Stationary electron swarms in electromagnetic fields

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Electron clouds rotating in axially symmetric magnetic fields have been known for a long time, but the agreement between theory and experiment is still very unsatisfactory. The discrepancy appears to be due to the interaction of electrons. Before approaching this difficult problem it is desirable to possess a more complete theory of stationary swarms without interaction. In the present paper the distribution density is calculated on the basis of classical statistical mechanics. It is shown that electrons injected at any point with very small initial velocities will distribute themselves with a density inversely proportional to the distance from the axis, in a certain annular space. Only the limits of this space, not the distribution inside it, will be dependent on the electric or magnetic fields. The uniform or nearly uniform distributions calculated by previous authors are singular solutions, inconsistent with any degree of statistical disorder. Other laws of density distribution can be realized by simultaneous injection of electrons at several points. These offer a possibility to realize dispersing electron lenses and corrected electron optical systems. It is shown that the ring current produced by the rotating electron cloud can reduce the magnetic field at the axis very considerably in devices of practicable dimensions. It appears also possible to produce clouds of free electrons with densities sufficient for observable optical effects.

The theory of the stationary motion of electron assemblies in electric and magnetic fields is of interest in two fields of applied electronics. It arises in the theory of the magnetron, an electronic device which so far has defied all attempts at a quantitative theoretical explanation. The second field is electron optics, which suffers from the fundamental limitation that lens correction is impossible so long as only electromagnetic fields free of space charge are employed. Hence the application of electron clouds offers one of the very few prospects for further development of electron optical devices, especially of the electron microscope.

The full-anode magnetron, a diode with a cylindrical anode and a coaxial filament as cathode, was first constructed and investigated by A. W. Hull in 1920, and has been explored theoretically and experimentally by numerous authors.* An especially thorough theoretical study is due to L. Brillouin (1941, 1942). These investigations have revealed a striking discrepancy between theory and the experimental results.† Hull's simple theory led to the conclusion that current could flow to the anode only above a certain critical voltage, which increases quadratically with the magnetic field intensity. Below this voltage the current ought to be cut off, and an electron cloud of uniform charge density

$$-\rho_H = \frac{eH^2}{8\pi mc^2} \tag{1}$$

should rotate in the magnetron like a solid body, with an angular velocity

$$\omega_H = \frac{eH}{2mc}, \tag{2}$$

* A. W. Hull (1921, 1924). F. B. Pidduck (1936). Extensive bibliography in A. F. Harvey (1943).

† See especially Harvey (1943, pp. 83–113) and also the Foreword by E. B. Moullin.

which is one-half of the 'Larmor frequency'. Though the experiments confirmed the existence of a critical voltage, the quantitative agreement was very unsatisfactory. Current starts flowing considerably below the critical voltage and reaches its saturation value only very gradually, whereas on Hull's theory, even making an allowance for the initial velocities of the electrons, the rise ought to be confined within a few tenths of a volt. Though later workers added some refinements to Hull's theory, they were unable to account for the width and shape of the cut-off characteristic. An exception is an investigation by E. G. Linder (1938*a, b*), who found experimentally that in a rotating electron cloud random motion develops with a very high electron temperature, of the order of 10 V. On this basis he could account for the width of the cut-off curves.

Several objections could be raised against Linder's results. He finds by probe measurements a particularly high electron temperature when the current is entirely cut off, which is a thermodynamical impossibility. His method of calculating the current from an electron cloud to a plate is also open to objection, as he uses a formula which is strictly valid only in the absence of a magnetic field. Nevertheless, it appears very likely that Linder has correctly located the root of the discrepancy. The new difficulty arises, however, that theory cannot account for the experimentally found rapid development of random motion in an electron cloud. This phenomenon is closely related to the inexplicably rapid establishment of electron temperature in gas discharges at very low pressures, which was discovered by I. Langmuir and H. Mott-Smith, and discussed in detail by Langmuir (1927, 1928). As the problem of electron interaction is one of extraordinary mathematical difficulty, it appears desirable to obtain first a reliable experimental estimate of the effect. The first step towards this must be a more complete investigation of the problem *without* interaction. It will be shown that no satisfactory basis for this has existed up to now, as the corrections which have to be applied to Hull's results are very much larger than those found by any of the later authors who tried to improve on them.

Hull's results (1) and (2) can be derived from the condition of zero radial current. In general a zero current would be the resultant of two equal and opposite currents, one flowing inwards, the other outwards. If the electrons start from the cathode with zero initial velocity, as assumed in the simple theory, reversal of the sense of motion can take place only at the outer boundary of the electron cloud. By formulating the condition of equal and opposite currents, combining this with the dynamical equations and substituting it into Poisson's equation, a differential equation for the space-charge distribution is obtained.* The remarkable result follows, that this equation has a solution free of singularities only in the case when both the opposite currents are zero, that is to say, if there is no radial electron motion at all. This means that one must imagine the electrons circling around the cathode in coaxial circular orbits. The question immediately arises how such a state of motion could ever establish itself. Some authors have thought that it could be justified, as they found that small radial velocities produce only insignificant deviations from Hull's law of

* Cf. e.g. Brillouin (1941).

distribution. It will be shown, however, that any initial velocities, however slight, require a fundamental modification of the picture, and that Hull's results must be considered as a singular solution of the problem, which could neither establish, nor maintain itself.

This is of great importance from the point of view of the other field mentioned at the beginning, the electron optical applications of space charges. Electron optical systems cannot be corrected at present for spherical aberration and other defects, because no dispersing electron lenses can be realized without (negative) space charges. A space charge according to equation (1) would be useless for this purpose. As explained above, this distribution satisfies the condition that electrons are everywhere in radial equilibrium. Hence, if an electron beam is shot through such an electron cloud, the concentrating effect of the magnetic field would exactly balance the dispersing effect of the space charge, and the resulting lens effect would be *nil*. But, as it will be shown that stationary electron clouds are possible in which radial equilibrium exists only in the average, but not at every radius, the prospect of electron optical applications is not closed. The problem will therefore be formulated in sufficient generality to cover both the magnetron and the electron optical applications.

The general method to be followed will be to treat the problem as one of statistical mechanics. First the distribution law will be derived on the basis of classical statistics, giving the density as a function of the position and of the electromagnetic potentials. The second step will be to substitute the density into Poisson's equation and find the density distribution, and the potential consistent with it, as a function of position. Fortunately, the first step leads to an extremely simple law, so that the second step and the discussion of the solution will present no difficulty.

1. The dynamical problem

Assume an axially symmetrical electromagnetic field, specified in cylindrical co-ordinates (z, r, θ) by an electrostatic potential $\phi(z, r)$ and a vector potential $A(z, r)$. As the vector potential has only a tangential component, it will not be necessary to write it as a vector, or to distinguish it by a suffix t like the other tangential components. The units will be Gaussian. The magnetic field follows from A by the relations

$$H_r = \mathrm{curl}_r A = -\frac{\partial A}{\partial z}, \quad H_z = \mathrm{curl}_z A = \frac{1}{r}\frac{\partial}{\partial r}(rA). \tag{3}$$

In the special case of a homogeneous magnetic field, $H_r = 0$, $H_z = H_0$

$$A = \tfrac{1}{2}H_0 r + C/r, \tag{4}$$

where C is an arbitrary constant. As it does not figure in the dynamical equations it can be put zero from the start.

Now assume a cathode in the form of a circular filament with negligible thickness, with a radius a at a position where the vector potential has the value A_0. Figure 1 illustrates an example. The cathode is placed in, or near to, a position of zero magnetic flux. In the case illustrated, the field at the axis has the law

$$H(0,z) = \text{const.}\,(\tfrac{1}{2}+\tan^{-1}z),$$

so that at great distance from the cathode, in either direction, the field becomes homogeneous, the field at the right being three times stronger than the field at the left, which plays only the part of an auxiliary, with the purpose of adjusting A_0 to a suitable (small) value. The magnetic field is illustrated by the field lines, i.e. the meridians of the tubes of constant flux $\Psi = 2\pi rA = \text{const.}$ The flux increases by equal steps from one line to the next. The field line passing through the cathode is shown in dotted lines. It approaches a radius r_0 at the far right, where the field becomes homogeneous. This arrangement is suitable for electron optical applications, as it leaves the axis free. The cathode has to be placed at or near the line $\Psi = 0$, as otherwise the electrons could not get near the axis.

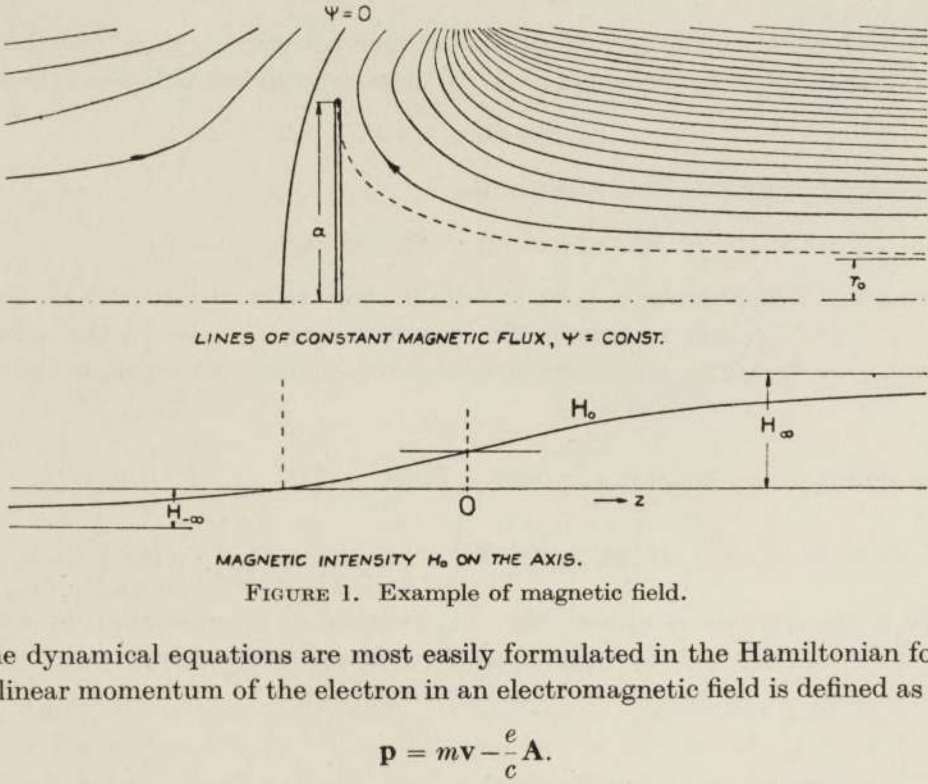

FIGURE 1. Example of magnetic field.

The dynamical equations are most easily formulated in the Hamiltonian form.* The linear momentum of the electron in an electromagnetic field is defined as

$$\mathbf{p} = m\mathbf{v} - \frac{e}{c}\mathbf{A}. \tag{5}$$

In an electromagnetic field of rotational symmetry it is advantageous to write the components of the linear momentum p_z, p_r and p_θ/r, where p_θ is the angular momentum. The Hamiltonian is

$$\mathscr{H} = \frac{1}{2m}\left[p_z^2+p_r^2+\left(\frac{1}{r}p_\theta+\frac{e}{c}A\right)^2\right]-e\phi. \tag{6}$$

The potential ϕ is to be measured from the cathode as zero level.

* R. Becker (1933, p. 96). The equation (5) is due to K. Schwarzschild (1903).

The problem has two integrals, $\mathscr{H}$ = const. and p_θ = const. Both can be immediately obtained from the canonical equations. Marking values relating to the cathode with a suffix 0, the momentum integral is written

$$p_\theta = r\left(mv_t - \frac{e}{c}A\right) = a\left(mv_{t0} - \frac{e}{c}A_0\right). \tag{7}$$

From this point on it will be convenient to measure the velocity in potential units. Therefore v is replaced by

$$u = (\sqrt{[m/2e]})\, v. \tag{8}$$

It will also be convenient to introduce a quantity, which could be called the vector potential relative to the cathode, defined by

$$\mathscr{A} = \sqrt{\left(\frac{e}{2mc^2}\right)}\left(A - \frac{a}{r}A_0\right). \tag{9}$$

$r\mathscr{A}$ is a measure of the flux which passes between the cathode (a, z_0) and the circle (r, z). $\mathscr{A}$ has the same dimensions as u. With these new symbols equation (7) becomes

$$ru_t - au_{t0} = r\mathscr{A}. \tag{7·1}$$

The energy integral is, in the new units,

$$u_z^2 + u_r^2 + u_t^2 - \phi = (u_r^2 + u_z^2 + u_t^2)_0. \tag{10}$$

A condition which electrons have to fulfil in order to have access to an element in phase space can now be formulated by eliminating u_{t0} between the equations (7·1) and (10). It will be convenient to introduce a dimensionless parameter

$$x = r/a \tag{11}$$

for the radius, i.e. to measure r in units a. Thus

$$Q \equiv u_z^2 + u_r^2 - (x^2-1)\left[u_t - \frac{x^2}{x^2-1}\mathscr{A}\right]^2 - \left[\phi - \frac{x^2}{x^2-1}\mathscr{A}^2\right] = u_{z0}^2 + u_{r0}^2 > 0. \tag{12}$$

This is the *criterion of accessibility*. Its geometrical interpretation in velocity space is shown in figure 2. $Q = 0$ is a quadric surface of rotational symmetry relative to the u_t-axis. Its type is given by the following table:

x	<1	>1
$\phi > x^2\mathscr{A}^2/(x^2-1)$	real ellipsoid	one-shell hyperboloid
$\phi < x^2\mathscr{A}^2/(x^2-1)$	imaginary ellipsoid	two-shell hyperboloid

This classification is illustrated in figure 3 in the special case of a homogeneous magnetic field, in which

$$\mathscr{A} = K\left(x - \frac{1}{x}\right), \quad \frac{x^2}{x^2-1}\mathscr{A}^2 = K^2(x^2-1), \tag{13}$$

where $K = a\sqrt{(eH^2/8mc^2)}$.

Now consider electrons which have left the cathode in the velocity interval u_0, u_0+du_0. They will arrive at some point (z, r) with a velocity u, and in an interval du, connected with the initial data by the relations

$$u^2 = u_0^2+\phi, \quad u\,du = u_0\,du_0. \tag{14}$$

Combining this with equation (12) it is seen that electrons which have access to (z, r) will all be contained in that segment of the spherical shell with radii u, $u+du$ which is (seen from the axis) *outside* the quadric surface $Q = 0$. This shell segment is the accessible volume in velocity space. It may be noted that it gives also a measure of the volume of momentum space, as, by equation (5),

$$\frac{\partial(p_x, p_y, p_z)}{\partial(v_x, v_y, v_z)} = m^3 = \text{const.} \tag{15}$$

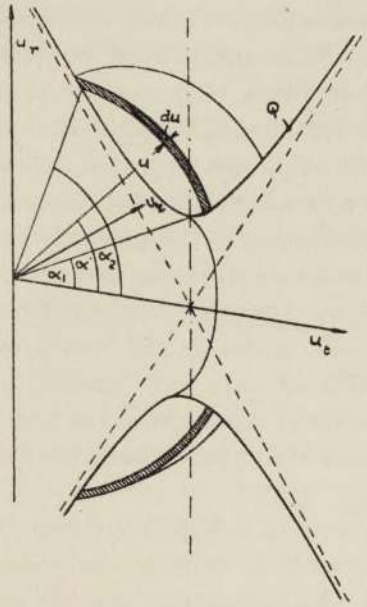

FIGURE 2. Construction of the accessible volume of momentum space.

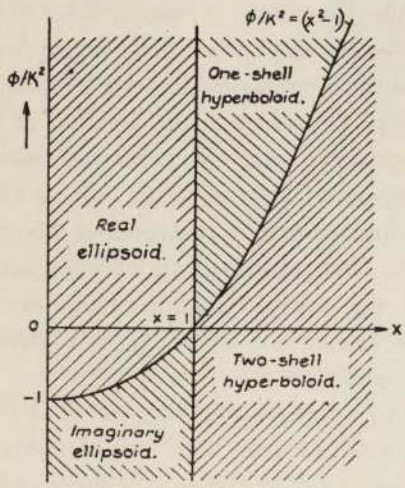

FIGURE 3. The quadric surface Q.

2. The law of velocity distribution

By Liouville's theorem the density D of a group of electrons in phase space is the same as the original density which this group had at the cathode. Hence if it were known that the whole accessible volume of momentum or (velocity) space calculated in the previous section would be actually filled with electrons, then the velocity distribution at any point would also be known, and from this the density in configuration space could be calculated. This condition is fulfilled in an important special case, in that of the infinitely long magnetron with cylindrical cathode, the only one considered by previous authors. In this case there is no further limitation of the accessible volume, as it is easy to see that a trajectory continued backwards from any point of it must actually reach the cylindrical cathode at some point.

In the general case Liouville's theorem by itself is not sufficient to determine the velocity distribution, for two reasons. First, in the case of limited cathodes in general, only a part of the accessible volume will be really filled. This difficulty arises especially in the case of an 'ordered' flow of electrons. Hull's distribution with electrons circling in circular orbits is an extreme example of such an 'ordered' flow. But this uncertainty will exist also in all cases of motion in 'short trajectories', i.e. if the electrons return to the cathode after spending only a short time in the cloud. There is, however, also a second sort of difficulty, which arises especially in the case of 'long trajectories'. The same group of electrons might return—and by the theorem of Poincaré and Zermelo *will* return—an indefinite number of times to the same element in phase space, if it is allowed to spend a long time in the cloud, and the number of times this will happen is not known. Both difficulties are resolved if this time can be assumed to be very long, by the *ergodic theorem*—long known as the ergodic hypothesis, until G. D. Birkhoff and J. v. Neumann proved it—according to which the whole accessible phase volume will be filled, with uniform density.* Moreover, this result is independent of the initial conditions, in the sense that though originally (at the cathode) D might have been a function not only of u_0 but also of the direction, ultimately, over long intervals it can be a function of u only.

One can therefore apply the law of uniform distribution in the accessible part of phase space to both the principal cases under consideration, to the magnetron and to the space-charge electron lens, as shown in figure 1, though for different reasons. In the first case it is justified by Liouville's theorem, in the second by the ergodic theorem. The assumption of very long trajectories appears well justified in this second case, as the cathode can be made very small in comparison to the rest of the volume, which the electrons will traverse in general many times until they return to the cathode.

It may again be emphasized that the problem is to be treated dynamically as a *one-electron problem*, and that electron interaction must be expected to make itself particularly felt in the case of long trajectories. It may be repeated that the present purpose is not to establish a generally valid theory of magnetrons and similar devices, but to build a basis for such a theory by first investigating the conclusions of a theory without interaction.

3. The law of space-charge distribution

On the basis of this assumption the volume of the accessible momentum (or velocity) space associated with any point is a measure of the space-charge density at this point. To calculate this, one puts, as in figure 2,

$$u_t^2 = u^2 \cos^2\alpha, \quad u_z^2 + u_r^2 = u^2 \sin^2\alpha. \tag{16}$$

* It appears that the present application is proof against the weighty arguments which have been advanced against considering the ergodic or the quasi-ergodic theorem as the basis of classical statistical mechanics. Cf. R. C. Tolman (1938), p. 70.

Substitution into equation (10) (with the equality sign) gives the following equation for the intersections of the sphere of radius u with the quadric $Q = 0$:

$$(u\cos\alpha - \mathscr{A})^2 = \left(\frac{a}{r}\right)^2 (u^2 - \phi) = \left(\frac{a}{r}u_0\right)^2. \tag{17}$$

Physically this means that the extreme electrons reaching a point will be those which started at the cathode in tangential direction. The limits are (u_0 being considered always as positive)

$$u\cos\alpha = \mathscr{A} \pm \frac{a}{r}u_0. \tag{17·1}$$

Four cases must now be distinguished, which give four different laws for the space-charge distribution:

(*a*) The equation (17) has two real roots, α_1, α_2. The condition for this is

$$\left(\mathscr{A} + \frac{a}{r}u_0\right)^2 < u^2 = \phi + u_0^2. \tag{a}$$

If this condition is fulfilled (as in figure 2), the volume of the shell segment is, apart from a factor 2π,

$$(\cos\alpha_1 - \cos\alpha_2)\, u^2 du.$$

Combined with (17·1), which gives

$$u(\cos\alpha_1 - \cos\alpha_2) = 2\frac{a}{r}u_0, \tag{17·2}$$

and taking into consideration that $u\,du = u_0 du_0$, one obtains for the charge density, apart from a constant factor, which may be included in the phase density D,

$$d\rho = \frac{a}{r} D(u_0)\, u_0^2 du_0. \tag{18}$$

This means that if electrons are emitted only in an infinitesimal energy interval ('microcanonic assembly') the density will be inversely proportional to the distance from the axis, independently of the electric and magnetic fields, which determine only the limits inside which it is valid, but do not interfere with the distribution itself. This strikingly simple law holds also if the primary emission is not homogeneous, up to a certain maximum initial velocity. This law is the most important, but for completeness the other cases will also be discussed.

(*b*) The equation (17) has only one real root. The condition for this is

$$\left(\mathscr{A} - \frac{a}{r}u_0\right)^2 < \phi + u_0^2 < \left(\mathscr{A} + \frac{a}{r}u_0\right)^2. \tag{b}$$

In this case the spherical segment becomes a spherical cap. The accessible volume is

$$(1 - \cos\alpha)\, u^2 du,$$

and the density will follow the law

$$d\rho = \left[\sqrt{(\phi+u_0^2)}-\mathscr{A}+\frac{a}{r}u_0\right]D(u_0)\,u_0^2du_0. \tag{19}$$

Unlike (*a*) this depends explicitly on ϕ and $\mathscr{A}$.

If equation (17) has no real roots, this can mean two things:

(*c*) The whole volume of the spherical shell is accessible. The law of density is

$$d\rho = \sqrt{(\phi+u_0^2)}\,D(u_0)\,u_0^2du_0. \tag{20}$$

This is the case if the whole sphere is outside the quadric, which in this case must be an ellipsoid.

(*d*) If, on the other hand, the whole sphere is inside the quadric, the accessible volume is zero, and so is the density. Therefore in the following this case will be referred to as (*o*).

One must now determine the range of validity of these laws, in terms of ϕ, $\mathscr{A}$, u_0 and r (or x). The discussion, though elementary, is rather complicated, and only the results will be given. The velocity ranges in which different solutions (*a*)–(*d*) are valid are separated by roots of the equation

$$\left(\mathscr{A}+\frac{a}{r}u_0\right)^2 = \phi+u_0^2,$$

which are
$$u_1,\,u_2 = \frac{x}{x^2-1}\left[\mathscr{A}\pm\sqrt{\{\mathscr{A}^2-(x^2-1)(\phi-\mathscr{A}^2)\}}\right]. \tag{21}$$

Of these roots, if they exist, the smaller one is called u_1 and the larger one u_2. It will also be convenient to introduce the abbreviation

$$\kappa = x^2/(x^2-1).$$

The following table gives the intervals of u_0 in which the laws (*a*), (*b*), (*c*) and (*d*) are valid:

$x<1$	$\phi>\mathscr{A}^2$	$\mathscr{A}^2>\phi>\kappa\mathscr{A}^2$	$\kappa\mathscr{A}^2>\phi$
(*a*)	$0-u_2$	—	—
(*b*)	—	u_1-u_2	—
(*c*)	$u_2-\infty$	$u_2-\infty$	—
(*o*)	—	$0-u_1$	$0-\infty$

$x>1$	$\phi>\kappa\mathscr{A}^2$	$\kappa\mathscr{A}^2>\phi>\mathscr{A}^2$	$\mathscr{A}^2>\phi$
(*a*)	$0-\infty$	$0-u_1,\ u_2-\infty$	$u_2-\infty$
(*b*)	—	u_1-u_2	u_1-u_2
(*c*)	—	—	—
(*o*)	—	—	$0-u_1$

This table is illustrated in figure 4 in the particular case $a = r_0$, i.e. if the cathode is in the homogeneous part of the field. In this case the curves which divide the field are

$$\phi/K^2 = \left(x-\frac{1}{x}\right)^2, \quad \phi/K^2 = x^2-1,$$

as given by equation (13). In the general case the curve $\phi = \kappa\mathscr{A}^2$ has a different shape, as it goes to $\pm\infty$ at $x = 1$, but the relative position of the two curves remains the same, and nothing essential is changed.

The table discloses the extraordinary variety of equilibrium shapes of rotating electron swarms which could be obtained with large initial velocities. It may be somewhat surprising that the initial velocity adds to the manifoldness of the problem, as one might think that lowering of the cathode potential would be equivalent to an increase of u_0. But u_0 appears in the equations not only in the combination $\phi + u_0^2$, but also in the combination au_0/r. This is a consequence of the momentum integral, and means physically that a velocity u_0 at the radius a is 'worth' a velocity au_0/r at the radius r. Things become simple only at small initial velocities, as in this range there obtains either the simple hyperbolical law (a) or the law (o).

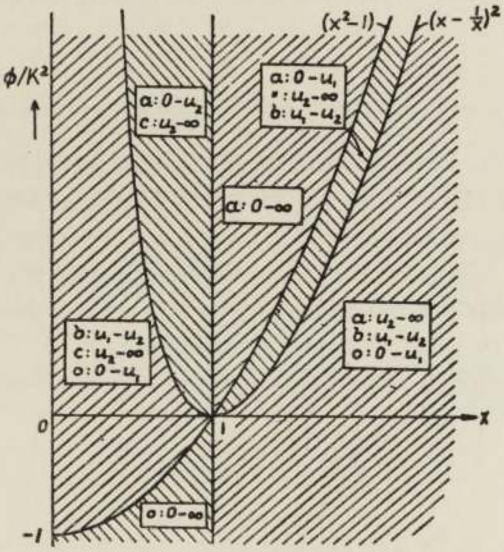

FIGURE 4. Domains of solutions a, b, c, o.

Before discussing law (a) in detail, it may be interesting to compare it with the corresponding law in a two-dimensional magnetron, that is, with an electron swarm in which ergodic disorder would exist only in the two dimensions r and θ, but not in the z-direction. This would be the case if the field were absolutely cylindrical, and if the electrons had definite, e.g. zero, initial components u_{z0}. In this case the phase volume changes into an area, which is

$$(\alpha_2 - \alpha_1)\, u\, du = \left[\cos^{-1}\frac{\mathscr{A} - au_0/r}{\sqrt{(\phi + u_0^2)}} - \cos^{-1}\frac{\mathscr{A} + au_0/r}{\sqrt{(\phi + u_0^2)}}\right] u\, du, \tag{22}$$

and series development gives the approximate formula for the density

$$\rho \sim (\alpha_1 - \alpha_2) \cong \frac{1}{\mathscr{A} + au_0/r}\left(\sqrt{[\phi - \mathscr{A}^2 + 2au_0\mathscr{A}/r]} - \sqrt{[\phi - \mathscr{A}^2 - 2au_0\mathscr{A}/r]}\right). \tag{22·1}$$

If $\phi - \mathscr{A}^2 \gg (2a/r)\, u_0 \mathscr{A}$, this can be further simplified and gives

$$\rho = \text{const.}/r\sqrt{(\phi - \mathscr{A}^2)}. \tag{22·2}$$

This is a law entirely different from any of the previous ones. Such sudden changes of law with the number of degrees of freedom are familiar in classical statistical mechanics, and are a warning to be cautious in the application of statistical principles. But in the present case there is good reason to think that the axial degree of freedom can be considered as 'fully excited'. In the case of the magnetron with infinite cylindrical cathode this follows without any application of statistics from Liouville's theorem, and the arrangement in figure 1 cannot be realized without departing from cylindricity.

It may be mentioned that a further step in the same direction, i.e. restricting the disorder of the system, would lead to Hull's solution, with the corrections applied by Brillouin.

4. Self-consistent stationary electron swarms

The main result of the foregoing discussion is that for very small initial velocities the density will follow the simple law $1/r$ in the region in which $\phi > \mathscr{A}^2$, and will be zero outside it. The more complicated laws (b) and (c) come into action only where $\phi \simeq \mathscr{A}^2$. Hence, apart from two sheaths near the edges of the electron cloud, the $1/r$ law will obtain not only for electrons emitted in an infinitesimal energy interval, but also for electrons emitted by thermionic cathodes.

In order to obtain the potential distribution one must substitute this law into Poisson's equation

$$\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{C}{r}, \tag{23}$$

where C is a constant, to be determined later. The solution will be discussed only in the cylindrical case $\partial^2\phi/\partial z^2 = 0$, which applies also approximately to arrangements like the one in figure 1, at some distance from the ends where the magnetic intensity falls off. Here the electron cloud can be prevented from escaping by two sufficiently negative end-plates.

The general solution in the cylindrical case is

$$\phi = Cr + c_1 + c_2 \log r/r_0, \tag{24}$$

where c_1 and c_2 are constants, and r_0 is as defined in figure 1.

The electron cloud has, in the present approximation, sharp boundaries at $r = r_1$ inside and $r = r_2$ outside. At both edges the velocity is purely tangential. The boundary conditions are

$$\phi(r_1) = \mathscr{A}^2(r_1), \quad \phi(r_2) = \mathscr{A}^2(r_2), \tag{25}$$

and in addition at the inner edge $\quad d\phi/dr = 0, \quad$ (26)

if there is no charge inside r_1. This condition is valid in the case of figure 1 because there is no electrode at the axis, but it is valid also in the case of the cylindrical magnetron, as the field strength at the cathode must be zero in the equilibrium condition. There are therefore three conditions for the five constants C, c_1, c_2, r_1 and r_2, and two degrees of freedom are left over. One of these corresponds to the anode potential, which can have any value below the critical potential, and the other to the degree of saturation. The total charge in the cloud may assume any value up to a certain maximum. This will be reached if $r_1 = r_0$ and $\phi(r_0) = 0$, which means that no further electrons can reach the cloud from the cathode.

Having assumed ϕ independent of z, the same assumption must now be made for $\mathscr{A}$. Hence, neglecting the magnetic effect of the rotating electrons, which will be dealt with later, a homogeneous magnetic field must be assumed, writing

$$\mathscr{A}^2 = \frac{eH^2}{8mc^2} r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2. \tag{27}$$

Substituting this into the boundary conditions (25) and (26), the following expression is obtained for the space-charge density at a radius r:

$$\rho(r) = \tfrac{1}{4}\rho_H \frac{(r_2^2 - r_1^2)\,[1-(r_0^2/r_1 r_2)^2]}{r[r_2 - r_1(1+\log r_2/r_1)]}, \tag{28}$$

where ρ_H is Hull's space-charge density, as defined in equation (1). This is the solution of the problem, with two free parameters r_1 and r_2. It is illustrated in figure 5 in the special case $r_2/r_0 = 3$ for various radii r_1, which correspond to various degrees of saturation. The total charge contained between r_1 and r_2 is marked on the curves in percentages of the maximum charge.

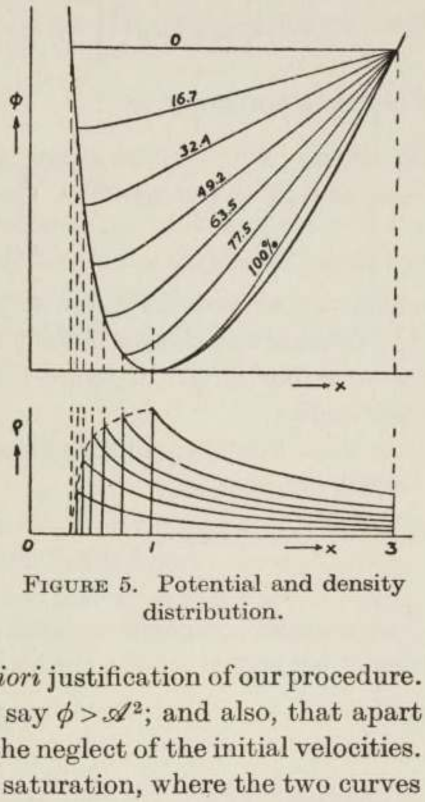

FIGURE 5. Potential and density distribution.

It may be noted that this result gives an *a posteriori* justification of our procedure. The law (a) has been assumed to hold, that is to say $\phi > \mathscr{A}^2$; and also, that apart from the edges ϕ exceeds $\mathscr{A}^2$ sufficiently to justify the neglect of the initial velocities. This assumption is now verified, except at 100 % saturation, where the two curves come rather close together, near the inner edge r_0.

In the case of maximum saturation, i.e. $r_1 = r_0$, the density at the outer edge becomes, according to equation (28),

$$\rho(r_2) = \tfrac{1}{4}\rho_H \frac{[1-(r_0/r_2)^2]^2}{1-(r_0/r_2)\log(r_2/r_0)}, \tag{28·1}$$

and for large ratios r_2/r_0 this approaches *one-quarter of Hull's value*. Applying this to the cut-off characteristic of the cylindrical magnetron an interesting difference arises between the present and previous theories. The sudden jump of current when the critical voltage is exceeded ought to be only about one-quarter of the saturation value, instead of the whole amount. There are, indeed, measured magnetron characteristics which show this phenomenon,* but it would be premature to consider this as a confirmation of the theory. The present theory cannot account—and no one-electron theory possibly could account—for the discrepancy of the voltage at which current starts to flow. Also, the present theory is valid only so long as the cloud is static, i.e. so long as there is no current. There is the possibility that even small currents might modify the calculated distribution very considerably.

The density at the inner edge r_1 in the case of saturation is about $\frac{1}{4}\rho_H r_2/r_0$, which can far exceed ρ_H. But this is not necessarily the largest density which can be maintained in a magnetron of prescribed outer radius. For very large ratios r_2/r_1, the density at the inner edge may be written approximately

$$\rho(r_1) \cong \tfrac{1}{4}\rho_H \frac{r_2}{r_1}\left[1-\left(\frac{r_0^2}{r_1 r_2}\right)^2\right] = \tfrac{1}{4}\rho_H \left(\frac{r_2}{r_0}\right)^2 \left(\frac{r_0^2}{r_1 r_2}\right)\left[1-\left(\frac{r_0^2}{r_1 r_2}\right)^2\right]. \tag{28·2}$$

This has a maximum at $$r_1 = \sqrt{(3)}\, r_0^2/r_2,$$

i.e. when r_1 is 1·73 times larger than the smallest radius which electrons can reach at all at a given outer radius. The value of the maximum is

$$\rho(r_1)_{\text{max.}} = 0{\cdot}094\rho_H (r_2/r_0)^2, \tag{28·3}$$

which can exceed the value of $\rho(r_1)$ at maximum saturation by any amount if r_2/r_0 is made sufficiently large. This effect begins to develop at about $r_2/r_0 = 4$. In the example shown in figure 5 the largest density is still reached at maximum saturation.

In what limits would such an electron cloud act as a dispersing lens? The radial electric gradient is

$$\frac{d\phi}{dr} = C + c_2/r = C(1-r_1/r) = -4\pi\rho(r_1)\, r_1(1-r_1/r). \tag{29}$$

This must be compared with the gradient in the case of Hull's distribution, which would just suffice to keep the electrons in equilibrium

$$\left(\frac{d\phi}{dr}\right)_H = -2\pi\rho_H r.$$

Substituting in equation (29) the maximum value of $\rho(r_1)$ from equation (28·3), the following criterion is obtained for a dispersing lens:

$$\frac{d\phi/dr}{(d\phi/dr)_H} = 0{\cdot}188\left(\frac{r_2}{r_1}\right)^2 \frac{r_1}{r}\left(1-\frac{r_1}{r}\right) > 1. \tag{30}$$

A dispersing effect will therefore always exist in a certain interval of r so long as $r_2/r_1 > 4{\cdot}6$. But this effect is confined between two limiting radii, between which it reaches a maximum value.

* Cf. Harvey (1943, p. 105, figure 43a).

This is not the law required of a dispersing lens, even less of a lens which could be used to correct the aberrations of ordinary electron lenses. In a correcting lens the density is required to start at the axis with a value exceeding Hull's value, and to increase approximately parabolically with the distance from it.

It appears likely that a distribution similar to the required one would establish itself in any arrangement of the type as shown in figure 1, unless it is constructed with extraordinary precision. At some distance from the cathode, where the magnetic field has become appreciably homogeneous, the 'axis' of the magnetic field loses its significance, and the momentum integral is valid only if the electric field is rigorously rotationally symmetric and perfectly alined with the cathode. Very small departures from these ideal conditions will cause the charge distribution to depart appreciably from the calculated form. The hollow tube inside the radius r_1 will fill up with electrons, and one may expect that within certain limits the real distribution will approach the desired shape.

5. Extended electron source

These processes may be followed to a certain point by a simple extension of the foregoing theory, if one considers cathodes of a certain radial extension instead of filament cathodes with vanishing thickness and perfect alinement.

The density contributed by the emission of a cathode of vanishing thickness, which emits in the limits u_0, $u_0 + du_0$, was found to be

$$d\rho = \frac{a}{r} D(u_0)\, u_0^2 du_0. \tag{18}$$

Now assume that u_0 and du_0 go to zero, whilst the product $D(u_0)\, u_0^2 du_0$ divided by the radial extension da of the cathode approaches a finite limit. It will be convenient to express a by r_0 (as defined in figure 1) and include it in a function $F(r_0)$, with which the density due to the emission of an infinitesimal strip of the cathode is written

$$d\rho = \frac{1}{r} F(r_0)\, dr_0. \tag{31}$$

If the cathode has constant emission density, the function $F(r_0)$ is proportional to the cathode area corresponding to the limits r_0, $r_0 + dr_0$, multiplied with the radius a.

Assume in the following, to simplify matters, that the cathode extends to $r_0 = 0$, i.e. to the flux line $\psi = 0$. Investigating only a region near the axis in which only a part of the cathode is active, the area outside a maximum radius $r_{om.}$ is cut off from contributing to the density by the momentum integral. This gives the following law for the potential and density distribution:

$$-4\pi\rho = \frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = \frac{C}{r}\int_0^{r_{om.}} F(r_0)\, dr_0. \tag{32}$$

The radius $r_{\text{om.}}$ is determined by $\phi = \mathscr{A}^2$, and in a homogeneous magnetic field (equation (27)) this gives

$$r^2_{\text{om.}} = r^2 + \frac{r}{K}\sqrt{\phi}. \tag{33}$$

As a particularly simple example assume $F(r_0) = \text{const.}$ This gives a density proportional to

$$\sqrt{\left(1+\frac{1}{Kr}\sqrt{\phi}\right)}.$$

Putting $\phi \sim r^2$ this becomes a constant, and the equation (32) is satisfied. Hence, in order to realize a dispersing lens in which at least near the axis the radial force is exactly proportional to the radius, the cathode must be arranged in such a way as to make $F(r_0) = \text{const.}$ This problem can easily be solved once the magnetic field is given. How far such a lens would in fact possess the desired qualities, and how far electron interaction would interfere with its performance, only experiment can show.

6. Magnetic effect of the rotating space charge

Up to now, in the examples though not in the general formulae, a homogeneous magnetic field has been assumed. One must now check under what conditions this assumption is justified.

The rotating electron swarm represents a ring current of intensity $i_t = \rho\bar{v}_t$. By a well-known proposition regarding the centre of gravity of a spherical shell segment

$$\bar{v}_t = \tfrac{1}{2}v(\cos\alpha_1 + \cos\alpha_2),$$

and introducing again u instead of v, from equation (17·1) the simple result

$$\bar{u}_t = \mathscr{A} \tag{34}$$

is obtained.

Outside the radius r_0 the cloud rotates in the direction of the vector potential A, inside it in the opposite direction. It may be noted that the maximum departures from the average $\mathscr{A}$ are $\pm au_0/r$. Hence on approaching the axis one finds increasing tangential velocity differences, at the same time as increasing electron density, so that electron interaction ought to play a particularly prominent part in this region.

The ring current has such a sign that outside r_0 it opposes the current which produces the magnetic field, whereas inside r_0 it increases the field. For large ratios r_2/r_0 the shielding effect far predominates.

Consider now only the middle region of the magnetron, in which the magnetic field is parallel to the z-direction. The law of its distribution is

$$-\operatorname{curl} H = \frac{dH_z}{dr} = -\frac{4\pi}{c}i_t = -\frac{4\pi}{c}\rho v_t. \tag{35}$$

Using equations (3) and (34) H_z and v_t is expressed by means of the vector potential A, and denoting differentiation with respect to r by primes,

$$A'' + \frac{1}{r}A' - \frac{1}{r^2}A = \frac{1}{\lambda^2}\left[A - \frac{a}{r}A_0\right]\left(\frac{\rho}{\rho_H}\right), \tag{36}$$

is obtained, where
$$\lambda^2 = 2\frac{mc^2}{eH} = \frac{1}{2}\left(\frac{c}{\omega_H}\right)^2. \tag{37}$$

λ is a characteristic length, which is $1/2\pi\sqrt{2} = 0{\cdot}1125$ of the vacuum wave-length associated with the frequency ω_H. Numerically

$$\lambda H = 2410\,\text{cm.gauss}. \tag{37·1}$$

The effect of the ring current will be appreciable only in large magnetrons in which the radius is of the order of λ.

Now write the charge density in the form

$$\rho = \rho_H \frac{\rho(r_2)}{\rho_H}\frac{r_2}{r} = \rho_H \frac{R}{r}. \tag{38}$$

The characteristic radius R will approach $\frac{1}{4}r_2$ for large ratios r_2/r_0. The equation (36) now becomes

$$A'' + \frac{1}{r}A' - \left(\frac{1}{r^2} + \frac{R}{\lambda^2 r}\right)A = -\frac{Ra}{(\lambda r)^2}A_0. \tag{36·1}$$

This has the particular solution $A = aA_0/r$,

which is of no importance, as it does not contribute to the magnetic field. The solution of the homogeneous equation is

$$A = Z_2(i\sqrt{[4Rr/\lambda^2]}), \tag{39}$$

i.e. a cylindrical function of the second order with imaginary argument. The solution is therefore a sum, with constant coefficients of the modified Bessel and Hankel functions $I_2(\xi)$ and $K_2(\xi)$, if the symbol ξ is introduced for the dimensionless parameter $\sqrt{(4Rr/\lambda^2)}$.

For the constants there are two boundary conditions. The first is that at the outer radius r_2 the field intensity H_z must assume a prescribed value. Substituting equation (39) into equation (3) and using the well-known relation between cylindrical functions

$$Z'_n + \frac{n}{\xi}Z_n = Z_{n-1},$$

then
$$\xi_2[C_1 I_1(\xi_2) + C_2 K_1(\xi_2)] = 2r_2 H(r_2), \tag{40}$$

where ξ_2 is the value of ξ corresponding to r_2.

The second boundary condition is not so obvious, as the field intensity at the inner radius r_1 is unknown. The condition is obtained by the following consideration. Though the solution does not extend to the axis, but only to the radius r_1, it ought to be possible to continue it, by adding ring currents inside, according to the same law, without modifying the solution outside r_1, as currents inside a radius have no effect on the field outside it. This process of continuation leads at the axis to a magnetic field of the same direction as the outer field, which goes to infinity like $1/r$. This gives the condition from which the coefficient C_2 of the modified Hankel function K_2 can be determined.

In the following, to simplify the discussion, only the case $A_0 = 0$ will be considered, i.e. the cathode arranged on the line $\psi = 0$, in which case the electrons arrive at the axis with zero tangential velocity and no singularity arises. In this case the whole mass of electrons rotates in one direction, so as to oppose the outer field by its magnetic effect. The solution in this case is given by the modified Bessel function $I_2(\xi)$, and the magnetic intensity follows the law

$$\frac{H(r)}{H(r_2)} = \sqrt{\left(\frac{r_2}{r}\right)} \frac{I_1(\sqrt{[4Rr/\lambda^2]})}{I_1(\sqrt{[4Rr_2/\lambda^2]})}. \tag{41}$$

For small arguments $I_1(\xi)/\xi$ approaches the limit $\frac{1}{2}$. Hence the magnetic field intensity at the axis becomes, if H is written for $H(r_2)$,

$$H_0 = \tfrac{1}{2}H\xi_2/I_1(\xi_2), \tag{41·1}$$

and as in this case $R = \frac{1}{4}r_2$, this can be written

$$H_0 = \tfrac{1}{2}H(r_2/\lambda)/I_1(r_2/\lambda). \tag{41·2}$$

The following table, calculated from the data of Jahnke & Emde (*Funktionentafeln*), shows the shielding effect of the ring current for magnetrons of different radius, measured in units of λ:

r_2/λ	0·25	0·5	1	2	3	4
H_0/H %	99·3	97·0	88·5	62·9	38·9	20·5
r_2/λ	5	6	7	8	9	10
H_0/H %	10·2	4·78	2·24	1·00	0·437	0·187

The effect becomes therefore very strong in large magnetrons, not too large to be practicable. With $H = 5000$ gauss, for instance, λ is a little less than 0·5 cm.; hence with a radius of 5 cm. it ought to be possible to reduce the magnetic field at the axis to less than 100 gauss. According to the present theory, therefore, it ought to be possible to produce extraordinary concentrations of free electrons, far larger than ever produced experimentally, and to study them under favourable conditions in a relatively weak magnetic field. It may well be possible to produce concentrations sufficient for observable optical effects. But it must be borne in mind that the higher the electron density the stronger the interaction, and the more remote the present theory must be from reality.

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The electrical conductivity of an ionized gas in a magnetic field, with applications to the solar atmosphere and the ionosphere

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The methods of Chapman and Enskog are used to discuss conduction of electricity and diffusion currents in an ionized gas with several constituents, in a transverse magnetic field. The free-path formula for the conductivity is compared with that derived by the exact methods. The two formulae are identical in form if a correction is applied to the usual free-path method; this correction robs the method of much of its simplicity. The uncorrected free-path method, however, gives correct results for the electron contribution to the conductivity in all practical cases; and for the ion contribution if a large number of neutral molecules are present—e.g. in the earth's upper atmosphere, about 5×10^5 times the number of ions (of both signs).

Numerical values are given for the conductivity in the sun's outer layers and in the earth's upper atmosphere. Mechanical forces due to currents induced in moving material are shown to be very important in the sun, and in the F-layer of the earth's atmosphere. The solar results are used to discuss the motion of solar prominences and eruptions. In the earth's atmosphere, the observed collision frequencies of electrons are shown to imply upper limits for ion-densities in the E and F layers. The integral conductivities of the E and F layers are estimated, and it is shown that, on the dynamo theory of the lunar variation of the earth's magnetic field, tidal oscillations in these layers must be between 100 and 1000 times as great as those at the ground. Diamagnetism and drift currents are shown to make negligible contributions to the lunar and solar variations of the earth's magnetic field.

In an Appendix, the applicability of Boltzmann's equation to strongly ionized gases is discussed.

1. Introduction

In discussing diffusion, and the conduction of electricity, in an ionized gas in a transverse magnetic field, some authors have used the free-path method, and others the more exact 'velocity-distribution' method, originally developed by Chapman