Open Mathematics

Research Article

Qing-Bo Cai*, Guorong Zhou, and Junjie Li*

Statistical approximation properties of λ-Bernstein operators based on q-integers

https://doi.org/10.1515/math-2019-0039
Received December 31, 2018; accepted March 12, 2019

Abstract: In this paper, we introduce a new generalization of λ-Bernstein operators based on q-integers, we obtain the moments and central moments of these operators, establish a statistical approximation theorem and give an example to show the convergence of these operators to $f(x)$. It can be seen that in some cases the absolute error bounds are smaller than the case of classical q-Bernstein operators to $f(x)$.

Keywords: q-integers, λ-Bernstein operators, Statistical convergence, basis function

MSC: 41A10; 41A25; 41A36

1 Introduction

A generalization of Bernstein polynomials based on q-integers was proposed by Lupaş in 1987 in [1]. However, the Lupaş q-Bernstein operators are rational functions rather than polynomials. The Phillips q-Bernstein polynomials were introduced by Phillips in 1997 in [2]. After that, there are many papers mentioning the approximation properties of positive linear operators in the area of q-calculus due to its applications in the field of approximation theory, such as generalized q-Bernstein polynomials [3–6], Durrmeyer type q-Bernstein operators [7–9], Kantorovich type q-Bernstein operators [10–12] and so on.

As we know, Phillips [2] defined the following q-Bernstein operators:

$$B_n(f;x) = \sum_{i=0}^{n} b_{n,i}(x;q) f\left(\frac{[i]_q}{[n]_q}\right), \tag{1}$$

where $b_{n,i}(x;q)$ $(i = 0, 1, \ldots, n)$ are q-Bernstein basis functions and defined as

$$b_{n,i}(x;q) = \begin{bmatrix} n \\ i \end{bmatrix}_q x^i \prod_{s=0}^{n-i-1} \left(1 - q^s x\right). \tag{2}$$

Motivated by above work, in this paper, we introduce λ-Bernstein operators based on q-integers as

$$\widetilde{B}_{n,q,\lambda}(f;x) = \sum_{i=0}^{n} \widetilde{b}_{n,i}(x;q,\lambda) f\left(\frac{[i]_q}{[n]_q}\right), \tag{3}$$

***Corresponding Author: Qing-Bo Cai:** School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China, E-mail: qbcai@126.com
Guorong Zhou: School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China, E-mail: goonchow@email.edu
***Corresponding Author: Junjie Li:** School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China, E-mail: lijunjie@xmut.edu.cn

Open Access. © 2019 Qing-Bo Cai et al., published by De Gruyter. This work is licensed under the Creative Commons Attribution alone 4.0 License.

where

$$\begin{cases} \widetilde{b}_{n,0}(x;q,\lambda) = b_{n,0}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,1}(x;q), \\ \widetilde{b}_{n,i}(x;q,\lambda) = b_{n,i}(x;q) + \lambda \left(\frac{[n]_q - 2[i]_q + 1}{[n]_q^2 - 1} b_{n+1,i}(x;q) - \frac{[n]_q - 2q[i]_q - 1}{[n]_q^2 - 1} b_{n+1,i+1}(x;q) \right), \\ \qquad (i = 1, 2, \ldots, n-1) \\ \widetilde{b}_{n,n}(x;q,\lambda) = b_{n,n}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,n}(x;q), \end{cases} \tag{4}$$

$b_{n,i}(x;q)$ are defined in (2), $\lambda \in [-1, 1]$, $n \geq 2$, $x \in [0, 1]$ and $0 < q \leq 1$.

Obviously, when $\lambda = 0$, $\widetilde{B}_{n,q,0}(f;x)$ turn out to be q-Bernstein operators defined in (1); when $\lambda = 0$, $q = 1$, $\widetilde{B}_{n,1,0}(f;x)$ turn out to be classical Bernstein operators; when $q = 1$, $\widetilde{B}_{n,1,\lambda}(f;x)$ turn out to be the λ-Bernstein operators which are studied by Cai, et al. in [13].

Details of q-integers can be found in [14, 15]. For any fixed real number $0 < q \leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where $[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$ Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \ldots [1]_q, & k = 1, 2, \ldots, \\ 1, & k = 0, \end{cases},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (n \geq k \geq 0).$$

Now we recall the concepts of statistical convergence.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),$$

provided the limit exists [16], where χ_K is the characteristic function of K.

A sequence $x := \{x_k\}$ is called statistically convergent to a number L, if for every $\epsilon > 0$, $\delta(k \in \mathbb{N} : |x_k - L| \geq \epsilon) = 0$ [17]. This convergence is denoted as $st - \lim_{n\to\infty} x_n = L$.

We also recall the following theorem given by Gadjiev and Orhan [18].

Theorem 1.1. *(See [18]) If the sequence of positive linear operators $A_n : C_M[a, b] \to C[a, b]$ satisfies the conditions $st - \lim_{n\to\infty} ||A_n(e_v; \cdot) - e_v||_{C[a,b]} = 0$, with $e_v(t) = t^v$ for $v = 0, 1, 2$, then for any function $f \in C_M[a, b]$, we have*

$$st - \lim_{n\to\infty} ||A_n(f; \cdot) - f||_{C[a,b]} = 0,$$

where $C_M[a, b]$ denotes the space of all functions f which are continuous in $[a, b]$ and bounded on the all positive axis.

The aims of this paper are to introduce a new generalization of λ-Bernstein operators based on q-integers and give the statistical approximation properties of these operators using the concept of statistical convergence.

The rest of this paper is organized as follows. In section 2, we give some lemmas which are needed to prove our main results. In section 3, we obtain the statistical approximation theorem and give an example to show that, in some cases, the absolute error bounds of these operators defined in (3) to $f(x)$ are smaller than the bounds of classical q-Bernstein operators to $f(x)$.

2 Some lemmas

In the sequel, let $e_v(t) = t^v$ for $v = 0, 1, 2$. In order to obtain the main results, we need the following lemmas:

Lemma 2.1. *Let $\lambda \in [-1,1]$, $x \in [0,1]$ and $q \in (0,1]$, for the operators $\widetilde{B}_{n,q,\lambda}(f;x)$, we have*

$$\widetilde{B}_{n,q,\lambda}(e_0;x) = 1. \tag{5}$$

Proof. Using $[n]_q = 1 + q[n-1]_q$, we have

$$\begin{aligned}\sum_{i=0}^{n} \widetilde{b}_{n,i}(x;q,\lambda) &= \sum_{i=0}^{n} b_{n,i}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,1}(x;q) + \lambda\frac{[n]_q - 2[1]_q + 1}{[n]_q^2 - 1} b_{n+1,1}(x;q)\\ &\quad -\lambda\frac{[n]_q - 2q[1]_q - 1}{[n]_q^2-1} b_{n+1,2}(x;q) + \lambda\frac{[n]_q - 2[2]_q + 1}{[n]_q^2-1} b_{n+1,2}(x;q)\\ &\quad -\lambda\frac{[n]_q - 2q[2]_q - 1}{[n]_q^2-1} b_{n+1,3}(x;q) + \ldots + \lambda\frac{[n]_q - 2[n-1]_q + 1}{[n]_q^2-1} b_{n+1,n-1}(x;q)\\ &\quad -\lambda\frac{[n]_q - 2q[n-1]_q - 1}{[n]_q^2-1} b_{n+1,n}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,n}(x;q)\\ &= \sum_{i=0}^{n} b_{n,i}(x;q) + \frac{2q[n-1]_q + 1 - [n]_q}{[n]_q^2-1}\lambda b_{n+1,n}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,n}(x;q)\\ &= \sum_{i=0}^{n} b_{n,i}(x;q) = 1.\end{aligned}$$

Then we can obtain (5) since $\widetilde{B}_{n,q,\lambda}(e_0;x) = \sum_{k=0}^{n} \widetilde{b}_{n,k}(x;q,\lambda)$. □

Lemma 2.2. *Let $\lambda \in [-1,1]$, $x \in [0,1]$ and $q \in (0,1]$, for the operators $\widetilde{B}_{n,q,\lambda}(f;x)$, we have*

$$\begin{aligned}\widetilde{B}_{n,q,\lambda}(e_1;x) &= x + \frac{[n+1]_q x\left(1-x^n\right)\lambda}{[n]_q\left([n]_q-1\right)} - \frac{2[n+1]_q x\lambda}{[n]_q^2-1}\left[\frac{1-x^n}{[n]_q} + qx\left(1-x^{n-1}\right)\right]\\ &\quad + \frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left[1 - \prod_{s=0}^{n}\left(1-q^s x\right) - x^{n+1} - [n+1]_q x\left(1-x^n\right)\right]\\ &\quad + \frac{\lambda}{[n]_q^2-1}\left\{2[n+1]_q x^2\left(1-x^{n-1}\right) - \frac{2[n+1]_q x}{q[n]_q}\left(1-x^n\right)\right.\\ &\quad \left. + \frac{2}{q[n]_q}\left[1 - \prod_{s=0}^{n}\left(1-q^s x\right) - x^{n+1}\right]\right\}.\end{aligned} \tag{6}$$

Proof. From (3), we have

$$\begin{aligned}\widetilde{B}_{n,q,\lambda}(e_1;x) &= \sum_{i=0}^{n} \widetilde{b}_{n,i}(x;q,\lambda)\frac{[i]_q}{[n]_q}\\ &= \sum_{i=1}^{n-1} \frac{[i]_q}{[n]_q}\widetilde{b}_{n,i}(x;q,\lambda) + \widetilde{b}_{n,n}(x;q,\lambda)\\ &= \sum_{i=1}^{n-1} \frac{[i]_q}{[n]_q}\left[b_{n,i}(x;q) + \lambda\left(\frac{[n]_q - 2[i]_q + 1}{[n]_q^2-1} b_{n+1,i}(x;q) - \frac{[n]_q - 2q[i]_q - 1}{[n]_q^2-1} b_{n+1,i+1}(x;q)\right)\right]\\ &\quad + b_{n,n}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,n}(x;q)\\ &= \sum_{i=0}^{n} \frac{[i]_q}{[n]_q} b_{n,i}(x;q) + \lambda\sum_{i=0}^{n} \frac{[i]_q}{[n]_q}\frac{[n]_q - 2[i]_q + 1}{[n]_q^2-1} b_{n+1,i}(x;q)\\ &\quad -\lambda\sum_{i=1}^{n-1} \frac{[i]_q}{[n]_q}\frac{[n]_q - 2q[i]_q - 1}{[n]_q^2-1} b_{n+1,i+1}(x;q).\end{aligned}$$

As we know, for the q-Bernstein operators [2], we have $\sum_{i=0}^{n}\frac{[i]_q}{[n]_q}b_{n,i}(x;q)=x$, then

$$\begin{aligned}\widetilde{B}_{n,q,\lambda}(e_1;x) &= x+\frac{[n+1]_q\lambda}{[n]_q\left([n]_q-1\right)}\sum_{i=1}^{n}\frac{[i]_q}{[n+1]_q}\frac{[n+1]_q!}{[i]_q![n+1-i]_q!}x^i\prod_{s=0}^{n-i}(1-q^sx)\\&-\frac{2[n+1]_q\lambda}{[n]_q^2-1}\sum_{i=1}^{n}\frac{[i]_q^2}{[n]_q[n+1]_q}\frac{[n+1]_q!}{[i]_q![n+1-i]_q!}x^i\prod_{s=0}^{n-i}(1-q^sx)\\&-\frac{[n+1]_q\lambda}{[n]_q\left([n]_q+1\right)}\sum_{i=1}^{n-1}\frac{[i]_q}{[n+1]_q}\frac{[n+1]_q!}{[i+1]_q![n-i]_q!}x^{i+1}\prod_{s=0}^{n-i-1}(1-q^sx)\\&+\frac{2q[n+1]_q\lambda}{[n]_q^2-1}\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q[n+1]_q}\frac{[n+1]_q!}{[i+1]_q![n-i]_q!}x^{i+1}\prod_{s=0}^{n-i-1}(1-q^sx)\\&:= x+\triangle_1+\triangle_2+\triangle_3+\triangle_4.\end{aligned}\tag{7}$$

Firstly, we have

$$\begin{aligned}\triangle_1 &= \frac{[n+1]_q\lambda}{[n]_q\left([n]_q-1\right)}\sum_{i=1}^{n}\frac{[i]_q}{[n+1]_q}\frac{[n+1]_q!}{[i]_q![n+1-i]_q!}x^i\prod_{s=0}^{n-i}(1-q^sx)\\&= \frac{[n+1]_q\lambda x}{[n]_q\left([n]_q-1\right)}\sum_{i=0}^{n-1}b_{n,i}(x;q)=\frac{[n+1]_q x\left(1-x^n\right)\lambda}{[n]_q\left([n]_q-1\right)}.\end{aligned}\tag{8}$$

Secondly, since $[i]_q^2=[i]_q+q[i]_q[i-1]_q$, we get

$$\begin{aligned}\triangle_2 &= -\frac{2[n+1]_q\lambda}{[n]_q^2-1}\sum_{i=1}^{n}\frac{[i]_q^2}{[n]_q[n+1]_q}\frac{[n+1]_q!}{[i]_q![n+1-i]_q!}x^i\prod_{s=0}^{n-i}(1-q^sx)\\&= \frac{-2[n+1]_q\lambda}{[n]_q\left([n]_q^2-1\right)}\sum_{i=1}^{n}\frac{[i]_q}{[n+1]_q}\frac{[n+1]_q!}{[i]_q![n+1-i]_q!}x^i\prod_{s=0}^{n-i}(1-q^sx)\\&+\frac{-2q[n+1]_q\lambda}{[n]_q^2-1}\sum_{i=1}^{n}\frac{[i]_q[i-1]_q}{[n+1]_q[n]_q}\frac{[n+1]_q!}{[i]_q![n+1-i]_q!}x^i\prod_{s=0}^{n-i}(1-q^sx)\\&= \frac{-2[n+1]_q\lambda x}{[n]_q\left([n]_q^2-1\right)}\sum_{i=0}^{n-1}b_{n,i}(x;q)+\frac{-2q[n+1]_qx^2\lambda}{[n]_q^2-1}\sum_{i=0}^{n-2}b_{n-1,i}(x;q)\\&= -\frac{2[n+1]_qx\lambda}{[n]_q^2-1}\left[\frac{1-x^n}{[n]_q}+qx\left(1-x^{n-1}\right)\right].\end{aligned}\tag{9}$$

Thirdly, since $[i]_q=[i+1]_q/q-1/q$, we have

$$\begin{aligned}\triangle_3 &= -\frac{[n+1]_q\lambda}{[n]_q\left([n]_q+1\right)}\sum_{i=1}^{n-1}\frac{[i]_q}{[n+1]_q}\frac{[n+1]_q!}{[i+1]_q![n-i]_q!}x^{i+1}\prod_{s=0}^{n-i-1}(1-q^sx)\\&= -\frac{[n+1]_q\lambda}{q[n]_q\left([n]_q+1\right)}\sum_{i=1}^{n-1}\frac{[i+1]_q}{[n+1]_q}\frac{[n+1]_q!}{[i+1]_q![n-i]_q!}x^{i+1}\prod_{s=0}^{n-i-1}(1-q^sx)\\&+\frac{\lambda}{q\left([n]_q+1\right)[n]_q}\sum_{i=1}^{n-1}b_{n+1,i+1}(x;q)\\&= -\frac{[n+1]_qx\lambda}{q[n]_q\left([n]_q+1\right)}\sum_{i=1}^{n-1}b_{n,i}(x;q)+\frac{\lambda}{q\left([n]_q+1\right)[n]_q}\sum_{i=1}^{n-1}b_{n+1,i+1}(x;q)\\&= -\frac{[n+1]_qx\lambda}{q[n]_q\left([n]_q+1\right)}\left[1-\prod_{s=0}^{n-1}(1-q^sx)-x^n\right]\end{aligned}$$

$$+\frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left[1-\prod_{s=0}^{n}\left(1-q^sx\right)-[n+1]x\prod_{s=0}^{n-1}\left(1-q^sx\right)-x^{n+1}\right]$$
$$=\frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left[1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}-[n+1]_qx\left(1-x^n\right)\right]. \tag{10}$$

Finally, since $[i]_q^2=[i]_q[i+1]_q/q-[i]_q/q$, we have

$$\begin{aligned}\triangle_4 &= \frac{2q[n+1]_q\lambda}{[n]_q^2-1}\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q[n+1]_q}\frac{[n+1]_q!}{[i+1]_q![n-i]_q!}x^{i+1}\prod_{s=0}^{n-i-1}\left(1-q^sx\right)\\
&= \frac{2[n+1]_q\lambda}{[n]_2-1}\sum_{i=0}^{n-2}b_{n-1,i}(x;q)-\frac{2[n+1]_qx\lambda}{q[n]_q\left([n]_q^2-1\right)}\sum_{i=1}^{n-1}b_{n,i}(x;q)\\
&\quad+\frac{2\lambda}{q[n]_q\left([n]_q^2-1\right)}\sum_{i=1}^{n-1}b_{n+1,i+1}(x;q)\\
&= \frac{\lambda}{[n]_q^2-1}\left\{2[n+1]_qx^2\left(1-x^{n-1}\right)-\frac{2[n+1]_qx}{q[n]_q}\left(1-x^n\right)\right.\\
&\quad\left.+\frac{2}{q[n]_q}\left[1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}\right]\right\}.\end{aligned} \tag{11}$$

Then, (6) can be obtained by (7)-(11). □

Lemma 2.3. *Let $\lambda\in[-1,1]$, $x\in[0,1]$, $n>1$ and $q\in(0,1]$, for the operators $\widetilde{B}_{n,q,\lambda}(f;x)$, we have*

$$\begin{aligned}&\widetilde{B}_{n,q,\lambda}(e_2;x)\\
&= x^2+\frac{x(1-x)}{[n]_q}+\frac{[n+1]_qx\lambda}{[n]_q\left([n]_q-1\right)}\left[qx\left(1-x^{n-1}\right)+\frac{1-x^n}{[n]_q}\right]-\frac{2[n+1]_q\lambda}{[n]_q\left([n]_q^2-1\right)}\\
&\times\left[\frac{x\left(1-x^n\right)}{[n]_q}+q(2+q)x^2\left(1-x^{n-1}\right)+q^3[n-1]_qx^3\left(1-x^{n-2}\right)\right]\\
&-\frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left\{[n+1]_qx^2\left(1-x^{n-1}\right)-\frac{[n+1]_qx\left(1-x^n\right)}{q[n]_q}\right.\\
&\left.+\frac{1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}}{q[n]_q}\right\}+\frac{2\lambda}{[n]_q\left([n]_q^2-1\right)}\\
&\times\left\{q[n-1]_q[n+1]_qx^3\left(1-x^{n-2}\right)-\frac{(1-q)[n+1]_qx^2\left(1-x^{n-1}\right)}{q}\right.\\
&\left.+\frac{[n+1]_qx\left(1-x^n\right)}{q^2[n]_q}-\frac{1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}}{q^2[n]_q}\right\}.\end{aligned} \tag{12}$$

Proof. From (3), we have

$$\begin{aligned}\widetilde{B}_{n,q,\lambda}(e_2;x) &= \sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2}\widetilde{b}_{n,i}(x;q,\lambda)\\
&= \sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2}b_{n,i}(x;q)+\lambda\sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2}\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}b_{n+1,i}(x;q)\\
&\quad-\lambda\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q^2}\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}b_{n+1,i+1}(x;q).\end{aligned}$$

For the q-Bernstein polynomials [2], we have $\sum_{i=0}^{n} \frac{[i]_q^2}{[n]_q^2} b_{n,i}(x;q) = x^2 + \frac{x(1-x)}{[n]_q}$, thus

$$\begin{aligned}
\widetilde{B}_{n,q,\lambda}(e_2;x) &= x^2 + \frac{x(1-x)}{[n]_q} + \frac{\lambda}{[n]_q - 1}\sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2} b_{n+1,i}(x;q) - \frac{2\lambda}{[n]_q^2 - 1}\sum_{i=0}^{n}\frac{[i]_q^3}{[n]_q^2} b_{n+1,i}(x;q) \\
&\quad - \frac{\lambda}{[n]_q + 1}\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q^2} b_{n+1,i+1}(x;q) + \frac{2q\lambda}{[n]_q^2 - 1}\sum_{i=1}^{n-1}\frac{[i]_q^3}{[n]_q^2} b_{n+1,i+1}(x;q) \\
&:= x^2 + \frac{x(1-x)}{[n]_q} + \triangle_5 + \triangle_6 + \triangle_7 + \triangle_8. \qquad (13)
\end{aligned}$$

We have

$$\begin{aligned}
\triangle_5 &= \frac{\lambda}{[n]_q - 1}\sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2} b_{n+1,i}(x;q) \\
&= \frac{[n+1]_q q x^2 \lambda}{[n]_q\left([n]_q - 1\right)}\sum_{i=0}^{n-2} b_{n-1,i}(x;q) + \frac{[n+1]_q x\lambda}{[n]_q^2\left([n]_q - 1\right)}\sum_{i=0}^{n-1} b_{n,i}(x;q) \\
&= \frac{[n+1]_q x\lambda}{[n]_q\left([n]_q - 1\right)}\left[qx\left(1 - x^{n-1}\right) + \frac{1 - x^n}{[n]_q}\right]. \qquad (14)
\end{aligned}$$

Since $[i]_q^3 = [i]_q + q(2+q)[i]_q[i-1]_q + q^3[i]_q[i-1]_q[i-2]_q$, we have

$$\begin{aligned}
\triangle_6 &= -\frac{2\lambda}{[n]_q^2 - 1}\sum_{i=0}^{n}\frac{[i]_q^3}{[n]_q^2} b_{n+1,i}(x;q) \\
&= -\frac{2[n+1]_q x\lambda}{[n]_q^2\left([n]_q^2 - 1\right)}\sum_{i=0}^{n-1} b_{n,i}(x;q) - \frac{2q(2+q)[n+1]_q x^2\lambda}{[n]_q\left([n]_q^2 - 1\right)}\sum_{i=0}^{n-2} b_{n-1,i}(x;q) \\
&\quad - \frac{2q^3[n-1]_q[n+1]_q x^3\lambda}{[n]_q\left([n]_q^2 - 1\right)}\sum_{i=0}^{n-3} b_{n-2,i}(x;q) \\
&= \frac{-2[n+1]_q\lambda}{[n]_q\left([n]_q^2 - 1\right)}\left[\frac{x\left(1 - x^n\right)}{[n]_q} + q(2+q)x^2\left(1 - x^{n-1}\right) + q^3[n-1]_q x^3\left(1 - x^{n-2}\right)\right]. \qquad (15)
\end{aligned}$$

Next, since $[i]_q^2 = [i+1]_q[i]_q/q - [i+1]_q/q^2 + 1/q^2$, we get

$$\begin{aligned}
\triangle_7 &= -\frac{\lambda}{[n]_q + 1}\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q^2} b_{n+1,i+1}(x;q) \\
&= -\frac{[n+1]_q x^2\lambda}{q[n]_q\left([n]_q + 1\right)}\sum_{i=0}^{n-2} b_{n-1,i}(x;q) + \frac{[n+1]_q x\lambda}{q^2[n]_q^2\left([n]_q + 1\right)}\sum_{i=1}^{n-1} b_{n,i}(x;q) \\
&\quad - \frac{\lambda}{q^2[n]_q^2\left([n]_q + 1\right)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x;q) \\
&= \frac{-\lambda}{q[n]_q\left([n]_q + 1\right)}\left\{[n+1]_q x^2\left(1 - x^{n-1}\right) - \frac{[n+1]_q x\left(1 - x^n\right)}{q[n]_q}\right. \\
&\quad \left. + \frac{1 - \prod_{s=0}^{n}\left(1 - q^s x\right) - x^{n+1}}{q[n]_q}\right\}. \qquad (16)
\end{aligned}$$

Finally, by some computations, we have

$$[i]_q^3 = [i+1]_q[i]_q[i-1]_q - \frac{1-q}{q^2}[i+1]_q[i]_q + \frac{[i+1]_q}{q^3} - \frac{1}{q^3}.$$

Then, we obtain

$$\triangle_8 = \frac{2q\lambda}{[n]_q^2 - 1}\sum_{i=1}^{n-1}\frac{[i]_q^3}{[n]_q^2} b_{n+1,i+1}(x;q)$$

$$= \frac{2q[n-1]_q[n+1]_q x^3\lambda}{[n]_q\left([n]_q^2-1\right)}\sum_{i=0}^{n-3} b_{n-2,i}(x;q) - \frac{2(1-q)[n+1]_q x^2\lambda}{q[n]_q\left([n]_q^2-1\right)}\sum_{i=0}^{n-2} b_{n-1,i}(x;q)$$

$$+\frac{2[n+1]_q x\lambda}{q^2[n]_q^2\left([n]_q^2-1\right)}\sum_{i=1}^{n-1} b_{n,i}(x;q) - \frac{2\lambda}{q^2[n]_q^2\left([n]_q^2-1\right)}\sum_{i=1}^{n-1} b_{n+1,i+1}(x;q)$$

$$= \frac{2\lambda}{[n]_q\left([n]_q^2-1\right)}\left\{q[n-1]_q[n+1]_q x^3\left(1-x^{n-2}\right) - \frac{(1-q)[n+1]_q x^2\left(1-x^{n-1}\right)}{q}\right.$$

$$\left.+\frac{[n+1]_q x\left(1-x^n\right)}{q^2[n]_q} - \frac{1-\prod_{s=0}^{n}\left(1-q^s x\right)-x^{n+1}}{q^2[n]_q}\right\}. \tag{17}$$

Using (13)-(17), we can obtain (12), Lemma 2.1 is proved. □

Remark 2.4. *When $\lambda = 0$, we get the moments of q-Bernstein operators (see [2]).*

Lemma 2.5. *For $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $q \in (0, 1]$, the operators $\widetilde{B}_{n,q,\lambda}(f; x)$ are positive linear operators.*

Proof. Indeed, we only need to prove $\widetilde{b}_{n,i}(t; q, \lambda) \geq 0$ for $0 \leq i \leq n$. For $i = 0$, we have

$$\begin{aligned}\widetilde{b}_{n,0}(x;q,\lambda) &= b_{n,0}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,1}(x;q)\\ &= \prod_{s=0}^{n-1}\left(1-q^s x\right) - \frac{\lambda}{[n]_q+1}[n+1]_q x\prod_{s=0}^{n-1}\left(1-q^s x\right)\\ &= \prod_{s=0}^{n-1}\left(1-q^s x\right)\left(1-\frac{[n+1]_q}{[n]_q+1}\lambda x\right) \geq 0,\end{aligned}$$

since $[n]_q + 1 \geq [n+1]_q$. Similarly, we can obtain $\widetilde{b}_{n,n}(x; q, \lambda) \geq 0$. Then we will prove $\widetilde{b}_{n,i}(x; q, \lambda) \geq 0$, where $1 \leq i \leq n-1$. Actually, by (4), we have

$$\begin{aligned}\widetilde{b}_{n,i}(x;q,\lambda) &= b_{n,i}(x;q) + \lambda\left[\frac{[n]_q-2[i]_q+1}{[n]_q^2-1} b_{n+1,i}(x;q) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1} b_{n+1,i+1}(x;q)\right]\\ &= b_{n,i}(x;q)\left\{1+\lambda\left[\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}\frac{[n+1]_q}{[n+1-i]_q}\left(1-q^{n-i}x\right) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}\frac{[n+1]_q}{[i+1]_q}x\right]\right\}.\end{aligned}$$

We need to prove

$$-1 \leq \frac{[n]_q-2[i]_q+1}{[n]_q^2-1}\frac{[n+1]_q}{[n+1-i]_q}\left(1-q^{n-i}x\right) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}\frac{[n+1]_q}{[i+1]_q}x \leq 1. \tag{18}$$

We discuss the above inequality in three cases as follows:

Case 1: For $1 \leq [i]_q \leq \min\left\{\frac{[n]_q-1}{2q}, \frac{[n]_q+1}{2}\right\}$, we have

$$0 \leq \frac{[n+1]_q}{[n+1-i]_q}\frac{[n]_q-2[i]_q+1}{[n]_q^2-1},\ \frac{[n+1]_q}{[i+1]_q}\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1} \leq 1.$$

Then we can obtain (18) by the fact that $0 \leq 1 - q^{n-i}x$, $0 \leq x \leq 1$.

Case 2: For $\max\left\{\frac{[n]_q+1}{2}, \frac{[n]_q-1}{2q}\right\} \leq [i]_q \leq [n-1]_q$, we have

$$-1 \leq \frac{[n+1]_q}{[n+1-i]_q}\frac{[n]_q-2[i]_q+1}{[n]_q^2-1},\ \frac{[n+1]_q}{[i+1]_q}\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1} \leq 0.$$

Then we can get (18) for $0 \leq 1 - q^{n-i}x$, $0 \leq x \leq 1$.

Case 3: For $\min\left\{\frac{[n]_q+1}{2}, \frac{[n]_q-1}{2q}\right\} < [i]_q < \max\left\{\frac{[n]_q+1}{2}, \frac{[n]_q-1}{2q}\right\}$.

If $\frac{[n]_q-1}{2q} < [i]_q < \frac{[n]_q+1}{2}$, we have

$$0 < \frac{[n]_q-2[i]_q+1}{[n]_q^2-1}\frac{[n+1]_q}{[n+1-i]_q},\ -\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}\frac{[n+1]_q}{[i+1]_q} \leq \frac{1}{2};$$

If $\frac{[n]_q+1}{2} < [i]_q < \frac{[n]_q-1}{2q}$, we have

$$-\frac{1}{2} \le \frac{[n]_q - 2[i]_q + 1}{[n]_q^2 - 1} \frac{[n+1]_q}{[n+1-i]_q}, \ -\frac{[n]_q - 2q[i]_q - 1}{[n]_q^2 - 1} \frac{[n+1]_q}{[i+1]_q} < 0.$$

Then we can get(18) in both two subcases. Therefore, we get the proof of Lemma 2.5. □

3 Main results

Let us give the following statistical approximation theorem for the operators (3).

Theorem 3.1. *Let $q = \{q_n\}$ be a sequence satisfying $st - \lim_{n\to\infty} q_n = 1$ for $0 < q_n < 1$. Then, for all $f \in C[0,1]$, $\lambda \in [-1,1]$, $n > 1$ and the operators $\widetilde{B}_{n,q_n,\lambda}(f;x)$, we have*

$$st - \lim_{n\to\infty} \left\| \widetilde{B}_{n,q_n,\lambda}(f;\cdot) - f \right\| = 0. \tag{19}$$

Proof. By Lemma 2.1, it is clear that

$$st - \lim_{n\to\infty} \left\| \widetilde{B}_{n,q_n,\lambda}(e_0;\cdot) - e_0 \right\|_{C[0,1]} = 0. \tag{20}$$

In order to obtain the desired result, using Lemma 2.5 and Theorem 1.1, we only need to prove the following equalities

$$st - \lim_{n\to\infty} \left\| \widetilde{B}_{n,q_n,\lambda}(e_v;\cdot) - e_v \right\|_{C[0,1]} = 0, \quad v = 1, 2. \tag{21}$$

For $v = 1$, using Lemma 2.2, we have

$$\begin{aligned}\left|\widetilde{B}_{n,q_n,\lambda}(e_1;x) - e_1(x)\right| &\le \frac{[n+1]_{q_n}}{[n]_{q_n}\left([n]_{q_n}-1\right)} + \frac{2[n+1]_{q_n}}{[n]_{q_n}^2-1}\left(\frac{1}{[n]_{q_n}}+1\right) + \frac{1}{q_n[n]_{q_n}\left([n]_{q_n}+1\right)} \\ &\quad + \frac{1+[n+1]_{q_n}}{q_n[n]_{q_n}\left([n]_{q_n}+1\right)} + \frac{2}{[n]_{q_n}^2-1}\left([n+1]_{q_n} + \frac{[n+1]_{q_n}}{q_n[n]_{q_n}} + \frac{1}{q_n[n]_{q_n}}\right) \\ &\le \frac{q_n[n+1]_{q_n} + 2q_n + 4}{q_n[n]_{q_n}\left([n]_{q_n}-1\right)} + \frac{5}{[n]_{q_n}-1} + \frac{2}{q_n[n]_{q_n}\left([n]_{q_n}^2-1\right)},\end{aligned}$$

since $[n+1]_{q_n} \le [n]_{q_n} + 1$, we obtain

$$\left|\widetilde{B}_{n,q_n,\lambda}(e_1;x) - e_1(x)\right| \le \frac{6}{[n]_{q_n}-1} + \frac{3q_n+4}{q_n[n]_{q_n}\left([n]_{q_n}-1\right)} + \frac{2}{q_n[n]_{q_n}\left([n]_{q_n}^2-1\right)}.$$

For a given $\epsilon > 0$, let us define the following sets

$$U = \left\{k : \left\|\widetilde{B}_{k,q_k,\lambda}(e_1;\cdot) - e_1\right\|_{C[0,1]} \ge \epsilon\right\};\ U_1 = \left\{k : \frac{6}{[k]_{q_k}-1} \ge \frac{\epsilon}{3}\right\};$$

$$U_2 = \left\{k : \frac{3q_k+4}{q_k[k]_{q_k}\left([k]_{q_k}-1\right)} \ge \frac{\epsilon}{3}\right\};\ U_3 = \left\{k : \frac{2}{q_k[k]_{q_k}\left([k]_{q_k}^2-1\right)} \ge \frac{\epsilon}{3}\right\}.$$

It is obvious that $U \subset U_1 \cup U_2 \cup U_3$, which implies that

$$\begin{aligned}&\delta\left\{k \le n : \left\|\widetilde{B}_{k,q_k,\lambda}(e_1;\cdot) - e_1\right\|_{C[0,1]} \ge \epsilon\right\} \\ &\le \delta\left\{k \le n : \frac{6}{[k]_{q_k}-1} \ge \frac{\epsilon}{3}\right\} + \delta\left\{k \le n : \frac{3q_k+4}{q_k[k]_q\left([k]_{q_k}-1\right)} \ge \frac{\epsilon}{3}\right\} \\ &\quad +\delta\left\{k \le n : \frac{2}{q_k[k]_{q_k}\left([k]_{q_k}^2-1\right)} \ge \frac{\epsilon}{3}\right\}.\end{aligned} \tag{22}$$

Since $st-\lim_{n\to\infty} q_n = 1$, we have

$$st-\lim_{n\to\infty}\frac{6}{[n]_{q_n}-1}=0;\ st-\lim_{n\to\infty}\frac{3q_n+4}{q_n[n]_{q_n}\left([n]_{q_n}-1\right)}=0;$$
$$st-\lim_{n\to\infty}\frac{2}{q_n[n]_{q_n}\left([n]_q^2-1\right)}=0,$$

that is to say, the right hand side of (22) is zero. Hence, (21) is proved for $v=1$.

For $v=2$, by Lemma 2.3, we have

$$\begin{aligned}
&\left|\widetilde{B}_{n,q_n,\lambda}(e_2;x)-e_2(x)\right|\\
\le&\frac{1}{4[n]_{q_n}}+\frac{[n+1]_{q_n}}{[n]_{q_n}\left([n]_{q_n}-1\right)}\left(1+\frac{1}{[n]_{q_n}}\right)+\frac{2[n+1]_{q_n}}{[n]_{q_n}\left([n]_{q_n}^2-1\right)}\left(\frac{1}{[n]_{q_n}}+3+[n-1]_{q_n}\right)\\
&+\frac{1}{q_n[n]_{q_n}\left([n]_{q_n}+1\right)}\left([n+1]_{q_n}+\frac{[n+1]_{q_n}}{q_n[n]_{q_n}}+\frac{1}{q_n[n]_{q_n}}\right)+\frac{2}{[n]_{q_n}\left([n]_{q_n}^2-1\right)}\\
&\left([n-1]_{q_n}[n+1]_{q_n}+\frac{[n+1]_{q_n}}{q_n}+\frac{[n+1]_{q_n}}{q_n^2[n]_{q_n}}+\frac{1}{q_n^2[n]_{q_n}}\right)\\
\le&\frac{1}{4[n]_{q_n}}+\frac{5}{[n]_{q_n}-1}+\frac{1}{q_n[n]_{q_n}}+\frac{2}{[n]_{q_n}\left([n]_{q_n}-1\right)}+\frac{8}{[n]_{q_n}^2-1}+\frac{1}{q_n^2[n]_{q_n}^2}\\
&+\frac{2}{q_n[n]_{q_n}\left([n]_{q_n}-1\right)}+\frac{1}{[n]_{q_n}^2\left([n]_{q_n}-1\right)}+\frac{8}{[n]_{q_n}\left([n]_{q_n}^2-1\right)}+\frac{1}{q_n^2[n]_{q_n}^2\left([n]_{q_n}+1\right)}\\
&+\frac{2}{q_n^2[n]_{q_n}^2\left([n]_{q_n}-1\right)}+\frac{2}{[n]_{q_n}^2\left([n]_{q_n}^2-1\right)}+\frac{2}{q_n^2[n]_{q_n}^2\left([n]_{q_n}^2-1\right)}.
\end{aligned}$$

For a given $\epsilon>0$, we define the following sets

$$\begin{aligned}
V&=\left\{k:\left\|\widetilde{B}_{k,q_k,\lambda}(e_2;\cdot)-e_2\right\|_{C[0,1]}\ge\epsilon\right\};\\
V_1&=\left\{k:\frac{1}{4[k]_{q_k}}+\frac{5}{[k]_{q_k}-1}+\frac{1}{q_k[k]_{q_k}}\ge\frac{\epsilon}{4}\right\};\\
V_2&=\left\{k:\frac{2}{[k]_{q_k}\left([k]_{q_k}-1\right)}+\frac{8}{[k]_{q_k}^2-1}+\frac{1}{q_k^2[k]_{q_k}^2}+\frac{2}{q_k[k]_{q_k}\left([k]_{q_k}-1\right)}\ge\frac{\epsilon}{4}\right\};\\
V_3&=\left\{k:\frac{1}{[k]_{q_k}^2\left([k]_{q_k}-1\right)}+\frac{8}{[k]_{q_k}\left([k]_{q_k}^2-1\right)}+\frac{1}{q_k^2[k]_{q_k}^2\left([k]_{q_k}+1\right)}+\frac{2}{q_k^2[k]_{q_k}^2\left([k]_{q_k}-1\right)}\ge\frac{\epsilon}{4}\right\};\\
V_4&=\left\{k:\frac{2}{[k]_{q_k}^2\left([k]_{q_k}^2-1\right)}+\frac{2}{q_k^2[k]_{q_k}^2\left([k]_{q_k}^2-1\right)}\ge\frac{\epsilon}{4}\right\}.
\end{aligned}$$

Obviously, $V\subset V_1\cup V_2\cup V_3\cup V_4$, then one have

$$\begin{aligned}
&\delta\left\{k\le n:\left\|\widetilde{B}_{k,q_k,\lambda}(e_2;\cdot)-e_2\right\|_{C[0,1]}\ge\epsilon\right\}\\
\le&\ \delta\left\{k\le n:\frac{1}{4[k]_{q_k}}+\frac{5}{[k]_{q_k}-1}+\frac{1}{q_k[k]_{q_k}}\ge\frac{\epsilon}{4}\right\}\\
&+\delta\left\{k\le n:\frac{2}{[k]_{q_k}\left([k]_{q_k}-1\right)}+\frac{8}{[k]_{q_k}^2-1}+\frac{1}{q_k^2[k]_{q_k}^2}+\frac{2}{q_k[k]_{q_k}\left([k]_{q_k}-1\right)}\ge\frac{\epsilon}{4}\right\}\\
&+\delta\left\{k\le n:\frac{1}{[k]_{q_k}^2\left([k]_{q_k}-1\right)}+\frac{8}{[k]_{q_k}\left([k]_{q_k}^2-1\right)}+\frac{1}{q_k^2[k]_{q_k}^2\left([k]_{q_k}+1\right)}+\frac{2}{q_k^2[k]_{q_k}^2\left([k]_{q_k}-1\right)}\ge\frac{\epsilon}{4}\right\}\\
&+\delta\left\{k\le n:\frac{2}{[k]_{q_k}^2\left([k]_{q_k}^2-1\right)}+\frac{2}{q_k^2[k]_{q_k}^2\left([k]_{q_k}^2-1\right)}\ge\frac{\epsilon}{4}\right\}.
\end{aligned}\tag{23}$$

Since $st-\lim_{n\to\infty} q_n = 1$, we obtain the following equalities

$$st-\lim_{n\to\infty}\left(\frac{1}{4[n]_{q_n}}+\frac{5}{[n]_{q_n}-1}+\frac{1}{q_n[n]_{q_n}}\right)=0;$$

$$st-\lim_{n\to\infty}\left[\frac{2}{[n]_{q_n}\left([n]_{q_n}-1\right)}+\frac{8}{[n]_{q_n}^2-1}+\frac{1}{q_n^2[n]_{q_n}^2}+\frac{2}{q_n[n]_{q_n}\left([n]_{q_n}-1\right)}\right]=0;$$

$$st-\lim_{n\to\infty}\left[\frac{1}{[n]_{q_n}^2\left([n]_{q_n}-1\right)}+\frac{8}{[n]_{q_n}\left([n]_{q_n}^2-1\right)}+\frac{1}{q_n^2[n]_{q_n}^2\left([n]_{q_n}+1\right)}+\frac{2}{q_n^2[n]_{q_n}^2\left([n]_{q_n}-1\right)}\right]=0;$$

$$st-\lim_{n\to\infty}\left[\frac{2}{[n]_{q_n}^2\left([n]_{q_n}^2-1\right)}+\frac{2}{q_n^2[n]_{q_n}^2\left([n]_{q_n}^2-1\right)}\right]=0,$$

that is, the right hand side of (23) is zero. Therefore, (21) is proved for $v = 2$. Theorem 3.1 is proved. □

Finally, we give an example to show the convergence of $\widetilde{B}_{n,q,\lambda}(f;x)$ to $f(x)$ with different values of parameters, we also obtain the absolute error bounds in some cases.

Example 3.2. *Let $f(x) = 1 - cos(4e^x)$, the graphs of $\widetilde{B}_{n,q,-1}(f;x)$ and $\widetilde{B}_{n,q,1}(f;x)$ with different cases of n and q are shown in Figure 1. Figure 2 shows the graphs of $\widetilde{B}_{n,q,1}(f;x)$ and $\widetilde{B}_{n,q,0}(f;x)$ with $n = 10$ and $q = 0.9$. In Table 1, we give the absolute error bounds of $\widetilde{B}_{n,q,\lambda}(f;x)$ to $f(x)$.*

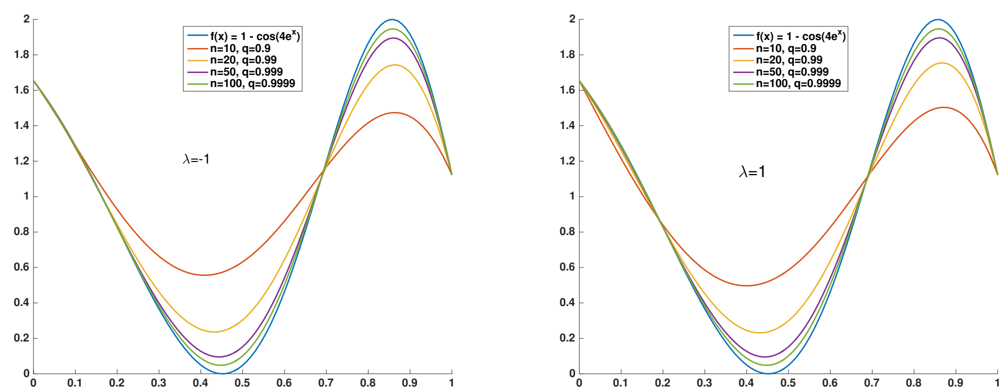

Figure 1: Convergence of $\widetilde{B}_{n,q,\lambda}(f;x)$ for $\lambda = -1$, $\lambda = 1$ and different values of q and n.

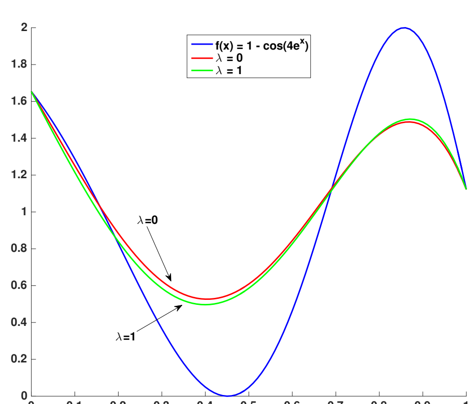

Figure 2: Convergence of $\widetilde{B}_{10,0.9,\lambda}(f;x)$ for $\lambda = 1$ and $\lambda = 0$.

Table 1: The errors of the approximation of $\widetilde{B}_{n,q,\lambda}(f;x)$ to $f(x)$.

		$\|f-\widetilde{B}_{n,q,\lambda}(f)\|_\infty$			
		$n=10$	$n=20$	$n=50$	$n=100$
$\lambda=-1$	$q=0.9$	0.586703	0.469244	0.425725	0.423995
	$q=0.99$	0.447241	0.259814	0.128908	0.082675
	$q=0.999$	0.438761	0.244595	0.107170	0.056663
	$q=0.9999$	0.437909	0.243088	0.105111	0.054359
$\lambda=0$	$q=0.9$	0.562861	0.459109	0.418998	0.417386
	$q=0.99$	0.432402	0.256685	0.128892	0.082737
	$q=0.999$	0.423813	0.241516	0.107168	0.056722
	$q=0.9999$	0.422953	0.240017	0.105112	0.054418
$\lambda=1$	$q=0.9$	0.539161	0.449016	0.412299	0.410807
	$q=0.99$	0.418391	0.253900	0.128901	0.082802
	$q=0.999$	0.409770	0.238817	0.107201	0.056785
	$q=0.9999$	0.408905	0.237324	0.105145	0.054481

One can see from Table 1 that in some cases, such as $n = 10, 20$ and $\lambda = 1$, the absolute error bounds of $||f-\widetilde{B}_{n,q,\lambda}(f)||_\infty$ are smaller than $||f-\widetilde{B}_{n,q,0}(f)||_\infty$, where $\widetilde{B}_{n,q,0}(f;x)$ are the classical q-Bernstein operators. So it is meaningful to consider the operators in the form (3).

Data availability statement
The data used to support the findings of this study are included within the article.

Acknowledgement: This work is supported by the National Natural Science Foundation of China (Grant No. 11601266), the Natural Science Foundation of Fujian Province of China (Grant Nos. 2016J05017, 2016J01040), the Project for High-level Talent Innovation and Entrepreneurship of Quanzhou (Grant No. 2018C087R), the Program for New Century Excellent Talents in Fujian Province University, Fujian Provincial Scholarship for Overseas Study and the High-level Personnel of Special Support Program of Xiamen University of Technology (YKJ15030R). We also thank Fujian Provincial Key Laboratory of Data-Intensive Computing, Fujian University Laboratory of Intelligent Computing and Information Processing and Fujian Provincial Big Data Research Institute of Intelligent Manufacturing of China.

References

[1] Lupaş A., A q-analogue of the Bernstein operator, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, 1987, 9, 85-92
[2] Phillips G. M., Bernstein polynomials based on the q-integers, Annals of Numerical Mathematics, 1997, 4, 511-518
[3] Nowak G., Approximation properties for generalized q-Bernstein polynomials, Journal of Mathematical Analysis and Applications, 2009, 350, 50-55
[4] Il'inskii A., Convergence of generalized Bernstein polynomial, Journal of Approximation Theory, 2002, 116, 100-112
[5] Wang H., Meng F., The rate of convergence of q-Bernstein polynomials, Journal of Approximation Theory 2005, 136, 151-158
[6] Mursaleen M., Khan A., Generalized q-Bernstein-Schurer operators and some approximation theorems, Journal of Function Spaces and Applications, Volume 2013, Article ID 719834, 7 pages, http://dx. doi. org/10. 1155/2013/719834
[7] Gupta V., Finta Z., On certain q-Durrmeyer type operators, Applied Mathematics and Computaion, 2009, 209, 415-420
[8] Gupta V., Some approximation properties of q-Durrmeyer operators, Applied Mathematics and Computation, 2008, 197, 172-178
[9] Gupta V., Wang H., The rate of convergence of q-Durrmeyer operators for $0 < q < 1$, Mathematical Methods in the Applied Sciences, 2008, 31, 1946-1955

[10] Mursaleen M., Khan F., Khan A., Approximation properties for King's type modified q-Bernstein-Kantorovich operators, Mathematical Methods in the Applied Sciences, 2015, 36(9), 1178-1197
[11] Agrawal P. N., Finta Z., Kumar A. S., Bernstein-Schurer-Kantorovich operators based on q-integers, Applied Mathematics and Computation, 2015, 256, 222-231
[12] Dalmanoğlu Ö., Doğru O., On statistical approximation properties of Kantorovich type q-Bernstein operators, Mathematical and Computer Modelling, 2010, 52, 760-771
[13] Cai Q. -B., Lian B. -Y., Zhou G., Approximation properties of λ-Bernstein operators, Journal of Inequalities and Applications, 2018, DOI: 10.1186/s13660-018-1653-7
[14] Gasper G., Rahman M., Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications. Cambridge University press, Cambridge, UK., 35, 1990
[15] Kac V. G., Cheung P., Quantum Calculus, Universitext, Springer-Verlag, New York, 2002
[16] Niven I., Zuckerman H. S., Montgomery H., An Introduction to the Theory of Numbers, 5th edition, Wiley, New York, 1991
[17] Fast H., Sur la convergence statistique, Colloq. Math., 1951, 2, 241-244
[18] Gadjiev A. D., Orhan C., Some approximation theorem via statistical convergence, Rocky Mountain J. Math., 2002, 32(1), 129-138
[19] Goldman R. N., Pyramid Algorithms: A dynamic programming approach to curves and surfaces for geometric modeling, Elsiver Inc. 2003