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STATIONARY GROUPS OF LINEAR DIFFERENTIAL EQUATIONS

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Dedicated to my dear teacher, Academician Otakar Borůvka, on the occasion of the 85th anniversary of his birthday.

I. INTRODUCTION

The most general form of a pointwise transformation that converts any linear homogeneous differential equation of the nth order, $n \geqq 2$, into an equation of the same type was derived by P. Stäckel in 1893 [12]. It consists in a change of the independent variable and in multiplying the dependent variable by a variable non-vanishing factor. When these transformations transform equations on their whole intervals of definition, we say that they are global.

Global transformations of the second order equations $y'' + p(x)\, y = 0$ including those that convert a given second order equation into itself ("dispersions") were deeply studied and completely described by O. Borůvka. He summarized his original methods and results till 1967 in the monograph [2].

For arbitrary n, $n \geqq 2$, the structure of global transformations was described, mainly by algebraic means, in [7]. Some results about stationary groups, i.e. groups formed by all global transformations of a linear differential equation of the nth order into itself, were obtained in [8] by using methods of the theory of functional equations.

In [11] J. Posluszny and L. A. Rubel characterized those transformations ("motions") of a linear differential equation of the nth order into itself that consist in a change of the independent variable only.

Here we give a complete list of possible groups of global transformations (in the most general form involving both the change of the dependent and independent variables) of a linear homogeneous differential equation of an arbitrary order n, $n \geqq 2$, into itself (in Theorem 1) and for each type of the group we characterize the corresponding equations (in Theorem 2). Examples of equations of each type are introduced in Theorem 3 and a brief account of possible groups with respect to the number of parameters, as announced in [10], is given in Theorem 4.

The method is based on the criterion of global equivalence [9] using also some ideas of G. H. Halphen [4], which restricts all stationary groups to subgroups of a 3-parameter group conjugated to the group of dispersions of a linear differential equation of the second order. Then the strong results of O. Borůvka [2] can be used together with some recent results of his [3] and of G. Blanton and J. A. Baker [1], including methods and results from algebra, topology and the theory of functional equations [7], [8]. An important role in the present characterization is played by Proposition 5 stating that only an iterative equation may admit a global transformation into itself whose increasing change of the independent variable intersects the identity.

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II. NOTATION AND DEFINITIONS

Let

$$P_n(y, x; I) = y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_0(x)\, y = 0 \quad \text{on} \quad I$$

and

$$Q_n(z, t; J) = z^{(n)} + q_{n-1}(t)\, z^{(n-1)} + \ldots + q_0(t)\, z = 0 \quad \text{on} \quad J$$

be linear differential equations of the nth order, $n \geqq 2$, with real continuous coefficients defined on open intervals I and $J \subset \mathbf{R}$, respectively, always considered with the leading coefficient equal to one.

In accordance with the most general form of (local) pointwise transformations due to P. Stäckel [12], and with respect to our requirement on global transformations that transform solutions on their whole intervals of definition, we say [10] that the equation $P_n(y, x; I)$ is *globally transformable* into $Q_n(z, t; J)$ if there exist functions f and h,

$$f\colon J \to \mathbf{R}, \quad f \in C^n(J), \quad f(t) \neq 0 \quad \text{on} \quad J,$$

$$h\colon J \to I, \quad h(J) = I, \quad h \in C^n(J), \ \mathrm{d}h(t)/\mathrm{d}t \neq 0 \quad \text{on} \quad J$$

such that

$$z(t) = f(t) \,.\, y(h(t)), \quad t \in J$$

is a solution of $Q_n(z, t; J)$ whenever y is a solution of $P_n(y, x; I)$.

We denote such a transformation by $T = \langle f, h \rangle$ and write $T * P_n(y, x; I) = Q_n(z, t; J)$, or briefly $T * P_n = Q_n$. The equations P_n and Q_n will be called *globally equivalent*, since "global transformability" is an equivalence relation.

For $n \geqq 2$, let $\mathbf{L}_n$ denote the set of all equations $P_n(y, x; I)$ having the coefficient p_{n-1} identically zero on I and the coefficient p_{n-2} from the class $C^{n-2}(I)$.

Consider a linear differential equation of the second order

$$u'' + p(x)\,u = 0 \quad \text{on} \quad I\,, \tag{1}$$

$p \in C^{n-2}(I)$, $n \geqq 2$, and two of its linearly independent solutions u_1, u_2. Define

$$y_i := u_1^{i-1} \,.\, u_2^{n-i}\,, \quad i = 1, \ldots, n\,.$$

We have $y_i \in C^n(I)$, the n-tuple $y_1, \ldots, y_n$ having a nonvanishing Wronskian. Hence the y_i's can be considered as solutions of a (unique) linear differential equation of the nth order, called the *iterative equation.* We shall denote this equation by

$$[p]_n\,(y, x; I) = 0\,, \quad \text{or briefly} \quad [p]_n = 0\,,$$

to express its dependence on the coefficient p in (1). The left hand side of the equation (with the unit leading coefficient) is called the *iterative operator* of the nth order *iterated* from (1) and has the form

$$[p]_n\,(y, x; I) = y^{(n)} + \binom{n+1}{3} p(x)\, y^{(n-2)} + 2\binom{n+1}{4} p'(x)\, y^{(n-3)} + \ldots\,,$$

see, e.g., [6].

According to O. Borůvka [2], the differential equation (1) is of a *finite type m, m* integer, $m \geqq 1$, if it possesses solutions with m zeros on the interval I but none with $m + 1$ zeros. In this case (1) is of the *general kind* if there are two linearly independent solutions with $m - 1$ zeros on I, otherwise (1) is of the *special kind.* If (1) is not of a finite type then it is either *one-side oscillatory* or *both-side oscillatory on I.*

Two linear differential equations, (1) and

$$v^{\cdot\cdot} + q(t)\,v = 0 \quad \text{on} \quad J \tag{2}$$

are of the *same character* if either

(i) both are of the same finite type m, and of the same kind (therefore both general or both special), or
(ii) both are one-side oscillatory, or
(iii) each is both-side oscillatory.

The *Kummer equation* associated with two linear differential equations (1) and (2) is the third order nonlinear equation

$$\{h, t\} + p(h(t))\,.\,h^{\cdot 2}(t) = q(t)\,, \quad t \in J\,, \tag{$p; q$}$$

where $\{h, t\}$ is the *Schwarzian derivative* of h at t, i.e.,

$$\{h, t\} := \frac{1}{2}\frac{h^{\cdot\cdot\cdot}(t)}{h^{\cdot}(t)} - \frac{3}{4}\frac{h^{\cdot\cdot 2}(t)}{h^{\cdot 2}(t)}\,.$$

Again in accordance with O. Borůvka [2] we introduce the *fundamental group F* as the set of all functions $f: \mathbf{R} \to \mathbf{R}, f \in C^0(\mathbf{R})$, satisfying

$$\tan f(t) = \frac{c_{11} \tan t + c_{12}}{c_{21} \tan t + c_{22}}$$

on $\mathbf{R}$ whenever the relation is defined, where $c_{11}, c_{12}, c_{21}, c_{22} \in \mathbf{R}$ and $|c_{11}c_{22} - c_{12}c_{21}| = 1$, with composition as the group operation.

Let G_0 and G_1 be two groups whose elements are (some) bijections of intervals I_0 and I_1 onto themselves, respectively. We say that the groups G_0 and G_1 are C^n-conjugate, if there exists a bijection φ of I_0 onto I_1, $\varphi \in C^n(I_0)$, such that

$$G_1 = \{\varphi f \varphi^{-1};\ f \in G_0\}\ .$$

Necessarily $\mathrm{d}\varphi(x)/\mathrm{d}x \neq 0$ on I_0, if $n \geqq 1$.

III. FORMER RESULTS

Let us recall some of the earlier results of O. Borůvka [2], and also some from [7], [8] and [9].

Lemma 1 ([7]). *The set of all global transformations between every pair of equations from a class of globally equivalent equations forms a Brandt groupoid with respect to the composition rule.*

The set of all global transformations of a linear differential equation P_n into itself, i.e.,

$$\{T;\ T * P_n = P_n\}\ ,$$

forms a group, $G(P_n)$, called *the stationary group of* P_n.

Lemma 2 ([7]). *Let* $T * P_n = Q_n$. *Then*

$$G(P_n) = TG(Q_n)\, T^{-1}\ ;$$

in other words: Stationary groups of any pair of globally equivalent equations are conjugate.

Lemma 3 ([9]). *Each equation* $Q_n(z, t; J) = 0$ *with* $q_{n-1} \in C^{n-1}(J)$ *and* $q_{n-2} \in C^{n-2}(J)$ *can be globally transformed onto an equation from* $\mathbf{L}_n$, *written in the form*

(3) $$[p]_n\,(y, x; I) + r_{n-3}(x)\, y^{(n-3)} + \ldots + r_0(x)\, y = 0 \quad \text{on} \quad I\ ,$$

where $p \in C^{n-2}(I)$, $r_i \in C^0(I)$ *for* $i = 0, 1, \ldots, n-3$.

Lemma 4 ([9]). *Let* $p_{n-1} = 0$ *in* $P_n(y, x; I)$ *and* $T * P_n(y, x; I) = Q_n(z, t; J)$, $T = \langle f, h \rangle$. *Then* $q_{n-1} = 0$ *in* $Q_n(z, t; J)$ *if and only if*

$$f(t) = c \,.\, |\mathrm{d}h(t)/\mathrm{d}t|^{(1-n)/2} \,,$$

c being a nonzero constant and $h \in C^{n+1}(J)$.

Lemma 5 ([9]). *Let an iterative equation* $[p]_n (y, x; I) = 0$ *be transformed by* $T = \langle c \,.\, |h^{\cdot}|^{(1-n)/2}, h \rangle$, $c = \text{const.} \neq 0$, *into an equation* Q. *Then* Q *is again an iterative equation,* $[q]_n(z, t; J) = 0$, *where* q *is the coefficient in the second order equation* $v^{\cdot\cdot} + q(t)\, v = 0$ *on* J *obtained from* $u'' + p(x)\, u = 0$ *on* I *by the transformation* $\langle |h^{\cdot}|^{-1/2}, h \rangle$.

Lemma 6 ([9]). *If the equation*

$$P_n = [p]_n (y, x; I) + r_{n-3}(x)\, y^{(n-3)} + \ldots + r_0(x)\, y = 0 \quad \text{on} \quad I$$

can be globally transformed by a transformation $T = \langle f, h \rangle$ *into*

$$Q_n = [q]_n (z, t; J) + s_{n-3}(t)\, z^{(n-3)} + \ldots + s_0(t)\, z = 0 \quad \text{on} \quad J \,,$$

then there exists a function $h \in C^{n+1}(J)$, $\mathrm{d}h(t)/\mathrm{d}t \neq 0$ *on* J, *such that the following three conditions are satisfied:*

A. $f(t) = c \,.\, |\mathrm{d}h(t)/\mathrm{d}t|^{(1-n)/2}$, $c = \text{const.} \neq 0$.

B. *The equation* $u'' + p(x)\, u = 0$ *on* I *is globally transformed by the transformation*

$$\langle |\mathrm{d}h(t)/\mathrm{d}t|^{-1/2}, h \rangle$$

into the equation $v^{\cdot\cdot} + q(t)\, v = 0$ *on* J.

C. $r_{n-3}(h(t)) \,.\, h^{\cdot 3}(t) = s_{n-3}(t)$ *on* J,
$r_{n-4}(h(t)) \,.\, h^{\cdot 4}(t) = s_{n-4}(t)$ *on* $\{t \in J;\ s_{n-3}(t) = 0\}$,
$r_{n-5}(h(t)) \,.\, h^{\cdot 5}(t) = s_{n-5}(t)$ *on* $\{t \in J;\ s_{n-3}(t) = s_{n-4}(t) = 0\}$,
etc.

Lemma 7 ([2]). *Let* $t_0 \in J$, $h_0 \in I$, $h^{\cdot}_0 (\neq 0)$, $h^{\cdot\cdot}_0$ *be arbitrary. Then there is precisely one maximal solution h of the Kummer equation* $(p; q)$ *defined on the interval* $J^* \subset \subset J$ *with the Cauchy initial conditions*

$$h(t_0) = h_0 \,, \quad h^{\cdot}(t_0) = h^{\cdot}_0 \,, \quad h^{\cdot\cdot}(t_0) = h^{\cdot\cdot}_0 \,;$$

where maximal is used in the sense that every solution of $(p; q)$ *satisfying the same initial conditions is a portion of h.*

Moreover, the transformation $\langle |h^{\cdot}|^{-1/2}, h\rangle$ *globally transforms the equation*

$$u'' + p(x)\,u = 0 \quad on \quad I^* = h(J^*) \subset I$$

onto

$$v^{\cdot\cdot} + q(t)\,v = 0 \quad on \quad J^* .$$

Lemma 8 ([2]). *Let* $\langle |h^{\cdot}|^{-1/2}, h\rangle$ *globally transform* (1) *into* (2). *Then the function* h *is a solution of the Kummer equation* $(p; q)$, h *is a bijection of* J *onto* I *and* $\mathrm{d}h(t)/\mathrm{d}t \neq 0$ *on* J. *If* $p \in C^n(I)$ *and* $q \in C^n(J)$, *then* $h \in C^{n+3}(J)$.

Lemma 9 ([2]). *The second order linear differential equations* (1) *and* (2) *are globally equivalent if and only if they are of the same character.*

Lemma 10 ([2]). *There are precisely the following classes of globally equivalent second order linear differential equations* (1):

both-side oscillatory equations;
one-side oscillatory equations;
equations of the finite type m and of special kind,

$$m = 1, 2, \ldots ;$$

equations of the finite type m and of general kind,

$$m = 1, 2, \ldots .$$

In the corresponding order here is the list of possible representations (canonical equations) for each of the above classes:

$$u'' + u = 0 \quad \text{on} \quad (-\infty, \infty),$$
$$u'' + u = 0 \quad \text{on} \quad (0, \infty),$$
$$u'' + u = 0 \quad \text{on} \quad (0, m\pi), \qquad m = 1, 2, \ldots ;$$
$$u'' + u = 0 \quad \text{on} \quad (0, m\pi - \pi/2), \quad m = 1, 2, \ldots ;$$

Lemma 11 ([2]). *The stationary group of the equation*

$$u'' + u = 0 \quad \text{on} \quad (-\infty, \infty)$$

is formed by the transformations $\langle c\,.\,|f'|^{-1/2}, f\rangle$, $c =$ const. $\neq 0$, *where* f *belongs to the fundamental group* F. *All elements of* F *are precisely all the maximal solutions of the Kummer equation* $(+1; +1)$, *i.e.*,

$$(+1; +1) \qquad\qquad \{f, x\} + f'^2(x) = 1 .$$

Lemma 12 ([7] and [8]). *Let* $T * P_n(y.\, x; I) = P_n(y, x; I)$, *where* $T = \langle f, h\rangle$ *with* $h(x) \neq x$ *for each* $x \in I$. *Then the equation* $P_n(y, x; I)$ *is globally equivalent* *to a linear nth order differential equation with periodic coefficients of the same period defined on the whole interval* $(-\infty, \infty)$.

The stationary group of a second order linear differential equation of the form (1) is formed by (some of the) transformations

$$\langle c \, . \, |h'|^{-1/2}, h \rangle \, , \quad c = \text{const.} \neq 0 \, .$$

The set of all these h's forms again a group with respect to composition, which was introduced by O. Borůvka [2] as the *group of dispersions* (*of the first kind*) of the equation (1).

The stationary group $G(P_n)$ of an equation P_n from $\mathbf{L}_n$ is formed by (some of the) transformations of the form

$$\langle c \, . \, \{h'|^{(1-n)/2}, h \rangle \, , \quad c = \text{const.} \neq 0 \, ,$$

see Lemma 4. The set of all the h's forms again a group with respect to composition of functions, which will be denoted by $G_0(P_n)$.

IV. PREPARATORY RESULTS

Proposition 1. *Let* $T * P_n = P_n \in \mathbf{L}_n$, *where* $P_n(y, x; I)$ *is written in the form* (3), *i.e., as*

$$[p]_n (y, x; I) + r_{n-3}(x) \, y^{(n-3)} + \ldots + r_0(x) \, y = 0 \quad on \quad I \, ,$$

with $p \in C^{n-2}(I)$, $r_i \in C^0(I)$ *for* $i = 0, 1, \ldots, n-3$.

Then $T = \langle c \, . \, |h'|^{(1-n)/2}, h \rangle$, $h \in C^{n+1}(I)$, *and* h *satisfies the Kummer equation* $(p; p)$:

$$\{h, x\} + p(h(x)) \, . \, h'^2(x) = p(x) \, , \quad x \in I \, .$$

Proof follows from Lemma 6 (conditions A and B) and from Lemma 8, where $p \equiv q \in C^{n-2}(I)$.

Proposition 2. *Up to conjugacy there are the following groups of dispersions of the second order linear differential equations of the form* (1):

1. *All functions* $f \colon \mathbf{R} \to \mathbf{R}$ *of the form*

$$f(x) = \text{Arctan} \frac{a \tan x + b}{c \tan x + d} \, , \quad x \in \mathbf{R} \, ,$$

where $a, b, c, d \in \mathbf{R}$, $ad - bc = \pm 1$, *that is a three-parameter group both with increasing and decreasing functions, called the "fundamental group" in Borůvka's terminology.*

2a. *All functions* $f\colon (0, \infty) \to (0, \infty)$ *of the form*

$$f(x) = \operatorname{Arctan} \frac{a \tan x}{c \tan x + 1/a}, \quad x \in (0, \infty),$$

$a, c \in \mathbf{R}$, $a \neq 0$, *forming a two-parameter group of increasing functions.*

2b. *All functions* $f\colon (0, m\pi) \to (0, m\pi)$, *m being a positive integer,*

$$f(x) = \operatorname{Arctan} \frac{a \tan x}{c \tan x \pm 1/a}, \quad x \in (0, m\pi),$$

$a, c \in \mathbf{R}$, $a \neq 0$, *that is a two-parameter group with both increasing and decreasing functions.*

3a. *All functions* $f\colon (0, m\pi - \pi/2) \to (0, m\pi - \pi/2)$, *m being a positive integer, of either the form*

$$f(x) = \operatorname{Arctan} (k \tan x), \quad x \in (0, m\pi - \pi/2),$$

or of the form

$$f(x) = \operatorname{Arctan} (k \cot x), \quad x \in (0, m\pi - \pi/2),$$

$k \in \mathbf{R}$, $k > 0$, *forming a one-parameter group of both increasing and decreasing functions.*

Remark. The group operation is always the composition of functions. The function Arctan means the branch of $\arctan x + m\pi$ that makes the function f continuous. It was proved in [2] that this is always possible and f is then even analytic.

Proof (see also [2]). Due to Lemmae 2 and 10, it is sufficient to consider groups of dispersions of the equation $u'' + u = 0$ on the following intervals:

1. $(-\infty, \infty)$. According to [2], the corresponding group of dispersions is called *the fundamental group* and the explicit formula of its elements is given in the Proposition under 1. Each of the elements is a bijection of $(-\infty, \infty)$ onto $(-\infty, \infty)$ and satisfies the Kummer equation $(+1, +1)$ on $(-\infty, \infty)$.

2a. $(0, \infty)$. According to [2], all elements of the group of dispersions in this case satisfy the Kummer equation $(+1; +1)$ on $(0, \infty)$ and, at the same time, they are bijections of $(0, \infty)$ onto $(0, \infty)$. Hence the elements are the restrictions of the functions

$$f(x) = \operatorname{Arctan} \frac{a \tan x + b}{c \tan x + d}, \quad ad - bc = \pm 1,$$

on $(0, \infty)$, for which $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$, or $\lim_{x \to 0+} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 0$. The first case implies $b = 0$ and $ad - bc = 1$, the second case is impossible. Hence $b = 0$ and $d = 1/a$ for $a \neq 0$ gives exactly the functions introduced under $2a$ in the Proposition.

2b. $(0, m\pi)$, $m = 1, 2, \ldots$. The same argument as in the case 2a leads to considering restrictions of elements of the fundamental group to the interval $(0, m\pi)$ that are bijections of the interval onto itself. For increasing dispersions we get

$$b = 0 \quad \text{and} \quad d = 1/a \quad \text{for} \quad a \neq 0 ,$$

for decreasing dispersions

$$b = 0 \quad \text{and} \quad d = -1/a \quad \text{for} \quad a \neq 0 ,$$

exactly the functions given in 2b of the Proposition.

3a. $(0, m\pi - \pi/2)$, $m = 1, 2, \ldots$. Analogous argument as above gives for increasing dispersions

$$b = 0 \quad \text{and} \quad \lim_{x \to m\pi - \pi/2} \operatorname{Arctan} \frac{a \tan x}{c \tan x + 1/a} = m\pi - \pi/2 ,$$

or

$$\lim_{x \to m\pi - \pi/2} \operatorname{Arctan} \frac{a \sin x}{c \sin x + \frac{1}{a} \cos x} = m\pi - \pi/2 , \quad \text{or} \quad c = 0 ;$$

hence $f(x) = \operatorname{Arctan} \frac{a \tan x}{1/a} = \operatorname{Arctan}(k \tan x)$, $x \in (0, m\pi - \pi/2)$, $k = a^2 > 0$.

For decreasing dispersions we obtain

$$d = 0 \quad \text{and} \quad a = 0 ,$$

or

$$f(x) = \operatorname{Arctg} \frac{b}{1/b \tan x} = \operatorname{Arctan}(k \cot x) , \quad k > 0 ,$$

which completes the proof of the Proposition.

Proposition 3. *The group $G_0(P_n)$ of each equation $P_n \in \mathbf{L}_n$ is C^{n+1}-conjugate to a subgroup of one of the groups listed in Proposition* 2.

Proof. Due to Lemmae 5, 6 and Proposition 2 we know that the functions h in Proposition 3 form a group which is conjugate to a subgroup of a group listed in Proposition 2 under 1, 2a, 2b, and 3a. Since

$$P_n(y, x; I) = [p]_n (y, x; I) + r_{n-3}(x)\, y^{(n-3)} + \ldots + r_0(x)\, y = 0 \quad \text{on} \quad I$$

with $p \in C^{n-2}(I)$, Lemmae 2 and 8 guarantee that this conjugacy is also a C^{n+1}-conjugacy.

Remark. In Proposition 3 we have mentioned subgroups of groups listed in Proposition 2, since the condition B in Lemma 6 is generally only sufficient for global transformations. To facilitate a classification of these subgroups we introduce the following notions.

Each element of the fundamental group F is uniquely determined as a solution f of the Kummer equation $(+1; +1)$ on $(-\infty, \infty)$ satisfying at a (fixed) point t_0 the Cauchy conditions

$$f(t_0) = f_0\,, \quad f'(t_0) = f_0' \neq 0\,, \quad f''(t_0) = f_0''\,.$$

To the solution we assign the point (f_0, f_0', f_0'') in the 3-dimensional euclidean space $\mathbf{R}^3$. The topology of this space induces a topology in the fundamental group F that becomes a topological group due to the continuous dependence of solutions of $(+1; +1)$ on initial values. This topology induces also a topology of each group listed in Proposition 2.

Proposition 4. *Each* $G_0(P_n)$ is C^{n+1}*-conjugate to a closed subgroup of a group listed in Proposition* 2.

Proof. According to Proposition 3, $G_0(P_n)$ is C^{n+1}-conjugate to a subgroup of a group listed in Proposition 2. This subgroup is determined by a finite number of restrictions involving continuous functions of the parameters a, b, c, d in the case that $u'' + p(x)\,u = 0$ on I is not both-side oscillatory. If not all (continuous) r_i of P_n written in the form (3) are identically zero, we have (a finite number of) further conditions involving continuous functions, caused by the fact that not only $[p]_n = 0$ must be transformed into itself, but the whole P_n is globally transformed into itself and the coefficients by $y, y', \ldots, y^{(n-3)}$ in the original and in the transformed equation must coincide. Due to the continuity of the functions in each of the finite number of the equations, the subgroup determined by these restrictions is closed.

Proposition 5. *Consider* $P_n(y, x; I) \in \mathbf{L}_n$ *written in the form* (3). *Let* $T = \langle f, h\rangle$, $\mathrm{d}h(x)/\mathrm{d}x > 0$ *on* I, *transform* P_n *into itself.*

If there exists $x_0 \in I$ *such that* $h(x_0) = x_0$ *and* $h \neq \mathrm{id}_I$, *then the equation* P_n *is an iterative equation, i.e.,* $r_{n-3} \equiv \ldots \equiv r_0 \equiv 0$.

Proof. Due to Lemma 4, $f = c\,.\,|h'|^{(1-n)/2}$, $h \in C^{n+1}(I)$. According to our suppositions, the set $S := \{x \in I;\ h(x) \neq x\}$ is open. Hence S is the union of at most denumerable number of open disjoint intervals. Because $h(x_0) = x_0$ for $x_0 \in I$ and $h \neq \mathrm{id}_I$, each of the intervals has at least one finite end-point in I. Denote by (a, b) one of the intervals. Without loss of generality, let $a \in I$. Then $h(a) = a$.

Applying Lemma 6, the condition C gives

$$r_{n-3}(h(x))\,.\,h'^3(x) = r_{n-3}(x) \quad \text{on} \quad I\,,$$

or

$$R_{n-3}(h(x)) - R_{n-3}(h(a)) = R_{n-3}(x) - R_{n-3}(a)$$

for

$$R_{n-3}(x) := \int_{x^*}^{x} |r_{n-3}(\sigma)|^{1/3}\,\mathrm{d}\sigma \in C^1(I)\,, \quad x^* \in I\,.$$

Hence

$$R_{n-3}(h(x)) = R_{n-3}(x) \quad \text{on} \quad I\,. \tag{4}$$

Let R_{n-3} be not a constant on (a, b), i.e., there exist $a_1, b_1 \in (a, b)$ such that $R_{n-3}(a_1) \neq R_{n-3}(b_1)$. Define two sequences $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ as follows:

if $h(a_1) > a_1$ and hence necessarily $h(b_1) > b_1$, then

$$a_{i+1} := h^{-1}(a_i) \quad \text{and} \quad b_{i+1} := h^{-1}(b_i)$$

for $i = 1, 2, \ldots$;

if $h(a_1) < a_1$ (hence also $h(b_1) < b_1$), then

$$a_{i+1} := h(a_i) \quad \text{and} \quad b_{i+1} := h(b_i)$$

for $i = 1, 2, \ldots$.

Both the sequences $\{a_i\}$ and $\{b_i\}$ are decreasing and converging to a. Due to (4),

$$R_{n-3}(a_i) = R_{n-3}(a_1) \neq R_{n-3}(b_1) = R_{n-3}(b_i)\,,$$

which contradicts $R_{n-3} \in C^1(I)$, since (finite) $a \in I$.

Hence R_{n-3} is constant on (a, b) which yields $r_{n-3}(x) = 0$ on (a, b). Thus $r_{n-3}(x) = 0$ on S.

Returning to condition C in Lemma 6, we get

$$r_{n-4}(h(x))\,.\,h'^4(x) = r_{n-4}(x) \quad \text{on} \quad S\,.$$

Since $h'(x) > 0$ on I, we may write

$$|r_{n-4}(h(x))|^{1/4}\,.\,h'(x) = |r_{n-4}(x)|^{1/4}\,,$$

and

$$R_{n-4}(h(x)) - R_{n-4}(h(a)) = R_{n-4}(x) - R_{n-4}(a)$$

for

$$R_{n-4}(x) := \int_{x^*}^{x} |r_{n-4}(\sigma)|^{1/4}\,d\sigma \in C^1(a, b)\,.$$

The above argument yields $R_{n-4} = \text{const.}$ on (a, b), hence $|r_{n-4}(x)| = 0 = r_{n-4}(x)$ on (a, b).

Analogously we get

$$r_i(x) = 0 \quad \text{on} \quad S = \{x \in I;\ h(x) \neq x\}$$

for $i = 0, 1, \ldots, n-3$.

If S is dense in I then the continuity of the r_i's gives $r_i = 0$ on I for all $i = 0, 1, \ldots, n-3$.

If S is not dense in I then there exists an interval $(c, d) \subset I$ such that $h(x) = x$ on (c, d). Take $x_1 \in (c, d)$. Evidently $h(x_1) = x_1$, $h'(x_1) = 1$, $h''(x_1) = 0$. Moreover, due to Proposition 1, h satisfies the Kummer equation $(p;\, p)$ on I, whose solutions are uniquely determined by the Cauchy conditions, Lemma 7. Since id_I satisfies the equation $(p;\, p)$ and the above initial conditions at x_1, we have $h = \mathrm{id}_I$, which was excluded from our considerations.

This completes the proof of Proposition 5.

Proposition 6. *Let P_n be not an iterative equation and let $h \in G_0(P_n)$, where $\mathrm{d}h(x)/\mathrm{d}x > 0$ and $h(x) \neq x$ on I. Increasing elements of $G_0(P_n)$ form its subgroup $G_0^+(P_n)$.*

Then either h is an element of an infinite cyclic group generated by a function h such that $h \in C^{n+1}(I)$, $\mathrm{d}h(x)/\mathrm{d}x > 0$ on I, $h(x) \neq x$ on I and $G_0^+(P_n)$ is the infinite cyclic group,

or P_n is globally equivalent to an equation from $\mathbf{L}_n$ with constant coefficients on $(-\infty, \infty)$ and $G_0^+(P_n)$ is C^{n+1}-conjugate to the group whose elements are functions

$$f_c\colon \mathbf{R} \to \mathbf{R}\,, \quad f_c(t) = t + c\,, \quad c \in \mathbf{R}\,.$$

Proof. If any two different elements of the group $G_0^+(P_n)$, say h and $\hat{h}$, intersect each other somewhere in I, i.e., $h(x_0) = \hat{h}(x_0)$ for $x_0 \in I$, then there is an element of the group, $h^{-1}\hat{h}$, that differs from id_I and satisfies $h^{-1}\,\hat{h}(x_0) = x_0$. According to Proposition 5, P_n is iterative. However, this was excluded from our considerations. Hence the elements of the group $G_0^+(P_n)$ can be naturally ordered. We obtain a fully ordered Archimedean group, because

$$\lim_{i \to \pm\infty} g^i(a)\,, \quad a \in I\,,$$

converges to both ends of the interval I for each element $g \in G_0^+(P_n)$, $g \neq \mathrm{id}_I$.

Due to [5], [2] and [1], there exists an order preserving isomorphism of $G_0^+(P_n)$ onto a subgroup of the additive group $\mathbf{R}$. This subgroup is either the infinite cyclic group generated by a nonzero number $b \in \mathbf{R}$ and formed by all numbers

$$\{ib;\ i = \ldots, -1, 0, 1, \ldots\}\,,$$

or it is dense in $\mathbf{R}$. In the first case h_1 is assigned some $n_1 b$, h_2 is assigned some $n_2 b$, and there exists an $h \in G_0^+(P_n)$ to which b is assigned, hence $h_1 = h^{n_1}$ and $h_2 = h^{n_2}$ for (positive or negative integers) n_1 and n_2. Since h is obtained by finite compositions of h_1, h_2, h_1^{-1}, and h_2^{-1}, $h \in C^{n+1}(I)$, $\mathrm{d}h(x)/\mathrm{d}x > 0$ and $h \in G_0^+(P_n)$. Hence $G_0^+(P_n)$ is an infinite cyclic group and h is its generator.

In the second case, when the subgroup of the additive group $\mathbf{R}$ is dense, the group $G_0^+(P_n)$ is isomorphic to the whole group $\mathbf{R}$, since it is closed (Proposition 4) and no different elements of $G_0^+(P_n)$ intersect each other. This isomorphism is order preserving. Each of the elements of $G_0^+(P_n)$ is of the class $C^{n+1}(I)$. Hence Theorem 1 from [1] guarantees the existence of a bijection of I onto $\mathbf{R}$, $\varphi \in C^{n+1}(I)$, such that

$$G_0^+(P_n) = \{\varphi^{-1}(\varphi(x) + c);\ c \in \mathbf{R}\}\,;$$

in other words, $G_0^+(P_n)$ is C^{n+1}-conjugate to the group formed by functions $f_c\colon \mathbf{R} \to \mathbf{R}$, $c \in \mathbf{R}$,

$$f_c(t) = t + c\,.$$

Consider the differential equation Q_n, $Q_n = T * P_n$, where $T = \langle |\varphi^{-1\prime}|^{(1-n)/2}, \varphi^{-1}\rangle$, $\varphi^{-1}\colon \mathbf{R} \to I$. The group $G_0^+(Q_n)$ is, according to Lemma 2, formed by all translations f_c, $c \in \mathbf{R}$. Hence Q_n, defined on $\mathbf{R}$, has only constant coefficients, Q.E.D.

Proposition 7. *Let* $h \in G_0(P_n)$, $P_n = P_n(y, x; I) \in \mathbf{L}_n$, $\mathrm{d}h(x)/\mathrm{d}x < 0$. *Then either* $hh = \mathrm{id}_I$, *or* P_n *is an iterative equation.*

Proof. Since $\mathrm{d}h(x)/\mathrm{d}x < 0$ and $h(I) = I$, there exists $x_0 \in I$ such that $h(x_0) = x_0$. Then $hh(x_0) = x_0$, $hh \in G_0(P_n)$ and hh is increasing. If $hh \neq \mathrm{id}_I$, then P_n is an iterative equation due to Proposition 5.

Proposition 8. *Let* $P_n \in \mathbf{L}_n$ *and* $h_1 \in G_0(P_n)$ *such that* $\mathrm{d}h_1(x)/\mathrm{d}x < 0$. *Then each* $h \in G_0(P_n)$ *with* $\mathrm{d}h(x)/\mathrm{d}x < 0$ *can be written in the form*

$$h = h_+ h_1\,,$$

where $h_+ \in G_0(P_n)$ *and* $\mathrm{d}h_+(x)/\mathrm{d}x > 0$.

Proof. Put $h_+ := hh_1^{-1}$. Evidently $h_+ \in G_0(P_n)$ and $\mathrm{d}h_+(x)/\mathrm{d}x > 0$.

Proposition 9. *For* $h \in C^{n+1}(I)$, $h(I) = I$, $\mathrm{d}h(x)/\mathrm{d}x < 0$, $hh = \mathrm{id}_I$, *consider the functional equation*

$$\alpha\, h(x) = -\alpha(x)\,, \quad x \in I\,,$$

with an unknown function $\alpha\colon I \to \mathbf{R}$. *There exists a solution of this equation,*

$$\alpha(x) = (x - h(x))/2\,,$$

which is of the class $C^{n+1}(I)$ *and* $\mathrm{d}\alpha(x)/\mathrm{d}x > 0$.

Proof. Since $h \in C^{n+1}(I)$, also $\alpha \in C^{n+1}(I)$ and $\mathrm{d}\alpha(x)/\mathrm{d}x = (1 - h'(x))/2 > 0$, because $h'(x) < 0$. Furthermore,

$$\alpha(h(x)) = (h(x) - hh(x))/2 = (h(x) - x)/2 = -\alpha(x)\,, \quad x \in I\,.$$

Remark. If $I = (c, d)$ in Proposition 9 then $\alpha(I) = (\tfrac{1}{2}(c - d), \tfrac{1}{2}(d - c))$.

Proposition 10. *If* $P_n = P_n(y, x; I) \in \mathbf{L}_n$ *is not an iterative equation and* $h \in G_0(P_n)$, $\mathrm{d}h(x)/\mathrm{d}x < 0$ *on* I, *then* P_n *can be globally transformed into* $Q_n(z; t, J)$ *whose* $G_0(Q_n)$ *contains* $-\mathrm{id}_J$.

Moreover, if the subgroup of $G_0(P_n)$ *formed by increasing functions from* $G_0(P_n)$ *is an infinite cyclic group with the generators* $x \pm a$, $a > 0$, $a \in \mathbf{R}$, *then* P_n *can be globally transformed into* Q_n *such that* $-\mathrm{id}_J \in G_0(Q_n)$ *and* $t \pm a$ *are generators of the subgroup of* $G_0(Q_n)$ *formed by increasing functions of* $G_0(Q_n)$.

Proof. Due to our suppositions, Proposition 7 yields $hh = \mathrm{id}_I$. For $t = \alpha(x)$ in Proposition 9 we have

$$\alpha h \alpha^{-1}(t) = -t \quad \text{on} \quad J = \alpha(I),$$

where $\alpha(x) = (x - h(x))/2 \in C^{n+1}(I)$, $\mathrm{d}\alpha(x)/\mathrm{d}x > 0$. Hence the transformation $\langle |\alpha^{-1\prime}|^{(1-n)/2}, \alpha^{-1} \rangle$ globally transforms P_n into $Q_n(z, t; J)$. Due to Lemma 2, $G_0(Q_n) \ni \ni \alpha h \alpha^{-1} = -\mathrm{id}_J$.

Moreover, let $x \pm a$, $a > 0$, $a \in \mathbf{R}$ be the generators of the subgroup of $G_0(P_n)$ with increasing functions. Evidently $x \mapsto h(x + a)$ is a decreasing element of $G_0(P_n)$. According to Proposition 8 we may write

$$h(x + a) = h_+(h(x)),$$

$h_+ \in G_0(P_n)$ with $\mathrm{d}h_+(x)/\mathrm{d}x > 0$. However, h_+ must be one of the generators, $x \pm a$. Because h is decreasing, $h_+(x) = x - a$. Hence

$$h(x + a) = h(x) - a.$$

Now

$$\alpha(x + a) = (x + a - h(x + a))/2 = (x - h(x))/2 + a = \alpha(x) + a$$

or

$$\alpha(\alpha^{-1}(t) + a) = t + a \in G_0(Q_n), \quad \text{Q.E.D.}$$

V. MAIN RESULTS

Theorem 1. *Let $P_n = P_n(y, x; I) \in \mathbf{L}_n$, $n \geqq 2$. The stationary group $G(P_n)$ is formed by all global transformations of the form*

$$\langle c|h'|^{(1-n)/2}, h \rangle,$$

where $c =$ const. $\neq 0$, and $h(I) = I$, $h \in C^{n+1}(I)$, $h'(x) = \mathrm{d}h(x)/\mathrm{d}x \neq 0$ on I, that globally transform the equation P_n into itself. For each group $G(P_n)$ the set of all h occurring in its elements forms the group $G_0(P_n)$ with respect to composition. Here is the list of all possible groups $G_0(P_n)$ up to C^{n+1}-conjugacy:

1. *Functions $f: \mathbf{R} \to \mathbf{R}$,*

$$f(x) = \mathrm{Arctan}\,\frac{a \tan x + b}{c \tan x + d}, \quad x \in \mathbf{R},$$

$a, b, c, d \in \mathbf{R}$, $ad - bc = \pm 1$,

a three-parameter group both with increasing and decreasing analytic functions, called the fundamental group in [2];

2a. $f: (0, \infty) \to (0, \infty)$,

$$f(x) = \operatorname{Arctan} \frac{a \tan x}{c \tan x + 1/a}, \quad x \in (0, \infty)$$

$a, c \in \mathbf{R}$, $a \neq 0$, *a two-parameter group of increasing analytic functions;*

2b. *for each positive integer m:*

$f: (0, m\pi) \to (0, m\pi)$,

$$f(x) = \operatorname{Arctan} \frac{a \tan x}{c \tan x \pm 1/a}, \quad x \in (0, m\pi),$$

$a, c \in \mathbf{R}$, $a \neq 0$, *a two-parameter group of both increasing and decreasing analytic functions;*

3a. *for each positive integer m:*

$f: (0, m\pi - \pi/2) \to (0, m\pi - \pi/2)$,

$$f(x) = \operatorname{Arctan}(k \tan x) \quad \textit{and} \quad f(x) = \operatorname{Arctan}(k \cot x), \quad x \in (0, m\pi - \pi/2),$$

$k \in \mathbf{R}$, $k > 0$, *a one-parameter group of both increasing and decreasing analytic functions;*

3b. *the functions* $f_c: \mathbf{R} \to \mathbf{R}$ *and* $g_c: \mathbf{R} \to \mathbf{R}$,

$$f_c(x) = x + c, \quad g_c(x) = -x + c, \quad \textit{for all} \quad c \in \mathbf{R};$$

3c. *the increasing functions from* 3b;

4a. *the functions* $f_k: \mathbf{R} \to \mathbf{R}$ *and* $g_k: \mathbf{R} \to \mathbf{R}$,

$$f_k(x) = x + k, \quad g_k = -x + k,$$

k ranging over all integers.

4b. *the increasing functions from* 4a;

5a. *the functions* $\mathrm{id}_{\mathbf{R}}$, $-\mathrm{id}_{\mathbf{R}}$;

5b. *only the identity on* $\mathbf{R}$, $\mathrm{id}_{\mathbf{R}}$.

Remark. Arctan x means the branch of arctan $x + m\pi$ that makes the functions f in 1 – 3 continuous; then f is even analytic.

Theorem 2. *If the equation* $P_n \in \mathbf{L}_n$, $n \geqq 2$, *is an iterative equation iterated from the second order linear differential equation* (1): $y'' + p(x)\, y = 0$, $p \in C^{n-2}(I)$, *and*

if the equation (1) *is both-side oscillatory then case* 1 *in Theorem* 1 *takes place,*

if (1) *is one-side oscillatory then* 2a *is valid,*

if (1) *is of a finite type m and special then* 2b *holds, and*

if (1) *is of a finite type m and general then* 3a *is true for the group* $G_0(P_n)$.

Case 3b *in Theorem* 1 *occurs when the equation* P_n *is not an iterative equation and is globally transformable into an equation* $Q_n(z, t; \mathbf{R})$ *with constant coefficients that are zeros by the* $(n - i)$*th derivatives with odd* i:

$$q_{n-i} \equiv 0 .$$

Case 3c *is valid when* P_n *is not an iterative equation and is globally transformable into an equation* $Q_n(z, t; \mathbf{R})$ *with constant coefficients and there is a coefficient by* $(n - i)$*th derivative with odd* i *that is not identically zero.*

Case 4a *is true if* P_n *is not an iterative equation, is not globally transformable into an equation with constant coefficients on* $\mathbf{R}$, *and is globally transformable into an equation* $Q_n(z, t; \mathbf{R})$ *with periodic coefficients of the same period, satisfying*

$$q_{n-i}(-t) = (-1)^i q_{n-i}(t) \tag{5}$$

for $t \in (-\infty, \infty)$, $i = 3, \ldots, n$.

Case 4b *takes place when* P_n *is of the form considered in* 4a, *but* (5) *is not satisfied.*

Case 5a *occurs when neither of the above cases takes place and* P_n *can be globally transformed into an equation* $Q_n(z, t; \mathbf{R})$ *whose coefficients satisfy* (5) *on* $\mathbf{R}$.

Case 5b *takes place when neither of the above cases is satisfied.*

Theorem 3. *Each of the cases listed in Theorem* 1 *actually occurs.* $G_0(P_n)$ *is* C^{n+1}*-conjugate to the group introduced in the case*:

1. *if* $n = 2$ *and* P_2 *is* $y'' + y = 0$ *on* $\mathbf{R}$,
   or P_n *is an iterative equation iterated from* P_2 *for an arbitrary* $n > 2$,

2a. *if* $n = 2$ *and* P_2 *is* $y'' + y = 0$ *on* $(0, \infty)$,
   or any equation iterated from P_2 *for an arbitrary* $n > 2$,

2b. *if* $n = 2$ *and* P_2 *is* $y'' + y = 0$ *on* $(0, m\pi)$,
   or any equation iterated from P_2 *for an arbitrary* $n > 2$,

3a. *if* $n = 2$ *and* P_2 *is* $y'' + y = 0$ *on* $(0, m\pi - \pi/2)$,
   or any equation iterated from P_2 *for an arbitrary* $n > 2$,

3b. *if* P_4 *is* $y^{\mathrm{IV}} + y = 0$ *on* $(-\infty, \infty)$,

3c. *if* P_3 *is* $y^{\mathrm{III}} + y = 0$ *on* $(-\infty, \infty)$,

4a. *if* P_4 *is* $y^{\mathrm{IV}} + (\cos 2\pi x)\, y = 0$ *on* $(-\infty, \infty)$,

4b. *if* P_4 *is* $y^{\mathrm{IV}} + y' + (\cos 2\pi x)\, y = 0$ *on* $(-\infty, \infty)$,

5a. *if* P_5 *is* $y^{\mathrm{V}} + y' + (\sinh x)\, y = 0$ *on* $(-\infty, \infty)$,

5b. *if* P_4 *is* $y^{\mathrm{IV}} + y' + (\sinh x)\, y = 0$ *on* $(-\infty, \infty)$.

Theorem 4. *There are* 5 *possible types of subgroups of increasing elements of* $G_0(P_n)$, $P_n \in \mathbf{L}_n$, $n \geqq 2$, *with respect to the number of parameters*:

1. *A three-parameter group that occurs when P_n is iterated from a both-side oscillatory second order equation.*
2. *A two-parameter group that occurs when P_n is iterated from an equation of the second order which is either one-side oscillatory or of finite type and special.*
3. *A one-parameter group in the case when P_n can be globally transformed into an equation with constant coefficients, and 1 and 2 do not occur.*
4. *An infinite cyclic group when P_n can be globally transformed into an equation with periodic coefficients of the same period on $(-\infty, \infty)$, and neither of the cases $1-3$ is valid.*
5. *The identity only when neither of the cases $1-4$ takes place.*

Remark. Each iterative equation P_n, especially each second order equation, has at least a one-parameter group of increasing elements of $G_0(P_n)$. With the exception of the case when P_n is iterated from a one-side oscillatory second order equation, each other iterative equation P_n has both increasing and decreasing elements in $G_0(P_n)$. The group in 3a for $m = 1$ is conjugate to the group 3b.

Proof of Theorems 1, 2, 3 and 4.

The form of global transformations of any equation $P_n \in \mathbf{L}_n$ into itself follows from Lemma 4.

First let us consider the increasing elements of $G_0(P_n)$.

Let $h \in G_0(P_n)$ be such that $h \neq \mathrm{id}_I$, $\mathrm{d}h(x)/\mathrm{d}x > 0$ on I, and $h(x_0) = x_0 \in I$. Then P_n is an equation iterated from a linear second order equation according to Proposition 5. Due to Lemma 5, $G_0(P_n)$ coincides with the group of dispersions of the linear second order equation and hence is C^{n+1}-conjugate to one of the groups introduced under 1, 2a, 2b, 3a according to its type and kind, as follows from Propositions 2 and 3.

Let P_n be not an iterative equation. Applying Proposition 6 we conclude that the subgroup of increasing elements of $G_0(P_n)$ is either C^{n+1}-conjugate to the group in 3c and then P_n can be globally transformed into an equation with constant coefficients on $(-\infty, \infty)$, or it is C^{n+1}-conjugate to the group 4b, and then P_n can be globally transformed into an equation with periodic coefficients with the same period on $(-\infty, \infty)$ due to Lemma 12.

If neither of the above cases occurs then the subgroup of increasing elements of $G_0(P_n)$ consists of the identity id_I only.

It remains to consider decreasing elements of $G_0(P_n)$. For the cases 1, 2a, 2b, 3a this has already been done. Hence, let P_n be not an iterative equation.

Let the subgroup of increasing elements of $G_0(P_n)$ be C^{n+1}-conjugate to the group in 3c, and let $h \in G_0(P_n)$ be such that $\mathrm{d}h(x)/\mathrm{d}x < 0$. We know that P_n can be globally transformed into an equation Q_n with constant coefficients on $(-\infty, \infty)$; the subgroup of increasing elements of $G_0(Q_n)$ is exactly the group in 3c. All decreasing elements of $G_0(P_n)$ are transformed by conjugacy into decreasing elements of $G_0(Q_n)$, see Lemma 2. Let $g \in G_0(Q_n)$, $\mathrm{d}g(t)/\mathrm{d}t < 0$, $g: \mathbf{R} \to \mathbf{R}$. Since Q_n is not an iterative equation, the condition C in Lemma 6 gives

$$r_{n-i}(g(t)) \cdot g^{\cdot i}(t) = r_{n-i}(t) \quad \text{on} \quad (-\infty, \infty),$$

for the first nonzero constant $r_{n-i} = r_{n-i}(t) \neq 0$. Hence

$$|g^{\cdot}(t)| = 1 \quad \text{on} \quad (-\infty, \infty).$$

Because g is decreasing, $g(t) = -t + c_0$ for a suitable $c_0 \in \mathbf{R}$. Since $G_0(Q_n)$ is a group, $-t + c = g_c(t) \in G_0(Q_n)$ for all $c \in \mathbf{R}$. Hence, if the subgroup of increasing elements of $G_0(P_n)$ is C^{n+1}-conjugate to the group in 3c, and if $G_0(P_n)$ contains a decreasing element, then $G_0(P_n)$ is C^{n+1}-conjugate to the group in 3b, and P_n is globally transformable into an equation with constant coefficients on $(-\infty, \infty)$ for which $-\mathrm{id}_{\mathbf{R}} \in G_0(Q_n)$, i.e., $q_{n-i} \equiv 0$ for odd i.

Let the subgroup of increasing elements of $G_0(P_n)$ be C^{n+1}-conjugate to the group in 4b, and let $h \in G_0(P_n)$ be such that $\mathrm{d}h(x)/\mathrm{d}x < 0$. We have shown that P_n can be globally transformed into an equation Q_n with periodic coefficients of the same period on $(-\infty, \infty)$. Since $G_0(P_n)$ contains a decreasing element, the equation P_n can be transformed into an equation Q_n with periodic coefficients of the same period on $(-\infty, \infty)$ such that $-\mathrm{id}_{\mathbf{R}} \in G_0(Q_n)$, see Proposition 10. Thus (5) must be satisfied on $(-\infty, \infty)$.

If id_I is the only increasing element of $G_0(P_n)$, and $h \in G_0(P_n)$ with $\mathrm{d}h(x)/\mathrm{d}x < 0$ on I, then P_n can be globally transformed first into an equation P_n^* on $\mathbf{R}$, then into an equation $Q_n = Q_n(z, t; (-\infty, \infty))$ such that $-\mathrm{id}_{(-\infty,\infty)} \in G_0(Q_n)$, hence (5) is satisfied on $(-\infty, \infty)$. This follows from Propositions 7 and 9 if $Q_n := T * P_n^*$, where $T = \langle |\alpha^{-1\prime}|^{(1-n)/2}, \alpha^{-1} \rangle$.

This completes the proofs of Theorems 1 and 2.

The examples of equations in cases 1, 2a, 2b and 3a in Theorem 3 are obtained directly from the corresponding cases in Theorems 1, 2 and Lemma 10.

Due to Theorem 2 case 3b, the group in Theorem 1 case 3b is $G_0(P_4)$ for P_4: $y^{\mathrm{IV}} + y = 0$ on $(-\infty, \infty)$, since this equation is not iterated and has constant coefficients satisfying $p_{4-1} = p_{4-3} = 0$.

The equation P_3: $y^{\mathrm{III}} + y = 0$ on $(-\infty, \infty)$ in case 3c of Theorem 3 is not iterated and has constant coefficients on $(-\infty, \infty)$, but $p_{3-3} = p_0(x) = 1 \neq 0$. According to Lemma 6,

$$r_0(h(x)) \cdot h'^3(x) = r_0(x)$$

where $r_0 = p_0 = 1$ or $h'(x) = 1$. Hence $G_0(P_3)$ has no decreasing element and due to Theorem 2, $G_0(P_3)$ is the group in 3c of Theorem 1.

The equations $y^{\mathrm{IV}} + (\cos 2\pi x)\, y = 0$ and $y^{\mathrm{IV}} + y' + (\cos 2\pi x)\, y = 0$ on $(-\infty, \infty)$ serve as examples for cases 4a and 4b in Theorem 1, respectively, since neither of them is iterative, both have periodic coefficients on $(-\infty, \infty)$, and the relation (5) is satisfied for the former, while the latter has no decreasing elements in its group G_0. Neither of them can be transformed into an equation with constant coefficients.

Each $h \in G_0(P_5)$ for P_5: $y^V + y' + (\sinh x) y = 0$ on $(-\infty, \infty)$ satisfies $(h'(x))^4 = 1$. Hence h must be of the form $h(x) = \pm x + c$. However, only for $c = 0$ these functions transform P_5 into itself.

For each $h \in G_0(P_n)$, where P_4: $y^{IV} + y' + (\sinh x) y = 0$ on $(-\infty, \infty)$, the relation

$$(h'(x))^3 = 1$$

must be satisfied due to Lemma 6, condition C. Hence $h(x) = x + c$. Since $h \in G_0(P_4)$ only for $c = 0$, the proof of Theorem 3 is complete.

Theorem 4 summarizes the preceding results with respect to the increasing elements of the corresponding groups. It is a direct consequence of Theorems 1 and 2 if we recall that each equation iterated from a linear second order differential equation (1) of a finite type and general (case 3a) can be globally transformed into an equation with constant coefficients (not necessarily on $(-\infty, \infty)$) since each second order equation can be globally transformed into an equation with constant coeficients, cf. Lemma 10.

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