# Stationary Subspace Analysis

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# Outline

# Motivation

- Non-stationarities can be found in many real-world data, yet they challenge standard Machine Learning methods.
- Different training and test distributions:
  $\longrightarrow$ Problems to generalise.

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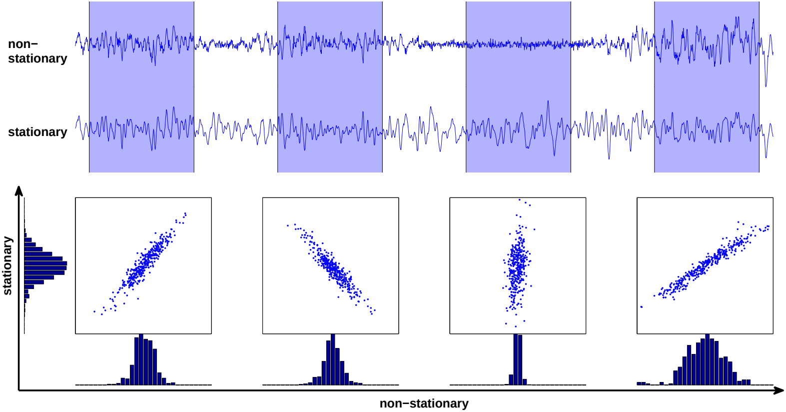
**Observation:**

Data generating systems are often only partly non-stationary.

- Getting rid of the non-stationary part might help.
- Understanding the nature of the non-stationarity is an interesting endeavour in its own right.

# Stationary and Non-stationary subspaces



# Generative Model

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- $d$ stationary source signals $s^{\mathfrak{s}}(t) \in \mathbb{R}^d$
- $D - d$ non-stationary source signals $s^{\mathfrak{n}}(t) \in \mathbb{R}^{(D-d)}$
- Observed signals: instantaneous linear superpositions of sources

$$x(t) = As(t) = \begin{bmatrix} A^{\mathfrak{s}} & A^{\mathfrak{n}} \end{bmatrix} \begin{bmatrix} s^{\mathfrak{s}}(t) \\ s^{\mathfrak{n}}(t) \end{bmatrix}$$

# Aim of Stationary Subspace Analysis

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## Goal

Given only $x(t)$, find an estimate $\hat{A}$ for the mixing matrix, such that $\hat{B} = \hat{A}^{-1}$ separates $\mathfrak{s}$-sources from $\mathfrak{n}$-sources.

$$\begin{bmatrix} \hat{s}^{\mathfrak{s}}(t) \\ \hat{s}^{\mathfrak{n}}(t) \end{bmatrix} = \hat{B}x(t) = \begin{bmatrix} \hat{B}^{\mathfrak{s}} \\ \hat{B}^{\mathfrak{n}} \end{bmatrix} x(t)$$

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Clearly, $\hat{A} = A$ is a solution. But are there other solutions?

# Symmetries and Invariances

Let's express the true $A^{\mathfrak{s}}$ and $A^{\mathfrak{n}}$ as linear combinations of the respective estimated subspaces

$$
\begin{aligned}
A^{\mathfrak{s}} &= \hat{A}^{\mathfrak{s}} M_1 + \hat{A}^{\mathfrak{n}} M_2 \\
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# Restriction to orthogonal demixing matrices

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**Result**

We can restrict our search for the mixing matrix to the space of orthogonal matrices even if the model allows general (i.e. non-orthogonal) mixing matrices.

# Measuring (Non-)Stationarity

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Given $N$ data sets, we will consider a set of $d$ estimated sources as stationary, if the joint distribution of these sources stays the same.

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### Gaussian Approximation

Consider only differences in the first two moments
$\rightarrow$ KL-Divergence between Gaussians (max. Entropy principle)

# The Optimization Problem

To stay on the manifold of orthogonal matrices: multiplicative updates with rotation matrices ($RR^\top = I$).

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## The loss function

$$L_B(R) = \sum_{i<j} \mathrm{KL}\left[\mathcal{N}(\hat{\mu}_i^{\mathfrak{s}}, \hat{\Sigma}_i^{\mathfrak{s}}) \,||\, \mathcal{N}(\hat{\mu}_j^{\mathfrak{s}}, \hat{\Sigma}_j^{\mathfrak{s}})\right]$$

with

$$\hat{\mu}_i^{\mathfrak{s}} = I^d RB\hat{\mu}_i \quad \text{and} \quad \hat{\Sigma}_i^{\mathfrak{s}} = I^d RB\hat{\Sigma}_i (I^d RB)^\top$$

denoting estimated mean and covariance of the i-*th* data set projected to the $\mathfrak{s}$-subspace and $I^d \in \mathbb{R}^{d\times D}$ the identity matrix truncated to the first $d$ rows.

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Manifold of all D-dimensional rotations:
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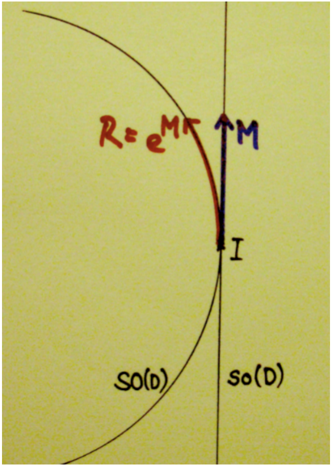

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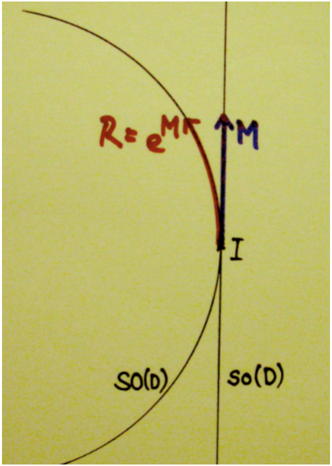
**From Group Theory:**

Every element of a Lie Group can be expressed as the exponential of an element from the corresponding Lie Algebra. (tangent space at $I$).



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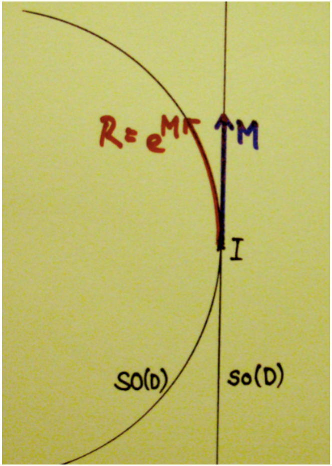
Linear space of all skew-symmetric matrices $M^\top = -M$:
Special Orthogonal Algebra $\mathfrak{so}$(D).



# Optimization in the Special Orthogonal Group

We express R as

$$R = \exp(M)$$

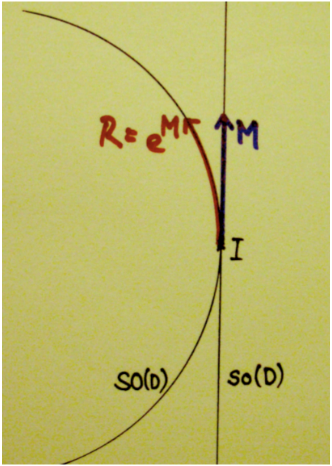
with $M^\top = -M$ and optimize the objective $L_B$ in terms of $M$.

**Interpretation of $M_{ij}$:**

Angle of rotation of axis i towards axis j

The gradient translates to:

$$\left.\frac{\partial L_B}{\partial M}\right|_{M=0} = \left(\frac{\partial L_B}{\partial R}\right) R^\top - R\left(\frac{\partial L_B}{\partial R}\right)^\top$$



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Thus the gradient has the shape

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$Z$ corresponds to rotations between $\mathfrak{s}$- and $\mathfrak{n}$-space.
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**Result**

The number of variables is reduced to $d(D-d)$.

# Simulations

## Experimental Setup

- $N$ covariance matrices and means are sampled that are stationary in the first $d$ coordinates.
- To each mean and covariance the same randomly sampled mixing matrix is applied.
- SSA is applied.
- The accuracy is measured as angle between the estimated $\mathfrak{n}$-subspace and the ground truth.

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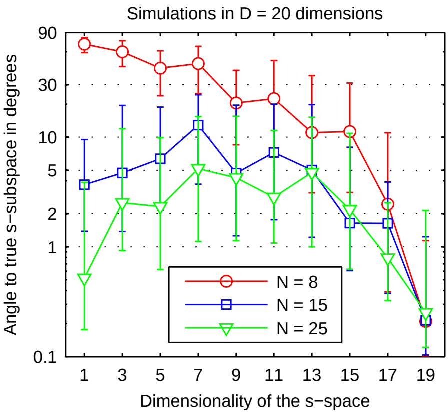
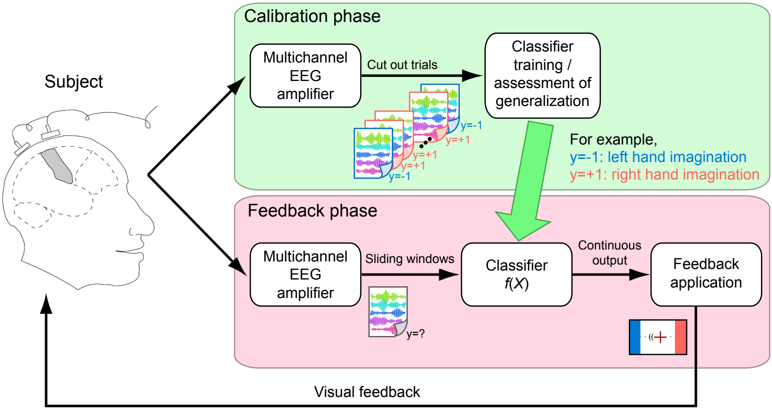



- Input space dimension $D = 20$
- Number of data sets $N = 8, 15, 25$
- Performance as median angle to the true subspace
- 100 repetitions, error bars 25% to 75% quantile

# BCI Experiment



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We induce changes in the strength of the $\alpha$-rhythm by extracting it from a separate artefact measurement session (using ICA) and superimpose it on the data (adaptation and test set) in varying strengths.

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- Divide Data into 3 parts:
  1. Training Set, used for running SSA, to train the classifier
  2. Adaptation Set, used for running SSA
  3. Test Set, used for evaluating the classifier
- Estimate $\hat{A}$ over the training and adaptation part
- Train Classifier (CSP/LDA) within the $\mathfrak{s}$-space in training set
- Performance: misclassification rate on the test set

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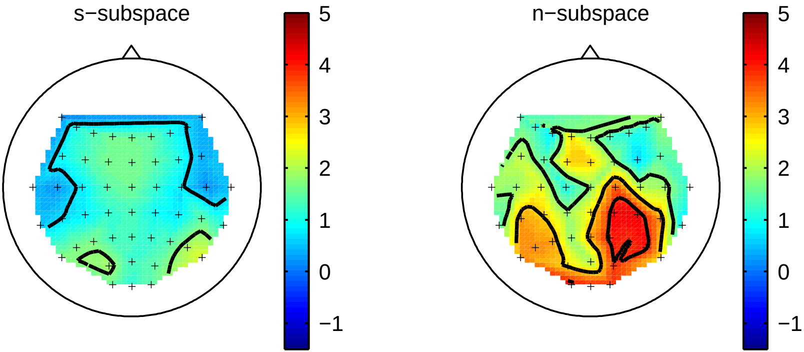


Relative power differences between training and test set.

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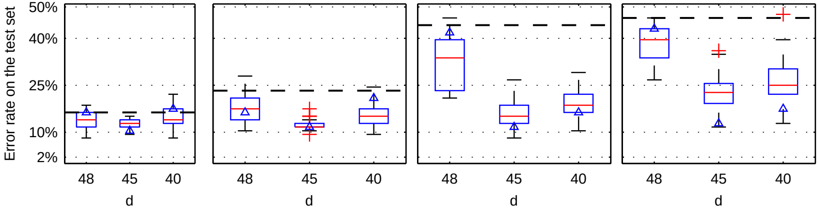


- Boxplots show distribution of the test error rates
- Dashed black line: Test error rate of the baseline method (using all data).
- Blue triangle: error rate on the subspace with minimum objective function value

# Conclusion

- We have presented an algorithm for decomposing a multivariate time-series into a stationary and a non-stationary component.
- We can restrict the search space to orthogonal transformations without limiting the applicability.
- Exploiting the underlying Lie-Group structure reduces the number of parameters and allows a stable and efficient optimization.
- Application to simulated and BCI data indicate that projecting out the $\mathfrak{n}$-sources can improve classification performance.

Thank You.