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Statistical Deferred Cesàro Summability and Its Applications to Approximation Theorems

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Abstract. Statistical $(C, 1)$ summability and a Korovkin type approximation theorem has been proved by Mohiuddine *et al.* [20] (see [S. A. Mohiuddine, A. Alotaibi and M. Mursaleen, Statistical summability $(C, 1)$ and a Korovkin type approximation theorem, *J. Inequal. Appl.* **2012** (2012), Article ID 172, 1-8). In this paper, we apply statistical deferred Cesàro summability method to prove a Korovkin type approximation theorem for the set of functions 1, e^{-x} and e^{-2x} defined on a Banach space $C[0, \infty)$ and demonstrate that our theorem is a non-trivial extension of some well-known Korovkin type approximation theorems. We also establish a result for the rate of statistical deferred Cesàro summability method. Some interesting examples are also discussed here in support of our definitions and results.

1. Introduction

In the study of sequence spaces, classical convergence has got numerous applications where the convergence of a sequence requires that almost all elements are to satisfy the convergence condition. That is, all the elements of the sequence need to be in an arbitrarily small neighborhood of the limit. However such restriction is relaxed in statistical convergence, where the validity of convergence condition is achieved only for a majority of elements. The notion of statistical convergence was introduced by Fast [11] and Steinhaus [30]. Recently, statistical convergence has been a dynamic research area due to the fact that it is more general than classical convergence and such theory is discussed in the study of Fourier Analysis, Number Theory and Approximation Theory. For more details, see [2], [5], [7], [10], [12], [14] and [15].

Let $\mathbb{N}$ be the set of natural numbers and let $K \subseteq \mathbb{N}$. Also let

$$K_n = \{k : k \leq n \text{ and } k \in K\}$$

and suppose that $|K_n|$ be the cardinality of K_n. Then the natural density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{|K_n|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n \text{ and } k \in K\}|,$$

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provided the limit exists.

A given sequence (x_n) is said to be statistically convergent to L if, for each $\epsilon > 0$, the set

$$K_\epsilon = \{k : k \in \mathbb{N} \text{ and } |x_k - L| \geq \epsilon\}$$

has zero natural density (see [11], [30]). That is, for each $\epsilon > 0$,

$$\delta(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n \text{ and } |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} - \lim_{n\to\infty} x_n = L.$$

Now we present an example to show that every convergent sequence is statistically convergent but the converse is not true in general.

Example 1.1. Let us consider the sequence $x = (x_n)$ by

$$x_n = \begin{cases} n & \text{when } n = m^2, \text{ for all } m \in \text{N} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

Then, it is easy to see that the sequence (x_n) is divergent in the ordinary sense, while 0 is the statistical limit of (x_n) since $\delta(K) = 0$, where $K = \{m^2, \text{ for all } m = 1, 2, 3, ...\}$.

In 2002, Móricz [21], introduced the fundamental idea of statistical $(C, 1)$ summability and recently Mohiuddine *et al.* [20] has established statistical $(C, 1)$ summability as follows.

Let us consider a sequence $x = (x_n)$, the $(C, 1)$ mean of the sequence is given by

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} x_k,$$

and (x_n) is said to be statistical $(C, 1)$ summable to L if, for each $\epsilon > 0$, the set

$$\{k : k \in \mathbb{N} \text{ and } |\sigma_k - L| \geq \epsilon\}$$

has zero Cesàro density. That is, for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n \text{ and } |\sigma_k - L| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} - \lim_{n\to\infty} \sigma_n = L \text{ or } C_1(\text{stat}) - \lim_{n\to\infty} x_n = L.$$

In the year 2008, Özarslan *et al.* [24] established certain results on statistical approximation for Kantorovich-type operators involving some special polynomials, and then Braha *et al.* [8] investigated a Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean. Very recently, Kadak *et al.* [16] has established some approximation theorems by statistical weighted $\mathcal{B}$-summability, and then Srivastava and Et [26] established a result on lacunary statistical convergence and strongly lacunary summable functions of order α. Furthermore, Srivastava *et al.* [28] has proved some interesting results on approximation theorems involving the q-Szàsz-Mirakjan-Kantorovich type operators via Dunkl's generalization.

Motivated essentially by the above-mentioned works, in view of establishing certain new approximation results, we now recall the deferred Cesàro $D(a_n, b_n)$ summability mean as follows:

Let (a_n) and (b_n) be sequences of non-negative integers such that

(i) $a_n < b_n$

and

(ii) $\lim_{n\to\infty} b_n = \infty$,

then the deferred Cesàro $D(a_n, b_n)$ mean is defined by (Agnew [1], p. 414),

$$D(a_n, b_n) = D(x_n) = \frac{x_{a_n+1} + x_{a_n+2} + x_{a_n+3} + ... + x_{b_n}}{b_n - a_n} = \sum_{k=a_n+1}^{b_n} x_k. \tag{1}$$

It is well known that, $D(a_n, b_n)$ is regular under conditions (i) and (ii) (see Agnew [1]).

Also very recently, Srivastava *et al.* [25] has introduced deferred weighted mean, $D_a^b(\overline{N}, p, q)$ as,

$$t_n = \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m x_m.$$

It will be interesting to see that, for $p_m = q_m = 1$, t_n is same as $D(x_n)$. Thus, deferred Cesàro mean is very fundamental in the study of such type of means. Here, we have considered the statistical summablity via deferred Cesàro mean in order to establish certain approximation theorems.

Let us now introduce the following definitions in support of our proposed work.

Definition 1.2. A sequence (x_n) is said to be statistical deferred Cesàro summable to L if, for every $\epsilon > 0$, the set

$$\{k : a_n < k \leq b_n \text{ and } |D(x_n) - L| \geq \epsilon\}$$

has deferred Cesàro density zero, that is,

$$\lim_{n\to\infty} \frac{1}{b_n - a_n} |\{k : a_n < k \leq b_n \text{ and } |D(x_n) - L| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} - \lim_{n\to\infty} D(x_n) = L \text{ or } DC_1(\text{stat}) - \lim_{n\to\infty} x_n = L.$$

Clearly, above definition can be viewed as the generalization of some existing definitions.

Remark 1.3. If $a_n = n - 1$ and $b_n = n$, then $D(n - 1, n)$ reduces to the identity transformation and also, if $a_n = 0$ and $b_n = n$, then $D(0, n)$ reduces to $(C, 1)$ transformation of x_n, which is often denoted as σ_n. Finally, if $a_n = n - 1$ and $b_n = n + t - 1$, then

$$D(n - 1, n + t - 1) = \sigma_{n,t} = \left(\frac{t + n}{t}\right)\sigma_{n+t-1} - \left(\frac{n}{t}\right)\sigma_{n-1}, \tag{2}$$

which is called the delayed arithmetic mean (see [32], p. 80).

Definition 1.4. A sequence (x_n) is said to be statistical delayed arithmetic summable to L if, for every $\epsilon > 0$, the set

$$\{k : n-1 < k \leq n+t-1 \ \text{ and } \ |\sigma_{n,t} - L| \geq \epsilon\}$$

has zero delayed arithmetic density, that is,

$$\lim_{n\to\infty} \frac{1}{t}|\{k : n-1 < k \leq n+t-1 \ \text{ and } \ |\sigma_{n,t} - L| \geq \epsilon\}| = 0.$$

In this case, we write

$$\text{stat} - \lim_{n\to\infty} \sigma_{n,t} = L \ \text{ or } \ DA_1(\text{stat}) - \lim_{n\to\infty} x_n = L.$$

Now, we present below an example to show that a sequence is statistically deferred Cesàro summable, whenever it is not statistically Cesàro summable.

Example 1.5. Let us consider the sequences $(a_n) = 2n$, $(b_n) = 4n$ and a sequence $x = (x_n)$ as

$$x_n = \begin{cases} \frac{n+1}{2} & (n \text{ is odd}) \\ \\ -\frac{n}{2} & (n \text{ is even}). \end{cases}$$

Clearly, we observe that (x_n) is neither convergent nor statistical convergent. Also it is not statistical Cesàro summable, but it is deferred Cesàro summable to 0, that is $\lim_{n\to\infty} D(2n, 4n) = 0$, implies (x_n) is statistical deferred Cesàro summable to 0.

The main object of this paper is to establish some important approximation theorems over the Banach space based on statistical deferred Cesàro summability which will effectively extend and improve most (if not all) of the existing results depending on the choice of sequences of the deferred Cesàro mean. Furthermore, we intend to estimate the rate of statistical deferred Cesàro summability and investigate Korovkin type approximation results.

2. A Korovkin Type Theorem

Several mathematicians have worked on extending or generalizing the Korovkin type theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces and so on. This theory is very useful in Real Analysis, Functional Analysis, Harmonic Analysis, Measure Theory, Probability Theory, Summability Theory and Partial Differential Equations. Recently, Mohiuddine [19] has obtained an application of almost convergence for single sequences in Korovkin-type approximation theorem and proved some related results. For the function of two variables, such type of approximation theorems are proved in [4] by using almost convergence of double sequences. Quite recently, in [22] and [23] the Korovkin type theorem is proved for statistical λ-convergence and statistical lacunary summability, respectively. For some recent work on this topic, we refer to [6], [9], [13], [17] and [29]. Recently, Mohiuddine *et al.* [20] have proved the Korovkin theorem on $C[0,\infty)$ by using the test functions 1, e^{-x} and e^{-2x}. In this paper, we generalize the result of Mohiuddine, Alotaibi and Mursaleen via the notion of statistical deferred Cesàro summability for the same test functions 1, e^{-x} and e^{-2x}. We also present an example to justify that our result is stronger than that of Mohiuddine, Alotaibi and Mursaleen (see [20]).

Let $C(X)$, be the space of all real valued continuous functions defined on $[0,\infty)$ under the norm $\|.\|_\infty$. Also, $C[0,\infty)$ is a Banach space. We have, for $f \in C[0,\infty)$, the norm of f denoted by $\|f\|$ is given by,

$$\|f\|_\infty = \sup_{x\in[0,\infty)} \{|f(x)|\}$$

with

$$\omega(\delta, f) = \sup_{0\leq|h|\leq\delta} \|f(x+h) - f(x)\|_\infty, \ f \in C[0,\infty).$$

The quantities $\omega(\delta, f)$ is called the modulus of continuity of f.

Let $L : C[0,\infty) \to C[0,\infty)$ be a linear operator. Then, as usual, we say that L is a positive linear operator provided that,

$$f \geq 0 \quad \text{implies} \quad L(f) \geq 0.$$

Also, we denote the value of $L(f)$ at a point $x \in [0,\infty)$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

The classical Korovkin theorem states as follows [18]:

Let $L_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators and let $f \in C[0,\infty)$. Then

$$\lim_{n\to\infty} \|L_n(f; x) - f(x)\|_\infty = 0 \Longleftrightarrow \lim_{n\to\infty} \|L_n(f_i; x) - f_i(x)\|_\infty = 0 \quad (i = 0,1,2),$$

where

$$f_0(x) = 1, \ \ f_1(x) = x \ \ \textit{and} \ \ f_2(x) = x^2.$$

The statistical Cesàro summability version for the theorem established by Mohiuddine *et al.* [20], states as follows.

Let $L_n : C[0,\infty) \to C[0,\infty)$ be a sequence of positive linear operators and let $f \in C[0,\infty)$. Then

$$C_1(stat) - \lim_{n\to\infty} \|L_n(f; x) - f(x)\|_\infty = 0 \Longleftrightarrow C_1(stat) - \lim_{n\to\infty} \|L_n(f_i; x) - f_i(x)\|_\infty = 0 \quad (i = 0,1,2),$$

where

$$f_0(x) = 1, \ \ f_1(x) = e^{-x} \ \ \textit{and} \ \ f_2(x) = e^{-2x}.$$

Now we prove the following theorem by using the notion of statistical deferred Cesàro summability.

Theorem 2.1. *Let $L_m : C[0,\infty) \to C[0,\infty)$ be a sequence of positive linear operators. Then for all $f \in C[0,\infty)$*

$$DC_1(stat) - \lim_{m\to\infty} \|L_m(f; x) - f(x)\|_\infty = 0, \tag{3}$$

if and only if

$$DC_1(stat) - \lim_{m\to\infty} \|L_m(1; x) - 1\|_\infty = 0, \tag{4}$$

$$DC_1(stat) - \lim_{m\to\infty} \|L_m(e^{-s}; x) - e^{-x}\|_\infty = 0 \tag{5}$$

and

$$DC_1(stat) - \lim_{m\to\infty} \|L_m(e^{-2s}; x) - e^{-2x}\|_\infty = 0. \tag{6}$$

Proof. Since each of $f_i(x) = \{1, e^{-x}, e^{-2x}\} \in C(X)$ $(i = 0, 1, 2)$ are continuous, the implication:

$$(3) \Longrightarrow (4) - (6)$$

is obvious. In order to complete the proof of the theorem we first assume that (4)-(6) hold true. Let $f \in C[X]$, then there exists a constant $\mathcal{K} > 0$ such that $|f(x)| \leq \mathcal{K}$, $\forall\, x \in X = [0, \infty)$.
Thus,

$$|f(s) - f(x)| \leq 2\mathcal{K}, \; s, x \in X. \tag{7}$$

Clearly, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s) - f(x)| < \epsilon \tag{8}$$

whenever $|e^{-s} - e^{-x}| < \delta$, for all $s, x \in X$.

Let us choose $\varphi_1 = \varphi_1(s, x) = (e^{-s} - e^{-x})^2$. If $|e^{-s} - x^{-x}| \geq \delta$, then we obtain:

$$|f(s) - f(x)| < \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x). \tag{9}$$

From equation (8) and (9), we get

$$|f(s) - f(x)| < \epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x),$$

$$\Rightarrow \; -\epsilon - \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x) \leq f(s) - f(x) \leq \epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x). \tag{10}$$

Now since $L_m(1; x)$ is monotone and linear, so by applying the operator $L_m(1; x)$ to this inequality, we have

$$L_m(1; x)\left(-\epsilon - \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x)\right) \leq L_m(1; x)(f(s) - f(x)) \leq L_m(1; x)\left(\epsilon + \frac{2\mathcal{K}}{\delta^2}\varphi_1(s, x)\right). \tag{11}$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$-\epsilon L_m(1; x) - \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x) \leq L_m(f; x) - f(x)L_m(1; x) \leq \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x). \tag{12}$$

But

$$L_m(f; x) - f(x) = [L_m(f; x) - f(x)L_m(1; x)] + f(x)[L_m(1; x) - 1]. \tag{13}$$

Using (12) and (13), we have

$$L_m(f; x) - f(x) < \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}L_m(\varphi_1; x) + f(x)[L_m(1; x) - 1]. \tag{14}$$

Now, estimate $L_m(\varphi_1; x)$ as,

$$\begin{aligned} L_m(\varphi_1; x) &= L_m((e^{-s} - e^{-x})^2; x) = L_m(e^{-2s} - 2e^{-x}e^{-s} + e^{-2x}; x) \\ &= L_m(e^{-2s}; x) - 2e^{-x}L_m(e^{-s}; x) + e^{-2s}L_m(1; x) \\ &= [L_m(e^{-2s}; x) - e^{-2x}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] + e^{-2x}[L_m(1; x) - 1]. \end{aligned}$$

Using (14), we obtain

$$\begin{aligned} L_m(f; x) - f(x) &< \epsilon L_m(1; x) + \frac{2\mathcal{K}}{\delta^2}\{[L_m(e^{-2s}; x) - e^{-2s}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] \\ &\qquad + e^{-2s}[L_m(1; x) - 1]\} + f(x)[L_m(1; x) - 1]. \\ &= \epsilon[L_m(1; x) - 1] + \epsilon + \frac{2\mathcal{K}}{\delta^2}\{[L_m(e^{-2s}; x) - e^{-2x}] - 2e^{-x}[L_m(e^{-s}; x) - e^{-x}] \\ &\qquad + e^{-2x}[L_m(1; x) - 1]\} + f(x)[L_m(1; x) - 1]. \end{aligned}$$

Since ϵ is arbitrary, we can write

$$\begin{aligned}|L_m(f;x)-f(x)| &\leq \epsilon+\left(\epsilon+\frac{2\mathcal{K}}{\delta^2}+\mathcal{K}\right)|L_m(1;x)-1| \\ &\quad +\frac{4\mathcal{K}}{\delta^2}|L_m(e^{-s};x)-e^{-x}|+\frac{2\mathcal{K}}{\delta^2}|L_m(e^{-2s};x)-e^{-2x}| \\ &\leq B\left(|L_m(1;x)-1|+|L_m(e^{-s};x)-e^{-x}|+|L_m(e^{-2s};x)-e^{-2x}|\right), \qquad (15)\end{aligned}$$

where

$$B=\max\left(\epsilon+\frac{2\mathcal{K}}{\delta^2}+\mathcal{K},\frac{4\mathcal{K}}{\delta^2},\frac{2\mathcal{K}}{\delta^2}\right).$$

Now replacing $L_m(f;x)$ by $\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}T_m(f;x)$ and then by $\Psi_m(f;x)$ in (15), we have for a given $r>0$, there exists $\epsilon>0$, such that $\epsilon<r$. Then, by setting

$$\Psi_m(x;r)=\left\{m:a_n<m\leq b_n \text{ and } \left|\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}T_m(f;x)-f(x)\right|\geq r\right\}$$

and for $i=0,1,2,$

$$\Psi_{i,m}(x;r)=\left\{m:a_n<m\leq b_n \text{ and } \left|\frac{1}{b_n-a_n}\sum_{m=a_n+1}^{b_n}T_m(f_i;x)-f_i(x)\right|\geq\frac{r-\epsilon}{3B}\right\},$$

we obtain,

$$\Psi_m(x;r)\leq\sum_{i=0}^{2}\Psi_{i,m}(x;r).$$

Clearly,

$$\frac{\|\Psi_m(x;r)\|_{C(X)}}{b_n-a_n}\leq\sum_{i=0}^{2}\frac{\|\Psi_{i,m}(x;r)\|_{C(X)}}{b_n-a_n}. \qquad (16)$$

Now, using the above assumption about the implications in (4)-(6) and by Definition 1.2, the right-hand side of (16) is seen to tend to zero as $n\to\infty$. Consequently, we get

$$\lim_{n\to\infty}\frac{\|\Psi_m(x;r)\|_{C(X)}}{b_n-a_n}=0 \quad (r>0).$$

Therefore, the implication (3) holds true.

This completes the proof of Theorem 2.1. □

Corollary 2.2. *Let $L_m : C[0,\infty)\to C[0,\infty)$ be a sequence of positive linear operators and let $f\in C[0,\infty)$. Then*

$$DA_1(stat)-\lim_{m\to\infty}\|L_m(f;x)-f(x)\|_\infty=0 \qquad (17)$$

if and only if

$$DA_1(stat)-\lim_{m\to\infty}\|L_m(1;x)-1\|_\infty=0, \qquad (18)$$

$$DA_1(stat)-\lim_{m\to\infty}\|L_m(e^{-s};x)-e^{-x}\|_\infty=0 \qquad (19)$$

and

$$DA_1(stat) - \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0. \tag{20}$$

Proof. By taking $a_n = n-1,\ \forall\ n$ and, $b_n = n+k-1,\ \forall\ n$ and proceeding in the similar line of Theorem 2.1, the proof of Corollary 2.2 is established. □

Remark 2.3. By taking $a_n = 0, b_n = n,\ \forall\ n$ in Theorem 2.1, one can obtain the statistical Cesàro summability version of Korovkin type approximation for the set of functions 1, e^{-x} and e^{-2x} established by Mohiuddine *et al.* [20].

Now we present below an illustrative example for the sequence of positive linear operators that does not satisfy the conditions of the Korovkin approximation theorems due to Mohiuddine *et al.* [20] and Boyanov and Veselinov [6] but satisfies the conditions of our Theorem 2.1. Thus, our theorem is stronger than the results established by both Mohiuddine *et al.* [20] and Boyanov and Veselinov [6].

Here we consider the operator

$$x(1+xD) \quad \left(D = \frac{d}{dx}\right)$$

which was used by Al-Salam [3] and, more recently, by Viskov and Srivastava [31] (see also the monograph by Srivastava and Manocha [27] for various general families of operators of this kind). Here, we use this operator over the Baskakov operators.

Example 2.4. Let $L_m : C[0,\infty) \to C[0,\infty)$ be defined by

$$L_m(f;x) = (1+x_m)x(1+xD)V_m(f;x), \tag{21}$$

where

$$V_m(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{m}\right)\binom{m-k+1}{k} x^k.(1+x)^{-n-k}$$

and (x_m) is a sequence defined in Example 1.5.

Now,

$$L_m(1;x) = [1+x_m]x(1+xD)1 = [1+x_m]x,$$

$$L_m(e^{-s};x) = [1+x_m]x(1+xD)(1+x-xe^{-\frac{1}{m}})^{-m}$$

$$= [1+x_m]x(1+x-xe^{-\frac{1}{m}})^{-m}\left(1 - mx(1-e^{-\frac{1}{m}})(1+x-xe^{-\frac{1}{m}})^{-1}\right),$$

$$L_m(e^{-2s};x) = [1+x_m]x(1+xD)(1+x^2-x^2e^{-\frac{1}{m}})^{-m}$$

$$= [1+x_m]x(1+x^2-x^2e^{-\frac{1}{m}})^{-m}\left(1 - 2mx^2(1-e^{-\frac{1}{m}})(1+x^2-x^2e^{-\frac{1}{m}})^{-1}\right).$$

So that, we obtain

$$DC_1(\text{stat}) - \lim_{m\to\infty} \|L_m(1;x) - 1\|_\infty = 0,$$

$$DC_1(\text{stat}) - \lim_{m\to\infty} \|L_m(e^{-s};x) - e^{-x}\|_\infty = 0$$

and

$$DC_1(\text{stat}) - \lim_{m\to\infty} \|L_m(e^{-2s};x) - e^{-2x}\|_\infty = 0,$$

that is, the sequence $L_m(f;x)$ satisfies the conditions (4)-(6). Therefore by Theorem 2.1, we have

$$DC_1(\text{stat}) - \lim_{m\to\infty} \|L_m(f;x) - f\|_\infty = 0.$$

Hence, it is statistically deferred Cesàro summable; however, since (x_m) is neither statistically convergent nor statistically Cesàro summable, so we conclude that earlier works under [20] and [6] is not valid for the operators defined by (21), while our Theorem 2.1 still works.

3. Rate of statistical deferred Cesàro summability

In this section, we study the rates of statistical deferred Cesàro summability of a sequence of positive linear operators $L(f;x)$ defined on $C[0,\infty)$ with the help of modulus of continuity.

We now presenting the following definition.

Definition 3.1. Let (u_n) be a positive non-increasing sequence. A given sequence $x = (x_m)$ is statistically deferred Cesàro summable to a number L with rate $o(u_n)$, if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{u_n(b_n - a_n)} |\{m : a_n < m \leq b_n \ \text{ and } \ |D(x_m) - L| \geq \epsilon\}| = 0.$$

In this case, we may write

$$x_m - L = DC_1(\text{stat}) - o(u_n).$$

We now prove the following basic lemma.

Lemma 3.2. *Let (u_n) and (v_n) be two positive non-increasing sequences. Let $x = (x_m)$ and $y = (y_m)$ be two sequences such that*

$$x_m - L_1 = DC_1(stat) - o(u_n)$$

and

$$y_m - L_2 = DC_1(stat) - o(v_n)$$

respectively. Then the following conditions hold true

(i) $(x_m + y_m) - (L_1 + L_2) = DC_1(stat) - o(w_n)$;

(ii) $(x_m - L_1)(y_m - L_2) = DC_1(stat) - o(u_n v_n)$;

(iii) $\lambda(x_m - L_1) = DC_1(stat) - o(u_n)$ *(for any scalar λ)*;

(iv) $\sqrt{|x_m - L_1|} = DC_1(stat) - o(u_n)$,

where

$$w_n = \max\{u_n, v_n\}.$$

Proof. In order to prove the condition (i), for $\epsilon > 0$ and $x \in [0, \infty)$, we define the following sets:

$$A_n(x;\epsilon) = \left|\{m : a_n < m \leq b_n \text{ and } |D(x_m) + D(y_m) - (L_1 + L_2)| \geq \epsilon\}\right|,$$
$$A_{0,n}(x;\epsilon) = \left|\left\{m : a_n < m \leq b_n \text{ and } |D(x_m) - L_1| \geq \frac{\epsilon}{2}\right\}\right|,$$

and

$$A_{1,n}(x;\epsilon) = \left|\left\{m : a_n < m \leq b_n \text{ and } |D(y_m) - L_2| \geq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, we have

$$A_n(x;\epsilon) \subseteq A_{0,n}(x;\epsilon) \cup A_{1,n}(x;\epsilon).$$

Moreover, since

$$w_n = \max\{u_n, v_n\},$$

by condition (3) of Theorem 2.1, we obtain

$$\frac{\|A_m(x;\epsilon)\|_\infty}{w_n(b_n - a_n)} \leq \frac{\|A_{0,n}(x;\epsilon)\|_\infty}{u_n(b_n - a_n)} + \frac{\|A_{1,n}(x;\epsilon)\|_\infty}{v_n(b_n - a_n)}. \tag{22}$$

Now, by conditions (4)-(6) of Theorem 2.1, we obtain

$$\frac{\|A_n(x;\epsilon)\|_\infty}{w_n(b_n - a_n)} = 0, \tag{23}$$

which establishes (i). Since the proofs of other conditions (ii)-(iv) are similar, we omit them. □

Further, we recall that the modulus of continuity of a function $f \in C[0, \infty)$ is defined by

$$\omega(f, \delta) = \sup_{|y-x|\leq\delta : x,y\in X} |f(y) - f(x)| \quad (\delta > 0).$$

Which implies that

$$|f(y) - f(x)| \leq \omega(f, \delta)\left(\frac{|x - y|}{\delta} + 1\right). \tag{24}$$

Now we state and prove a result in the form of the following theorem.

Theorem 3.3. *Let $[0, \infty) \subset \mathbb{R}$ and let $L_m : C[0, \infty) \to C[0, \infty)$ be a sequence of positive linear operators. Assume that the following conditions hold true:*

(i) $\|L_m(1; x) - 1\|_\infty = DC_1(stat) - o(u_n)$,

(ii) $\omega(f, \lambda_m) = DC_1(stat) - o(v_n)$,

where

$$\lambda_m = \sqrt{L_m(\varphi^2; x)} \text{ and } \varphi_1(y, x) = (e^{-y} - x^{-x})^2.$$

Then, for all $f \in C[0, \infty)$, the following statement holds true:

$$\|L_m(f; x) - f\|_\infty = DC_1(stat) - o(w_n), \quad (w_n = \max\{u_n, v_n\}). \tag{25}$$

Proof. Let $f \in C[0,\infty)$ and $x \in [0,\infty)$. Using (24), we have

$$\begin{aligned} |L_m(f;x) - f(x)| &\leq L_m(|f(y) - f(x)|;x) + |f(x)||L_m(1;x) - 1| \\ &\leq L_m\left(\frac{|e^{-x} - e^{-y}|}{\lambda_m} + 1; x\right)\omega(f,\lambda_m) + |f(x)||L_m(1;x) - 1| \\ &\leq L_m\left(1 + \frac{1}{\lambda_m^2}(e^{-x} - e^{-y})^2; x\right)\omega(f,\lambda_m) + |f(x)||L_m(1;x) - 1| \\ &\leq \left(L_m(1;x) + \frac{1}{\lambda_m^2}L_m(\varphi_x;x)\right)\omega(f,\lambda_m) + |f(x)||L_m(1;x) - 1|. \end{aligned}$$

Putting $\lambda_m = \sqrt{L_m(\varphi^2;x)}$, we get

$$\begin{aligned} \|L_m(f;x) - f(x)\|_\infty &\leq 2\omega(f,\lambda_m) + \omega(f,\lambda_m)\|L_m(1;x) - 1\|_\infty + \|f(x)\|\|L_m(1;x) - 1\|_\infty \\ &\leq \mathcal{M}\{\omega(f,\lambda_m) + \omega(f,\lambda_m)\|L_m(1;x) - 1\|_\infty + \|L_m(1;x) - 1\|_\infty\}, \end{aligned}$$

where

$$\mathcal{M} = \{\|f\|_\infty,\ 2\}.$$

Thus,

$$\begin{aligned} \left\|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x)\right\|_\infty &\leq \mathcal{M}\left\{\omega(f,\lambda_m)\frac{1}{b_n - a_n} + \omega(f,\lambda_m)\left\|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x)\right\|_\infty\right\} \\ &\quad + \mathcal{M}\left\{\left\|\frac{1}{b_n - a_n}\sum_{m=a_n+1}^{b_n} L_m(f;x) - f(x)\right\|_\infty\right\}. \end{aligned}$$

Now, by using the conditions (i) and (ii) of Theorem 3.3, in conjunction with Lemma 3.2, we arrive at the statement (25) of Theorem 3.3.

This completes the proof of Theorem 3.3. □

4. Concluding remarks and observations

In this concluding section of our investigation, we present several further remarks and observations concerning to various results which we have proved here.

Remark 4.1. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence given in Example 1.5. Then, since

$$DC_1(\text{stat}) - \lim_{m\to\infty} x_m \to 0 \text{ on } [0,\infty),$$

we have

$$DC_1(\text{stat}) - \lim_{m\to\infty} \|L_m(f_i;x) - f_i(x)\|_\infty = 0 \quad (i = 0,1,2). \tag{26}$$

Thus, we can write (by Theorem 2.1)

$$DC_1(\text{stat}) - \lim_{m\to\infty} \|L_m(f;x) - f(x)\|_\infty = 0 \quad (i = 0,1,2), \tag{27}$$

where

$$f_0(x) = 1,\ \ f_1(x) = e^{-x} \text{ and } f_2(x) = e^{-2x}.$$

However, since (x_m) is not ordinarily convergent and so also it does not converge uniformly in the ordinary sense. Thus, the classical Korovkin theorem does not work here for the operators defined by (21). Hence, this application clearly indicates that our Theorem 2.1 is a non-trivial generalization of the classical Korovkin-type theorem (see [18]).

Remark 4.2. Let $(x_m)_{m\in\mathbb{N}}$ be a sequence as given in Example 1.5. Then, since

$$DC_1(\text{stat}) - \lim_{m\to\infty} x_m \to 0 \text{ on } [0,\infty),$$

so (26) holds true. Now by applying (26) and Theorem 2.1, condition (27) holds true. However, since (x_m) does not statistical Cesàro summable, so Theorem 2.1 of Mohiuddine *et al.* (see [20]) does not work for our operator defined in (21). Thus, our Theorem 2.1 is also a non-trivial extension of Theorem 2.1 of Mohiuddine *et al.* [20] (see also [6] and [18]). Based upon the above results, it is concluded here that our proposed method has successfully worked for the operators defined in (21) and therefore it is stronger than the classical and statistical version of the Korovkin type approximation (see [20], [6] and [18]) established earlier.

Remark 4.3. Let us suppose that we replace the conditions (i) and (ii) in Theorem 3.3, by the following condition:

$$|L_m(f_i;x) - f_i| = DC_1(\text{stat}) - o(u_{n_i}) \quad (i = 0,1,2). \tag{28}$$

Then, since

$$L_m(\varphi^2;x) = e^{-2x}|L_m(1;x) - 1| - 2e^{-x}|L_m(e^{-x};x) - e^{-x}| + |L_m(e^{-2x};x) - e^{-2x}|,$$

we can write

$$L_m(\varphi^2;x) \leq M\sum_{i=0}^{2} |L_m(f_i;x) - f_i(x)|_\infty, \tag{29}$$

where

$$M = \{\|f_2\|_\infty + 2\|f_1\|_\infty + 1\}.$$

Now it follows from (28), (29) and Lemma 3.2 that,

$$\lambda_m = \sqrt{L_m(\varphi^2)} = DC_1(\text{stat}) - o(d_n), \tag{30}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

This implies,

$$\omega(f,\delta) = DC_1(\text{stat}) - o(d_n).$$

Now using (30) in Theorem 3.3, we immediately see that, for $f \in C[0,\infty)$,

$$L_m(f;x) - f(x) = DC_1(\text{stat}) - o(d_n). \tag{31}$$

Therefore, if we use the condition (28) in Theorem 3.3 instead of (i) and (ii), then we obtain the rates of statistical deferred Cesàro summability of the sequence of positive linear operators in Theorem 2.1.

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