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STATISTICAL AND THERMODYNAMICAL DESCRIPTIONS OF HADRON PRODUCTION IN e^+e^- ANNIHILATION

J. Engels and K. Schilling

Department of Theoretical Physics
University of Bielefeld

and

H. Satz *)

CERN - Geneva

A B S T R A C T

The statistical model and the thermodynamical (statistical bootstrap) model are formulated for multihadron production in e^+e^- annihilation. Asymptotic results are derived for both approaches, with particular attention to critical features subject to experimental test. Quantitative predictions of multiplicities, average secondary energies and inclusive single particle distributions are presented for storage ring centre-of-mass energies from 2 to 6 GeV.

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*) On leave from the Department of Theoretical Physics, University of Bielefeld, Germany.

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1. - INTRODUCTION

Forthcoming experiments at e^+e^- storage rings are expected to provide us with data on hadronic production rather different from those observed until now : within the validity of the one-photon approximation, we can hope to study hadronic systems of large mass ($M \lesssim 6$ GeV) but with a total spin one - to be contrasted with an average spin of around twenty for a similar system of this mass when produced in hadron-hadron interactions.

This novel and experimentally open situation has led to a considerable amount of theoretical speculation [1], covering almost the entire range of models consistent with energy conservation : thus predictions for the asymptotic multiplicity $\bar{N}$ of produced hadrons vary from $\bar{N}$ constant [2] to $\bar{N}$ increasing linearly with the centre-of-mass energy M of the e^+e^- system [3].

The aim of the present paper is to develop and study in some detail a model which forms one of the two extremes (it yields $\bar{N} \sim M$) : the thermodynamical model for high energy e^+e^- annihilation into hadrons. On the one hand, we want to investigate particularly the theoretical foundation of such a description in the framework of the statistical bootstrap formalism, in order to see which of the resulting features are critical for the approach ; on the other hand, we want to present quantitative predictions for physical secondaries at the expected (finite) machine energies.

As a preliminary study to our main problem, we shall derive the predictions one obtains by applying to e^+e^- annihilation the traditional statistical model of Fermi [4]. This approach assumed an equidistribution over all final states compatible with initial state kinematics - a hadronic ideal gas. As is well known, however, hadron-hadron collision experiments soon indicated a strong jet structure in particle production (transverse momentum bound). Hence the statistical model in an unmodified form is certainly not in accord with hadronic production data. As it is even to-day not clear whether the unbounded longitudinal or the bounded transverse momentum distributions of secondaries reflect the essence of hadron dynamics, a comparison of simple phase space with e^+e^- annihilation data will certainly be of interest. Moreover, the Fermi model, as the simplest possible approach, proves quite useful in discussing novel features introduced by the statistical bootstrap condition.

The application of the thermodynamical [5] or statistical bootstrap [6] model to e^+e^- annihilation is rather straightforward, if one accepts the hadron-like character of (timelike) photons. The basic building blocks of the thermodynamical model are fireballs : energetic hadronic matter at rest, or in other words, hadronic systems of large mass M and low spin j, with $j/M \to 0$ as $M \to \infty$. The fundamental property of fireballs is that they decay, as the consequence of the statistical bootstrap condition, into secondaries of asymptotically bounded average energy. In theymodynamic language this is equivalent to the existence of a highest temperature T_o. From applications to the transverse momentum distribution in inclusive hadron-hadron interactions one finds $T_o \simeq 160$ MeV. With $j = 1$ and a mass of about 6 GeV, an energetic virtual photon would be a fireball par excellence and could provide the most unambiguous test for the basis of the thermodynamical model and related questions [7] – always under the assumption that a hadron-like photon is a valid extrapolation into deep timelike regions.

Considerations of a somewhat related nature were first proposed by Bjorken and Brodsky [3], who generalize the transverse momentum restriction $e^{-a|p_T|}$ in hadron-hadron interactions to an energy bound e^{-ap_o} for all secondaries produced in e^+e^- annihilation. This, however, will in itself not lead to a bound on the avarage energy per secondary, as suggested in Ref. 3) ; since $\exp - a\sum_1^N p_{io} = \exp - 2M$ gives a common factor to all transitions, independent of particle number, it cannot influence average quantities such as the multiplicity $\bar{N}$ or the average secondary energy $\bar{w}$. The bound on $\bar{w}$ derived in 3) is in fact due to an ad hoc assumption about the distribution over particle numbers – an assumption which can be understood, as we shall see later on, only in the framework of a statistical bootstrap scheme.

Let us finally note for all such thermodynamical considerations a point, minor in principle, but not unimportant for comparison with data : in models with a finite asymptotic average energy $\bar{w}$ per secondary, the zero mass limit never coincides with the high energy limit – in contrast to models with unbounded $\bar{w}$. Although the qualitative features in the thermodynamical model remain as $m \to 0$, actual predictions are significantly altered.

We begin in Section 2 with the presentation of the statistical model for $e^+e^- \to$ hadrons, followed by that of the thermodynamical model as obtained from statistical bootstrap arguments. Our main objective in Section 2 will be a discussion of the theoretical aspects of the models and of their asymptotic $(M \to \infty)$ results ; quantitative predictions for finite energies, obtained by numerical calculations, will be presented in Section 3.

2. - STATISTICAL AND THERMODYNAMICAL MODELS FOR e^+e^- ANNIHILATION

The cross-section for the production of N hadrons (which we shall from now on take to be pions unless otherwise stated) in e^+e^- annihilation at centre-of-mass energy $\sqrt{s}$ can be written as [8])

$$\sigma_N(s) = \frac{4\pi\alpha}{s^{3/2}}\,\Gamma_N(s) \tag{1}$$

where $\Gamma_N(s)$ denotes the integrated decay width of the virtual photon into N pions

$$\Gamma_N(s) = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{d^3p_i}{2p_{i0}}\,\delta^{(4)}\Big(\sum_1^N p_i - Q\Big)\,\rho(q_1,q_2;p_1,\dots,p_N) \tag{2}$$

with $Q = q_1 + q_2$, $Q^2 = s$ characterizing the initial state. The production dynamics are now contained in the N particle momentum space weight ρ. With the total decay width

$$\Gamma_{\gamma\to \text{hadrons}}(s) = \sum_{N=2}^{\infty}\Gamma_N(s) \equiv \Gamma_{tot}(s) \tag{3}$$

we have

$$\sigma_{tot}(s) = \frac{4\pi\alpha}{s^{3/2}}\,\Gamma_{tot}(s) \tag{4}$$

as the total cross-section for hadron production.

2.1. - The statistical model

In the absence of information about the decay dynamics, the simplest assumption is to take ρ constant

$$\rho(q_1,q_2;p_1,\dots,p_N) = c\,\mathcal{H}^N \tag{5}$$

i.e., to consider an equidistribution in momentum space. The normalization function c can depend only on $s = Q^2$; a determination of this dependence (the γ hadron coupling) is certainly not possible in statistical or thermodynamical approaches, which determine only relative weights of different final states and do not give absolute predictions. We thus write

$$\Gamma_N(s) = \frac{s^{3/2}}{4\pi\alpha}\, \sigma_{tot}(s) \frac{G_N(s)}{\sum_{i=2}^{\infty} G_i(s)} \tag{6}$$

with

$$G_N(s) = \frac{\varkappa^N}{N!} \int \prod_{i=1}^{N} \frac{d^3 p_i}{2 p_{i0}}\, \delta^{(4)}\left(\sum_{i=1}^{N} p_i - Q\right) \tag{7}$$

denoting the N particle phase space. For large s (extreme relativistic limit), the phase space (7) becomes

$$G_N(s) \simeq \frac{\pi}{2} \frac{(s\pi/2)^{N-2}}{(N-1)!\,(N-2)!} \frac{\varkappa^N}{N!} \tag{8}$$

Summing (6) over all N gives us the total phase space, which asymptotically takes the form [9)]

$$G(s) \simeq \frac{(\pi\varkappa s/2)^{2/3}}{\sqrt{3}\,\pi^2 s^2}\, e^{3[\pi\varkappa s/2]^{1/3}} \tag{9}$$

The average number of produced hadrons (multiplicity) can for large s easily be obtained from (3), (5) and (9)

$$\begin{aligned} \bar{N}(s) &= \varkappa \frac{\partial \log G(s)}{\partial \varkappa} \\ &\simeq [\pi\varkappa s/2]^{1/3} \end{aligned} \tag{10}$$

It thus increases as a power, but less than linearly, in the photon mass ; consequently the average energy per secondary

$$\bar{\omega}(s) \simeq \frac{\sqrt{s}}{\bar{N}} \simeq [2/\pi\varkappa]^{1/3}\, s^{1/6} \tag{11}$$

grows asymptotically without bound. The normalized inclusive single particle distribution for secondary energy q_o

$$F(q_o,s) = \varkappa^2 \frac{\partial G([Q-q]^2)}{\partial \varkappa} \Big/ G(s) \tag{12}$$

becomes for large photon mass $\sqrt{s} = \sqrt{Q^2}$

$$F(q_o,s) \simeq \varkappa (\pi\varkappa s/2)^{1/3} (1 - \frac{2q_o}{\sqrt{s}})^{-1} e^{-2[\pi\varkappa/2]^{1/3} \frac{q_o}{s^{1/6}}} \tag{13}$$

and hence does not lead to a non-zero scaling limit ; for fixed $x \equiv 2q_o/\sqrt{s}$

$$F(x,s) \simeq \varkappa(\pi\varkappa s/2)^{1/3} (1-x)^{-1} e^{-x[\pi\varkappa s/2]^{1/3}} \tag{14}$$

vanishes exponentially.

We close our survey of the asymptotic behaviour of the statistical model with two comments.

1) - The qualitative features of most of these results are rather insensitive to the specific form of momentum space. Had we chosen

$$G_N^F(s) = \frac{\varkappa_F^N}{N!} \int \prod_{i=1}^{N} d^3p_i \, \delta^{(4)}(\sum_1^N p_i - Q) \tag{15}$$

instead of (6), then, e.g., the multiplicity would grow like [9] $\bar{N}_F \sim s^{3/8}$ instead of $s^{1/3}$, leaving the general picture essentially unchanged. We might note here also the case

$$G_N^P(s) = \frac{\varkappa_P^N}{N!} \int \prod_{i=1}^{N} \left\{ \frac{d^3p_i}{2p_{io}} \frac{1}{p_{io}^2} \right\} \delta^{(4)}(\sum_1^N p_i - Q) \tag{16}$$

which, although without any jet structure, leads to

$$\begin{aligned} \bar{N}(s) &\sim \ln s \\ F(q_o,s) &\sim (1-x)^{\text{const.}} , \quad x \equiv 2q_o/\sqrt{s} \end{aligned} \tag{16a}$$

i.e., to features otherwise found in uncorrelated jet [10] or parton model [11] considerations. It is thus clear that more detailed quantities than multiplicities or single particle spectra have to be measured to distinguish these models from a purely statistical description.

2) - We have not taken into account momentum space restrictions due to the given spin one of the photon, as it can be shown [12] that the resulting effects become unimportant at high energies.

2.2. - The thermodynamical (statistical bootstrap) model

Instead of the single centre decay (Fig. 1) discussed above, consider now a cascade decay with two body intermediate steps (Fig. 2), each consisting of one pion and one excited hadronic system (fireball). The decay width is then given by

$$\begin{aligned} \Gamma_N(s) \sim & \int \frac{d^3p_1}{2p_{10}} d^4P_1 \, \delta^{(4)}(p_1+P_1-Q) \, |\langle P_1 p_1 | Q \rangle|^2 \, * \\ & \int \frac{d^3p_2}{2p_{20}} d^4P_2 \, \delta^{(4)}(p_2+P_2-P_1) \, |\langle P_2 p_2 | P_1 \rangle|^2 \, * \cdots * \\ & \int \frac{d^3p_{N-1}}{2p_{N-1,0}} \frac{d^3p_N}{2p_{N,0}} \, \delta^{(4)}(p_{N-1}+p_N-P_{N-2}) \, |\langle p_{N-1} p_N | P_{N-2} \rangle|^2 \end{aligned} \tag{17}$$

Now if all couplings are constant, $|\langle P_i p_i | P_{i-1} \rangle|^2 = \lambda$, $i = 2,3,\ldots,N$, then $\Gamma_N(s)$ with proper normalization can be written

$$\begin{aligned} \Gamma_N(s) &= \Gamma_{tot}(s) \frac{\tau_N(s)}{\sum_{i=2}^{\infty} \tau_i(s)} \\ \tau_N(s) &= \lambda^{N-1} \int \prod_{i=1}^{N} \frac{d^3p_i}{2p_{i0}} \, \delta^{(4)}\left(\sum_1^N p_i - Q\right) \end{aligned} \tag{18}$$

The essential difference between (6) and (18) is the absence of the Boltzmann factor 1/N! in the latter ; as we shall see now, it is this difference which yields in one case secondaries with unbounded, in the other with bounded average energy.

Consider first the case of mass zero secondaries, for which all phase space expressions are solvable in closed form. We obtain from (18)

$$\tau_N(s) = \lambda^{N-1} \frac{\pi}{2} \frac{(\pi s/2)^{N-2}}{(N-1)!\,(N-2)!} \tag{19}$$

and hence for the sum over all N

$$\tau(s) = (\lambda\pi/2s)^{1/2}\, I_1(2\sqrt{\lambda\pi s/2}) \tag{20}$$

where $I_1(x)$ is the Bessel function of order one and pure imaginary argument. For large s Eq. (20) gives

$$\tau(s) \simeq \frac{\lambda\pi^{1/2}}{4} (\lambda\pi s/2)^{-3/4}\, e^{2\sqrt{\lambda\pi s/2}} \tag{21}$$

which leads to an average particle number

$$\bar{N}(s) = 1 + \lambda \frac{\partial \log \tau(s)}{\partial\lambda} = \sqrt{\lambda\pi s/2}\,\left(1 + O\left(\tfrac{\ln s}{s}\right)\right) \tag{22}$$

increasing linearly with the photon mass $\sqrt{s}$. In contrast to (11) we thus obtain with

$$\bar{\omega} \simeq \frac{\sqrt{s}}{\bar{N}} \simeq \sqrt{\frac{2}{\pi\lambda}} \tag{23a}$$

an asymptotically bounded average energy per secondary ; the actual value of the bound depends on the coupling constant λ. The inclusive single particle distribution [cf., (12)] at high energy and for $c_0 \ll \sqrt{s}$ is easily obtained from (21) ; as expected from the above, we have with

$$F(q_0, s) \simeq \lambda\sqrt{\lambda\pi s/2}\,(1-x)^{-1/4}\, e^{-\sqrt{2\pi\lambda}\,q_0}$$

$$x = 2q_0/\sqrt{s} \tag{23b}$$

an exponential cut-off in secondary energy which is independent of the photon mass.

Let us now rederive these results using thermodynamic arguments. We define the grand canonical partition function

$$Z(\beta, \lambda) \equiv \int d^4Q \; e^{-\beta_\mu Q^\mu} \, \tau(Q^2) \tag{24}$$

with $\beta^2 = \beta_\mu \beta^\mu > 0$, $\beta_0 > 0$; from (3) and (19) we then have

$$Z(\beta, \lambda) = \frac{\lambda \varphi^2(\beta)}{1 - \lambda \varphi(\beta)} \tag{25}$$

with

$$\varphi(\beta) \equiv \int \frac{d^3p}{2p_0} e^{-\beta_\mu p^\mu} = \frac{2\pi}{\beta^2} \tag{26}$$

For the particularly simple case considered here ($m = 0$, linear chain decay) we could now invert the Laplace transform (24) to obtain $\tau(Q^2)$ for large Q^2 ; as this is, however, practically very difficult in more complicated cases, we shall instead pursue here a more general approach. The exponential increase in $\sqrt{s}$ of $\tau(s)$ leads to a singularity of $Z(\beta, \lambda)$ at that value $\beta = \beta_0$ for which

$$\varphi(\beta_0) = 1/\lambda \tag{27}$$

and from (24) and (27) we see that the allowed range of β is

$$\beta_0 = \sqrt{2\pi\lambda} < \beta < \infty \tag{28}$$

In thermodynamic language, β is an inverse temperature, $\beta = 1/kT$, with k denoting the Boltzmann constant. Equation (27) thus predicts the existence of a highest temperature

$$T_0 = 1/k\sqrt{2\pi\lambda} \tag{29}$$

for which the partition function becomes singular. Using (24), we can now define an average total energy

$$\bar{E} = -\frac{\partial}{\partial\beta}\log Z(\beta,\lambda) \tag{30}$$

in terms of the inverse temperature β. Requiring $\bar{E}$ to be equal to the actual energy $\sqrt{s}$ yields an energy-temperature relation [13] ("Stefan-Boltzmann law")

$$\sqrt{s} = -\bar{N}(\beta,\lambda)\,\frac{\partial \log \varphi(\beta)}{\partial\beta} \tag{31}$$

with

$$\bar{N} = 1 + \lambda\frac{\partial}{\partial\lambda}\log Z(\beta,\lambda) \tag{32}$$

For the average energy per secondary we then have

$$\bar{\omega} \simeq \frac{\sqrt{s}}{\bar{N}} = -\frac{\partial\log\varphi(\beta)}{\partial\beta} = \frac{2}{\beta} \tag{33}$$

while (31) and (32) give

$$\beta^2 = 2\pi\lambda\left[1 + \frac{1}{\sqrt{\pi\lambda s/2}} + O\left(\frac{1}{\lambda s}\right)\right] \tag{34}$$

Inserting this value of β in (32) and (33) reproduces our previous results (22) and (23) : as $s\to\infty$, the results of the grand canonical formulation converge to the exact calculation.

We have thus seen that the bound on the average energy per secondary is expressed in thermodynamic language as a bound on the temperature, $T \leq T_o$. If we choose $kT_o = 160$ MeV, as found in inclusive hadron-hadron interactions, then the only free parameter λ is fixed and we obtain

$$\bar{\omega}_\infty = 320 \text{ MeV} \tag{35}$$

as the asymptotic energy of a (mass zero) secondary. As already indicated above, this bound will be different for massive secondaries, to which we shall return further down.

Now we want to investigate the relation between our cascade decay scheme (18) and statistical bootstrap arguments by essentially rewriting (18) as a bootstrap condition. In the cascade, we require an excited hadronic state (the photon) to decay into an object of similar nature (fireball) plus one pion. Denote the density of states of the initial fireball of mass $\sqrt{Q^2}$ by $\tau(Q^2)$; we then want the decay product fireball of mass M_1 to be described by $\tau(M_1^2)$. We thus obtain the bootstrap equation [14]

$$\tau(Q^2) = \delta_0(Q^2-m^2) + \lambda \int \frac{d^3p}{2p_0} d^4P\, \tau(P^2)\, \delta^{(4)}(p+P-Q) \tag{36}$$

with

$$\delta_0(x^2-m^2) = \theta(x_0)\, \delta(x^2-m^2)$$

$$\tau(x^2) = \theta(x^2)\, \theta(x_0-m)\, \tilde{\tau}(x^2) \tag{37}$$

From (36) we obtain by Laplace transformation $(\beta^2 > 0,\ \beta_0 > 0)$ for the partition function

$$Z(\beta,\lambda) \equiv \int d^4Q\, e^{-\beta_\mu Q^\mu} \left[\tau(Q^2) - \delta_0(Q^2-m^2)\right] \tag{38}$$

the relation

$$Z(\beta,\lambda) = \frac{\lambda\varphi^2(\beta)}{1-\lambda\varphi(\beta)} \tag{39}$$

with $\varphi(\beta)$ as defined in (26). As the partition function (39) obtained from (36) is identical to the one, (25), obtained from (18), we conclude that the cascade decay (18) is in fact the solution of the bootstrap condition (36).

Relation (36) is, however, not the most general bootstrap condition ; this is obtained [5),6),15),16)] by allowing unrestricted decay of the photon into any number of fireballs and/or pions (cf. Fig. 3) :

$$\tau(Q^2) = \delta_0(Q^2-m^2) + \sum_{\ell=2}^{\infty} \frac{B^{\ell-1}}{\ell!} \int \prod_{i=1}^{\ell} \left\{ d^4k_i\, \tau(k_i^2) \right\} \delta^{(4)}\left(\sum_1^{\ell} k_i - Q\right) \tag{40}$$

where B^{ℓ} denotes the one to ℓ fireball coupling. The Laplace transformation of this "full" bootstrap condition gives

$$\begin{aligned} Z(\beta, B) &\equiv \int d^4Q\, e^{-\beta_\mu Q^\mu}\, \tau(Q^2) \\ &= \varphi(\beta) + \frac{1}{B}\left[e^{BZ} - 1 - BZ \right] \end{aligned} \tag{41}$$

The resulting $Z(\beta, B)$ as functional of φ can be shown [16),17)] to have a square root branch point at

$$B\varphi(\beta) = 2\ln 2 - 1 \equiv z_0 \tag{42}$$

Instead of the pole (39), it is now this singularity which determines the maximum temperature $kT_0 = \beta_0^{-1}$. From (26) and (42) we obtain

$$kT_0 = \beta_0^{-1} = \sqrt{z_0/2\pi B} \tag{43}$$

for the connection between T_0 and the interaction parameter B, instead of the relation (28)/(29) for the linear cascade. With the choice

$$B/\lambda = z_0 \tag{44}$$

we thus have the same maximum temperature in both cases.

The solution to Eq. (40) can moreover be written [15)]

$$\tau(Q^2) = \sum_{\ell=1}^{\infty} g_\ell\, B^{\ell-1} \int \prod_{i=1}^{\ell} \frac{d^3p_i}{2p_{i0}}\, \delta^{(4)}\left(\sum_1^{\ell} p_i - Q\right) \tag{45}$$

where the g_ℓ are determined by expressing the solution to Eq. (41) in the form

$$Z(\beta, B) = \sum_{\ell=1}^{\infty} g_\ell \left[B\varphi(\beta) \right]^\ell \tag{46a}$$

As a consequence the g_ℓ obey the recursion relation

$$g_{\ell+1} = \frac{-1}{\ell+1} \left[\ell g_\ell - 2 \sum_{k=1}^{\ell} k\, g_k\, g_{\ell+1-k} \right] \tag{46b}$$
$$g_1 = 1$$

which for large ℓ has the solution [16]

$$g_\ell \sim z_0^{-\ell+1}\, \ell^{-3/2} \tag{46c}$$

From (44) and (46c) we thus see that the full bootstrap (46a) and the linear cascade (39) lead to the same exponential increase in level density

$$\tau(M) = M^\nu e^{M/kT_0} \tag{47}$$

the additional $\ell^{-\frac{3}{2}}$ in (46) yields only a different value of ν. This asymptotic "equivalence" between the two cases is not as surprising as it may first seem, since the linear cascade was in fact shown [6] to be the dominant decay mode in the full bootstrap.

In Section 3 we shall return to non-asymptotic calculations using the g_ℓ from (46b) ; we note here already that we want to consider predictions both from the full bootstrap and from the linear chain. As long as we have no specific dynamical model, it is not clear whether a given bootstrap scheme (if at all applicable) describes only the asymptotic limit or the approach thereto as well. By investigating two thermodynamical schemes converging at high energies, we have some measure of the range of predictions from such approaches possible at finite energies.

Up to now we have in all calculations treated only secondaries of zero mass ; before taking up the case of massive secondaries, let us briefly comment on the approach of Bjorken and Brodsky [3], who consider explicitly only the $m = 0$ case. As already noted in the Introduction, all inclusive quantities remain unchanged if one replaces (18) by

$$G_N^{BB}(s) = \lambda_{BB}^{N-1} \int \prod_{i=1}^{N} \left\{ \frac{d^3p_i}{2p_{i0}} e^{-a p_{i0}} \right\} \delta^{(4)}\left(\sum_1^N p_i - Q\right) \tag{48}$$

which is the form proposed by Ref. 3). The critical assumption is, as we have seen in comparing (16) and (18), the absence of the Boltzmann factor $1/N!$ in (18) and (48) ; it is this absence, and not the momentum space damping, which provides the bound on the average secondary energy ; the form (48) with a $1/N!$ leads to secondaries with unbounded average energies. Note, however, the difference between (18) and (48) for certain exclusive measurements : the single particle distribution for an N body final state should according to (48) already exhibit exponential energy damping

$$F_N(q_0, M)/F_N(0, M) \sim e^{-a q_0} \left(1 - \frac{2q_0}{M}\right)^{2N-4} \tag{49}$$

whereas with (18) we have

$$F_N(q_0, M)/F_N(0, M) \sim \left(1 - \frac{2q_0}{M}\right)^{2N-4} \tag{50}$$

since the exponential damping of q_0 here arises only after summation over all N.

Let us now extend our considerations to include massive secondaries. In the formulation as given here this is achieved simply by replacing $\varphi(\beta)$ in (26) by the corresponding function for non-vanishing mass

$$\varphi_m(\beta) \equiv \int \frac{d^3p}{2p_0} e^{-\beta_\mu p^\mu} = \frac{2\pi m}{\beta} K_1(m\beta) \tag{51}$$

where $K_1(x)$ denotes the Hankel function of order one and imaginary argument. The determining equation (27)/(42) between maximum temperature and interaction parameter then becomes

$$\frac{2\pi m\lambda}{\beta_0} K_1(m\beta_0) = 1 \tag{52}$$

for the linear chain and

$$\frac{2\pi m B}{\beta_0} K_1(m\beta_0) = z_0 \tag{53}$$

for the full bootstrap. In either case we have, by fixing $kT_0 = \beta_0^{-1} = 160$ MeV, determined the only open parameter in the model.

As all other arguments remain valid, we now obtain from (33) and (57)

$$\bar{\omega}_\infty = \frac{2}{\beta_0} + m \frac{K_0(m\beta_0)}{K_1(m\beta_0)} \tag{54}$$

which in the case of pions yields

$$\bar{\omega}_\infty = 414 \text{ MeV} \tag{55}$$

as the asymptotic energy per secondary ; the corresponding multiplicity grows as

$$\bar{N}(s) = 2.4 \sqrt{s} \tag{56a}$$

with increasing photon mass.

The extension of our thermodynamical description to μ different types of secondaries is easily obtained by generalizing Eq. (18) to

$$\tau_{N_1 \cdots N_\mu}(s) = \frac{N!}{N_1! \cdots N_\mu!} \lambda_1^{N_1} \cdots \lambda_\mu^{N_\mu} \int \prod_{i=1}^{N} \frac{d^3p_i}{2p_{i0}} \delta^{(4)}\left(\sum_1^N p_{i0} - Q\right) \tag{56b}$$

$$N = \sum_{i=1}^{\mu} N_i \tag{56c}$$

where N_i denotes the number of particles of type i and λ_i the corresponding coupling constant. The combinatorial factor in (56b) counts the number of different possible orderings of the decay chain with fixed $N_1, \ldots, N_\mu$, assuming all particles to obey Boltzmann statistics. Summing (56b) over all N_i, $i = 1, \ldots, \mu$, and Laplace transforming the result gives us as generalization of (25) the form

$$Z(\beta, \lambda_1, \cdots, \lambda_\mu) = \frac{1}{1 - \sum_1^{\mu} \lambda_i \varphi_i(\beta)} \tag{56d}$$

so that the maximum temperature $kT_0 = \beta_0^{-1}$ and hence one of the μ parameters λ_i is determined by

$$\sum_{i=1}^{\mu} \lambda_i \varphi_i(\beta, m_i) = 1 \tag{56e}$$

Since

$$\bar{N}_i = \lambda_i \frac{\partial}{\partial \lambda_i} \log Z(\beta, \lambda_1, \cdots, \lambda_\mu) \qquad i = 1, 2, \cdots, \mu \tag{56f}$$

we can fix the remaining $\mu - 1$ constants by using the asymptotic multiplicity ratios

$$R_i = \bar{N}_i / \bar{N}_1 \quad , \qquad i = 2, 3, \cdots, \mu \tag{56g}$$

as given input. Going from the linear chain to the full bootstrap is achieved by multiplying $\tau_{N_1 \ldots N}$ in Eq. (56b) by $g_N z_0^N$, as evident from Eqs. (44) and (45).

Further details concerning physical particles, conservation laws for discrete quantum numbers, etc., will be taken up in Section 3 ; we close this section with some comments on the asymptotic results obtained so far.

2.3. - Critical aspects of asymptotic behaviour

In the Table we summarize the essential results of Section 2, concerning the asymptotic predictions of statistical and thermodynamical descriptions of e^+e^- annihilation into hadrons.

As is evident, the thermodynamical approach can easily be subjected to critical empirical tests. The maximum temperature T_o is related directly to the hadronic level density, which one would expect to be universal, i.e., independent of production mechanism. Hence average pion energy in e^+e^- annihilation should with increasing photon mass approach from below an asymptotic value of about 400-450 MeV *). Thus if one should observe at storage ring energies around 6 GeV average pion energies of around one GeV, then this would constitute a serious difficulty for the thermodynamical approach. A failure of this approach, on the other hand, would seem to rule out the only scheme - besides the dual resonance model, which moreover appears closely related [7] - proposed up to now to explain, rather than only accommodate, the transverse momentum bound in hadronic interactions. Whatever the results of experiments will be, from this point of view they will provide significant information.

The statistical description, in contrast, is much less specific. If we allow the possibility of factorized power law weights in momentum space, e.g.,

$$G_N^{(r)}(s) = \frac{\varkappa_r^N}{N!} \int \prod_{i=1}^{N} \left\{ \frac{d^3p_i}{2p_{i0}} \, p_{i0}^{r} \right\} \delta^{(4)}\left(\sum_1^N p_i - Q\right) \tag{57}$$

where $r=0$ yields the covariant form (6), $r=1$ the Fermi version (15), then for multiplicities, average secondary energies or single particle spectra it can accommodate any behaviour <u>except</u> that predicted by the thermodynamical model. As we have already indicated, the measurement of these quantities will therefore not be sufficient to distinguish between parton model [11] and statistical considerations. For this it will be necessary to first establish experimentally the jet structure predicted by the former, i.e., to measure two or more particle correlations. It should be noted that, although not listed in the Table, a statistical approach can also lead to bounded asymptotic multiplicities : if in (57) $r < -2$, $\bar{N}$ will become constant at high energies.

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*) This value could be exceeded significantly only if one should observe - in contrast to large angle hadron-hadron data - very strong production of heavy secondaries ; cf. Section 3.

3. - FINITE ENERGY PREDICTIONS

In this section we shall present the essential predictions of the thermodynamical approach to e^+e^- annihilation for the energy range to be investigated in storage ring experiments at SLAC and DESY, i.e., $\sqrt{s} \equiv E = 2$-6 GeV. For comparison we shall also show some results of the conventional statistical model at these energies.

As above, we choose always for the maximum temperature the value $kT_o = 160$ MeV, observed in large angle hadron production at the ISR [18]. Charge conservation, more generally isospin, is introduced into the model by weighting the various final state charge configurations according to their statistical isospin weight factors [19], normalized over all possible charge configurations of a given N particle final state. To do this, it is necessary to specify the relative coupling strength $g_V/g_S \equiv r$ of hadrons to the isovector and isoscalar component of the photon, which are assumed not to interfere. It turns out, however, that for $E \geq 2$ GeV, the inclusive results considered here depend very little (less than 1% variation in $\bar{N}^{ch}$) on r and are in fact very close to the asymptotic predictions $\bar{N}_{\pi^\pm} = \bar{N}_{\pi^0}$, $\bar{N}_{K^\pm} = \bar{N}_{K^0} = \bar{N}_{\bar{K}^0}$, etc. The main impact of r comes through G parity conservation, which allows the isovector photon component to couple only to an even, the isoscalar component only to an odd number of pions in the final state. At high energies, however, pure pion final states and hence G parity conservation become less and less important. All results presented in the following were calculated with $r = 1$.

The necessary phase space integrations were performed by Monte Carlo methods using the standard Fowl [20]. Allowing only pionic final states yields through Eq. (52), i.e., for the linear chain model, the value $\lambda_\pi = 9.45$ GeV^{-2}. The resulting average multiplicity of charged pions $\bar{N}^{ch}_\pi$, is shown in Fig. 4. It is seen to be a straight line, which can be parametrized by

$$\bar{N}^{ch}_{\pi} = .6 + \frac{E}{.414} \qquad (58)$$

with E in GeV, in accord with the asymptotic average pion energy of 414 MeV found in (55). The very early start ($E \geq 1$ GeV) of asymptotic behaviour is modified, as we shall see, by the presence of other final state particles besides pions.

As next simplest case, we allow pions and kaons in the final state, enforcing strangeness conservation for the latter. This is assured by writing

$$\tau(s) = \sum_{N_\pi, N_K} \binom{N_\pi + N_K}{N_\pi} \lambda_\pi^{N_\pi} \lambda_K^{2N_K} *$$

$$\int \prod_{i=1}^{N_\pi} \frac{d^3 p_i^\pi}{2 p_{i0}^\pi} \prod_{j=1}^{2N_K} \frac{d^3 p_j^K}{2 p_{j0}^K} \delta^{(4)}\left(\sum_1^{N_\pi} p_i^\pi + \sum_1^{2N_K} p_j^K - Q\right) \tag{59}$$

which yields

$$Z(\beta, \lambda_\pi, \lambda_K) = \left[1 - \lambda_\pi \varphi(\beta, m_\pi) - \lambda_K^2 \varphi(\beta, m_K)\right]^{-1} \tag{60}$$

One of the parameters λ_π and λ_K is determined as in (56e) by the relation defining the asymptotic temperature

$$1 = \lambda_\pi \varphi(\beta_0, m_\pi) + \lambda_K^2 \varphi^2(\beta_0, m_K) \tag{61}$$

the other by requiring the pion to kaon multiplicity to have the value observed at large angle ISR experiments [18]

$$\bar{N}_{\pi^+} : \bar{N}_{K^+} = 100 : 7 \tag{62}$$

The resulting values

$$\lambda_\pi = 9.03 \text{ GeV}^{-2} \tag{63}$$

$$\lambda_K = 11.82 \text{ GeV}^{-2} \tag{64}$$

were then used to calculate the multiplicity distributions and spectra shown in Figs. 4 to 9. We notice in Fig. 4 that the average number of charged pions now increases slower with E than in the previous case where only pions are allowed as secondaries - an effect due to the opening of phase space for kaon production. At 6 GeV, the presence of kaons has some 20%

effect on the average number of charged pions. The average energies per pion or kaon, calculated by integrating the energy spectra (in Figs. 5 and 6 we show representative results at 4 and 6 GeV), slowly converge to their asymptotic values of 414 and 754 MeV, respectively, as seen in Fig. 7.

The model can now be further refined by including baryon pair and associated production as well. We have tried to assess the influence of such processes by including nucleon-antinucleon production at the rate observed in ISR data [18)]

$$\bar{N}_{\pi^+} : \bar{N}_{\bar{p}} = 100 : 2.5 \tag{65}$$

The pair production restriction is taken into account as for kaons, and the determining equation for the maximum temperature becomes now

$$1 = \lambda_\pi \varphi(\beta_0, m_\pi) + \lambda_K^2 \varphi_K^2(\beta_0, m_K) + \lambda_N^2 \varphi_N^2(\beta_0, m_N) \tag{66}$$

yielding together with (62) and (65)

$$\lambda_\pi = 8.86\ \mathrm{GeV}^{-2}\ ;\ \lambda_K = 11.7\ \mathrm{GeV}^{-2}\ ;\ \lambda_N = 85.4\ \mathrm{GeV}^{-2} \tag{67}$$

We see in Fig. 4 that the inclusion of nucleon-antinucleon pair production reduces the average charged pion multiplicity by about 5% at 6 GeV.

All results so far were for the linear chain version of the model. To have some feeling for variations, possible at finite energies in different decay schemes, we have also calculated the charged pion multiplicity for the full bootstrap (40), allowing pions and kaons in the final state. The resulting values of $\bar{N}_\pi^{ch}$ at 6 GeV are somewhat (5%) lower than those from the linear chain, as seen in Fig. 10.

Let us now compare the behaviour of our thermodynamical considerations with that of the conventional statistical model (6/7). Adjusting the open parameter $\varkappa$ of the latter to reproduce the charged particle multiplicities observed in e^+e^- annihilation at 2 GeV [2)] gives $\varkappa = 12\ \mathrm{GeV}^{-2}$; the resulting charged pion multiplicity is shown in Fig. 11. As expected from (10), it increases much slower with E than the corresponding statistical bootstrap predictions.

In closing we show in Fig. 12, for pions and kaons in the final state, the predictions of $\bar{N}^{ch}_{\pi}$ from linear chain, full bootstrap and from the statistical model together with all presently available data [21),22)]. It is clearly seen that more accurate and higher energy data are needed before any conclusions can be drawn.

ACKNOWLEDGEMENT

It is a pleasure to thank R. Hagedorn for stimulating discussions.

	Multiplicity $\bar{N}$	Average secondary energy $\overline{w}$	Single particle spectrum $F(q_o, M)$
Statistical model, covariant	$M^{2/3}$	$M^{1/3}$	$e^{-const.(\varkappa/M)^{\frac{1}{3}}q_o}$
Statistical model, Fermi	$M^{3/4}$	$M^{1/4}$	$e^{-const.(\varkappa/M)^{\frac{1}{4}}q_o}$
Statistical model, Eq. (16)	$\ln M$	$\frac{M}{\ln M}$	$(1-x)^{const.}$ $x \equiv 2q_o/M$
Thermodynamical model	M	const	$e^{-const.q_o}$

TABLE : The asymptotic behaviour of statistical and thermodynamical models for e^+e^- annihilation into hadrons ; M denotes the photon mass, q_o the centre-of-mass energy of the observed secondary.

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FIGURE CAPTIONS

Figure 1 Statistical decay.

Figure 2 Linear cascade decay.

Figure 3 Full bootstrap decay.

Figure 4 Charged pion multiplicity as function of e^+e^- energy, for allowed final states consisting of pions only (-----), pions and kaons (———), and pions, kaons and nucleons (-.-.-.-).

Figure 5 Energy distributions for secondary pions and kaons at 4 GeV.

Figure 6 Energy distributions for secondary pions and kaons at 6 GeV.

Figure 7 Average pion energy as function of e^+e^- energy for full bootstrap (-----) and linear cascade (———).

Figure 8 Multiplicity distribution for negative pions, allowing pions and kaons in the final state, at an e^+e^- energy of 4 GeV.

Figure 9 Multiplicity distribution for negative pions, allowing pions and kaons in the final state, at an e^+e^- energy of 6 GeV.

Figure 10 Charged pion multiplicity, allowing pions and kaons on the final state, for full bootstrap (-----) and linear cascade.

Figure 11 Charged pion multiplicity, allowing pions and kaons in the final state, for linear chain (———) and conventional statistical model (-----).

Figure 12 Charged pion multiplicity, allowing pions and kaons in the final state, for linear cascade (———), full bootstrap (-----) and conventional statistical model (-.-.-), compared with present experimental data for charged multiplicity [Refs. 20) and 21)].

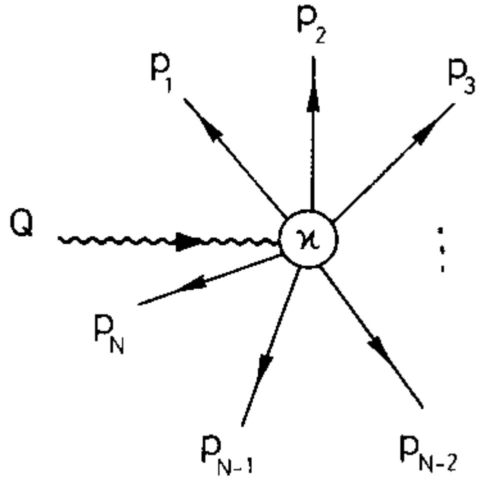


FIG.1

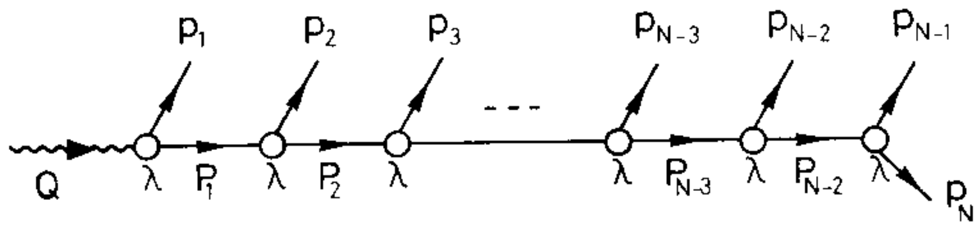


FIG.2

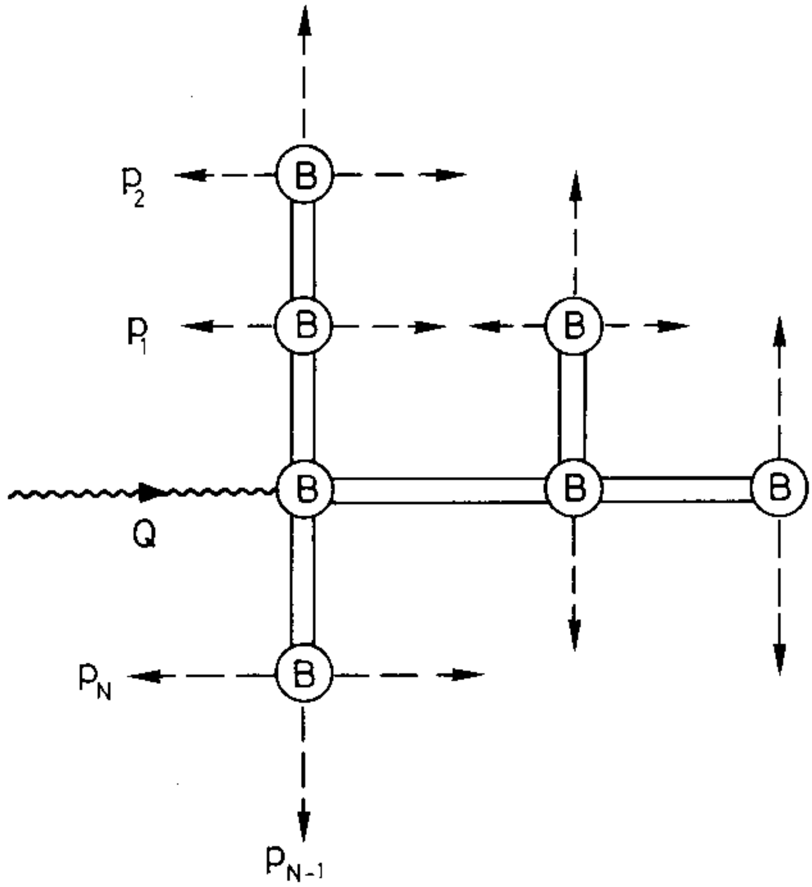


FIG. 3

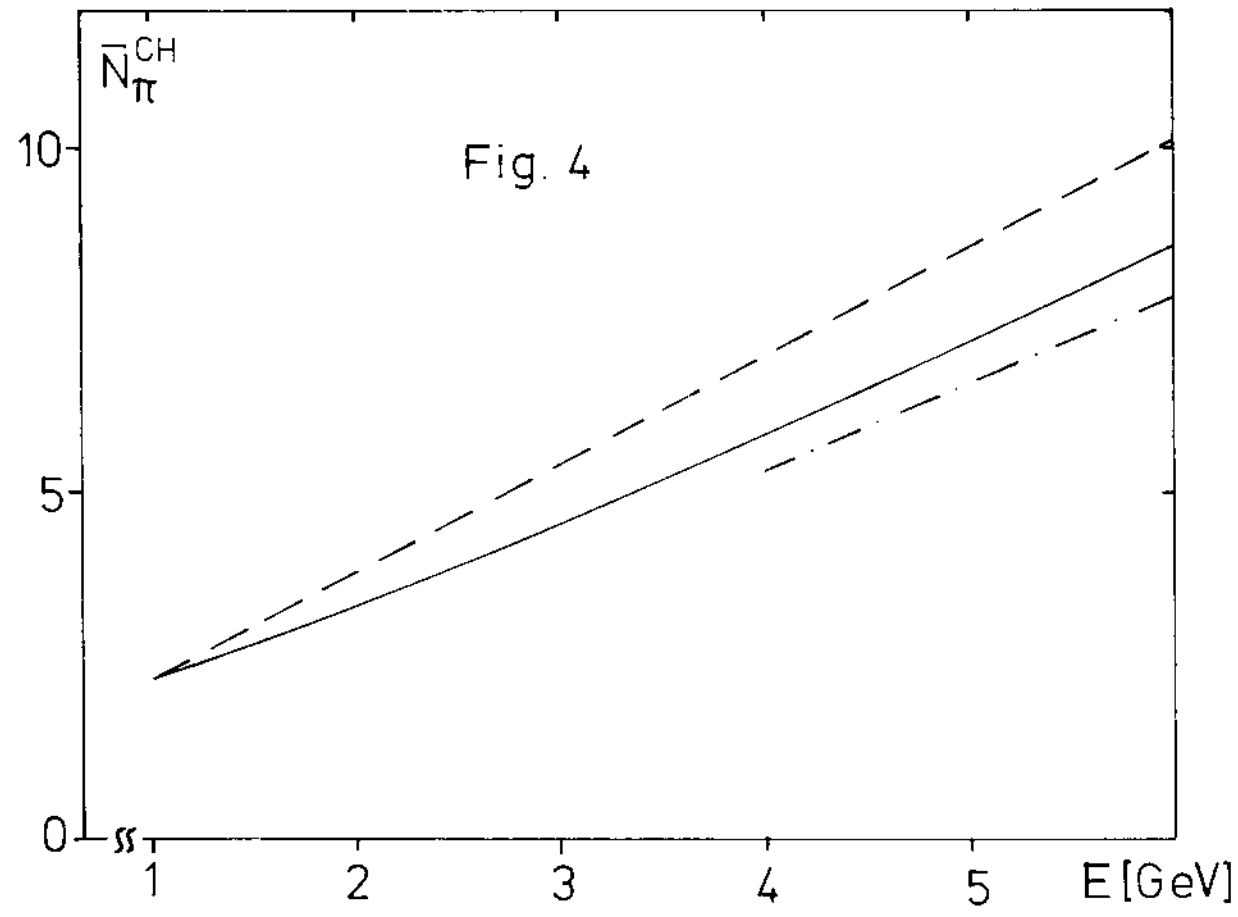
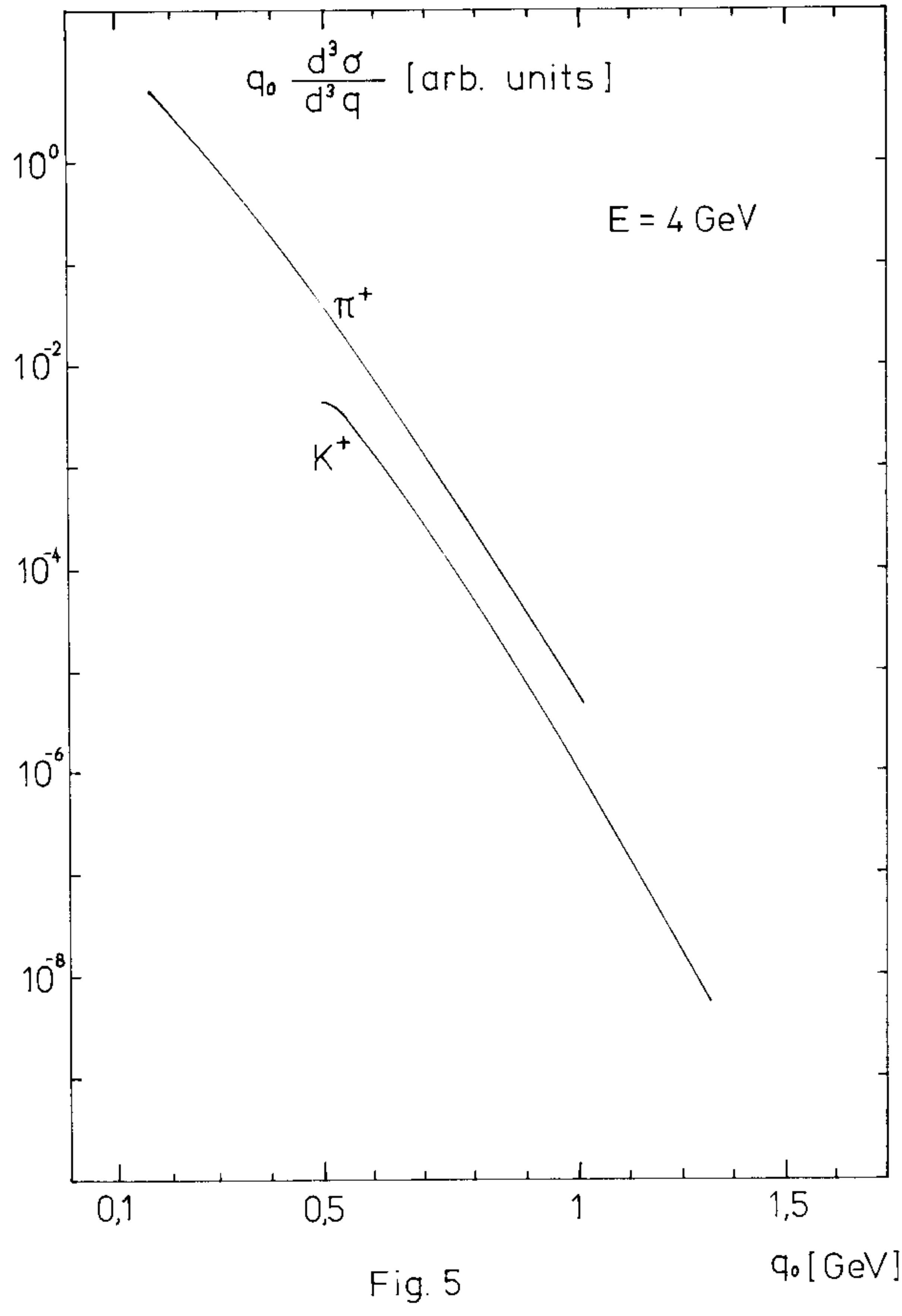
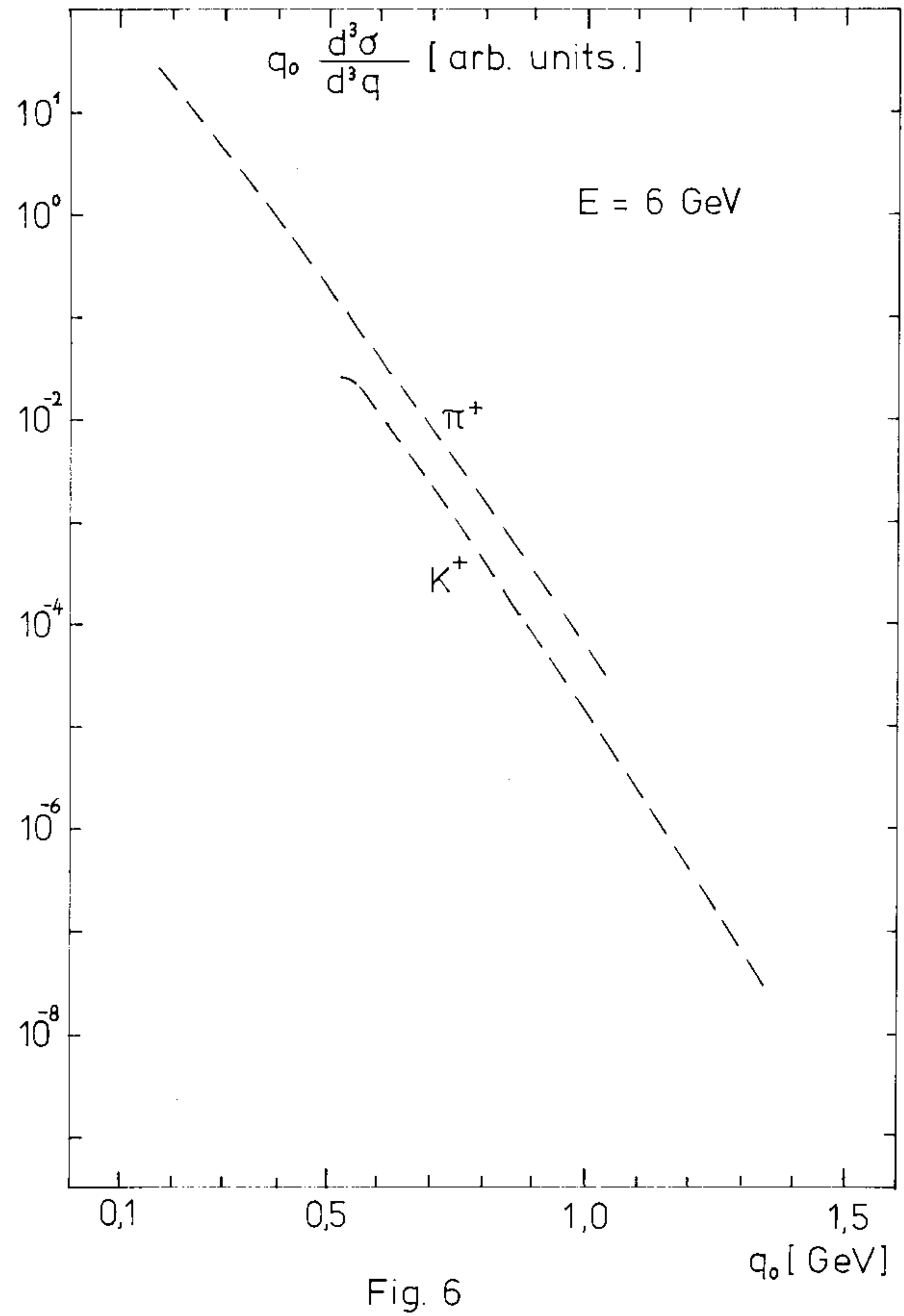


Fig. 4



$q_0 \frac{d^3\sigma}{d^3q}$ [arb. units]
E = 4 GeV
π^+
K^+
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
0,1
0,5
1
1,5
q_0 [GeV]

Fig. 5



$q_o \frac{d^3\sigma}{d^3q}$ [arb. units.]
E = 6 GeV
π^+
K^+
10^1
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
0,1
0,5
1,0
1,5
q_o [GeV]

Fig. 6

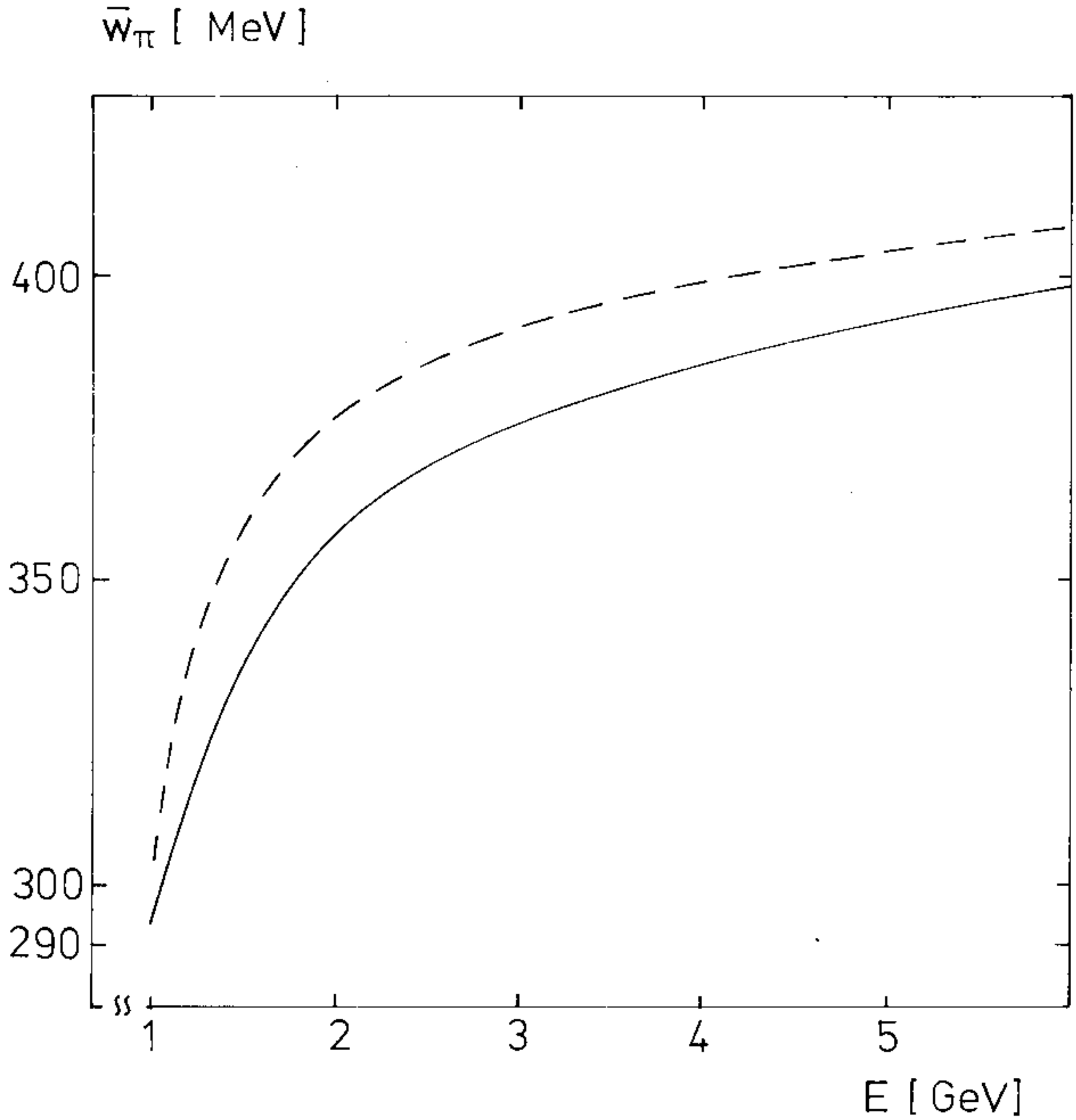


Fig. 7

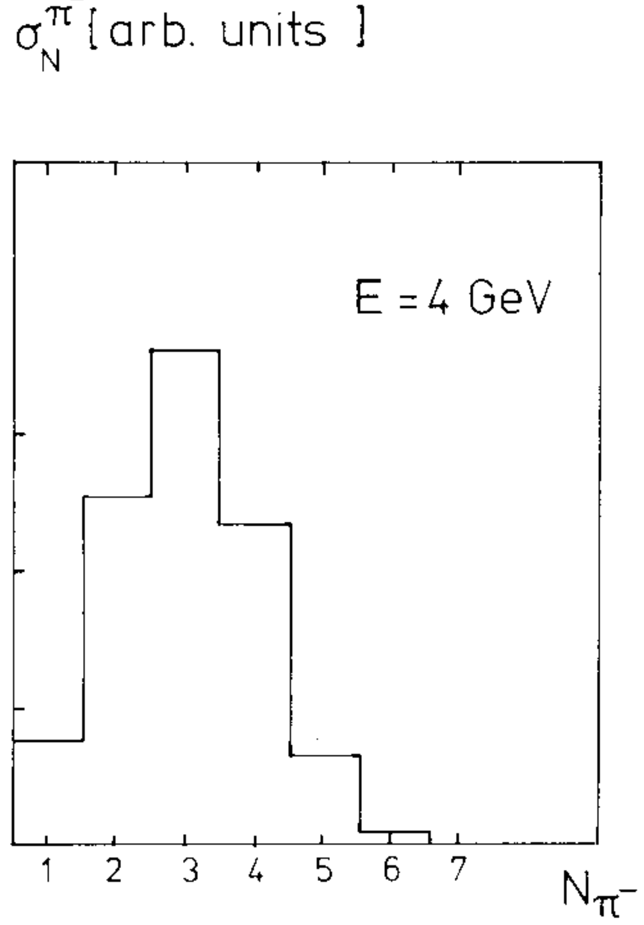


Fig. 8

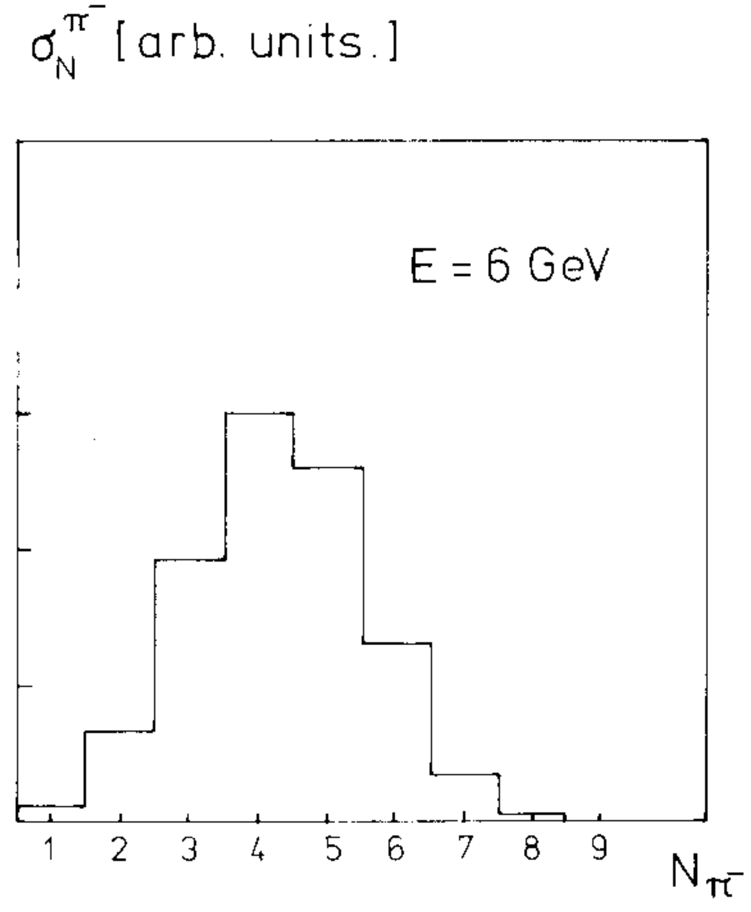


Fig. 9

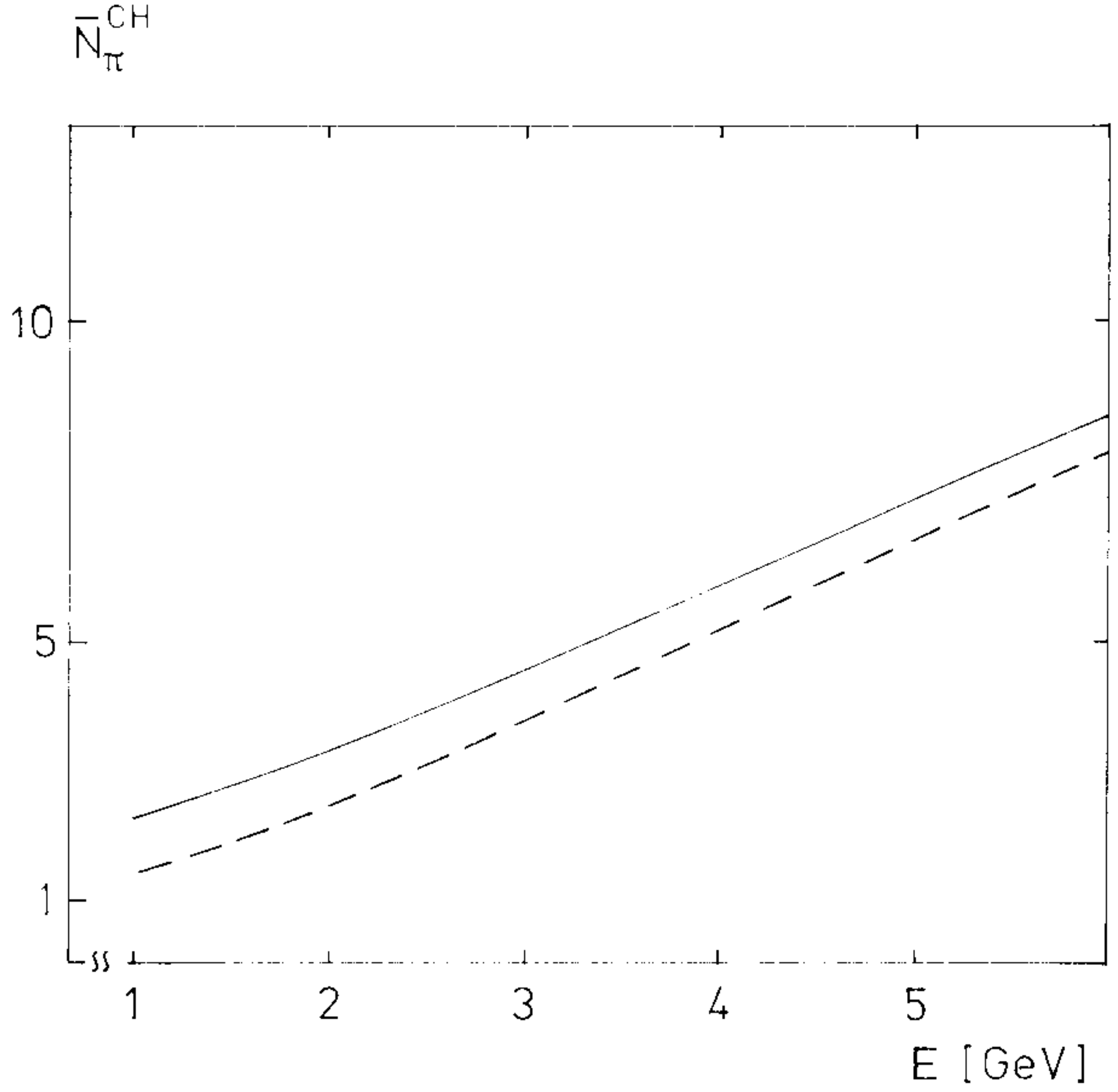


Fig. 10

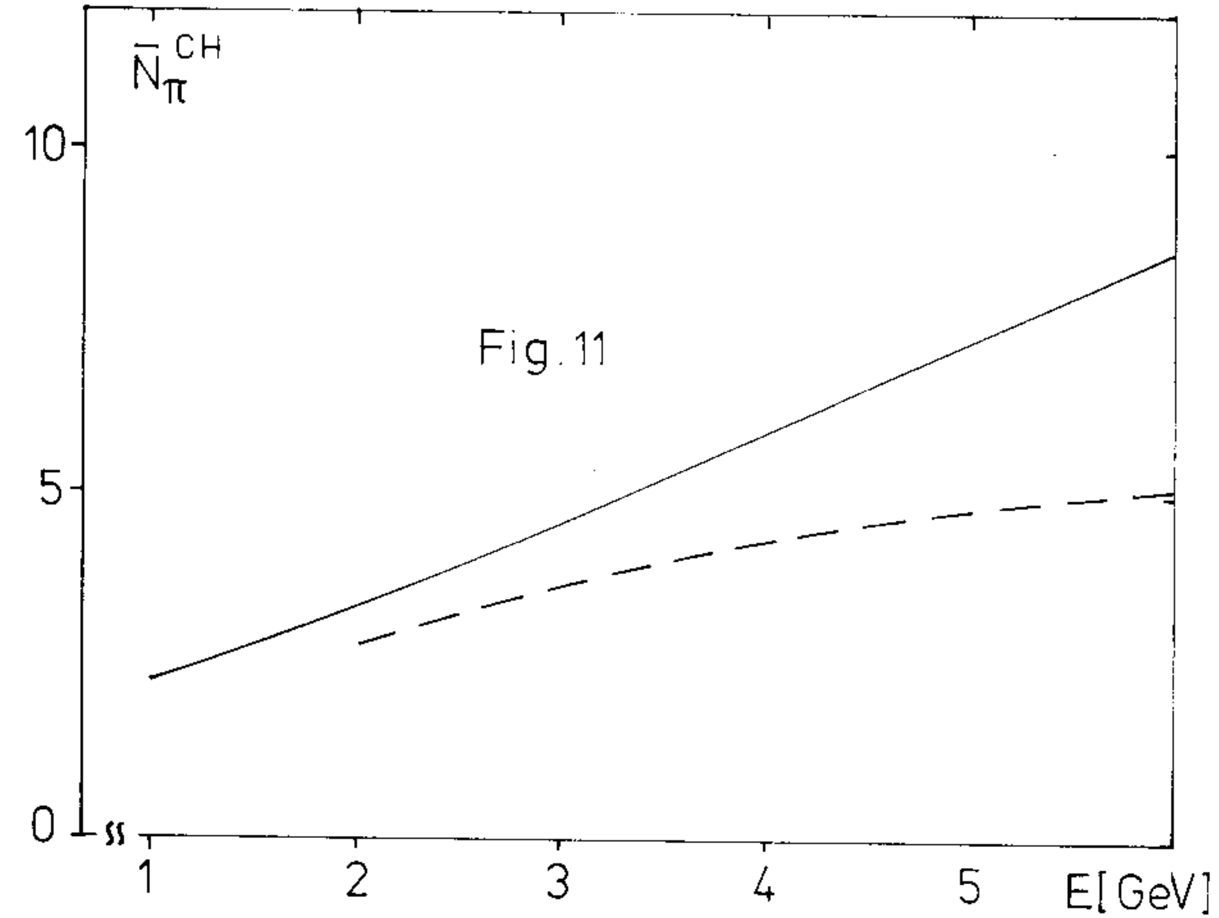


Fig. 11

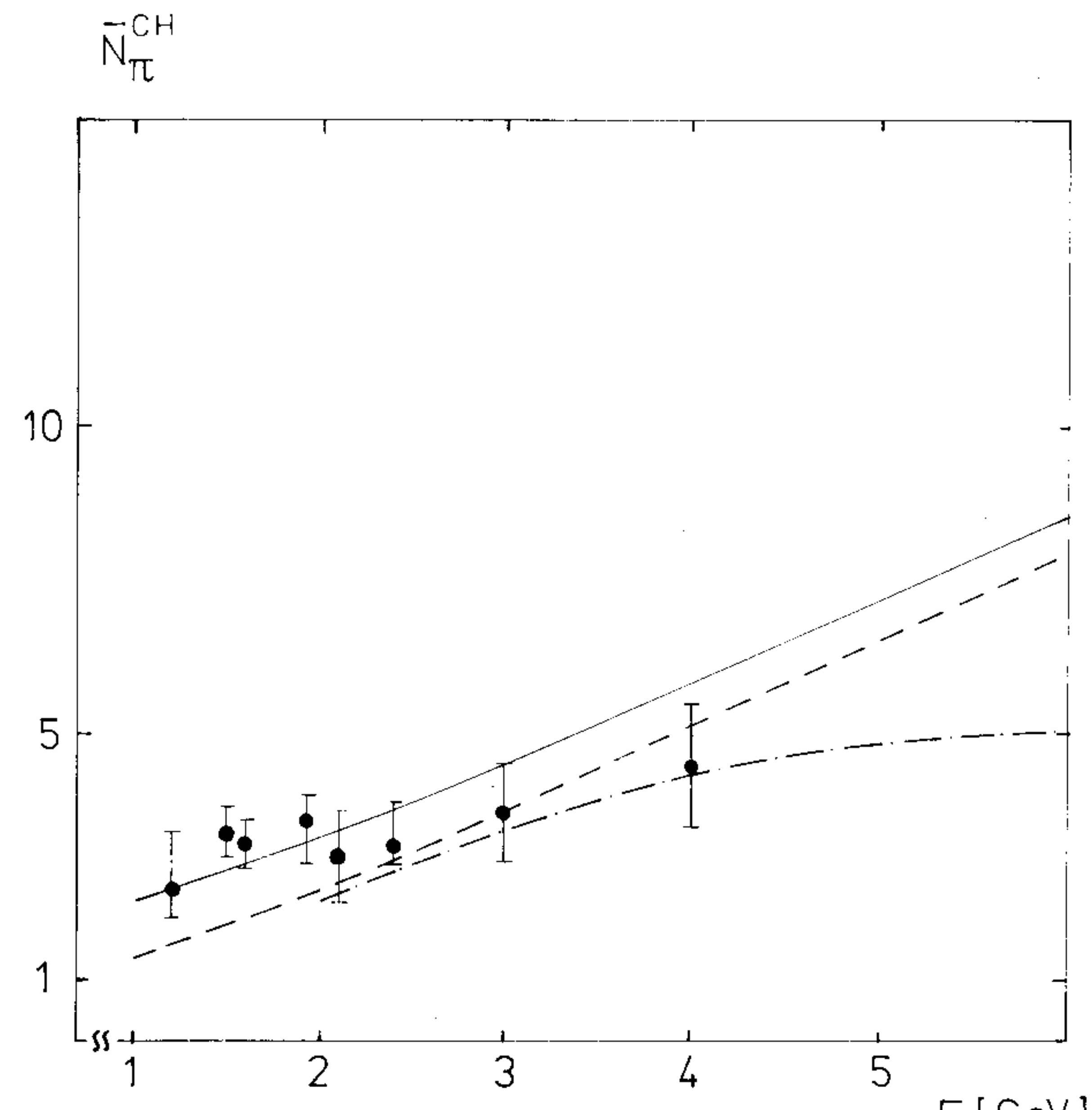


Fig. 12

Archives

Ref.TH.1674-CERN

STATISTICAL AND THERMODYNAMICAL DESCRIPTIONS OF HADRON PRODUCTION IN e^+e^- ANNIHILATION

J. Engels, H. Satz and K. Schilling

E R R A T U M

Unfortunately Figs. 5 and 6 contain a wrong abscissa scale. Actually we have plotted the natural and not the decadic logarithm of $q_o d^2\sigma/d^3q$.

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