb{R}$ be a non-negative continuous function with compact support. Define $\hat{f} = f * \psi$ by*

$$\hat{f}(x) = \int_G f(hx)\psi(h)\, dm_G(h).$$

The function $\hat{f}$ is G-continuous.

Proof. For $g \in G$ we have

$$\begin{aligned}
\|\hat{f} \circ g - \hat{f}\|_\infty &= \text{ess-}\sup_{x\in X} \left| \int_G f(hgx)\psi(h)\, dm_G(h) - \int_G f(hx)\psi(h)\, dm_G(h) \right| \\
&= \text{ess-}\sup_{x\in X} \left| \int_G f(hx)\psi(hg^{-1})\, dm_G(h) - \int_G f(hx)\psi(h)\, dm_G(h) \right| \\
&\le \text{ess-}\sup_{x\in X} \int_G |f(hx)||\psi(hg^{-1}) - \psi(h)|\, dm_G(h) \\
&\le \|f\|_\infty \int_G |\psi(hg^{-1}) - \psi(h)|\, dm_G(h).
\end{aligned}$$

Thus $g \to e$ implies $\|\hat{f} \circ g - \hat{f}\|_\infty \to 0$ and $\hat{f}$ is G-continuous. □

We let $\{\psi_n : n = 1, 2, \dots\}$ be a fixed approximate identity. This means that there is a decreasing sequence V_n of precompact neighborhoods of e in G with $\cap_{n=1}^\infty V_n = \{e\}$, and $\psi_n : G \to \mathbb{R}$ is a sequence of nonnegative continuous functions with supp $\psi_n \subset V_n$ and $\int_G \psi_n\, dm_G = 1$ for $n = 1, 2, \dots$.

8.7. **Corollary.** *The bounded G-continuous functions are dense in $L^2(X, \mu)$.*

Proof. It is easy to check that a sequence $\{\psi_n : n = 1, 2, \dots\}$ as above is an approximate identity in $L^2(X, \mu)$; i.e. $\|f * \psi_n - f\|_2 \to 0$ for every bounded $f \in L^2(X, \mu)$. Now apply lemma 8.6. □

8.8. **Proposition.** *Let $(X, \mathcal{X}, \mu, G)$ be a G-system and $0 \ne f = \mathbf{1}_A \in L^\infty(X, \mu)$. Let $\psi_n : G \to \mathbb{R}$ be an approximate identity in $L^2(X, \mu)$ as above. Then*

$$\lim_{n\to\infty} \|f * \psi_n\|_\infty = \|f\|_\infty = 1.$$

Proof. With no loss in generality we can assume that (X, μ, G) is a topological model. By the regularity of the measure μ we can also assume (by passing to a subset) that A is closed. Again by regularity of μ, given $\epsilon > 0$ we can choose an open neighborhood U of A in X such that $\mu(U \setminus A) < \epsilon$. Since

$$\lim_{h\to e} \mu(hA \bigtriangleup A) = \lim_{h\to e} \|\mathbf{1}_{hA} - \mathbf{1}_A\|_1 = 0,$$

we can choose a neighborhood $V = V^{-1}$ of e in G such that for all $h \in V$

$$(i) \quad \|\mathbf{1}_{hA} - \mathbf{1}_A\|_1 = \mu(hA \bigtriangleup A) < \epsilon \quad \text{and} \quad (ii) \quad hA \subset U.$$

Let $\psi = \psi_n$ be a member of the approximate identity which satisfies $\mathrm{supp}\,(\psi) \subset V$.

Set

$$\hat{f}(x) = f * \psi = \int_G f(hx)\psi(h)\, dm_G(h) = \int_G f(hx)\, dp(h),$$

where $dp = \psi \cdot dm_G$, a probability measure on G. By (ii), if $x \notin U$ then $f(hx) = \mathbf{1}_A(hx) = 0$ for every $h \in V$ and it follows that $\hat{f}(x) = 0$. Thus

$$\int_X \hat{f}\, d\mu(x) = \int_U \hat{f}\, d\mu(x). \tag{8.2}$$

By Fubini and the estimation (i),

$$\int_X \hat{f}\, d\mu(x) = \int_X \mu(hA)\, dp(h) \ge \mu(A) - \epsilon. \tag{8.3}$$

For $\delta > 0$ let

$$D = D_\delta = \{x \in U : \hat{f}(x) < 1 - \delta\}.$$

Then

$$\begin{aligned}
\mu(A) - \epsilon &\le \int_X \hat{f}\, d\mu(x) = \int_U \hat{f}\, d\mu(x) \\
&= \int_D \hat{f}\, d\mu(x) + \int_{U\setminus D} \hat{f}\, d\mu(x) \\
&\le (1-\delta)\mu(D) + \mu(U \setminus D) \\
&= -\delta\mu(D) + \mu(U) \\
&\le -\delta\mu(D) + \mu(A) + \epsilon,
\end{aligned}$$

hence $\mu(D) \le \frac{2\epsilon}{\delta}$. Fixing δ at the outset we choose ϵ so that, say, $\frac{2\epsilon}{\delta} \le \frac{1}{2}\mu(A)$, and then for sufficiently large n,

$$\hat{f}(x) = f * \psi_n(x) \ge 1 - \delta,$$

for every x in the set $U \setminus D$ whose measure $\mu(U \setminus D) \ge \frac{1}{2}\mu(A) > 0$. This completes the proof of the proposition. ☐

Sat and absolute contractibility are equivalent

8.9. **Theorem.** *Let $(X, \mathcal{X}, \mu, G)$ be a G system, then μ is SAT iff it is absolutely contractible.*

Proof. Let (Y, ν, G) be a compact model and let $f \in L_\infty(\nu)$ with $\|f\|_\infty = 1$ and $\epsilon > 0$ be given. Set $A = \{y \in Y : |f(y)| \geq 1 - \epsilon\}$. Then $\nu(A) > 0$ and we now assume that also $\nu(A_+) > 0$ where $A_+ = \{y \in Y : f(y) \geq 1 - \epsilon\}$. By assumption there is a sequence $g_n \in G$ such that $\lim_{n\to\infty} \nu(g_n^{-1} A_+) = 1$. Hence

$$\begin{aligned} P_\nu f(g_n) &= \int_{g_n^{-1}A_+} f(g_n y)\, d\nu(y) + \int_{g_n^{-1}A_+^c} f(g_n y)\, d\nu(y) \\ &\geq (1-\epsilon)\nu(g_n^{-1}A_+) - \nu(g_n^{-1}A_+^c) \to 1 - \epsilon. \end{aligned}$$

Hence $\limsup_{n\to\infty} P_\nu f(g_n) \geq 1 - \epsilon$. Similarly when $\nu(A_-) > 0$ with $A_- = \{y \in Y : f(y) \leq -1 + \epsilon\}$ we get $\limsup_{n\to\infty} |P_\nu f(g_n)| \geq 1 - \epsilon$. Thus the $LUC(G)$ norm $\|P_\nu f\| \geq 1 - \epsilon$ and as this holds for every ϵ we get $\|P_\nu f\| = 1$. Thus $P_\nu : L^\infty(Y, \nu) \to LUC(G)$ is an isometry. In particular $P_\nu : C(Y) \to LUC(G)$ is an isometry and by theorem 8.2, (Y, ν, G) is contractible.

Conversely, assume that μ is absolutely contractible and let $A \in \mathcal{X}$ be a set with positive μ measure. Write $f = \mathbf{1}_A$. By proposition 8.8, given $\epsilon > 0$, we can choose $\psi = \psi_n : G \to \mathbb{R}$, a function in the approximate identity, such that for $\hat{f} = f * \psi$,

$$\big| \|\hat{f}\|_\infty - 1 \big| = \big| \|\hat{f}\|_\infty - \|f\|_\infty \big| < \epsilon. \tag{8.4}$$

By lemma 8.6, the function $\hat{f}$ is G-continuous and by theorem 4.3 there is a topological model (Y, ν, G) for $(X, \mathcal{X}, \mu, G)$ in which the $L^\infty(Y, \nu)$ function corresponding to $\hat{f}$, say F, is in $C(Y)$. By assumption the measure ν on Y is contractible and thus by theorem 8.2, $P_\nu(F) = P_\mu(\hat{f}) \in LUC(G)$ satisfies

$$\|P_\mu(\hat{f})\| = \|F\| = \|\hat{f}\|_\infty.$$

Let $g \in G$ satisfy

$$\|P_\mu(\hat{f})\| < P_\mu(\hat{f})(g) + \epsilon. \tag{8.5}$$

Now

$$\begin{aligned} P_\mu(\hat{f})(g) &= \int_X \hat{f}(gx)\, d\mu(x) \\ &= \int_X \int_G f(hgx)\psi(h)\, dm(h)\, d\mu(x) \\ &= \int_G \psi(h) \left(\int_X f(hgx)\, d\mu(x) \right) dm(h) \\ &= \int_G \psi(h) P_\mu(f)(hg)\, dm(h), \end{aligned}$$

and, since $\psi \geq 0$ and $\int_G \psi\, dm = 1$, it follows that for some $h \in G$

$$P_\mu(f)(hg) > P_\mu(\hat{f})(g) - \epsilon. \tag{8.6}$$

Collecting the estimations (8.4), (8.5) and (8.6) we get

$$P_\mu(f)(hg) > P_\mu(\hat{f})(g) - \epsilon > \|P_\mu(\hat{f})\| - 2\epsilon = \|\hat{f}\|_\infty - 2\epsilon > 1 - 3\epsilon.$$

Explicitly

$$P_\mu(f)(hg) = \int_X f(hgx)\, d\mu(x) = \mu(hgA) > 1 - 3\epsilon$$

and the proof is complete. □

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