Stationary solutions, intermediate asymptotics and large time behaviour of type II Streater's models

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Abstract

In this paper, we study type II Streater's models. These models describe the coupled evolution of the density of a cloud of particles in an external potential and a temperature, preserving the energy, with eventually a nonlocal Poisson coupling. We introduce an entropy and consider in a bounded domain, or in an unbounded domain with a confining external potential, the stationary solutions (with given mass and energy), for which we have existence and uniqueness results. The entropy is reinterpreted as a relative entropy which controls the convergence to the stationary solutions. We consider also the whole $\mathbb{R}^d$ space problems without exterior potential using time-dependent rescalings and show the existence of intermediate asymptotics in special cases.

Key words and phrases: Streater's models, drift-diffusion systems, relative entropy, asymptotic behavior of solutions, confinement, convex functional, *a priori* estimates, mass/charge, energy, existence and uniqueness of stationary solutions, rate of convergence, time-dependent rescalings, intermediate asymptotics.

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The models we study in this paper describe systems of Brownian particles in the presence of an exterior potential and/or a self-consistent potential given by a Poisson coupling. We refer to the papers by Streater [26, 27, 28] for the derivation of the models in the case of an exterior potential only (they generalize the Smoluchowski system proposed in the beginning of the twentieth century), and to a paper by Biler, Krzywicki and Nadzieja [7] for the interacting particles case, some mathematical comments and a list of open problems.

Systems governing the evolution of the density of a cloud of particles consist of two or three equations. The first takes into account the (Brownian) diffusion of particles and their collective motion caused by the gradient of a potential. The second equation represents the balance of heat and involves terms connected with thermal diffusion, convection and heat production. The potential is either a given external one, or is generated by the particles themselves. Mathematically, this leads to the third equation which is a Poisson equation for the Coulombic or Newtonian potential depending on the character of the interaction between the particles (charged or massive).

The fundamental property that these models share is that they preserve mass or charge, the energy, and that they are compatible with the second law of thermodynamics. Our main tool in this study will be the entropy, which yields the only natural *a priori* estimate that we can use to control the behaviour of solutions for large times and the structure of the set of the stationary solutions. From a mathematical point of view, the problem is to analyze convexity properties of these entropy functions. The entropy was, of course, known before in the context of parabolic equations but has probably been underestimated in the study of simpler models for which other estimates controlling the relaxation were easy to build.

Here we first use a variational approach to characterize the stationary solutions which has already been exploited in simpler cases in [18, 17, 12, 20] and which basically allows to characterize a stationary solution as the unique minimum of a strictly convex functional. The paper [17] by Desvillettes and Dolbeault contains a very similar situation (for the stationary solutions of the Vlasov-Poisson-Boltzmann system) where the problem is given for fixed mass and energy. Actually, this convex functional plays an important role also for the evolution problem, since it is strongly related to the entropy functional. Moreover, when one deals with problems in unbounded domains, it can be shown that the growth condition on the external potential is equivalent to asking that the convex functionals used to characterize the stationary states or the entropy are bounded from below (see [11, 19] for a discussion of these results and for the notion of confinement in the context of kinetic equations, which can be extended in a straightforward manner to parabolic problems).

The special form of the stationary solutions and the above remarks on various convex functionals allow us to rewrite the entropy as a relative entropy. Entropy methods have been introduced in the context of kinetic equations (where very few estimates were at hand) to understand the behaviour of the solutions: it was already well known by Boltzmann that such an entropy was governing the long time behaviour of the solutions (*H theorem* for the Boltzmann equation). By relative entropy, we simply mean that the difference of the entropy of the solution and the possible asymptotic (stationary) state can be put in a form involving convex functions that control the convergence of global solutions in some appropriate norms, usually in L^1 (see for instance [3] for a recent application of these ideas).

Recently, Toscani made the link between these ideas and parabolic problems (heat equation) [29, 30, 31] and his results have been extended and systematized in [2] in a linear context (with a PDE approach; in the context of probabilistic methods, see the papers by Bakry and Emery, and the other references quoted in [2]) and also adapted to nonlinear diffusions in [16, 14, 13, 22, 23].

It turns out that the method is quite robust and can also be used if the problem has a nonlinear nonlocal Poisson coupling ([1, 4, 5] and [6] for an earlier analysis) and works for nonlinear diffusion as well (see [5]), or even if the evolution of the density is coupled with an equation for the temperature, as we shall see below.

Here we will focus on the existence and uniqueness of stationary solutions and the control of the convergence of the solutions of the Cauchy problem to these stationary solutions. Note that the existence of solutions to the Cauchy problem is only partially known and a proof of a global existence result is open and seems very difficult. The reasons are first that the temperature enters into the equations with a negative power, and then, that the second equation of the system is of nonclassical type (for related energy-transport models with similar difficulties, see [15]). For all the (formal) computations involving integrations by parts, we shall therefore assume that all the functions involved are in the right functional spaces. We will also formulate the problem of finding intermediate asymptotics (described by self-similar solutions of linear diffusion equations) by relative entropy methods after convenient time-dependent rescalings (see [21, 16, 4, 5] for various examples of time-dependent rescalings). This approach has also been developed recently and is quite powerful, but in the case of Streater's models, some estimates on the relative entropy production term are missing (except for radially symmetric solutions) and the method fails to provide a satisfactory answer, *e.g.*, for the convergence rate. Note also that the rate of convergence in the confined case or in the case of intermediate asymptotics after rescaling is mainly open.

Note, once again, that the existence of solutions for the evolution problem not being known in general, all our results about this problem are formal. However, they will be made precise at the level of the stationary solutions.

1 The models

In a connected domain $\Omega \subset \mathbb{R}^d$ consider the system

$$\begin{cases} u_t = \nabla \cdot \left[\kappa(\nabla u + \frac{u}{\theta}(\epsilon \nabla \phi + \zeta \nabla \phi_0))\right] \\ (u\theta)_t = \nabla \cdot (\lambda \nabla \theta) + \nabla \cdot [\kappa(\theta \nabla u + \epsilon u \nabla \phi + \zeta u \nabla \phi_0)] \\ \qquad\qquad + (\epsilon \nabla \phi + \zeta \nabla \phi_0) \cdot \left[\kappa(\nabla u + \frac{u}{\theta}(\epsilon \nabla \phi + \zeta \nabla \phi_0))\right] \\ -\Delta \phi = u \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} \partial_n u + \frac{u}{\theta}(\epsilon \partial_n \phi + \zeta \partial_n \phi_0) = 0 & \text{(no mass flux)} \\ \partial_n \theta = 0 & \text{(no heat flux)} \end{cases} \tag{2}$$

where ∂_n denotes the normal outgoing derivative on the boundary $\partial\Omega$, and for ϕ we consider Dirichlet boundary conditions

$$\phi = 0 \quad \text{(perfect conductor)} \tag{3}$$

when Ω bounded and when $d \geq 3$ (for Ω unbounded, (3) means that ϕ is asymptotically equal to 0 at infinity). When $\Omega = \mathbb{R}^2$, ϕ will be explicitly given by $\phi = -\frac{1}{2\pi} \log |x| * u$. In order to cover all the possible cases with or without external potential and Poisson coupling, we shall assume that $\langle \epsilon, \zeta \rangle$ takes the values $\langle 0, 0 \rangle$, $\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$ or $\langle 1, 1 \rangle$. We assume that ϕ_0 is an external potential, the coefficients κ and λ are nonnegative functions which may depend on x, u, θ, ϕ, and which can take the value 0 only at $\theta = 0$. This covers the classical choice in the Smoluchowski equations (predecessors of those of Streater): $\kappa(\theta) = \theta \geq 0$. In order to simplify the presentation and computations in this paper, we will actually assume that κ and λ are actually positive on $\mathbb{R}^+$. Note that for being consistent with the third law of thermodynamics, $\kappa(\theta) = o(\theta)$ as $\theta \to 0$ has to be assumed. In all that follows, we assume that $\langle u, \theta, \phi \rangle$ is a smooth solution corresponding to initial data whose mass (or rather charge), energy and entropy (see below) are at least initially well defined.

System (1) will be called a type II Streater's model in opposition with type I Streater's models like

$$\begin{cases} u_t = \nabla \cdot \left[\kappa(\nabla u + \frac{u}{\theta}(\epsilon \nabla \phi + \zeta \nabla \phi_0))\right] \\ \theta_t = \nabla \cdot (\lambda \nabla \theta) + (\epsilon \nabla \phi + \zeta \nabla \phi_0) \cdot \left[\kappa(\nabla u + \frac{u}{\theta}(\epsilon \nabla \phi + \zeta \nabla \phi_0))\right] \\ -\Delta \phi = u, \end{cases} \tag{4}$$

which have probably better properties as far as the Cauchy problem is concerned, but for which the study of the long time asymptotics is apparently more difficult. Indeed, for this model considered in the whole space $\mathbb{R}^d$ we expect that the temperature θ will decay like a solution of the heat equation, but the term $-\int_{\mathbb{R}^d} \log \theta$ in the entropy is divergent for such an asymptotic temperature θ.

In this paper we consider the case of electric repulsion of particles which corresponds to the $-$ sign in the Poisson equation, the third equation of the system (4). The choice of the $+$ sign corresponds to the gravitational attraction of the particles and leads to much more difficult problems, especially when the global in time existence of solutions is considered (*cf.* [7, 9, 10]).

Some related results for stationary solutions have been obtained in [8, 9, 24, 25] using an approach based on the Leray-Schauder theory instead of variational methods.

2 Case with confinement

First we consider the case when either Ω is bounded or when the external potential ϕ_0 grows sufficiently at infinity to prevent the runaway of the system particles, which means that stationary solutions may exist or equivalently that the entropy is bounded from below. In the following we will refer to this situation as the case "with confinement" in opposition with whole space cases without exterior potential where the solution converges locally to 0 because of the diffusion.

2.1 A priori estimates

Total mass (or rather charge)

$$M = \int_\Omega u \, dx$$

and energy

$$E = \int_\Omega u \left(\theta + \zeta \phi_0 + \frac{\epsilon}{2} \phi \right) \, dx$$

are (formally) preserved. Note that if $\nabla \phi \in L^2(\Omega)$,

$$E = \int_\Omega u \left(\theta + \zeta \phi_0 \right) \, dx + \frac{\epsilon}{2} \int_\Omega |\nabla \phi|^2 \, dx.$$

This is the case if Ω is bounded or if $d \geq 3$. When $d = 2$, $\nabla \phi$ does not decay fast enough at infinity to be square integrable. We define the entropy by

$$W = \int_\Omega u \log \left(\frac{u}{\theta} \right) \, dx.$$

Our main assumption on the external potential ϕ_0 is

$$e^{-\phi_0/T} \in L^1(\Omega) \tag{5}$$

for some $T > 0$. Note that if ϕ_0 is smooth and if Ω is bounded, this assumption is always satisfied. If Ω is unbounded, the meaning of (5) is that ϕ_0 is confining at temperature T (see [19]). Particular interesting situations arise when ϕ_0 is bounded either from below or from above. Note also that the nonnegativity of ϕ_0 is actually equivalent to assuming that ϕ_0 is bounded from below (except for the boundary conditions, where the assumption $\phi_0 \geq 0$ plays no role) since adding a constant to ϕ_0 does not change the solution of the Poisson-Boltzmann equation (see below), and changes the energy by just a constant. This assumption is moreover natural as soon as ϕ_0 is smooth (and confining in the unbounded case).

Proposition 2.1 *Let $M > 0$ and consider a potential ϕ_0 satisfying (5) for some $T > 0$. For every measurable functions $u \geq 0$, $\theta \geq 0$ such that $\int_\Omega u \, dx = M$, we have*

$$\int_\Omega u \left(\log \left(\frac{u}{\theta} \right) + \theta + \frac{1}{T} \phi_0 \right) \, dx \geq M \left[1 + \log \left(\frac{M}{\int_\Omega e^{-\phi_0/T} \, dx} \right) \right] .$$

Proof. This estimate is given by the inequality $-\log\theta+\theta-1\geq 0$ for any $\theta\geq 0$, and by the Jensen inequality applied to the convex function $t\mapsto t\log t$

$$\int_\Omega \frac{u}{e^{-\phi_0/T}}\log\left(\frac{u}{e^{-\phi_0/T}}\right)\frac{e^{-\phi_0/T}}{\int_\Omega e^{-\phi_0/T}dx}dx \geq \frac{\int_\Omega u dx}{\int_\Omega e^{-\phi_0/T}dx}\log\left(\frac{\int_\Omega u dx}{\int_\Omega e^{-\phi_0/T}dx}\right).$$

□

Proposition 2.2 *If Ω is of class C^1 and if $\langle u,\theta,\phi\rangle$ is a smooth solution of (1)-(2)-(3), then the entropy W is decreasing*

$$\frac{dW}{dt}=-\int_\Omega \lambda\frac{|\nabla\theta|^2}{\theta^2}\,dx-\int_\Omega \kappa u\left|\frac{\nabla u}{u}+\frac{1}{\theta}(\epsilon\nabla\phi+\zeta\nabla\phi_0)\right|^2\,dx\,. \qquad (6)$$

Another property which will be useful in the remainder of the paper is the lower bound for the term $\int_\Omega u\phi\,dx$ for all solutions to (1)-(2)-(3) and this for all dimensions $d\geq 2$. More precisely,

Lemma 2.3 *Let ϕ_0 an external potential which is bounded from below in Ω and satisfies (5) for some $T>0$. Moreover, in dimension $d=2$ let us assume that either Ω is bounded or that $\Omega=\mathbb{R}^2$ and*

$$\lim_{|x|\to+\infty}\frac{\log|x|}{\phi_0(x)}=0\,. \qquad (7)$$

Then, for all $a,b>0$, there is a constant C such that for any solution $\langle u,\phi,\theta\rangle$ of (1)-(2)-(3), $u,\theta\geq 0$, we have

$$a\int_\Omega u\phi_0\,dx+b\int_\Omega u\phi\,dx\geq C\,. \qquad (8)$$

and we call $C_{a,b}$ the optimal constant in (8).

Proof. For $d\geq 3$, the proof is immediate, since $\phi=\sigma_d\,|x|^{2-d}*u$, $\sigma_d>0$. In this case, $C_{a,b}=0$ for all $a,b>0$. A similar argument holds when Ω is bounded, in all dimensions. When $\Omega=\mathbb{R}^2$, $\phi=-(2\pi)^{-1}\log|x|*u$ and so,

$$\begin{aligned}\int_{\mathbb{R}^2} u\phi\,dx &= -\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log|x-y|u(x)u(y)\,dx\,dy\\ &\geq -\frac{M^2\log R}{2\pi}-\frac{1}{2\pi}\iint_{|x-y|>R}\log|x-y|u(x)u(y)\,dx\,dy.\end{aligned} \qquad (9)$$

The lemma will be proved if we show that for every $\varepsilon > 0$, there is some $R > 0$ such that

$$\eta(R) := \sup_{|x-y|>R} \frac{\log|x-y|}{\phi_0(x)+\phi_0(y)} \to 0 \quad \text{as} \quad R \to +\infty\,, \tag{10}$$

since then, we would have

$$\int_{\mathbb{R}^2} u\phi\, dx \geq -\frac{M^2 \log R}{2\pi} - \frac{\eta(R)}{\pi}\left(M^2\,|\inf \phi_0| + M \int_{\mathbb{R}^2} u\phi_0\, dx\right), \tag{11}$$

which proves (8) by taking R large enough.

In order to show (10), define

$$\epsilon(R) := \sup_{|x|>R} \frac{\log|x|}{\phi_0(x)} \searrow 0 \quad \text{as} \quad R \to +\infty\,. \tag{12}$$

Assume that R is large enough and that for some $x, y \in \mathbb{R}^2$, $|x-y| > R$ and $|x| > R/2$. Then, $\phi_0(x)+\phi_0(y) \geq \phi_0(x) + \inf\phi_0 \geq \phi_0(x)/2$. Moreover, if $|y| \leq |x|(|x|-1)$,

$$\frac{\log|x-y|}{\phi_0(x)+\phi_0(y)} \leq \frac{2\,\log(|x|+|y|)}{\phi_0(x)} \leq \frac{4\log|x|}{\phi_0(x)} \leq 4\epsilon(R/2)\,, \tag{13}$$

while if $|y| > |x|(|x|-1)$, then $|y| > R/2$ and

$$\frac{\log|x-y|}{\phi_0(x)+\phi_0(y)} \leq \frac{\log(|x|+|y|)}{\phi_0(x)+\phi_0(y)} \leq \frac{\log|x|+\log|y|}{\phi_0(x)+\phi_0(y)} \leq \epsilon(R/2)\,. \tag{14}$$

Therefore, for R large enough,

$$\eta(R) \leq 4\epsilon(R/2)\,,$$

which ends the proof. □

2.2 Stationary solutions with given charge and energy

We are interested in existence and uniqueness results for the stationary problem corresponding to solutions with given charge and energy. We will distinguish four cases depending on the values of ϵ, ζ which can be 0 or 1 with our notations. First we state a preliminary result which is a straightforward consequence of Proposition 2.2.

Corollary 2.4 *Any smooth stationary solution of (1) with given $M > 0$ and $E > 0$ is such that θ is a positive constant and u, ϕ are determined by the nonlinear Poisson-Boltzmann equation*

$$-\Delta\phi = M \frac{e^{-(\phi+\phi_0)/\theta}}{\int_\Omega e^{-(\phi+\phi_0)/\theta}\, dx} =: u \ , \tag{15}$$

where θ is given by the energy relation

$$E = M\theta + \int_\Omega \left(\phi + \frac{1}{2}\phi_0\right) u \, dx.$$

2.2.1 Case $\langle \epsilon, \zeta \rangle = \langle 0, 0 \rangle$, Ω bounded

This particular case is the simplest one since u and θ are independently stationary solutions of the heat equation. In a bounded domain u and θ are constants, with $u \equiv M/|\Omega|$ and $\theta \equiv E/M$. Note that this case can be also viewed as a special case of the next one (with $\zeta = 1$ and $\phi_0 \equiv 0$).

2.2.2 Case $\langle \epsilon, \zeta \rangle = \langle 0, 1 \rangle$

The stationary solution is obviously given by

$$u = u_{\infty,\theta} \equiv M \frac{e^{-\phi_0/\theta}}{\int_\Omega e^{-\phi_0/\theta}\, dx}, \tag{16}$$

where θ is a constant determined by the condition

$$E = E(\theta) = M\theta + \int_\Omega \phi_0 u_{\infty,\theta} \, dx \ . \tag{17}$$

Note that adding a constant to ϕ_0 does not change $u_{\infty,\theta}$, so that we may assume $\phi_0 \geq 0$ as soon as it is bounded from below (the condition $\min \phi_0 = 0$ normalizes the energy).

Proposition 2.5 *Equation (17) with $u_{\infty,\theta}$ given by (16) has at most one solution. Moreover, if we define the numbers*

$$E_\pm = \lim_{\pm(\theta - T_\mp) \searrow 0} \left(M\theta + \int_\Omega \phi_0 u_{\infty,\theta} \, dx \right) \ , \textit{ with}$$
$$T_- = \inf\{\theta > 0 \ : \ E(\theta) > -\infty\} \textit{ and } T_+ = \sup\{\theta > 0 \ : \ E(\theta) < +\infty\} \ ,$$

then, the solution of (17) exists if and only if $E \in (E_-, E_+)$. If ϕ_0 is bounded from below, then $T_- = 0$, $E_- = \inf \phi_0$. Moreover, if ϕ_0 is bounded both from above and from below (which is possible only if Ω is bounded), then $T_+ = E_+ = +\infty$.

Proof. If (5) is not satisfied for any T, no stationary solution exists. In the following, we shall therefore assume that (5) is satisfied for some $T>0$. The function

$$E(\theta)=M\theta+M\,\frac{\int_\Omega \phi_0 e^{-\phi_0/\theta}\;dx}{\int_\Omega e^{-\phi_0/\theta}\;dx}$$

is nondecreasing. Indeed, because of the Lebesgue dominated convergence theorem and $e^{-\phi_0/\theta}=e^{-(\phi_0)_+/\theta}\leq e^{-(\phi_0)_+/T}$ (where ψ_+ denotes the positive part of ψ) for any $\theta<T$, the function $\theta\mapsto E(\theta)$ is of class C^1 and

$$\frac{E'(\theta)}{M}=1+\frac{1}{\theta^2}\frac{\int_\Omega \phi_0^2 e^{-\phi_0/\theta}\;dx\cdot\int_\Omega e^{-\phi_0/\theta}\;dx-\left(\int_\Omega \phi_0 e^{-\phi_0/\theta}\;dx\right)^2}{\left(\int_\Omega e^{-\phi_0/\theta}\;dx\right)^2}\geq 1>0$$

by the Cauchy-Schwarz inequality and the above formula is indeed correct if all the integrals involved are well defined.

Assume now that ϕ_0 is bounded from below. To prove that $T_-=0$ and $E_-=\inf\phi_0$, let us consider the quantity

$$\Delta(\theta)=\frac{1}{M}\int_\Omega \phi_0 u_{\infty,\theta}\;dx\geq\inf\phi_0$$

for $\theta<T$: $p=p(\theta)=T/\theta>1$ and $\lim_{\theta\searrow 0}p(\theta)=+\infty$. If we note $\rho=e^{-\phi_0/T}$, then $\phi_0=-T\log\rho$ and by the Jensen inequality

$$\begin{aligned}
\Delta(\theta) &= -T\frac{\int_\Omega \rho^p\log\rho\;dx}{\int_\Omega\rho^p\;dx}\\
&= -\frac{T}{p-1}\frac{\int_\Omega\rho\;dx}{\int_\Omega\rho^p\;dx}\cdot\int_\Omega\rho^{p-1}\log(\rho^{p-1})\cdot\frac{\rho\;dx}{\int_\Omega\rho\;dx}\\
&\leq -\frac{T}{p-1}\frac{\int_\Omega\rho\;dx}{\int_\Omega\rho^p\;dx}\cdot(R\log R)\Big|_{R=\frac{\int_\Omega\rho^p\;dx}{\int_\Omega\rho\;dx}}\\
&= -\frac{T}{p-1}\log\left(\frac{\int_\Omega\rho^p\;dx}{\int_\Omega\rho\;dx}\right)\\
&= \frac{T\theta}{T-\theta}\log\left(\int_\Omega\rho\;dx\right)-\frac{T^2}{T-\theta}\log\left(\|\rho\|_{L^{p(\theta)}(\Omega)}\right)\,.
\end{aligned}$$

Since $\lim_{\theta\searrow 0}\|\rho\|_{L^{p(\theta)}(\Omega)}=\|\rho\|_{L^\infty(\Omega)}=e^{-\frac{\inf\phi_0}{T}}$, $\Delta(\theta)\to\inf\phi_0$ as $\theta\searrow 0$.

If ϕ_0 is also bounded from above, then we have

$$M(\theta+\inf\phi_0)\leq E(\theta)\leq M(\theta+\sup\phi_0)\quad\text{for all}\quad\theta>0\,.$$

Hence, $T_+ = E_+ = +\infty$. □

Example If ϕ_0 is not bounded from below (respectively from above), we may have $T_- > 0$ (respectively $T_+ < +\infty$). Consider for instance $\phi_0(x) = \log|x|$ in the unit ball $\Omega = B(0,1) \subset \mathbb{R}^d$. A straightforward computation gives $T_- = 1/d$ and $\Delta(\theta) = -(d - 1/\theta)^{-1} \to -\infty$ as $\theta \searrow 1/d$.

2.2.3 Case $\langle \epsilon, \zeta \rangle = \langle 1, 0 \rangle$, Ω bounded

Assume that $\langle u, \theta, \phi \rangle$ is a stationary solution: according to (6),

$$\frac{\nabla u}{u} + \frac{1}{\theta} \nabla \phi = 0$$

a.e. with respect to the measure $u(x)\,dx$. Then $\Delta\theta = 0$ means that θ is a constant, and $\psi = \phi/\theta = C - \log u$ is a solution of the Poisson-Boltzmann equation

$$-\theta \Delta \psi = M \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx}, \tag{18}$$

where θ is determined by the energy

$$E = M\theta + \frac{1}{2} \|\nabla \psi\|^2_{L^2(\Omega)} \theta^2 .$$

(Note that here we are assuming that Ω is bounded and in this case, $\nabla\psi$ is in $L^2(\Omega)$ and the term $\|\nabla\psi\|_{L^2(\Omega)}$ makes sense). For given M and E, ψ is the solution of

$$-\sigma(\|\nabla \psi\|_{L^2(\Omega)})\, \Delta \psi = \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx} , \quad \psi_{|\partial\Omega} = 0, \tag{19}$$

where $\sigma(t) = (-1 + \sqrt{1 + \chi t^2})/t^2$, $\chi = 2E/M^2$.

Theorem 2.6 [17] *If Ω is a bounded domain of class C^1, then (19) has a unique solution for any $M > 0$ and $E > 0$.*

We refer to [17] for a complete proof and only sketch the main steps:
1) Any solution ψ of (19) is a critical point of the functional

$$J[\psi] = F(\|\nabla \psi\|_{L^2(\Omega)}) + \log \left(\int_\Omega e^{-\psi_+}\,dx \right)$$

where

$$F(t) = \left[\sqrt{1 + \chi t^2} - \log\left(1 + \sqrt{1 + \chi t^2} \right) \right] .$$

2) J is a strictly convex functional since $F'(t) = t\sigma(t)$ is nonnegative and

$$F''(t) = \frac{\chi}{\sqrt{1+\chi t^2}\,(1+\sqrt{1+\chi t^2})} > 0\ .$$

3) The minimum of J is reached by some ψ because of the following estimate.

Lemma 2.7 [17] *If Ω is a bounded domain of class C^1, there exists a constant $C \in \mathbb{R}$ such that for any function $\psi \in H^1_0(\Omega)$,*

$$\log\left(\int_\Omega e^{-\psi}\,dx\right) \geq C - 2\log\left(\|\nabla\psi\|_{L^2(\Omega)}\right)(1 + o(\|\nabla\psi\|_{L^2(\Omega)}))$$

as $\|\nabla\psi\|_{L^2(\Omega)} \to \infty$.

The main ideas of the proof of this lemma are the following: consider a function $u \geq 0$ in $H^1_0(\Omega)$, $\Omega_\lambda = \{x \in \Omega\ :\ d = d(x, \partial\Omega) \leq 1/\lambda\}$, $v = u/\lambda$:

$$\int_\Omega e^{-u}\,dx \geq \int_{\Omega_\lambda} e^{-v/d}\,dx \geq |\Omega_\lambda| e^{-|\Omega_\lambda|^{-1/2}\|v/d\|_{L^2(\Omega)}}$$

according to the Jensen inequality. Then using the Hardy inequality

$$\left\|\frac{v}{d}\right\|_{L^2(\Omega)} \leq \frac{1}{\lambda} K(\Omega)\,\|\nabla u\|_{L^2(\Omega)}$$

and $|\Omega_\lambda| \sim \lambda^{-1}|\partial\Omega|$ as $\lambda \to +\infty$, an optimization on λ gives the result. See [17] for more details. In two dimensions, the Moser-Trudinger inequality (*cf.* its use in, *e.g.*, [9] (15.1)) gives a better (optimal) result: $\log\left(\int_\Omega e^{-\psi}\right) \geq C - (8\pi)^{-1}\log(\|\nabla\psi\|_2)$.

□

2.2.4 Case $\langle\epsilon, \zeta\rangle = \langle 1, 1\rangle$

In this case, we are able to obtain existence results for the stationary problem associated to (1)-(2)-(3). The uniqueness of stationary solutions remains an open problem, except in very particular cases.

Theorem 2.8 *Let $M > 0$ and*

$$E > E_{\phi_0,\Omega} := \inf\left\{\int_\Omega \left(u\phi_0 + \frac{u\phi}{2}\right) dx\,;\ u \in L^1_+(\Omega),\ \int_\Omega u\,dx = M\right\}, \quad (20)$$

where ϕ is given by (1)-(2)-(3). Assume furthermore that $d \geq 2$ and that ϕ_0 is an external potential which is bounded from below in Ω and satisfies (5) for some T_0 strictly larger than $\overline{T} := (E - C_{1,1/2})/M$. If $d = 2$, assume also that either Ω is bounded or $\Omega = \mathbb{R}^2$ and ϕ_0 satisfies (7). Then, there exists at least one $u \in L^\infty_+(\Omega)$ such that the Poisson-Boltzmann equation (15) has a solution.

Remark 2.9 *Note that when $d \geq 3$ or when Ω is bounded, $\overline{T} = E/M$ and $E_{\phi_0,\Omega} \geq 0$. The actual value of $E_{\phi_0,\Omega} \geq 0$ strongly depends on the geometry of Ω and ϕ_0.*

Proof. As in [20], the proof is based on a variational argument involving directly the entropy. The main difference is that the critical level is defined as a max-min level instead of simply being the minimal level of a convex functional.

Consider the functional

$$W[u,\theta] = \int_\Omega u \log\left(\frac{u}{\theta}\right)\, dx$$

which is well defined on the set

$$X = \left\{ \langle u,\theta \rangle \in L^1_+(\Omega) \times L^\infty_+(\Omega) : \int_\Omega u\, dx = M,\ \int_\Omega u\left(\theta + \phi_0 + \frac{1}{2}\phi\right) dx = E \right\}.$$

On the set X, and for any $\mu \in \mathbb{R}$, $W[u,\theta]$ coincides with the functional

$$\tilde{W}[u,\theta] = W[u,\theta] + \mu\Big(\int_\Omega u\left(\theta + \phi_0 + \frac{1}{2}\phi\right)\, dx - E\Big).$$

But for any $\mu > 0$ and for any $u \geq 0$,

$$\tilde{W}[u,\theta] \geq \tilde{W}[u,\overline{\theta}],$$

where $\overline{\theta}$ is a constant such that

$$u\left(-\frac{1}{\overline{\theta}} + \mu\right) = 0 \quad \text{a.e. },$$

or, in other words, $\overline{\theta} = 1/\mu$, $u(x)dx$ a.e. Thus

$$\inf_{(u,\theta)\in X} W[u,\theta] = \inf_{(u,\overline{\theta})\in \overline{X}} \mathcal{W}[u,\overline{\theta}]$$

where

$$\mathcal{W}[u,\overline{\theta}] = W[u,\overline{\theta}] + \left(\int_\Omega u\left[1+\frac{1}{\overline{\theta}}\left(\phi_0+\frac{1}{2}\phi\right)\right]\,dx - \frac{E}{\overline{\theta}}\right)$$

and

$$\overline{X} = \left\{\langle u,\overline{\theta}\rangle \in L^1_+(\Omega)\times \mathbb{R}^*_+ : \int_\Omega u\,dx = M,\ \int_\Omega u\left[1+\frac{1}{\overline{\theta}}\left(\phi_0+\frac{1}{2}\phi\right)\right]\,dx = \frac{E}{\overline{\theta}}\right\}.$$

Consider now the set

$$\widetilde{X} = \left\{\langle u,\overline{\theta}\rangle \in L^1_+(\Omega)\times \mathbb{R}^*_+ : \int_\Omega u\,dx = M,\ \int_\Omega u\left[1+\frac{1}{\overline{\theta}}\left(\phi_0+\frac{1}{2}\phi\right)\right]dx \le \frac{E}{\overline{\theta}}\right\},$$

which corresponds to relaxing the constraint on the energy of u and its projection on $\mathbb{R}^*_+$, Θ. Notice that $\Theta \subset (0,\overline{T}]$.

The functional $\mathcal{W}$ is convex in u. Moreover, for any fixed positive $\overline{\theta}\in\Theta$, $u\mapsto \mathcal{W}[u,\overline{\theta}]$ is bounded from below. Indeed, with the notations of Lemma 2.3, if $u\in L^1_+(\Omega)$, we have

$$\mathcal{W}[u,\overline{\theta}] \ge \int_\Omega u\log\frac{u}{\overline{\theta}}\,dx + \frac{1}{T_0}\int_\Omega u\phi_0\,dx + M\overline{\theta} + C_{1/\overline{\theta}-1/T_0,1/2} - \frac{E}{\overline{\theta}}\,.$$

and then we conclude by using Proposition 2.1.

Let us fix $\overline{\theta}\in\Theta$ and consider a minimizing sequence $\{u_n\}$ for the functional $u\mapsto\mathcal{W}[u,\overline{\theta}]$ in $\{u\in L^1_+(\Omega)\,;(u,\overline{\theta})\in\widetilde{X}\}$. According to the Dunford-Pettis criterion, u_n converges (up to the extraction of a subsequence) to some $\overline{u}\in L^1_+(\Omega)$ weakly in $L^1(\Omega)$ and

$$\liminf_{n\to+\infty}\mathcal{W}[u_n,\overline{\theta}] \ge \mathcal{W}[\overline{u},\overline{\theta}]\,,$$

because of the convexity of $\mathcal{W}$. Moreover,

$$\int_\Omega \overline{u}\,dx = M\ ;\quad \int_\Omega \overline{u}\left[1+\frac{1}{\overline{\theta}}\left(\phi_0+\frac{1}{2}\phi\right)\right]\,dx \le \frac{E}{\overline{\theta}}\,.$$

But the Euler-Lagrange equation satisfied by $\overline{u}$ is exactly the Poisson-Boltzmann equation with $\theta=\overline{\theta}$, and a Lagrange multiplier

$$\lambda = -1+\log\left(\frac{\overline{\theta}}{M}\right)+\log\left(\int_\Omega e^{-\frac{1}{\overline{\theta}}(\phi+\phi_0)}\,dx\right)$$

corresponding to the constraint $\int_\Omega \overline{u}\,dx = M$. Hence, $\overline{u}$ does not depend on the considered subsequence, and we can denote it by $u_{\overline{\theta}}$.

At this point, we may notice that the function $\overline{\theta} \mapsto \mathcal{W}[u_{\overline{\theta}}, \overline{\theta}]$ is nondecreasing on any open interval (a, b) contained in Θ. Moreover, if $\overline{\theta}_0 \in (a, b)$,

$$\frac{d}{d\overline{\theta}}\mathcal{W}[u, \overline{\theta}]_{|_{\overline{\theta}=\overline{\theta}_0}} = 0 \quad \Longleftrightarrow \quad \int_\Omega u \left[1 + \frac{1}{\overline{\theta}_0}\left(\phi_0 + \frac{1}{2}\phi\right)\right]\, dx = \frac{E}{\overline{\theta}_0}\,.$$

Next, let us define

$$I := \sup_{\overline{\theta} \in \Theta} \mathcal{W}[u_{\overline{\theta}}, \overline{\theta}] < +\infty\,.$$

There are two possibilities: either I is not achieved in Θ or it is achieved by some $\overline{\theta}_M \in \Theta$. In the latter case, if we prove that $\langle u_{\overline{\theta}_M}, \overline{\theta}_M \rangle \in \overline{X}$, the proof is finished. Then, by contradiction, assume that

$$\int_\Omega u_{\overline{\theta}_M}\left[1 + \frac{1}{\overline{\theta}_M}\left(\phi_0 + \frac{1}{2}\phi\right)\right]\, dx < \frac{E}{\overline{\theta}_M}\,. \tag{21}$$

Then, $\langle u_{\overline{\theta}_M}, \overline{\theta} \rangle \in \widetilde{X}$ for $\overline{\theta}$ close enough to $\overline{\theta}_M$. This, (21) and the monotonicity of $\overline{\theta} \mapsto \mathcal{W}[u_{\overline{\theta}}, \overline{\theta}]$ contradicts the maximizing character of $\overline{\theta}_M$.

The only case remaining to be considered is when I is not achieved in Θ. This means that there is a maximizing sequence for I in Θ, $\{\overline{\theta}_n\}$, such that

$$\lim_{n \to +\infty} \mathcal{W}[u_{\overline{\theta}_n}, \overline{\theta}_n] = \sup_{\overline{\theta} \in \Theta} \mathcal{W}[u_{\overline{\theta}}, \overline{\theta}]\,,$$

but such that $\overline{\theta}_\infty := \lim_n \overline{\theta}_n \notin \Theta$. But this is impossible. Indeed, we can apply again the Dunford-Pettis criterion, to extract a subsequence, still denoted by $\{\overline{\theta}_n\}$, such that $u_{\overline{\theta}_n}$ converges weakly in $L^1(\Omega)$ to some $u \in L^1_+(\Omega)$ such that $\int_\Omega u\, dx = M$. Moreover, by lower semicontinuity, $\langle u, \overline{\theta}_\infty \rangle \in \widetilde{X}$. A contradiction. So, this case can never happen and the proof is finished. $\square$

The question of the uniqueness remains largely open. In special cases however, it is possible to give some results.

Example. Case $d = 1$, $\phi_0 = |x|$, $\Omega = \mathbb{R}$.

The solution of the Poisson-Boltzmann equation (15) can be rescaled according to $\phi(x)/\theta = \psi(x/\theta)$, and ψ is now a solution of

$$-\Delta\psi = M\frac{e^{-\psi-|x|}}{\int_{\mathbb{R}} e^{-\psi-|y|}\, dy} = \theta u(\theta x)\,.$$

Therefore, ψ does not depend on θ and has a unique nonnegative solution such that $\psi(0) = 0$, $\lim_{x\to\pm\infty} \psi'(x) = \pm\frac{M}{2}$ (for instance). The equation

$$E = \theta \left(M + \frac{M}{2} \frac{\int_{\mathrm{I\!R}} \psi e^{-\psi-|x|}\, dx}{\int_{\mathrm{I\!R}} e^{-\psi-|x|}\, dx} + M \frac{\int_{\mathrm{I\!R}} |x| e^{-\psi-|x|}\, dx}{\int_{\mathrm{I\!R}} e^{-\psi-|x|}\, dx} \right)$$

then determines θ uniquely for any given $M > 0$ and $E > 0$.

Example. Case $d \geq 3$, $\phi_0 = |x|^\beta/\beta$, $\Omega = \mathrm{I\!R}^d$.

In this case, one could try to see whether uniqueness actually holds for some values of E, M and β by using a different variational argument.

As in the case $d = 1$, the solution of the Poisson-Boltzmann equation (15) can be rescaled according to $\phi(x)/\theta = \psi(\theta^{-1/\beta} x)$, and ψ is now a solution of

$$-\theta^{1+\frac{d-2}{\beta}} \Delta\psi = M \frac{e^{-\psi-|x|^\beta/\beta}}{\int e^{-\psi-|y|^\beta/\beta}\, dy} = \theta^{\frac{d}{\beta}} u(\theta^{1/\beta} x)\ . \tag{22}$$

The Rellich-Pokhozaev identity, which is obtained by multiplying (22) by $(x \cdot \nabla\psi)$ and integrating by parts, gives

$$M\theta^{-1-\frac{d-2}{\beta}} \frac{\int |x|^\beta e^{-\psi-|x|^\beta/\beta}\, dx}{\int e^{-\psi-|x|^\beta/\beta}\, dx} = dM\theta^{-1-\frac{d-2}{\beta}} + \left(\frac{d}{2} - 1\right) \|\nabla\psi\|^2_{L^2(\mathrm{I\!R}^d)}\ .$$

The temperature θ is therefore determined by the condition

$$E = M\theta \left(1 + \frac{d}{\beta}\right) + \theta^{2+\frac{d-2}{\beta}} \left(\frac{1}{2} + \left(\frac{d}{2} - 1\right)\frac{1}{\beta}\right) \|\nabla\psi\|^2_{L^2(\mathrm{I\!R}^d)}\ .$$

Let $a = M(1 + \frac{d}{\beta})$, $b = \frac{1}{2} + \left(\frac{d}{2} - 1\right)\frac{1}{\beta}$, $\alpha = 2 + \frac{d-2}{\beta}$ and $t = \|\nabla\psi\|_{L^2(\mathrm{I\!R}^d)}$; $\theta = \theta(t)$ is implicitly defined as a function of t by the equation

$$E = a\theta + bt^2\theta^\alpha \tag{23}$$

with the condition $\theta > 0$. Let $F(t) = \int_0^t (\theta(s))^{1+\frac{d-2}{\beta}}\, s\, ds$. Any critical point of the functional

$$J[\psi] = F\left(\|\nabla\psi\|_{L^2(\mathrm{I\!R}^d)}\right) + \log\left(\int_{\mathrm{I\!R}^d} e^{-\psi-|x|^\beta/\beta}\, dx\right)$$

is a solution of equation (22). Hence, uniqueness of stationary solutions is equivalent to uniqueness of critical points for the functional J.

2.3 Relative entropy and convergence to the stationary solution in a bounded domain

Consider a solution $\langle u, \theta, \phi\rangle$ of (1) and denote by $\langle u_\infty, \theta_\infty, \phi_\infty\rangle$ a stationary solution with same charge and energy, such that

$$\lim_{t\to+\infty} W[u(t,.), \theta(t,.), \phi(t,.)] = W[u_\infty, \theta_\infty, \phi_\infty] \; .$$

Such a solution exists because

$$\begin{aligned}\lim_{t\to+\infty} \quad & \int_t^{t+1} d\tau \Big[\int_\Omega \lambda \frac{|\nabla\theta|^2}{\theta^2}(\tau,x)\; dx \\ & + \int_\Omega \kappa u(\tau,x) \left| \frac{\nabla u}{u}(\tau,x) + \frac{1}{\theta(\tau,x)}(\epsilon\nabla\phi(\tau,x) + \zeta\nabla\phi_0(x)) \right|^2 dx \Big] = 0\end{aligned}$$

means that at least for an increasing unbounded sequence $(t_n)_{n\in\mathbb{N}}$, the sequence $\langle u, \theta, \phi\rangle(t_n + .,.)$ converges to some $\langle u_\infty, \theta_\infty, \phi_\infty\rangle$. We may define the relative entropy by

$$\Sigma[u,\theta,\phi] = W[u,\theta,\phi] - W[u_\infty, \theta_\infty, \phi_\infty] \; , \tag{24}$$

and using successively the fact that θ_∞ is a constant, the Poisson-Boltzmann equation (19) to compute the term involving $\log(u_\infty)$, and the conservation of mass and energy: $E[u,\theta,\phi] = E[u_\infty, \theta_\infty, \phi_\infty]$, we prove that

$$\begin{aligned}\Sigma[u,\theta,\phi] \;\; = \;\; & \int_\Omega u \log\left(\frac{u}{u_\infty}\right)\; dx + \int_\Omega (u - u_\infty)\log(u_\infty)\; dx \\ & - \int_\Omega u \log\left(\frac{\theta}{\theta_\infty}\right)\; dx + \frac{1}{\theta_\infty}(E[u,\theta,\phi] - E[u_\infty, \theta_\infty, \phi_\infty]) \; ,\end{aligned}$$

and so, we can write

$$\Sigma[u,\theta,\phi] = \int_\Omega s_1\left(\frac{u}{u_\infty}\right) u_\infty\, dx + \int_\Omega s_2\left(\frac{\theta}{\theta_\infty}\right) u\, dx + \frac{1}{2\theta_\infty}\int_\Omega |\nabla\phi - \nabla\phi_\infty|^2\; dx$$

where $s_1(t) = t\log t + 1 - t$ and $s_2(t) = t - 1 - \log t$ are two nonnegative strictly convex functions such that $s_1(1) = s_2(1) = 0$.

Proposition 2.10 *Let $\langle u, \theta, \phi\rangle$ be a smooth solution of (1) in a bounded domain Ω. With the above notations, $\Sigma[u,\theta,\phi]$ is nonnegative, decreasing, $\lim_{t\to+\infty} \frac{d}{dt}\Sigma[u,\theta,\phi](t) = 0$ and*

$$\begin{aligned}& \frac{1}{M}\,\|u - u_\infty\|^2_{L^1(\Omega)} + \frac{2}{\theta_\infty}\,\|\nabla\phi - \nabla\phi_\infty\|^2_{L^2(\Omega)} \\ & \qquad + C[\theta,u](t)\,\|\theta - \theta_\infty\|^2_{L^1(\Omega, u(t,x)\mathbf{1}_{\{\theta<\ell\theta_\infty\}}\mathrm{dx})} \le 4\,\Sigma[u,\theta,\phi](t) \; ,\end{aligned} \tag{25}$$

for all $\ell \geq 1$, *where* $C[\theta, u] = \left(M\theta_\infty^2 \cdot \max\{1, \ell\}\right)^{-1}$.

Proof. The proof of this proposition follows from (6): $\Sigma[u, \theta, \phi](t)$ is bounded from below and

$$\frac{d}{dt}\Sigma[u,\theta,\phi](t) = -\int_\Omega \lambda \frac{|\nabla\theta|^2}{\theta^2}\,dx - \int_\Omega \kappa u \left|\frac{\nabla u}{u} + \frac{1}{\theta}(\epsilon\nabla\phi + \zeta\nabla\phi_0)\right|^2 dx \leq 0\ ,$$

and this inequality is strict unless $u \equiv u_\infty$: the only possible limit of $\frac{d}{dt}\,\Sigma[u, \theta, \phi](t)$ is zero. The bound (25) is given by the Csiszár-Kullback inequality, which can be stated as follows (see [16] for a proof and further references). □

Lemma 2.11 [16] *Assume that* Ω *is a domain in* $\mathbb{R}^d$ *and that* s *is a convex nonnegative function on* $\mathbb{R}^+$ *such that* $s(1) = 0$ *and* $s'(1) = 0$*. If* μ *is a nonnegative measure on* Ω*, and if* f*,* g *are nonnegative measurable functions on* Ω *with respect to* μ*, then*

$$\int_\Omega s\left(\frac{f}{g}\right) g\,d\mu \geq \frac{K}{\max\{\int_\Omega f\,d\mu, \int_\Omega g\,d\mu\}}\|f-g\|^2_{L^1(\Omega,d\mu)}$$

where the constants K*,* K_1 *and* K_2 *are defined by* $K = \frac{1}{4}\cdot\min\{K_1, K_2\}$*,* $K_1 = \min_{\eta\in[0,1]} s''(\eta)$ *and* $K_2 = \min_{\theta\in[0,1],h\geq 0} s''(1+\theta h)(1+h)$*, provided that all the above integrals are finite.* □

Corollary 2.12 *Any smooth solution* $\langle u, \theta, \phi\rangle$ *of (1) converges as* $t \to +\infty$ *to a unique stationary solution* $\langle u_\infty, \theta_\infty, \phi_\infty\rangle$ *with same charge and energy.*

This corollary answers the question raised in Section 4.2 of [7]. The result does not mean that the stationary problem has a unique solution for any given charge and energy, but that the limit for a given solution is unique. In other words, the limit does not depend on the sequence $(t_n)_{n\in\mathbb{N}}$ used to define the limit $\langle u_\infty, \theta_\infty, \phi_\infty\rangle$.

3 The whole space problem without exterior potential

We consider now the case $\Omega = \mathbb{R}^d$ and $\zeta = 0$. No stationary solution exists, but a time-dependent rescaling like the one in [4, 5] provides a convenient framework to study the intermediate asymptotics with relative entropy methods. This can be done for special functions κ and λ only.

3.1 Time-dependent rescalings

Assume that κ and λ are homogeneous in u: there exist constants m and p such that $\kappa = |u|^{m-1}\tilde{\kappa}(\theta)$, $\lambda = |u|^{p-1}\tilde{\lambda}(\theta)$, and consider like in [16, 4, 5] the time-dependent rescaling given by

$$\begin{cases} u(t,x) = R^{-d}\bar{u}(\log(R(t)), \frac{x}{R}) \\ \theta(t,x) = \bar{\theta}(\log(R(t)), \frac{x}{R}) \\ \phi(t,x) = \bar{\phi}(\log(R(t)), \frac{x}{R}) \end{cases} \tag{26}$$

where R is the solution of $\dot{R} = R^{-d(m-1)-1}$, $R(0) = 1$. The rescaled functions (we omit the bar $^-$ in the remainder of this section and use the same notations as for the unscaled functions) are solutions of the system

$$\begin{cases} u_t = \nabla \cdot \left[\kappa(\nabla u + \epsilon \frac{u}{\theta}\nabla\phi) + xu\right] \\ (u\theta)_t = e^{\beta t}\nabla\cdot(\lambda\nabla\theta) + \nabla\cdot\left[\kappa(\theta\nabla u + \epsilon u\nabla\phi) + xu\theta\right] \\ \qquad\qquad +\epsilon\nabla\phi\cdot\left[\kappa(\nabla u + \epsilon\frac{u}{\theta}\nabla\phi)\right] \\ -\Delta\phi = e^{-(d-2)t}u \end{cases} \tag{27}$$

with $\beta = d(m-p+1)$. The initial data are the same as for the unscaled problem.

In the following, we shall assume that $\kappa \equiv 1$ ($m = 1$): $R(t) = \sqrt{1+2t}$, and $\lambda \equiv 1$: $\beta = d$.

3.2 Radial problem with Poisson coupling

The energy $E = \int u(\theta + \phi/2)\, dx$ for the rescaled problem (27) with $\kappa \equiv 1$ ($m = 1$), $R(t) = \sqrt{1+2t}$, $\lambda \equiv 1$, and $\beta = d$, is preserved and a direct computation shows that

$$\frac{d}{dt}\int u\frac{|x|^2}{2}\, dx = -\int x\cdot\left(\nabla u + xu + \frac{u}{\theta}\nabla\phi\right)\, dx\ ,$$

$$\frac{d}{dt}\int u\log\left(\frac{u}{\theta}\right) dx = -e^{dt}\int|\nabla\log\theta|^2 dx - \int u\left|\frac{\nabla u}{u} + \frac{1}{\theta}\nabla\phi\right|^2 dx - \int x\cdot\nabla u\, dx\ .$$

Thus $\Sigma(t) = \int\left(u\log(u/u_\infty) - u\log(\theta/\theta_\infty)\right)\, dx$, with $u_\infty = M\frac{e^{-|x|^2/2}}{(2\pi)^{d/2}}$, is such that

$$\frac{d\Sigma}{dt} = -e^{dt}\int\frac{|\nabla\theta|^2}{\theta^2}\, dx - \int u\left|\frac{\nabla u}{u} + x + \frac{1}{\theta}\nabla\phi\right|^2\, dx + \int\frac{u}{\theta}\left(x\cdot\nabla\phi\right)\, dx\ .$$

Proposition 3.1 *With the above notations, if $\langle u, \theta, \phi\rangle$ is a radially symmetric (in x) solution of (27), then $d\Sigma/dt \leq 0$ and $\langle u, \theta, \phi\rangle$ converges as $t \to +\infty$ to $\langle u_\infty, \theta_\infty, 0\rangle$.*

Proof. The term $\int u\,\theta^{-1}\,(x\cdot\nabla\phi)\,dx$ is negative because $\nabla\phi(x) = x\phi'(|x|)$ and $\phi'(r) = -r^{1-d}\int_0^r u(\rho,t)\rho^{d-1}\,d\rho \leq 0$. The remainder of the reasoning is standard. □

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