Stationary probability density functions: An exact result

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An exact expression is obtained for the probability density function (pdf) of any quantity measured in a general stationary process, in terms of conditional expectations of time derivatives of the signal. This expression indicates that the conditional expectations of both the time derivative squared and of the second time derivative influence the shape of the pdf, including its tails. A previous result of Ching [Phys. Rev. Lett. **70**, 283 (1993)] for temperature measurements in turbulent flows corresponds to the particular case when the latter quantity is linear.

There has been considerable recent interest in the shapes of probability density functions (pdf's) in turbulent flows, with experimental results coming from Chicago,[1-3] Cornell,[4,5] Yale,[6] and elsewhere.[7] These experiments have prompted several theoretical works attempting to describe and explain the observed shapes.[8-13]

In the present work an exact expression is obtained for the pdf of a stationary process in terms of its conditional time derivatives. This expression does not depend on any physics of the process and sheds further light on the shape of pdf's especially their tails. By invoking a linearity assumption, we recover a previous result of Ching,[13] which successfully describes the pdf's of temperature and temperature differences (except for differences with very short time separation) observed in several experiments.

In the experiments cited above, a physical variable (e.g., temperature T) is measured as a function of time t at a fixed spatial location in a statistically stationary flow. In such circumstances $T(t)$ is a smooth stationary random process, with mean $\langle T\rangle$ and variance σ_T. We analyze the standardized process

$$X(t)\equiv[T(t)-\langle T\rangle]/\sigma_T. \tag{1}$$

With x being the sample-space variable, the one-time pdf of $X(t)$, $P(x)$, can be written as the expectation of the *fine-grain pdf $p(x;t)$:*

$$P(x)=\langle p(x;t)\rangle, \tag{2}$$

where

$$p(x;t)\equiv\delta[X(t)-x], \tag{3}$$

see, e.g., Ref. 14, Eq. (2.67). Differentiating Eq. (3) twice with respect to time we obtain

$$\begin{aligned}\ddot{p}&=-\ddot{X}\frac{\partial p}{\partial x}+\dot{X}^2\frac{\partial^2 p}{\partial x^2}\\ &=-\frac{\partial}{\partial x}[p\ddot{X}]+\frac{\partial^2}{\partial x^2}[p\dot{X}^2],\end{aligned} \tag{4}$$

where an overdot indicates the time derivative and the second step follows because $X(t)$ is independent of x.

Now for any stationary random process $Q(t)$ (e.g., $\ddot{X}$ or $\dot{X}^2$) we have

$$\langle p(x;t)Q(t)\rangle=P(x)\langle Q(t)|X(t)=x\rangle \tag{5}$$

[see, e.g., Ref. 14, Eq. (2.150)] where $\langle Q(t)|X(t)=x\rangle$ is the conditional expectation of Q, which is a function of x only, and henceforth is written $\langle Q|x\rangle$. Hence the mean of Eq. (4) is

$$0=-\frac{d}{dx}[P(x)\langle\ddot{X}|x\rangle]+\frac{d^2}{dx^2}[P(x)\langle\dot{X}^2|x\rangle]. \tag{6}$$

The solution to Eq. (6) is our principal result: the one-time pdf of $X(t)$ is given by

$$P(x)=\frac{C_1}{\langle\dot{X}^2|x\rangle}\exp\left(\int_0^x\frac{\langle\ddot{X}|x'\rangle}{\langle\dot{X}^2|x'\rangle}dx'\right), \tag{7}$$

where the constant C_1 is determined by the normalization condition $\int_{-\infty}^{\infty}P(x)dx=1$ (as is the constant C_2 introduced below). This is a very general result that applies to any stationary random process subject to two technical requirements: that $X(t)$ is twice continuously differentiable, and that $P(x)$ decreases sufficiently rapidly as $|x|$ tends to infinity [so that, when Eq. (6) is integrated once, the constant of integration is zero].

With the nondimensional conditional expectations defined by

$$q(x)\equiv\langle\dot{X}^2|x\rangle/\langle\dot{X}^2\rangle \tag{8}$$

and

$$r(x)\equiv\langle\ddot{X}|x\rangle/\langle\dot{X}^2\rangle, \tag{9}$$

Eq. (7) can alternatively be written

$$P(x)=\frac{C_2}{q(x)}\exp\left(\int_0^x\frac{r(x')}{q(x')}dx'\right). \tag{10}$$

Ching's result[13] corresponds to the case when $\langle\ddot{X}|x\rangle$ is a linear function of x. The stationarity and standardization of X constrain this linear relation to be

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$$r(x) = -x, \tag{11}$$

and hence Eq. (7) becomes

$$P(x) = \frac{C_2}{q(x)} \exp\left(\int_0^x \frac{-x'\,dx'}{q(x')} \right). \tag{12}$$

[It may be observed that for a Gaussian process $r(x)$ is given by Eq. (11), and $q(x)$ is unity. Hence Eq. (10) yields the Gaussian distribution.]

Several comments are required to relate the above equations [Eqs. (7)–(12)] to previous work and to experimental observations.

(1) It is again emphasized that Eq. (7) does not depend on the physics of the process. It is an exact relation for any smooth stationary process with a pdf decaying sufficiently rapidly for large fluctuations.

(2) The combination of Eqs. (7) and (11) provides an alternative derivation of Ching's result, Eq. (12). The linearity assumption embodied in Eq. (11) is exactly equivalent to Ching's "fluctuation–dissipation" assumption:

$$(2n-1)\langle X^{2n-2}\dot{X}^2\rangle = \langle X^{2n}\rangle\langle \dot{X}^2\rangle. \tag{13}$$

Equation (11) probably is a simpler, more understandable statement of the assumption.

(3) The Sinai–Yakhot formula[8] has the same mathematical form as Eq. (12) but with $q(x)$ being the (normalized conditional expectation of the dissipation (i.e., the square of the *spatial* gradient of temperature). The conditional expectation of the dissipation will be close to that of the time derivative squared when some form of Taylor's frozen flow hypothesis[15] is valid. However, the physical assumption leading to the formula is quite different. Specifically, the Sinai–Yakhot formula was derived for a homogeneous, decaying field (with no forcing) evolving by the convection–diffusion equation, whereas the current development assumes only stationarity, and Eq. (12) follows from an additional linearity assumption, Eq. (11).

(4) Jayesh and Warhaft[5] report measurements of temperature pdf's in grid-generated turbulence. They find that these pdf's are well described by Eq. (12), with $q(x)$ obtained from time derivatives [i.e., Eq. (9)]. These authors invoked Taylor's hypothesis to interpret $q(x)$ as the normalized conditional dissipation, and hence interpreted Eq. (12) as the Sinai–Yakhot formula. They are careful to point out, however, that the requirements for the application of this formula are not fulfilled in the flow studied. It is now apparent that the observed agreement between their measured pdf's and Eq. (12) is more simply explained as being a consequence of Eqs. (7) and (11). However, as pointed out by one of us,[13] the fact that Eq. (11) is a good approximation for data in various different physical situations suggests that there may be a universality in turbulence.

(5) The (qualified) success of Ching's formula[5,13] in describing experimentally observed pdf's suggests that Eq. (11) can provide a good approximation for $\langle \ddot{X}|x\rangle$ even for strongly non-Gaussian processes. In contrast, the corresponding approximation for $\langle \dot{X}^2|x\rangle$ [i.e., $\langle \dot{X}^2|x\rangle = \langle \dot{X}^2\rangle$ or $q(x)=1$] is known to be inaccurate for non-Gaussian processes (although, as may be expected, it holds reasonably well for turbulent temperature data when the pdf is Gaussian). Measurements of $\langle \ddot{X}|x\rangle$ for different quantities and under different flow conditions can be used to assess the accuracy and range of validity of Eq. (11).

While $\langle \dot{X}^2|x\rangle$ has been measured in several experiments, $\langle \ddot{X}|x\rangle$ has not. In Fig. 1 we show

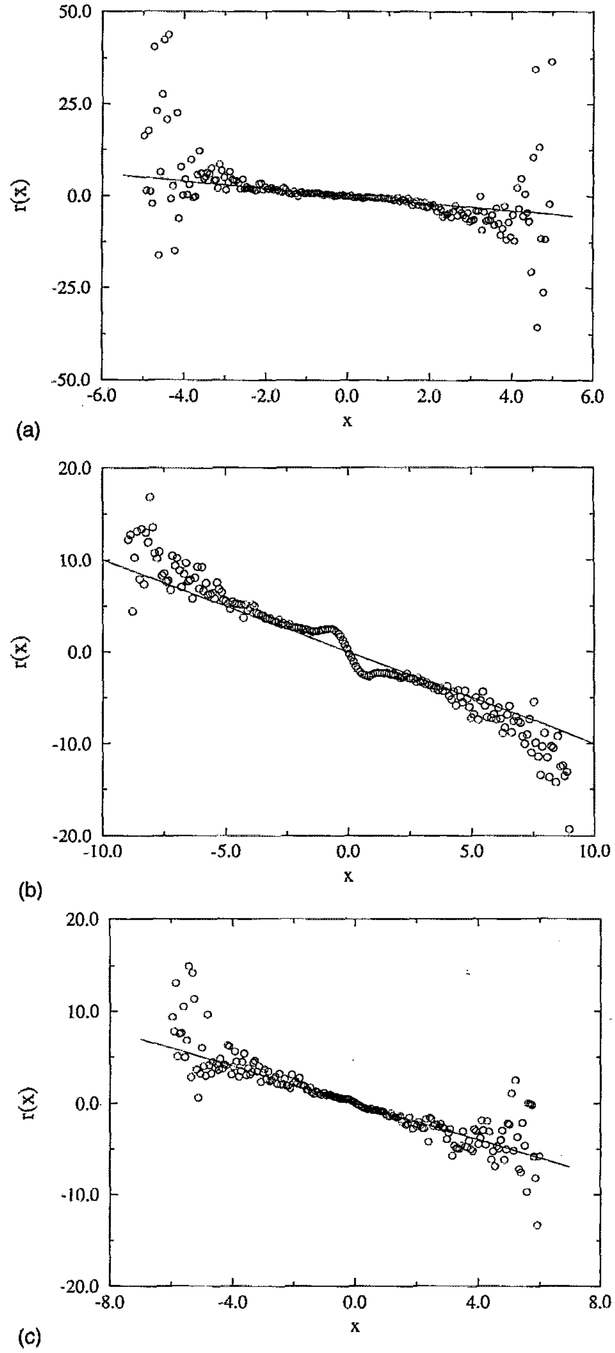

FIG. 1. The quantity $r(x) \equiv \langle \ddot{X}|x\rangle/\langle \dot{X}^2\rangle$ is measured using the Chicago convection data with Rayleigh number equal to 5.8×10^{14}. $X(t)$ is the standardized ($\langle X\rangle = 0$, $\langle X^2\rangle = 1$) (a) temperature fluctuation, (b) temperature time derivative, and (c) temperature difference with time separation equals 64 sampling intervals (turnover time of the flow is about 4000 sampling intervals). The solid line is the linearity assumption Eq. (11) or equivalently the "fluctuation–dissipation" assumption Eq. (13).

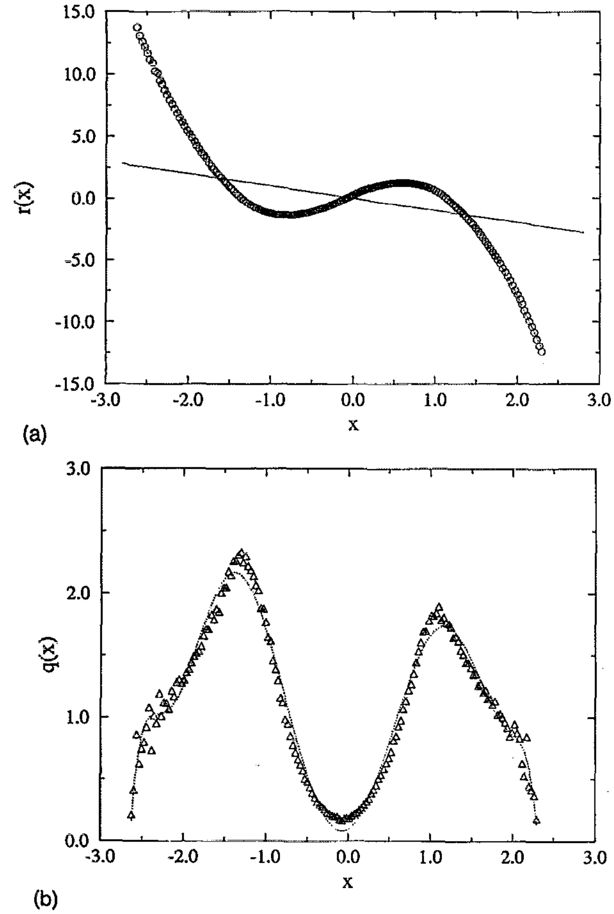

FIG. 2. The conditional expectations of the second time derivative and the time derivative squared using (standardized) $\bar{x}$-coordinate data from the Lorenz model [Eqs. (17)–(19)] (a) $r(x)$ vs x. The solid line is again Eq. (11) while the dotted line is a fifth-order polynomial fit. (b) $q(x)$ vs x. The dotted line is a tenth-order polynomial fit which clearly does not fit the data well.

$r(x)\equiv\langle\ddot{X}|x\rangle/\langle\dot{X}^2\rangle$ plotted against x using the Chicago convection data with Rayleigh number equal to 5.8×10^{14}. The solid line is Eq. (11) which indeed describes the data quite well. Even for the temperature difference with the shortest time separation (which essentially is the temperature time derivative), Eq. (11) works well for large $|x|$ [see Fig. 1(b)]. Both Figs. 1(a) and 1(b) suggest that $r(x)$ may vary more strongly than linearly for large $|x|$. However, as is evident from the figures, there is inevitably substantial statistical uncertainty in these tails.

While it appears that Eq. (11) provides a reasonable approximation to $\langle\ddot{X}|x\rangle$ for much of the turbulence data presented here, it is equally apparent that it is far from universal. It clearly provides a poor approximation for small $|x|$ in Fig. 1(b). A more striking example is provided by the Lorenz model:[16]

$$\dot{\bar{x}}=-\sigma\bar{x}+\sigma y, \tag{14}$$

$$\dot{y}=-\bar{x}z+\bar{r}\,\bar{x}-y, \tag{15}$$

$$\dot{z}=\bar{x}y-bz, \tag{16}$$

with $\sigma=10$, $\bar{r}=28$, and $b=8/3$. In Fig. 2(a) we plot $r(x)$ with X being the standardized $\bar{x}$ coordinate in the Lorenz model. The solid line is again Eq. (11) which does not agree with the data this time. Instead, the data are very well approximated by a fifth-order polynomial which is shown as the dotted line. Figure 2(b) shows $q(x)\equiv\langle\dot{X}^2|x\rangle/\langle\dot{X}^2\rangle$ for the Lorenz model, which cannot be well approximated by a polynomial.We have checked the correctness of Eq. (7) in the case when Eq. (11) does not hold.

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[1] F. Heslot, B. Castaing, and A. Libchaber, "Transitions to turbulence in helium gas," Phys. Rev. A **36**, 5870 (1987).

[2] B. Castaing, G. Gunaratne, F. Heslot, L. Kadanoff, A. Libchaber, S. Thomae, X.-Z. Wu, S. Zaleski, and G. Zanetti, "Scaling of hard thermal turbulence in Rayleigh–Bénard convection," J. Fluid Mech. **204**, 1 (1989).

[3] M. Sano, X.-Z. Wu, and A. Libchaber, "Turbulence in helium-gas free convection," Phys. Rev. A **40**, 6421 (1989).

[4] Jayesh and Z. Warhaft, "Probability distribution of a passive scalar in grid-generated turbulence," Phys. Rev. Lett. **67**, 3503 (1991).

[5] Jayesh and Z. Warhaft, "Probability distribution, conditional dissipation, and transport of passive temperature fluctuations in grid-generated turbulence," Phys. Fluids A **4**, 2292 (1992).

[6] R. R. Prasad and K. R. Sreenivasan, "The measurement and interpretation of fractal dimensions of the scalar interface in turbulent flows," Phys. Fluids A **2**, 792 (1990); R. R. Prasad and K. R. Sreenivasan, "Quantitative three-dimensional imaging and the structure of passive scalar fields in fully turbulent flows," J. Fluid Mech. **216**, 1 (1990).

[7] J. P. Gollub, J. Clarke, M. Gharib, B. Lane, and O. N. Mesquita, "Fluctuations and transport in a stirred fluid with a mean gradient," Phys. Rev. Lett. **67**, 3507 (1991).

[8] Y. G. Sinai and V. Yakhot, "Limiting probability distributions of a passive scalar in a random velocity field," Phys. Rev. Lett. **63**, 1962 (1989).

[9] V. Yakhot, "Probability distributions in high-Rayleigh-number Bénard convection," Phys. Rev. Lett. **63**, 1965 (1989).

[10] A. Pumir, B. I. Shraiman, and E. D. Siggia, "Exponential tails and random advection," Phys. Rev. Lett. **66**, 2984 (1991).

[11] H. Chen, S. Chen, and R. H. Kraichnan, "Probability distributions of a stochastically advected scalar field," Phys. Rev. Lett. **63**, 2657 (1989); R. H. Kraichnan, "Models of intermittency in hydrodynamic turbulence," Phys. Rev. Lett. **65**, 575 (1990).

[12] Z.-S. She, "Physical model of intermittency in turbulence: Near-dissipation-range non-Gaussian statistics," Phys. Rev. Lett. **66**, 600 (1991); Z.-S. She and S. A. Orszag, "Physical model of intermittency in turbulence: Inertial-range non-Gaussian statistics," *ibid.* **66**, 1701 (1991).

[13] E. S. C. Ching, "Probability densities of turbulent temperature fluctuations," Phys. Rev. Lett. **70**, 283 (1993).

[14] S. B. Pope, "PDF methods for turbulent reactive flows," Prog. Energy Combust. Sci. **11**, 1991 (1985).

[15] G. I. Taylor, "The spectrum of turbulence," Proc. R. Soc. London Ser. A **164**, 476 (1938).

[16] E. N. Lorenz, "Deterministic nonperiodic flow," J. Atmos. Sci. **20**, 130 (1963).