Statistical clustering of temporal networks through a dynamic stochastic block model

Catherine Matias and Vincent Miele

CNRS - Université Pierre et Marie Curie, Paris
catherine.matias@math.cnrs.fr
http://cmatias.perso.math.cnrs.fr/

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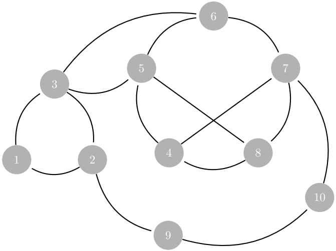
Outline

Clustering dynamic networks I

$t = t_1$



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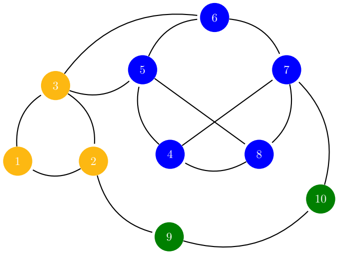

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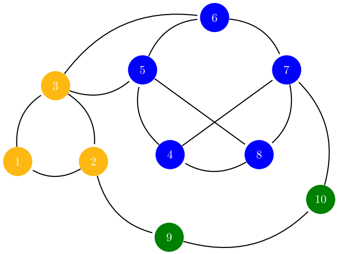
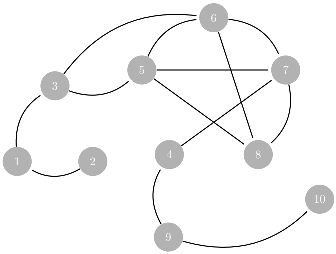


$t = t_2$



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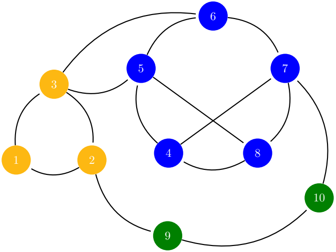

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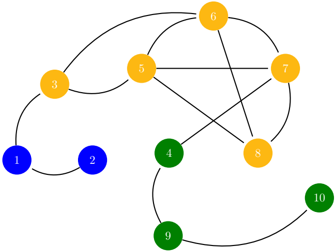
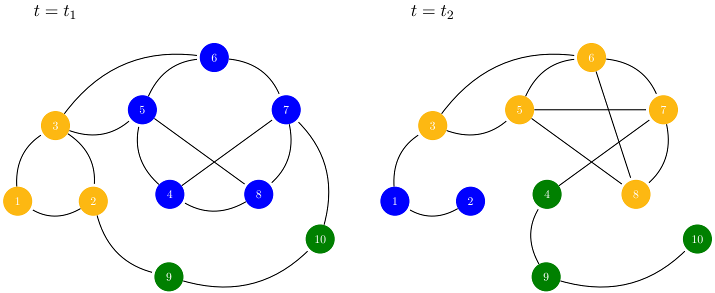


Clustering dynamic networks I



Issues

- Deal with the label switching across time.
- See the evolution of individual nodes: who is changing group between 2 time points?

Our goal: smooth recovery of the clusters across time.

Clustering dynamic networks II

Discrete time networks

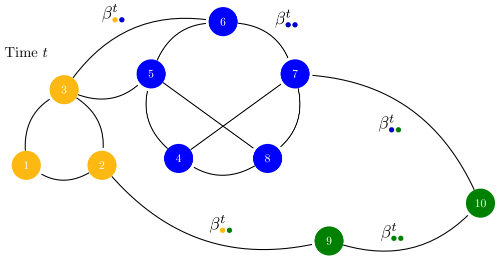
- We observe a sequence $Y^1, \ldots, Y^T$ of adjacency matrices,
- $\forall t, Y^t = (Y^t_{ij})_{1 \leq i,j \leq N}$ may contain either binary, discrete or continuous values.

Nodes clustering

- Clusters model heterogeneity in nodes interactions,
- They summarize information through a finite number of behaviors.
- Many different approaches: spectral algorithms, community detection (e.g. based on modularity criterion), model-based clustering (e.g. latent space models, SBM)

Here, we choose to focus on the Stochastic block model (SBM) for undirected graphs, with no self-loops.

Static part modeling: SBM - binary case

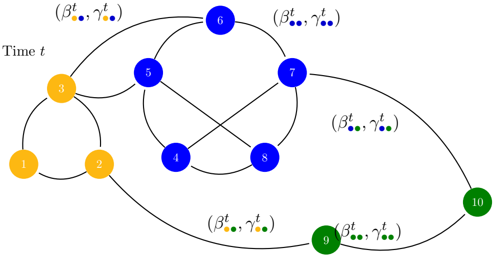


$n = 10, Q = 3,$
$Z_5^t = \bullet,$
$Y_{12}^t = 1, Y_{15}^t = 0$

Binary case; parameter $\boldsymbol{\beta}^t = (\beta_{ql}^t)_{1 \le q \le l \le Q}$

- Q groups (=colors •••).
- $\{Z_i^t\}_{1 \le i \le n}$ i.i.d. in $\{1, \dots, Q\}$ not observed.
- Observations: presence/absence of an edge at time t, given through adjacency matrix $\{Y_{ij}^t\}_{1 \le i < j \le n}$,
- Conditional on $\{Z_i^t\}$'s, the r.v. Y_{ij}^t are independent $\mathcal{B}(\beta^t_{Z_i^t Z_j^t})$.

Static part modeling: SBM - weighted case



$n = 10, Q = 3,$
$Z^t_5 = \bullet,$
$Y^t_{12} \in \mathbb{R}^s, Y^t_{15} = 0$

Weighted case; parameter $(\boldsymbol{\beta}^t, \boldsymbol{\gamma}^t) = (\beta^t_{ql}, \gamma^t_{ql})_{1\leq q \leq l \leq Q}$

- Latent variables: *idem*
- Observations: weights Y^t_{ij} , where $Y^t_{ij} = 0$ or $Y^t_{ij} \in \mathbb{R}^s \setminus \{0\}$,
- Conditional on the $\{Z^t_i\}$'s, the random variables Y^t_{ij} are independent with density

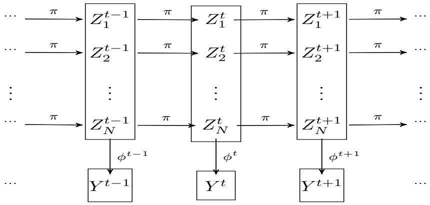
$$\phi(\cdot; \beta^t_{Z^t_i Z^t_j}, \gamma^t_{Z^t_i Z^t_j}) := (1 - \beta^t_{Z^t_i Z^t_j})\delta_0(\cdot) + \beta^t_{Z^t_i Z^t_j} f(\cdot, \gamma^t_{Z^t_i Z^t_j}),$$

(Assumption: f has continuous cdf at zero).

Dynamics: Markov chain on latent groups

Latent Markov chain

- Across individuals: $(Z_i)_{1\leq i\leq N}$ iid,
- Across time: Each $Z_i = (Z_i^t)_{1\leq t\leq T}$ is a stationary Markov chain on $\{1,\ldots,Q\}$ with transition $\boldsymbol{\pi} = (\pi_{qq'})_{1\leq q,q'\leq Q}$ and initial stationary distribution $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_Q)$.



Goal

Infer the parameter $\theta = (\boldsymbol{\pi},\boldsymbol{\beta},\boldsymbol{\gamma})$, recover the clusters $\{Z_i^t\}_{i,t}$ and follow their evolution through time.

Other very close works

[Yang *et al., 2011*] and [Xu and Hero, 2014] propose very close models (in the binary setup).

Main differences with our work

- We allow for both groups and parameters to vary with time and discuss valid assumptions for parameters' identifiability;
- We model binary as well as weighted graphs;
- We propose a model selection criterion for the number of clusters;
- We discuss a proper clustering index for measuring the classification performances taking into account label switching across time.

Identifiability

If both $(\beta^t, \gamma^t)_t$ and $(Z^t)_t$ can change, the parameters are not identifiable.

Main Assumption: Fixed diagonal connectivity parameters

$\forall q \in \mathcal{Q}, \forall t, t'$, we assume that

$$\begin{cases} \text{Binary case:} & \beta^t_{qq} = \beta^{t'}_{qq}, \\ \text{Weighted case:} & \gamma^t_{qq} = \gamma^{t'}_{qq}. \end{cases}$$

Results

- Under the above assumption (plus other classical assumptions), we prove identifiability (up to *a global* label switching) of the model's parameters.
- We underly that in the affiliation case, no current method can avoid label switching between time steps ! The parameters are not identifiable.

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Outline

Introduction and model

Inference

Simulations

Real data set

Variational Expectation Maximization (VEM) I

Complete data log-likelihood (here $Z_i^t = (Z_{i1}^t, \dots, Z_{iQ}^t)$).

$$\log \mathbb{P}_\theta(\mathbf{Y}, \mathbf{Z}) = \sum_{i=1}^{N} \sum_{q=1}^{Q} Z_{iq}^1 \log \alpha_q + \sum_{t=2}^{T} \sum_{i=1}^{N} \sum_{1 \le q, q' \le Q} Z_{iq}^{t-1} Z_{iq'}^t \log \pi_{qq'}$$
$$+ \sum_{t=1}^{T} \sum_{1 \le i < j \le N} \sum_{1 \le q, l \le Q} Z_{iq}^t Z_{jl}^t \log \phi(Y_{ij}^t; \beta_{ql}^t, \gamma_{ql}^t).$$

- Conditional expectation of latent $\mathbf{Z}$, given observations $\mathbf{Y}$ may not be exactly computed,
- Use instead a variational approximation

$$\mathbb{Q}_\tau(\mathbf{Z}) = \prod_{i=1}^{N} \mathbb{Q}_\tau(Z_i) = \prod_{i=1}^{N} \mathbb{Q}_\tau(Z_i^1) \prod_{t=2}^{T} \mathbb{Q}_\tau(Z_i^t | Z_i^{t-1}).$$

Variational Expectation Maximization (VEM) II

Let

$$J(\theta, \tau) := \mathbb{E}_{\mathbb{Q}_\tau}(\log \mathbb{P}_\theta(\mathbf{Y}, \mathbf{Z})) + \mathcal{H}(\mathbb{Q}_\tau)$$

and note that

$$\log \mathbb{P}_\theta(\mathbf{Y}) = J(\theta, \tau) + \mathcal{KL}(\mathbb{Q}_\tau \| \mathbb{P}_\theta(\mathbf{Z}|\mathbf{Y})).$$

VEM principle

Iterate the following steps

- `VE`-step: Compute $\tau^{(k+1)} = \text{Argmax}_\tau J(\theta^{(k)}, \tau)$,
- `M`-step: Compute $\theta^{(k+1)} = \text{Argmax}_\theta J(\theta, \tau^{(k+1)})$.

More details can be found in the paper ...

Model selection

ICL criterion

$$ICL(Q) = \log \mathbb{P}_{\hat{\theta}_Q}(\mathbf{Y}, \hat{\mathbf{Z}}) - \frac{1}{2}Q(Q-1)\log(NT) - pen(N, T, \boldsymbol{\beta}, \boldsymbol{\gamma}),$$

- the second penalty $pen(N, T, \boldsymbol{\beta}, \boldsymbol{\gamma})$ depends on the distribution ϕ ; we give expressions for classical cases (Bernoulli, Poisson, Gaussian, ...)
- Groups parameters $\boldsymbol{\pi}$ and connectivity parameters $(\boldsymbol{\beta}, \boldsymbol{\gamma})$ are not penalized in the same way (count the number of observations corresponding to these parameters).

Outline

Clustering performances I

Indexes

- Global ARI: Adjusted Rand Index on the whole classification $\{Z_i^t\}_{1 \le i \le N, 1 \le t \le T}$,
- Averaged ARI: mean value of ARI_t, computed for each t on the classification $\{Z_i^t\}_{1 \le i \le N}$. Easier ! Label switching between time steps !

Clustering performances II

Simulations setup

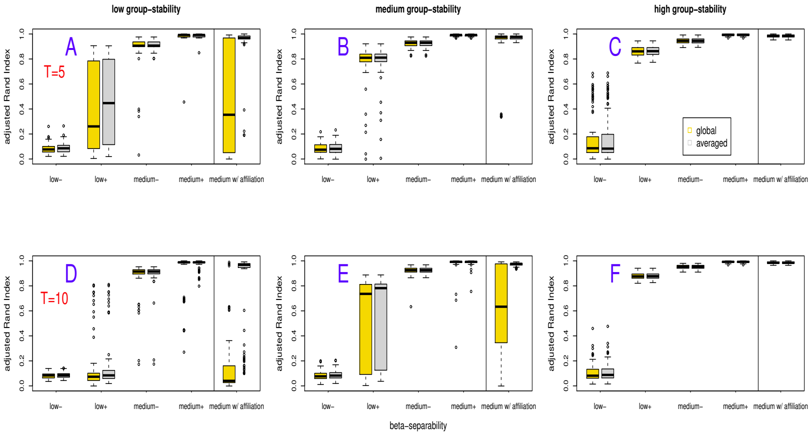
- Binary graphs, $N = 100$ nodes and $T \in \{5; 10\}$, 100 datasets,
- $Q = 2$ latent groups and $\boldsymbol{\pi} \in \{\boldsymbol{\pi}_{low}, \boldsymbol{\pi}_{med}, \boldsymbol{\pi}_{high}\}$

$$\boldsymbol{\pi}_{low} = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}; \boldsymbol{\pi}_{med} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}; \boldsymbol{\pi}_{high} = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}.$$

- Connectivity parameter $\boldsymbol{\beta}$

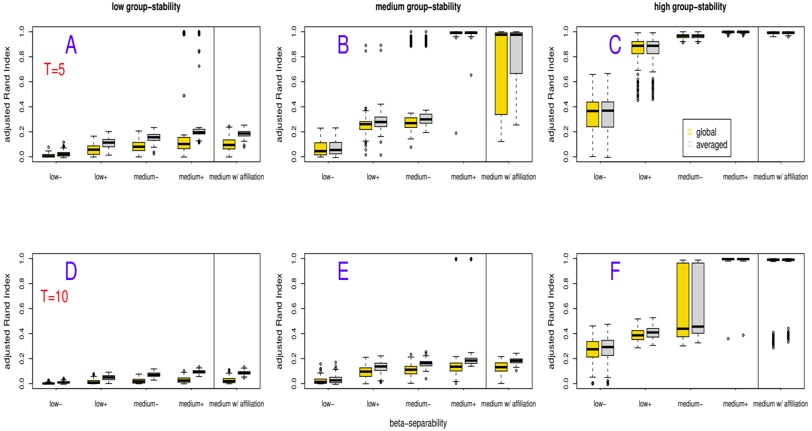
Difficulty	β_{11}	β_{12}	β_{22}
low-	0.2	0.1	0.15
low+	0.25	0.1	0.2
medium-	0.3	0.1	0.2
medium+	0.4	0.1	0.2
med w/ affiliation	0.3	0.1	0.3

Clustering performances III



Clustering performances IV

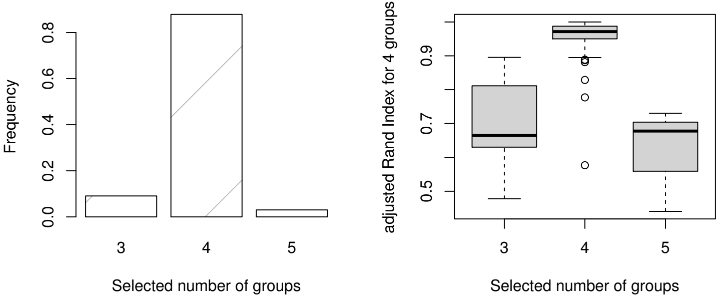
Yang *et al.*'s method with our initialization strategy



Model selection

Simulation setup

- Binary model, $Q = 4$ groups, $\pi_{qq} = 0.91$ and $\pi_{ql} = 0.03$ for $q \neq l$, 100 datasets
- We draw i.i.d. random variables $\{\epsilon_{ql}\}_{1 \leq q \leq l \leq 4} \in [-1, 1]$ and then choose $\beta_{qq} = 0.4 + \epsilon_{qq} 0.1$ and $\beta_{ql} = 0.1 + \epsilon_{ql} 0.1$ for $q \neq l$.



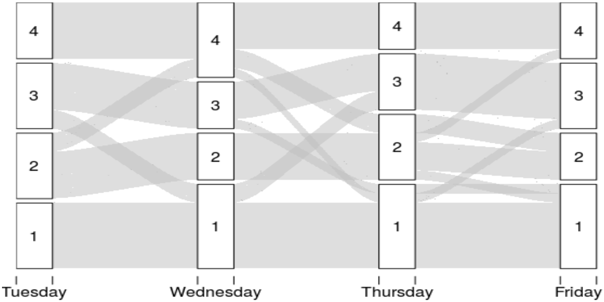
Outline

Encounters between high school students I

Fournet and Barrat, 2014, `http://www.sociopatterns.org/`

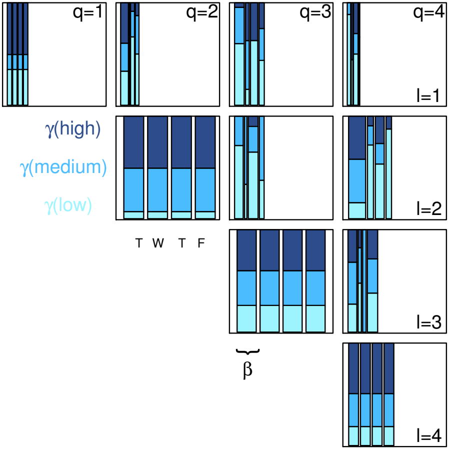
- Face-to-face encounters of high school students (wearable sensors), $T = 4$ days, $N = 27$ students,
- Discrete weight with 3 bins. Selection of $Q = 4$ groups.

Reconstructed dynamics



Encounters between high school students II

Estimated connectivity parameters



Conclusions

DynamicSBM

- Reconstruction of group's evolution through time
- Control of the label switching issue between different time steps
- Models binary or weighted datasets
- Model selection performed through ICL.

R package available at `http://lbbe.univ-lyon1.fr/dynsbm` and soon on the CRAN.
Preprint available at `http://arxiv.org/abs/1506.07464`

Thanks for your attention !

Extra short biblio

- Xu, K. and A. Hero.
  Dynamic stochastic blockmodels for time-evolving social networks.
  Selected Topics in Signal Processing, IEEE Journal of 8(4), 552–562, 2014.

- Yang, T., Y. Chi, S. Zhu, Y. Gong, and R. Jin.
  Detecting communities and their evolutions in dynamic social networks—a Bayesian approach.
  Machine Learning 82(2), 157–189, 2011.