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Statistical Effect of Interactions on Particle Creation in Expanding Universe

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The statistical effect of interactions which drives many-particle systems toward equilibrium is expected to change the qualitative and quantitative features of particle creation in expanding universe. To investigate this problem a simplified model called the finite-time reduction model is formulated and applied to the scalar particle creation in the radiation dominant Friedmann universe. The number density of created particles and the entropy production due to particle creation are estimated. The result for the number density is compared with that in the conventional free field theory. It is shown that the statistical effect increases the particle creation and lengthens the active creation period. As for the entropy production it is shown that it is negligible for scalar particles in the Friedmann universe.

§ 1. Introduction

There has been a considerable amount of work in recent years on the subject of quantum field theory in curved spacetime, and especially on the creation of particles in dynamical spacetime.[1] Most of the previous work, however, has been concerned with noninteracting quantum fields. When we take into account interactions of quantum fields, there appear some new phenomena. One of them is the multi-particle production from vacuum due to the interaction terms of Lagrangians. This phenomenon was recently studied by Birrel et al.[2] Another phenomenon is the thermalization of created particles and the associated entropy production. We study problems concerned with the latter phenomenon in this paper.

In stationary spacetime, a closed system containing many particles generally evolves toward an equilibrium state due to interactions after some characteristic time whatever its initial state is. This suggests that, in dynamical spacetime, the effect of interactions which drives the system toward equilibrium and the effect of the dynamical change of spacetime which causes the system deviate from equilibrium through particle creation will compete. Such non-equilibrium processes generally produce entropy. Thus we can expect the entropy production due to particle creation and the associated heating up of universe. Furthermore, each quantum state (or more precicely speaking, the density matrix) representing an equilibrium state specified by a set of values of thermodynamical parameters changes with time because the definition of particles changes with time in dynamical spacetime. Hence the statistical effect of interactions, which drives systems toward equilibrium, is also expected to change the qualitative and quantitative features of particle creation themselves.

The main purpose of this paper is to estimate these statistical effects of interactions on particle creation for scalar fields in Robertson-Walker universes, especially Friedmann universe. Since there exists no exact satisfactory formulation yet which derives statistical effects of interactions from first principles,[3] we study the problem by the following simplified model, which we call the finite-time reduction model. First we divide the cosmic time into a sequence of short time intervals $[t_j, t_{j+1}]$, whose length called a reduction time in this paper varies from one interval to another in general. At time $t=t_j$ the scalar fields are assumed to be in the thermal equilibrium state given by the temperature of the universe at that time. Then the fields evolve following the free field theory in the interval $[t_j, t_{j+1}]$, and at $t=t_{j+1}$ the quantum state of the fields suddenly changes to the new equilibrium state given by the temperature of the universe at $t=t_{j+1}$. Thereafter the same process is repeated.

This sudden change of the state is the central point of the model, and regarded as simulating the change of the state due to the statistical effect of interactions during each interval which actually occurs continuously. Thus in this model nature of interactions chiefly comes into the argument through the length of the reduction time.

A more precise formulation is given in § 2 with some discussion on associated various problems after a brief explanation of the motivation and the background of the model based on the generalized master equation. In § 3 we derive the main formulas which express the number and the entropy of created particles during each interval by the cosmic scale factor, the temperature and the reduction time at the interval. Then we apply these formulas to the radiation dominant Friedmann universe and estimate the total number and the total entropy production. In § 4 for the comparison with the result of § 3 we estimate the particle creation from vacuum in Friedmann universe following the conventional free field theory. The comparison is made in § 5 after the summarization of the result in § 3. Some cosmological implications are commented. Section 6 is devoted to discussion. The proofs of various mathematical formulas used in the text are given in the appendices. Throughout this paper the absolute units $c=\hbar=G=1$ are employed.

§ 2. Formulation

In order to explain the physical background of our phenomenological model, we first briefly review the argument which is used to show that a quantum system containing many particles evolves in time toward equilibrium. Let $H_T(t)=H(t)+H_I(t)$ be the Hamiltonian of the system and $D(t)$ be the density matrix, in the interaction picture with respect to the free Hamiltonian $H(t)$. We assume that the interaction of the particles composing the system is sufficiently weak. Then the time evolution of $D(t)$ is given by[4]

$$\frac{d}{dt}D(t)=i\mathcal{L}_I D(t), \tag{2·1}$$

where $\mathcal{L}_I(t)$ is the linear operator acting on operators defined by

$$\mathcal{L}_I\mathcal{O}\equiv-[H_I,\mathcal{O}] \tag{2·2}$$

for any operator $\mathcal{O}$. Let $\{|\nu_t\rangle\}$ be a complete set of eigenstates of $H(t)$ at time t, and let us define the projection operator $\mathcal{P}(t)$ by the action on an operator $\mathcal{O}$

$$\mathcal{P}(t)\,\mathcal{O}\equiv\sum_\nu|\nu_t\rangle\langle\nu_t|\,\mathcal{O}\,|\nu_t\rangle\langle\nu_t|. \tag{2·3}$$

In general $\mathcal{P}(t)$ should be replaced by a more general coarse grained operator. However, since such state-coarse-graining does not play an important role in the following, we assume the simplest form given by Eq. (2·3). After a simple calculation we can derive from Eq. (2·1) the following generalized master equation for $\tilde{D}(t)\equiv\mathcal{P}(t)D(t)$:[4]

$$\begin{aligned}\frac{d}{dt}\tilde{D}(t)=&\dot{\mathcal{P}}(t)\tilde{D}(t)\\&+[\dot{\mathcal{P}}(t)+i\,\mathcal{P}(t)\mathcal{L}_I(t)]\mathcal{V}(t,t_0)(1-\mathcal{P}(t_0))D(t_0)\\&+[\dot{\mathcal{P}}(t)+i\,\mathcal{P}(t)\mathcal{L}_I(t)]\int_{t_0}^{t}dt'\mathcal{V}(t,t')[-\dot{\mathcal{P}}(t')+i\mathcal{L}_I(t')]\tilde{D}(t'),\end{aligned} \tag{2·4}$$

where

$$\mathcal{V}(t_2,t_1)\equiv(1-\mathcal{P}(t_2))\Big[\,T\exp\int_{t_1}^{t_2}[-\dot{\mathcal{P}}(t)+i(1-\mathcal{P}(t))\mathcal{L}_I(t)]dt\Big](1-\mathcal{P}(t_1)), \tag{2·5}$$

in which T denotes the time-ordered product.

In the case in which the Hamiltonian is conservative, we can assume that $\mathcal{P}$ is time-independent. Then it follows from Eq. (2·4) that

$$\begin{aligned}\frac{d}{dt}\tilde{D}(t)=&i\,\mathcal{P}\mathcal{L}_I(t)\mathcal{V}(t,t_0)(1-\mathcal{P})D(t_0)\\&+i\,\mathcal{P}\mathcal{L}_I(t)\int_{t_0}^{t}dt'\mathcal{V}(t,t')i\mathcal{L}_I(t')\tilde{D}(t').\end{aligned} \tag{2·6}$$

In the lowest order with respect to $\mathcal{L}_I$, we can put $\mathcal{V}(t, t')\simeq 1-\mathcal{P}$ in the second term. Further we can show[5] that $\mathcal{L}_I(t)\mathcal{L}_I(t')$ vanishes for $t-t'\gg\tau_c$, in which τ_c is a characteristic time scale of the duration of particle collisions, if τ_c is much shorter than the relaxation time scale τ_r in which the density matrix $D(t)$ changes significantly. Thus by the time coarse graining of the scale τ_c, Eq. (2·6) can be written as

$$\frac{d}{dt}\tilde{D}(t)=\mathcal{R}(t)\tilde{D}(t)+i\mathcal{P}\mathcal{L}_I(t)\mathcal{V}(t, t_0)(1-\mathcal{P})D(t_0), \tag{2·7}$$

where

$$\mathcal{R}(t)\equiv i\mathcal{P}\mathcal{L}_I(t)\int_{-\infty}^{t}\mathcal{V}(t, t')i\mathcal{L}_I(t')dt'. \tag{2·8}$$

In the usual argument in the quantum statistical mechanics,[4] it is assumed that $(1-\mathcal{P})D(t_0)=0$. Then since $\mathcal{R}(t)$ is a dissipative operator, Eq. (2·7) suggests that $\tilde{D}(t)$ approaches one of the zero-eigenvalue states of $\mathcal{R}$ representing an equilibrium state, though this has not been exactly proved yet.

In ordinary systems, the assumption $(1-\mathcal{P})D(t_0)=0$ is acceptable since systems are regarded to be constantly measured with a certain spacetime accuracy. However, for the systems such as the quantum fields in dynamical spacetime we are interested in, this assumption is no longer valid. For example consider the situation in which a universe which has been stationary experiences an abrupt expansion during a short period and then becomes stationary again. In this case, since the definitions of particles in the two stationary regions do not coincide, the density matrix describing the state of quantum fields does not satisfy the condition $(1-\mathcal{P})D=0$ shortly after the abrupt expansion even if this condition was satisfied before the expansion. Nevertheless this system is expected to evolve toward some equilibrium state after a sufficiently long time as long as it contains many particles per unit volume.

This observation suggests that the operator $\mathcal{V}(t, t_0)$ has a damping property, namely, $\mathcal{V}(t, t_0)\sim 0$ for $t-t_0\gg\tau_r$.[3] This implies that the non-diagonal components of the density matrix with respect to $\{|\nu\rangle\}$ are smeared away by the interaction in the time scale τ_r. The central point of the arguments in the present paper is to assume that $\mathcal{V}(t, t')$ really has this property at least effectively and to investigate the influence of this assumption on the particle creation in expanding universe. Under this assumption Eq. (2·4) is written as

$$\frac{d}{dt}p_\nu(t)\simeq\sum_{\nu'}(\mathcal{R}_{\nu\nu'}(t)+\mathcal{C}_{\nu\nu'}(t))p_{\nu'}(t)$$

$$-\sum_{\nu'}\int_{t-\tau_r}^{t}dt' p_{\nu'}(t')\langle\nu_t|\dot{\mathcal{P}}(t)\mathcal{V}(t, t')\dot{\mathcal{P}}(t')(|\nu_{t'}\rangle\langle\nu_{t'}|)|\nu_t\rangle, \tag{2·9}$$

where

$$p_\nu(t)\equiv\langle\nu_t|D(t)|\nu_t\rangle\,, \tag{2·10}$$

and $\mathcal{R}_{\nu\nu'}(t)=\langle\nu_t|\mathcal{R}(t)|\nu_t'\rangle$, and $\mathcal{C}_{\nu\nu'}(t)$ denote the matrix components of the terms linear in $\dot{\mathcal{P}}$ in the last term of Eq. (2·4). These terms linear in $\dot{\mathcal{P}}$ represent the particle creation directly induced by the existence of interaction terms in the Hamiltonian. Since we are chiefly interested in the statistical effect of interactions on particle creation, we neglect these terms in the following. On the other hand, the last term of Eq. (2·9) quadratic in $\dot{\mathcal{P}}$ represents the genuine particle creation due to the change of the particle definition with time and survives even if $\mathcal{L}_I=0$. As is seen by the comparison with Eq. (2·4), this term represents the particle creation rate per unit time for the case in which the initial state is set at time $t-\tau_r$ so that the density matrix at that time satisfies the condition $(1-\mathcal{P})D=0$. Thus the statistical effect of interactions produced by the damping property of $\mathcal{V}(t,t')$ erases the history dependence of particle creation over time scale longer than τ_r, hence is expected to affect the essential features of particle creation.

Unfortunately we cannot solve Eq. (2·9) directly. Thereupon we instead study the problem by a simplified model formulated in the following which incorporates the essential features of Eq. (2·9). Since $\mathcal{L}_I(t)$ becomes important only for $t-t_0\gtrsim\tau_r$ in $\mathcal{V}(t,t')$, the last term of Eq. (2·9) can be interpreted as representing that the particle creation occurs freely during the time scale τ_r and then after this time scale the non-diagonal components of $D(t)$ produced by this particle creation become negligible through interactions. This means that we can incorporate the statistical effect of interactions by simply diagonalizing the density matrix with respect to $\{|\nu_t\rangle\}$ in every time interval τ_r. Hence we replace Eq. (2·9) by the following model. We divide the time into a sequence of intervals $[t_j, t_{j+1}]$ ($\delta_j t\equiv t_{j+1}-t_j\sim\tau_r$ is called the reduction time in this paper), and assume that the state of the system at $t=t_j$ is represented by a density matrix D_j which is diagonal with respect to $\{|\nu_{t_j}\rangle\}$. D_{j+1} is related with D_j by

$$D_{j+1}=\mathcal{P}(t_{j+1})\,V(j+1,j)D_j\,V(j+1,j)^{-1}\,, \tag{2·11}$$

where

$$V(j+1,j)\equiv T\exp-i\int_{t_j}^{t_{j+1}}H_I(t)dt\,. \tag{2·12}$$

Hence the change of the expectation value of a physical quantity $\mathcal{O}(t)$ which is diagonal with respect to $\{|\nu_t\rangle\}$ is given by

$$\begin{aligned}\delta_j\langle\mathcal{O}\rangle&\equiv\langle\mathcal{O}_{j+1}\rangle-\langle\mathcal{O}_j\rangle\\&=\delta_j\mathcal{O}+\mathrm{Tr}\,\mathcal{O}_{j+1}(V(j+1,j)D_jV(j+1,j)^{-1}-D_j),\end{aligned} \tag{2·13}$$

where

$$\langle \mathcal{O}_j \rangle \equiv \mathrm{Tr}\, \mathcal{O}_j D_j \,, \tag{2·14}$$

$$\delta_j \mathcal{O} \equiv \mathrm{Tr}\, \mathcal{O}_{j+1} D_j - \mathrm{Tr}\, \mathcal{O}_j D_j \,. \tag{2·15}$$

The first term $\delta_j \mathcal{O}$ in Eq. (2·13) represents the change of $\langle \mathcal{O} \rangle$ due to particle creation and the second term represents the change due to interactions. To study the particle creation following this model in detail we must specify the types of fields and the spacetime structure of universe, assign the well-defined operators to physical quantities concerned, and introduce further simplifications.

In the present paper we only consider scalar fields in a spatially flat Robertson-Walker[6] universe with a metric

$$ds^2 = R(\eta)^2(-d\eta^2 + d\boldsymbol{x}^2). \tag{2·16}$$

From now on we always work with the conformal time η unless otherwise stated, which is related with the physical time t by

$$t = \int^{\eta} R(\eta')\, d\eta' \,. \tag{2·17}$$

We assume that there exists sufficiently abundant background matter in the universe compared with the scalar particles concerned. Hence the interactions of the scalar particles with the background matter play the central role, and consequently the reduction time $\delta t (= R\delta\eta)$ is determined by these interactions. Further we assume that the thermal equilibrium of the scalar particles with the background matter is reached within the reduction time, and D_j is expressed as

$$D_j = Z_j^{-1} \exp[-(H_j - \mu_j N_j)/T_j];$$
$$Z_j = \mathrm{Tr} \exp[-(H_j - \mu_j N_j)/T_j], \tag{2·18}$$

where N is the number operator for the scalar fields, μ is the chemical potential of the scalar particles and T is the temperature of the universe. Then if the chemical reaction is sufficiently slow, Eq. (2·13) can be written as

$$\delta_j \langle \mathcal{O} \rangle = \delta_j \mathcal{O} + (\Lambda_j - \Gamma_j \langle \mathcal{O} \rangle_j)\delta_j t \,, \tag{2·19}$$

where Λ_j and Γ_j are the creation and the annihilation rate of the quantity $\langle \mathcal{O} \rangle$ through the chemical reactions. In contrast if the chemical reaction is very rapid, we can assume the chemical equilibrium at each time $t = t_j$ and $\delta_j \langle \mathcal{O} \rangle$ has no relation with $\delta_j \mathcal{O}$.

As the physical quantities we only consider the number and the energy of particles. One important problem we are interested in is how much the universe is heated up by the particle creation. To study this problem, as is shown below, it is convenient to take $Q = RH$ as the basic quantity in stead of H itself.

Suppose that the background matter is composed of only massless particles, and let H^{BG} and D^{BG} be their Hamiltonian and density matrix, respectively. Then, since the total energy is conserved through interactions, it follows from Eq. (2·13) that

$$\mathrm{Tr}(H_{j+1}D_j)+\mathrm{Tr}(H^{BG}_{j+1}D^{BG}_j)=\mathrm{Tr}(H_{j+1}D_{j+1})+\mathrm{Tr}(H^{BG}_{j+1}D^{BG}_{j+1}). \qquad (2\cdot 20)$$

Noting that $\mathrm{Tr}(H^{BG}_j D^{BG}_j)$ can be written as $CR_j{}^3T_j{}^4$ with a constant C, we can rewrite Eq. (2·20) as

$$\langle Q_{j+1}\rangle-\langle Q_j\rangle+C(R^4_{j+1}T^4_{j+1}-R_j{}^4T_j{}^4)=\delta_jQ\,, \qquad (2\cdot 21)$$

where $R_j=R(\eta_j)$. From this equation we find that the sum of δ_jQ over all intervals is related only with the initial and the final state of the universe. Furthermore the third term in the left-hand side, the background matter part, is directly related with the change of entropy of the background matter.[7] From these facts, we call δ_jQ the entropy production due to particle creation.

If the number production and/or the entropy production due to particle creation is sufficiently large, they affect the temperature and the expansion of the universe. In this case we must determine T_{j+1}, μ_{j+1} and R_{j+1} from T_j, μ_j and R_j step by step. The prescription for this is given by Eq. (2·19) for $\mathcal{O}=N$, Eq. (2·21) and one of the Einstein equations, $[(R_{j+1}-R_j)/\delta\eta]^2\simeq\dot{R}_j{}^2=(8\pi/3)R_j{}^4\rho_j$ (where ρ is the total energy density of the universe).[6]

The remaining task in the formulation of our model is to specify the operators H and N, and the eigenstates $\{|\nu_t\rangle\}$ explicitly. For that purpose we first recapitulate the fundamental formulas for the quantum free scalar fields in a Robertson-Walker universe. For the sake of simplicity we write down the formulas only for a single neutral scalar field in this section. If we want to discuss multiple fields and/or charged fields, we only have to sum up the quantities corresponding to each component field in the final expressions.

The Lagrangian density of a neutral scalar field ϕ in a curved spacetime with a metric $g_{\mu\nu}$ is given by[1]

$$\mathcal{L}=-\frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\left(m^2+\frac{\xi}{6}\mathcal{S}\right)\phi^2\right], \qquad (2\cdot 22)$$

where m is the mass of the field, $\mathcal{S}$ is the Ricci scalar of the metric $g_{\mu\nu}$, and ξ is the coupling parameter of the scalar field with the geometry. In the spatially flat Robertson-Walker universe given by (2·16), $\mathcal{L}$ is written as

$$\mathcal{L}=\frac{1}{2}R^2[\dot{\phi}^2-\partial_j\phi\partial_j\phi-(m^2R^2+\xi R^{-1}\ddot{R})\phi^2], \qquad (2\cdot 23)$$

where the dot denotes the differentiation with respect to the conformal time η.

We quantize ϕ following the usual canonical scheme.[1] In order to avoid the familiar difficulty[8] associated with infinite volume, we limit the field in a finite coordinate volume $V=L^3$ by the periodic boundary condition in the intermediate stage of calculation and take the $V\to\infty$ limit in the final expressions. Then we can express ϕ by creation and annihilation operators, $a_{\boldsymbol{k}}^\dagger$ and $a_{\boldsymbol{k}}$, as follows:

$$\phi(\eta,\boldsymbol{x})=R(\eta)^{-1}\sum_{\boldsymbol{k}}(f_{\boldsymbol{k}}(\eta)\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x})a_{\boldsymbol{k}}+f_{\boldsymbol{k}}(\eta^*)\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x}^*)a_{\boldsymbol{k}}^\dagger), \tag{2·24}$$

where $\boldsymbol{k}$ runs over the lattice points given by $(2\pi/L)\boldsymbol{n}$ with integer vectors $\boldsymbol{n}$, and

$$\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x})=V^{-1/2}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{2·25}$$

The mode functions $\{f_{\boldsymbol{k}}(\eta)\}$ in Eq. (2·24) are a set of solutions of the equation

$$\ddot{f}_{\boldsymbol{k}}+\Omega_k{}^2 f_{\boldsymbol{k}}=0\,; \tag{2·26}$$

$$\Omega_k{}^2=k^2+m^2R^2+(\xi-1)R^{-1}\ddot{R}\,, \qquad (k=|\boldsymbol{k}|) \tag{2·27}$$

and satisfy the normalization condition

$$f_{\boldsymbol{k}}\dot{f}_{\boldsymbol{k}}{}^*-f_{\boldsymbol{k}}{}^*\dot{f}_{\boldsymbol{k}}=i\,. \tag{2·28}$$

Each choice of a set $\{f_{\boldsymbol{k}}(\eta)\}$ determines one Fock representation of ϕ.

Since H and N should commute with each other in order that the chemical potential is well-defined, the specification of H entails that of N and $\{|\nu_\eta\rangle\}$. Namely it is the most natural to take as $\{|\nu_\eta\rangle\}$ the Fock basis diagonalizing $H(\eta)$ at time η, and as N the corresponding number operator. Thus we only have to give H. As is clear from the above consideration, H should be the canonical Hamiltonian. However, as is well known, this requirement does not determine H uniquely in dynamical spacetime due to the freedom of the choice of the canonical variable.[9] In spatially flat Robertson-Walker universe this freedom is represented by the momentum dependent canonical transformation

$$\tilde{\phi}_{\boldsymbol{k}}(\eta)=\lambda_{\boldsymbol{k}}(\eta)\phi_{\boldsymbol{k}}(\eta), \tag{2·29}$$

where $\phi_{\boldsymbol{k}}(\eta)=\int_V d^3x\phi(\eta,\boldsymbol{x})\mathcal{Y}_{\boldsymbol{k}}(\boldsymbol{x})$. Let us specify the mode functions $\{f_{\boldsymbol{k}(\eta')}(\eta)\}$ corresponding to the Fock basis at $\eta=\eta'$, $\{|\nu_{\eta'}\rangle\}$, by the ratio of $\dot{f}_{\boldsymbol{k}}(\eta')$ to $f_{\boldsymbol{k}}(\eta')$, $-i\mu_{\boldsymbol{k}}+\gamma_{\boldsymbol{k}}$ ($\mu_{\boldsymbol{k}}>0$ and $\gamma_{\boldsymbol{k}}$ is real):

$$\dot{f}_{\boldsymbol{k}(\eta')}=(-i\mu_{\boldsymbol{k}}+\gamma_{\boldsymbol{k}})f_{\boldsymbol{k}(\eta')} \quad \text{at} \quad \eta=\eta'\,. \tag{2·30}$$

Then we can easily show that the Fock representation which diagonalizes the canonical Hamiltonian corresponding to $\tilde{\phi}_{\boldsymbol{k}}$ is characterized by the following choice of $\mu_{\boldsymbol{k}}$ and $\gamma_{\boldsymbol{k}}$:[9]

$$\mu_{\boldsymbol{k}}{}^2=k^2+m^2R^2+\xi R^{-1}\ddot{R}-\lambda_{\boldsymbol{k}}{}^{-1}\ddot{\lambda}_{\boldsymbol{k}}-2R^{-1}\lambda_{\boldsymbol{k}}(R\lambda_{\boldsymbol{k}}{}^{-1})^{\cdot}+(R^{-1}\lambda_{\boldsymbol{k}})^2\dot{A}_{\boldsymbol{k}}\,, \tag{2·31}$$

$$\gamma_k = -R\lambda_k^{-1}(R^{-1}\lambda_k)^{\cdot}, \tag{2·32}$$

where A_k represents the contribution to the Hamiltonian from the total derivative term of the Lagrangian which can be added arbitrarily.

What is interesting is that we can limit the freedom of the choice of $\lambda_k(\eta)$ and $A_k(\eta)$ by the general conditions resulting from the requirement that our formalism is consistent. One condition is that the Fock bases $\{|\nu_\eta\rangle\}$ at different times should span the same Hilbert space. The other condition is the finiteness of the entropy production δQ at each time interval. The first condition imposes the following constraint on γ_k:[10)]

$$\gamma_k = o(k^{-1/2}). \qquad (k\sim\infty) \tag{2·33}$$

Applying this constraint to Eq. (2·32), we obtain

$$\lambda_k = R + o(k^{-1/2}). \qquad (k\sim\infty) \tag{2·34}$$

On the other hand, as is shown in Appendix A, the second condition yields the constraint on $\dot{A}_k$:

$$\dot{A}_k = o(1). \qquad (k\sim\infty) \tag{2·35}$$

Since we can unfortunately find no principle to determine the small-k behavior of λ_k and $\dot{A}_k$, we will take from now on the simplest choice, suggested by Eqs. (2·34) and (2·35):

$$\lambda_k = R, \qquad \dot{A}_k = 0. \tag{2·36}$$

Another problem associated with the specification of H is the well-known divergence problem.[11)] In general in curved space time the regularized Hamiltonian is composed of the normal material part and the vacuum polarization part.[12)] The latter part becomes important only in the era in which the spacetime curvature scale is of the order of Planck length.[13)] However, as will be discussed later, our model is not applicable to such an era. Thereupon we neglect the vacuum polarization part and assume that the normal-ordered Hamiltonian with respect to the Fock representations which diagonalize the Hamiltonian at each time yields a good approximation of the genuine regularized Hamiltonian henceforth.

To summarize, we take as H the canonical Hamiltonian obtained by taking $\tilde{\phi} = R\phi$ as the canonical variable. Then the mode functions $\{f_{k(j)}(\eta)\}$ corresponding to the Fock representation diagonalizing this Hamiltonian at $\eta = \eta_j$ are specified by the conditions

$$\mu_k = \Omega_k(\eta_j), \qquad \gamma_k = 0. \tag{2·37}$$

Let $a^\dagger_{k(j)}$ and $a_{k(j)}$ be the creation and the annihilation operators corresponding to these mode functions. Then the particle number operator at $\eta=\eta_j$, N_j, and the Hamiltonian at $\eta=\eta_j$, H_j, are expressed as follows:

$$N_j=\sum_k N_{k(j)}\,; \qquad N_{k(j)}=a^\dagger_{k(j)}a_{k(j)}\,, \tag{2·38}$$

$$H_j=R_j^{-1}\sum_k \Omega_k(\eta_j)N_{k(j)}\,. \tag{2·39}$$

Finally for later applications let us express δN and δQ in terms of the Bogoliubov transformation coefficients,[1] $\alpha_{k(j)}$ and $\beta_{k(j)}$, between the two Fock representations at $\eta=\eta_j$ and $\eta=\eta_{j+1}$. These coefficients are defined by the relation

$$f_{k(j+1)}(\eta)=\alpha_{k(j)}f_{k(j)}(\eta)+\beta_{k(j)}f_{k(j)}(\eta)^*\,, \tag{2·40}$$

which is equivalent to the relation

$$a_{k(j+1)}=\alpha^*_{k(j)}a_{k(j)}-\beta^*_{k(j)}a^\dagger_{k(j)}\,. \tag{2·41}$$

Substituting Eqs. (2·38), (2·39) and (2·41) into Eqs. (2·15) for N and Q, we obtain the following expressions for the basic quantities:

$$\delta_j N=\sum_k |\beta_{k(j)}|^2[1+2\,\mathrm{Tr}(D_jN_{k(j)})], \tag{2·42}$$

$$\delta_j Q=\sum_k[\Omega_k(\eta_{j+1})|\beta_{k(j)}|^2\{1+2\,\mathrm{Tr}(D_jN_{k(j)})\}+(\delta_j\Omega_k)\mathrm{Tr}(D_jN_{k(j)})], \tag{2·43}$$

where $\delta_j\Omega_k=\Omega_k(\eta_{j+1})-\Omega_k(\eta_j)$. The first terms in Eqs. (2·42) and (2·43) correspond to the particle creation from vacuum, and the second terms to the induced creation due to the existence of particles.[8] In contrast, the third term in Eq. (2·43) represents the kinematical entropy production due to the time dependence of $\Omega_k(\eta)$, and has no relation with particle creation.

§ 3. Particle creation in the finite-time reduction model

In this section we evaluate the number and the entropy production of created particles in expanding universe by the finite-time reduction model formulated in § 2. First of all we remark on the applicable range of this model. In this model it is assumed that newly created particles come in thermal equilibrium with the background matter of universe within the reduction time scale. This assumption holds only when the reduction time δt, which is the order of the characteristic time scale of interactions, is much shorter than the cosmic expansion time scale $\tau_{\exp}\equiv R/(dR/dt)$, namely, the condition

$$\delta t/\tau_{\exp}\ll 1 \tag{3·1}$$

is satisfied. Thus our model is applicable only to the period in which this condition is satisfied. In the period in which $\delta t/\tau_{\rm exp}\gg 1$ interactions of particles are effectively frozen and the particle creation should be evaluated by the conventional free field theory.

First we evaluate the particle creation during each reduction time interval $[\eta,\eta+\delta\eta]$ without specifying the expansion law of the universe. We omit the index of intervals from now on. In order to eliminate the artificial volume dependence, we rewrite Eqs. (2·42) and (2·43) into the density forms. Let us write δN and δQ as $V\delta(R^3 n_c)$ and $V\delta(R^4\rho_c)$. Further we replace $\sum_k$ by the integration $(2\pi)^{-3}V\int d^3k$. Then after a short calculation we obtain the following formulas:

$$\delta(R^3 n_c)=\frac{1}{2\pi^2}\int_0^\infty dkk^2|\beta_k|^2+\frac{1}{\pi^2}\int_0^\infty dkk^2|\beta_k|^2(e^{(\omega-\mu)/T}-1)^{-1}\,, \qquad (3\cdot2)$$

$$\delta(R^4\rho_c)=\frac{1}{2\pi^2}\int_0^\infty dkk^2\Omega_k|\beta_k|^2+\frac{1}{\pi^2}\int_0^\infty dkk^2\Omega_k|\beta_k|^2(e^{(\omega-\mu)/T}-1)^{-1}$$
$$+\frac{1}{2\pi^2}\int_0^\infty dkk^2(\delta\Omega_k)(e^{(\omega-\mu)/T}-1)^{-1}\,, \qquad (3\cdot3)$$

where

$$\omega^2=R^{-2}\Omega^2=R^{-2}k^2+m^2+(\xi-1)R^{-3}\ddot{R}\,. \qquad (3\cdot4)$$

Since we assume the condition (3·1), we only consider the lowest order effect in $\delta t/\tau_{\rm exp}$. In this approximation the Bogoliubov coefficient β_k in the interval $[\eta,\eta+\delta\eta]$ is given by (see Appendix A)

$$|\beta_k|^2\simeq\frac{1}{4}\frac{m^4R^2\dot{R}^2}{\Omega_k{}^6}\sin^2(\Omega_k\delta\eta), \qquad (3\cdot5)$$

where $\delta\eta=\delta t/R$. Substituting this equation into Eqs. (3·2) and (3·3), and changing the integration variable from k to $x=\Omega_k/mR$, we obtain

$$\delta(R^3n_c)=I_1+I_2\,; \qquad (3\cdot6)$$

$$I_1=\frac{1}{8\pi^2}\frac{m\dot{R}^2}{R}\int_1^\infty dx\frac{\sqrt{x-1}}{x^5}\sin^2(mR\delta\eta x), \qquad (3\cdot6\text{a})$$

$$I_2=\frac{1}{4\pi^2}\frac{m\dot{R}^2}{R}\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^5}\sin^2(mR\delta\eta x)(e^{mx/T}-1)^{-1}\,, \qquad (3\cdot6\text{b})$$

and

$$\delta(R^4\rho_c)=I_3+I_4+I_5\,; \qquad (3\cdot7)$$

$$I_3=\frac{1}{8\pi^2}m^2\dot{R}^2\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^4}\sin^2(mR\delta\eta x), \qquad (3\cdot7\text{a})$$

$$I_4=\frac{1}{4\pi^2}m^2\dot{R}^2\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^4}\sin^2(mR\delta\eta x)(e^{mx/T}-1)^{-1}\,, \tag{3·7b}$$

$$I_5=\frac{1}{\pi^2}m^4R^3\dot{R}\delta\eta\int_1^\infty dx\sqrt{x^2-1}(e^{mx/T}-1)^{-1}\,. \tag{3·7c}$$

We have set $\mu=0$ in these equations because we assume that the scalar particles are in chemical equilibrium with the background matter for $T\gg m$, and I_2, I_4 and I_5 can be neglected compared with I_1 and I_3 for $T\ll m$ in the first order approximation. It is not difficult to write down the corresponding formulas for the general case $\mu\neq 0$ in the following.

In order to estimate the integrals we consider the following three cases. The necessary mathematical formulas are given in Appendix B.

(i) $T>m$ and $mR\delta\eta<1$

From Eqs. (B·5), (B·9), (B·12), (B·13) and (B·15) it follows that

$$\frac{\delta(R^3n_c)}{\delta\eta}\simeq m^3R\dot{R}\left(\frac{1}{32\pi}+\frac{1}{24\pi^2}\frac{T}{m}\right)\delta\eta\,, \tag{3·8}$$

$$\frac{\delta(R^4\rho_c)}{\delta\eta}\simeq m^4R^2\dot{R}^2\left\{\frac{1}{8\pi^2}\left(\frac{1}{2}-\gamma-\ln(mR\delta\eta)\right)+\frac{1}{32\pi}\frac{T}{m}\right\}\delta\eta$$
$$+\frac{1}{12}m^4R^3\dot{R}\left(\frac{T}{m}\right)^2, \tag{3·9}$$

where γ is the Euler constant.

(ii) $T>m$ and $mR\delta\eta>1$

From Eqs. (B·4), (B·11) and (B·13) it follows that

$$\frac{\delta(R^3n_c)}{\delta\eta}\simeq\frac{m\dot{R}^2}{R}\left(\frac{1}{256\pi}+\frac{1}{60\pi^2}\frac{T}{m}\right)\frac{1}{\delta\eta}\,, \tag{3·10}$$

$$\frac{\delta(R^4\rho_c)}{\delta\eta}\simeq m^2\dot{R}^2\left(\frac{1}{48\pi^2}+\frac{1}{256\pi}\frac{T}{m}\right)\frac{1}{\delta\eta}+\frac{1}{12}m^4R^3\dot{R}\left(\frac{T}{m}\right)^2. \tag{3·11}$$

(iii) $T<m$ and $mR\delta\eta>1$

In this case the temperature-dependent terms become negligible due to the factor $(e^{mx/T}-1)^{-1}$ even in the chemical equilibrium case and only the first terms of Eqs. (3·10) and (3·11) survive:

$$\frac{\delta(R^3n_c)}{\delta\eta}\simeq\frac{1}{256\pi}\frac{m\dot{R}^2}{R}\frac{1}{\delta\eta}\,, \tag{3·12}$$

$$\frac{\delta(R^4\rho_c)}{\delta\eta}\simeq\frac{1}{48\pi^2}m^2\dot{R}^2\frac{1}{\delta\eta}\,. \tag{3·13}$$

To sum up the contributions from each time interval $[\eta_j, \eta_{j+1}]$ given by Eqs. (3·8)~(3·13) to obtain the total amount of the number and the entropy production of created particles, we must specify the expansion law of universe. As explained in § 2, exactly speaking, the change of the cosmic scale factor is coupled with the particle creation through the Einstein equation. Hence R and T (and μ if necessary) are determined step by step taking into account the effect of particle creation. However, when we only want to estimate whether such backreaction effect of particle creation is negligible or not, it is sufficient to calculate neglecting the backreaction on R and T and assuming that the background matter is closed itself. If the resultant total energy and the entropy production of created particles are negligible compared with the energy of the background matter, this assumption is justified and the corresponding result of calculation can be considered to give a good approximation. Following this idea we next evaluate the particle creation in the fixed radiation dominant Friedmann universe.

The cosmic scale factor $R(\eta)$ for the radiation dominant Friedmann universe is given by[6)]

$$R=\frac{1}{2}\eta\ , \tag{3·14}$$

normalizing R as unity at Planck time $t=1$ ($\eta=2$), where the physical time t is related with the conformal time η by

$$t=\frac{1}{4}\eta^2\ . \tag{3·15}$$

The temperature of the universe at time η is given by[6)]

$$T=T_*/R=2T_*/\eta\ , \tag{3·16}$$

where T_* is the temperature at Planck time, which is expressed by the number of particle species g at that time as

$$T_*=\frac{1}{2}\left(\frac{2\pi^3}{45}g\right)^{-1/4} \tag{3·17}$$

We assume that the reduction time δt is equal to the characteristic interaction time $\tau_{\rm int}$ of the scalar particles with the background matter. Furthermore we assume that the interactions are of the gauge field type with a universal coupling constant $\sim\sqrt{\alpha}$, following the recent trend of particle physics. Then $\tau_{\rm int}$ $(=1/\langle\sigma v\rangle n_{\rm BG}$; $n_{\rm BG}$ denotes the number density of the background particles interacting with the concerned scalar particles) is given as

$$\tau_{\text{int}} \simeq \begin{cases} \dfrac{1}{\alpha^2 T^{-2} \times h T^3} = \dfrac{1}{\alpha^2 h T} & \text{for } T \gg m\,, \quad (3\cdot 18\text{a}) \\ \dfrac{1}{\alpha^2 m^2 \times h T^3} = \dfrac{m^2}{\alpha^2 h T^3} & \text{for } T \ll m\,, \quad (3\cdot 18\text{b}) \end{cases}$$

where h denotes the number of particle species which have masses smaller than the temperature T and participate in the interactions. We assume that $h\alpha^2 \lesssim g\alpha^2 < 1$. Thus the condition (3·1) is in the present case expressed as

$$\tau_{\text{int}}/\tau_{\text{exp}} \simeq \begin{cases} \dfrac{1}{\alpha^2 h T_* \eta} \ll 1 & \text{for } T \gg m\,, \quad (3\cdot 19\text{a}) \\ \dfrac{m^2 \eta}{4\alpha^2 h T_*^3} \ll 1 & \text{for } T \ll m\,. \quad (3\cdot 19\text{b}) \end{cases}$$

In general the statistical weight h changes with time. In most cases we can safely put $h=g$ for $T>m$ and $h=h_m$ (the value of h at $T\sim m$) for $T<m$ in Eq. (3·19). Thus we find that our model is applicable in the period

$$\eta_i \equiv \frac{1}{\alpha^2 g T_*} \ll \eta \ll \frac{4\alpha^2 h_m T_*^3}{m^2} \equiv \eta_f\,. \qquad (3\cdot 20)$$

Especially from this we obtain the constraint on the mass:

$$m \ll M \equiv 2\alpha^2 \sqrt{g h_m}\, T_*^2 \simeq 0.4\alpha^2 \sqrt{h_m}\,. \qquad (3\cdot 21)$$

Taking $\alpha = 10^{-2}$ and $h_m = 10^2$, for example, Eq. (3·21) yields the condition $m \ll 5 \times 10^{15}$ GeV. Particles with mass $m > M$ never come in thermal equilibrium with the background matter and their creation can be evaluated by the conventional free field theory.

Now we evaluate the total amount of particle creation in the Friedmann universe. Since $\delta t = R\delta\eta \ll \tau_{\text{exp}}$, we can replace the summation of $\delta(R^3 n_c)$ and $\delta(R^4 \rho_c)$ at each time interval by the integration of $\delta(R^3 n_c)/\delta\eta$ and $\delta(R^4\rho_c)/\delta\eta$ in terms of η. According to the cases (i)~(iii) in the estimation of $\delta(R^3 n_c)$ and $\delta(R^4\rho_c)$ we divide the period (3·20) into three sub-periods. Then the total amounts of the number and the entropy of created particles in each sub-period are given as follows:

(i) $\eta_i \equiv (\alpha^2 g T_*)^{-1} \ll \eta < 2\alpha^2 h_1 T_*/m$

(h_1 is the value of h at the time $mR\delta\eta = 1$)

$$\Delta(R^3 n_c) \simeq \frac{1}{128\pi}\alpha^2 h_1 T_* m + \frac{1}{48\pi^2} T_* m\,, \qquad (3\cdot 22)$$

$$\Delta(R^4 \rho_c) \simeq \frac{1}{48\pi^2}\left(\frac{5}{6} - \gamma\right)\alpha^4 h_1^2 T_*^2 m + \frac{1}{128\pi}\alpha^2 h_1 T_*^2 m + \frac{1}{48}\alpha^4 h_1^2 T_*^4\,. \qquad (3\cdot 23)$$

In order that this sub-period exists, m must satisfy the condition

$$m<2a^4 g h_1 T_*^2 \simeq 0.4 a^4 h_1 \sqrt{g}\ . \tag{3·24}$$

(ii) $2a^2 h_1 T_*/m<\eta<2T_*/m$

$$\Delta(R^3 n_c)\simeq\frac{1}{512\pi}a^2 h_2 T_* m \ln\left(\frac{1}{a^2 h_1}\right)+\frac{1}{120\pi^2}T_* m\ , \tag{3·25}$$

$$\Delta(R^4\rho_c)\simeq\frac{1}{96\pi^2}a^2 h_2 T_*^2 m+\frac{1}{512\pi}a^2 h_2 T_* m \ln\left(\frac{1}{a^2 h_1}\right)+\frac{1}{48}T_*^4(1-a^4 h_1^2), \tag{3·26}$$

where h_2 is the mean value of h in this sub-period. In the case in which $m>2a^4 g h_1 T^{*2}$ and the sub-period (i) does not exist, we should replace $\ln(1/a^2 h_1)$ by $\ln(2a^2 g T_*^2/m)$ in Eqs. (3·25) and (3·26).

(iii) $2T_*/m<\eta\ll 4a^2 h_m T_*^3/m^2\equiv\eta_f$

$$\Delta(R^3 n_c)\simeq\frac{1}{1024\pi}a^2 h_m T_* m\ , \tag{3·27}$$

$$\Delta(R_4\rho_c)\simeq\frac{1}{96\pi^2}a^2 h_m T_*^2 m\ . \tag{3·28}$$

In order that this subperiod exists, m must satisfy the condition

$$m\ll 2a^2 h_m T_*^2\ . \tag{3·29}$$

Note that the major parts of $\Delta(R^3 n_c)$ and $\Delta(R^4\rho_c)$ in this subperiod are created around $T\sim m$ since $\delta(R^3 n_c)/\delta\eta$ and $\delta(R^4\rho_c)/\delta\eta$ are monotonic decreasing functions of η as is seen from Eqs. (3·12) and (3·13).

§ 4. Free particle creation in Friedmann universe

For the later comparison with the result in the finite reduction-time model, we evaluate the scalar particle creation in the radiation dominant Friedmann universe in the conventional free field theory.[1] We assume that the field is a neutral scalar field and that the definition of particles at each time is given by the simultaneous Hamiltonian diagonalizing one given in § 2. Furthermore in this section we only consider the particle creation from vacuum, i.e., we assume that the initial state of the field is the vacuum state at the starting point of the universe.

The equation for the mode functions (2·26) is expressed in this case from Eq. (3·14) as

$$\ddot{f}_k+\left(k^2+\frac{1}{4}m^2\eta^2\right)f_k=0\,. \tag{4·1}$$

Note that this equation has no singularity at $\eta=0$, the big-bang point of the universe. This is the reason why we can set the initial condition for the field at $\eta=0$. By setting $x=\sqrt{m}\eta$ and $a=-k^2/m$, Eq. (4·1) is transformed to

$$f_k''+\left(\frac{1}{4}x^2-a\right)f_k=0\,, \tag{4·2}$$

where the prime denotes the differentiation with respect to x. Solutions of Eq. (4·2) are written as linear combinations of the parabolic cylindrical functions $E(a,x)$ and $E(a,x)^*$ (see Ref. 14)). Let $X_k(\eta)$ be the solution of Eq. (4·1) expressed by $E(a,x)$ as

$$X_k(\eta)=(E(a,x)/E(a,0))^*\,, \tag{4·3}$$

and define σ_k by

$$\sigma_k\equiv i\dot{X}_k(0)=i\sqrt{m}(E'(a,0)/E(a,0))^*\,. \tag{4·4}$$

Then the Bogoliubov coefficients $\beta_k(\eta)$ between the mode functions at $\eta=0$ and η specified by Eq. (2·30) with $\mu_k{}^2=\Omega_k{}^2=k^2+m^2\eta^2/4$ are given by[10]

$$|\beta_k(\eta)|^2=\frac{|\mathcal{D}|^2}{16k\Omega_k(\mathrm{Re}\,\sigma_k)^2}\,, \tag{4·5}$$

where Re σ_k denotes the real part of σ_k and $\mathcal{D}$ is given by

$$\mathcal{D}=-i(\sigma_k-k)(\dot{X}_k{}^*+i\Omega_kX_k{}^*)-i(\sigma_k{}^*+k)(\dot{X}_k+i\Omega_kX_k). \tag{4·6}$$

The number density of created particles at time η is given by the first term of Eq. (3·2) in the present case. Unfortunately we cannot perform the k-integration exactly. Therefore we ought to be satisfied with a rough estimate obtained by replacing $|\beta_k|^2$ by its asymptotic expansion for $\eta\gg1/\sqrt{m}$. After a little elaborate calculation using the asymptotic formulas for $E(a,x)$ and $E'(a,x)$,[14] we find the following approximate expression for $|\beta_k|^2$:

$$|\beta_k|^2\simeq\begin{cases}\dfrac{m^4\eta^2}{256\pi k^6} & \text{for } k\gg m\eta\,, & (4·7\mathrm{a})\\[2mm] \dfrac{1}{8m^2\eta^4} & \text{for } m^{3/4}\eta^{1/2}<k\ll m\eta\,, & (4·7\mathrm{b})\\[2mm] \dfrac{m^{1/2}}{4k}g(a) & \text{for } k<m^{3/4}\eta^{1/2}\,, & (4·7\mathrm{c})\end{cases}$$

where

$$g(a)=\frac{4|\sigma_k/\sqrt{m}|\times|\sigma_k/\sqrt{m}-\sqrt{|a|}|}{(1+e^{2\pi a})^{1/2}(\mathrm{Re}\,\sigma_k/\sqrt{m})^2}\,. \qquad (4\cdot8)$$

Note that $g(a)$ is regular at $a=0$.

Substituting Eq. (4·7) into the first term of Eq. (3·2) and performing the integration, we obtain the estimate for the number density n of created particles at $\eta\gg 1/m$:

$$\begin{aligned} R^3 n &\equiv \frac{1}{2\pi^2}\int_0^\infty dkk^2|\beta_k|^2 \\ &\simeq \frac{1}{1536\pi^2}\frac{m}{\eta}+\frac{1}{192\pi^2}\frac{m}{\eta}+\frac{m^{3/2}}{16\pi^2}\int_0^{\sqrt{m}\,\eta} d(-a)g(a), \end{aligned} \qquad (4\cdot9)$$

where each term corresponds to the contribution from each range of k in Eq. (4·7) in order. Noting that $g(a)$ has the asymptotic behavior

$$g(a)\sim\frac{1}{128|a|^{7/2}}\,, \qquad (a\sim-\infty) \qquad (4\cdot10)$$

we can see that the particle creation stops for $\eta\gg 1/\sqrt{m}$ ($t\gg 1/m$), and eventually $R^3 n$ approaches a constant:

$$R^3 n \longrightarrow m^{3/2}\times\mathrm{const}\,. \qquad (\eta\to\infty) \qquad (4\cdot11)$$

This result for the total number of created particles coincides with the one obtained by Audretsh and Schäfer for a different initial condition,[15] though the spectra are different. We can find the era during which the particle creation is most active from the comparison with Frolov et al.'s calculation for the case $\eta\ll 1/\sqrt{m}$.[16] They showed for this case

$$R^3 n \simeq \frac{1}{24\pi^2}m^3 R^3\,. \qquad (4\cdot12)$$

The right-hand side of Eq. (4·9) is a decreasing function of η for $\eta\gg 1/\sqrt{m}$ and the right-hand side of Eq. (4·12) is an increasing function of η, and both give the same order of values at $\eta\sim 1/\sqrt{m}$, which is proportional to $m^{3/2}$. Thus we can conclude that, in the conventional free field theory, the particle creation of scalar particles in the Friedmann universe is most active around $\eta\sim 1/\sqrt{m}$, namely around the compton time $t\sim 1/m$.

§ 5. Statistical effect of interactions on particle creation

First we summarize the characteristic features of particle creation in the finite-time reduction model estimated in § 3. Equations (3·8)~(3·13) with Eqs. (3·14) and (3·16) show that the particle creation occurs most actively around

$mR\delta\eta \sim 1$ ($\eta \sim 2\alpha^2 h_1 T_*/m$) and makes its pace down slowly in the period (ii) ($2\alpha^2 h_1 T_*/m < \eta < 2T_*/m$), and ceases when $T \ll m$. Note that a comparable amount of particles are created in the period (iii) ($T < m$) compared with the previous periods, as is seen from Eqs. (3·22)~(3·28). Some implication of this fact will be discussed later. Another feature we can notice from these equations is that as for the number the contribution from the induced particle creation (the second terms in Eqs. (3·22) and (3·25)) is greater than that from the vacuum creation (the first terms) by the factor $1/\alpha^2 h$ in the stage $T > m$.

Concerning the total amount of created particles, both the number $\Delta(R^3 n_c)$ and the entropy production $\Delta(R^4 \rho_c)$ are proportional to the mass of the scalar particles (see the first and the second terms of Eqs. (3·22)~(3·26), and Eqs. (3·27) and (3·28)). Especially the entropy production due to particle creation is much smaller than the kinematical entropy production (the third terms of Eqs. (3·23) and (3·26)) under the constraint on the mass (3·20). This kinematical term is further smaller than $V^{-1}R\,\mathrm{Tr}(H^{\mathrm{BG}}D^{\mathrm{BG}}) = R^4 \rho^{\mathrm{BG}} \simeq (2\pi^2/15)gT_*^4$ since $\alpha^2 h \ll 1$. Therefore from Eq. (2·21) we can conclude that the entropy production due to particle creation, hence its effect on the temperature, is negligible for scalar particles in the radiation dominant Friedmann universe.

In the stage $T > m$, this means that the effect of particle creation on the time evolution of the universe is negligible since in this stage the temperature completely determines the energy density of the universe including the contribution of the scalar particles concerned. In contrast, in the stage $T < m$, the scalar particles go out of chemical equilibrium with the background matter within some time after $T \sim m$. Hence there is a possibility that the particles created after $T \sim m$ survive and make the universe matter dominant even when the universe is matter-anti-matter symmetric. In the real universe with a small baryon excess, however, we can show that such an effect does not become important in any stage. In fact, after short calculation, it is shown that the mass of the scalar particles must be larger than 10^9 GeV in order that the mass density of the surviving created particles exceeds the mass density due to the baryon asymmetry. We can show that these massive particles decay away before their energy density exceeds the radiation energy density unless the decay life is extraordinarily long. Therefore we can conclude that the created particles affect the cosmic expansion only through the entropy production even in the stage $T < m$, hence such effect is negligible.

Now we compare these features with those in the conventional free field theory discussed in § 4. The most important difference of them is the time variation of particle creation. As shown in § 4, the particle creation in the conventional theory occurs most actively around compton time $t \sim 1/m$ ($\eta = \eta_c \sim 1/m$). In contrast, it continues until after $T \sim m$ ($\eta = \eta_m \sim 2T_*/m$) in our model. Since $\eta_m \gg \eta_c$ in the mass range (3·21), we find that the period in which particle creation actively occurs is lengthened when the interactions of the particles with the background matter are taken into account.

Next we compare the number of created particles. Since we have assumed in § 4 that the field is initially at vacuum state, we should compare the result of § 4 with the first terms of Eqs. (3·22), (3·25) and (3·27) (the particle creation from vacuum). The total number of created particles is proportional to $m^{3/2}$ in the conventional case, and to m in our model. The latter is larger than the former from the condition (3·20). Further the first terms of $\Delta(R^3 n_c)$ in § 3 increase if the interaction is effectively made stronger by increasing the value of α. These observations show that the statistical effect of interactions increases the number of created particles, and the stronger the interactions are, the larger the number becomes.

Finally we comment on a cosmological implication of the lengthening of the active particle creation era in the finite-time reduction model. Recently a possibility has been actively studied by many authors that the cosmological baryon-anti-baryon asymmetry can be explained by the baryon-number nonconserving processes in the grand unified gauge theories.[17] In these arguments heavy bosons which decay through C and CP violating interactions and produce baryon number play an important role. Here we estimate the baryon number production due to the decay of such heavy bosons created in the $T < m$ stage.

Let us suppose that heavy bosons X with mass m_X produce a baryon number ΔB per one boson by decay. We assume that these particles are sufficiently heavy and come out of chemical equilibrium with the background matter soon after $T \sim m_X$. Then Eq. (3·27) yields the following estimate of the baryon-to-entropy ratio produced by the decay of X-bosons created after $T \sim m$:

$$n_B/n_\gamma \simeq \frac{(1/512\pi)\alpha_X^2 h_{m_X} T_* m_X}{(4\pi^2/45) g T_*^3} \times N_X \Delta B$$

$$\simeq 7\times 10^{-4} \alpha_X^2 \left(\frac{h_{m_X}}{g}\right)\left(\frac{m_*}{T_*}\right)^2 \left(\frac{m_X}{m_*}\right) N_X \Delta B\,, \tag{5·1}$$

where N_X is the number of species of X-bosons. If we put, for example, $\alpha_X^2 = 10^{-3}$, $h_m \simeq g = 200$, $m_X = 5\times 10^{-4} m_* \simeq 5\times 10^{15}$ GeV, $N_X \simeq 50$ and $\Delta B = 10^{-6}$, we obtain $n_B/n_\gamma \simeq 10^{-12}$. Though this value is a little smaller than the required value $n_B/n_\gamma = 10^{-8} \sim 10^{-10}$, this result indicates yet that the particle creation may play an important role in the problem of cosmological baryon number production, taking into account the roughness of our estimation of particle creation.

§ 6. Discussion

In this paper we have formulated the finite-time reduction model in order to study the statistical effect of interactions on particle creation in expanding universe, and applied it to scalar fields in the radiation dominant Friedmann universe. We have shown that the particle creation increases if this statistical effect is taken into account, and furthermore the active era of particle creation is much lengthened, compared with the result in the conventional free field theory. Especially we have shown that a comparable amount of particles are created in the period $T < m$ compared with the preceding period, and this result may play an important role in the problem of the cosmological baryon number production.

The most important new aspect opened by the finite-time reduction model is that we have become able to discuss the entropy production due to particle creation and its backreaction on the time evolution of universe. Unfortunately it has been shown that the entropy production due to the scalar particle creation in Friedmann universe is small compared with the entropy of the background matter, and hence its backreaction is negligible. However there is a possibility that this type of entropy production may play an important role in other situations,[18] such as in the exponentially expanding era of the universe suggested in connection with the GUT first-order phase transitions.[19]

In spite of these fascinating features of the finite reduction model, the validity of this model, in other words, the reality of the statistical effect of interaction asserted in § 2, is an open problem at present. In connection with this problem the recent arguments[19] in the measurement theory for quantum mechanics are very interesting. In these arguments it is suggested that uncontrolable interactions of macroscopic systems with the surroundings may play an essential role in the explanation of the wave packet reduction. Since the essential features of the problems are quite similar in the explanation of the wave packet reduction and in our problem, these arguments indicate that our problem might be solved in the same line as the problem of the measurement theory. Anyway the further progress in the fundamental physics is required to convince ourselves with the validity of our model completely.

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Appendix A

The purpose of this appendix is two-fold: one is to show that the condition of $\delta_j Q$ being finite leads to the choice $\dot{A}=0$ in Eq. (2·31) and the other is to derive the formula (3·5). Both are based on the study of the behavior of the Bogoliubov coefficients β_k. For simplicity we write the concerned interval as $[\eta_0, \eta_1]$ and express the values of various quantities at η_0 and η_1 by the suffices 0 and

1, respectively, and omit the momentum suffix k. Then, as shown in Ref. 10), the Bogoliubov coefficient in this interval is expressed as

$$|\beta|^2=\frac{|\mathcal{D}|^2}{16\mu_0\mu_1\Omega_0^2}, \tag{A·1}$$

where

$$\mathcal{D}=\mathcal{B}_1\sqrt{\frac{\Omega_0}{\Omega_1}}[\mathcal{D}_1\sin\Phi+\mathcal{D}_2\cos\Phi+i(\mathcal{D}_3\sin\Phi+\mathcal{D}_4\cos\Phi)], \tag{A·2}$$

$$\mathcal{D}_1=\mu_1\frac{\dot{\Omega}_0}{\Omega_0}+\mu_0\frac{\dot{\Omega}_1}{\Omega_1}+2\mu_0\frac{\dot{\mathcal{B}}_1}{\mathcal{B}_1}, \tag{A·2a}$$

$$\mathcal{D}_2=2(\mu_1\Omega_0-\mu_0\Omega_1\mathcal{B}_1^{-2}), \tag{A·2b}$$

$$\mathcal{D}_3=2(\Omega_0\Omega_1\mathcal{B}_1^{-2}-\mu_0\mu_1)+\frac{1}{2}\frac{\dot{\Omega}_0}{\Omega_0}\frac{\dot{\Omega}_1}{\Omega_1}-\frac{\dot{\mathcal{B}}_1}{\mathcal{B}_1}\frac{\dot{\Omega}_0}{\Omega_0}, \tag{A·2c}$$

$$\mathcal{D}_4=\Omega_0\frac{\dot{\Omega}_1}{\Omega_1}-\Omega_1\frac{\dot{\Omega}_0}{\Omega_0}\mathcal{B}_1^{-2}-2\Omega_0\frac{\dot{\mathcal{B}}_1}{\mathcal{B}_1}, \tag{A·2d}$$

in which we have put $\gamma=0$ following the argument in § 2. Φ is given by

$$\Phi=\int_{\eta_0}^{\eta_1}\mathcal{B}(\eta)^{-2}\Omega(\eta)d\eta, \tag{A·3}$$

and $\mathcal{B}(\eta)$ is the solution of the nonlinear equation

$$\Omega^{-1}(\Omega^{-1}\dot{\mathcal{B}})^{\cdot}+(\Lambda+1-\mathcal{B}^{-4})\mathcal{B}=0;\quad \mathcal{B}(\eta_0)=1,\quad \dot{\mathcal{B}}(\eta_0)=0, \tag{A·4}$$

where

$$\Lambda=\frac{5}{16}\Omega^{-6}[(\Omega^2)^{\cdot}]^2-\frac{1}{4}\Omega^{-4}(\Omega^2)^{\cdot\cdot}. \tag{A·5}$$

By setting $u=\mathcal{B}-1$, the integral equation corresponding to Eq. (A·4) is written as

$$u(\eta)=-\frac{1}{2}\int_{\eta_0}^{\eta}\sin\left(2\int_{\eta'}^{\eta}\Omega(\eta'')d\eta''\right)$$
$$\times[(1+u(\eta'))\Lambda(\eta')+u(\eta')q(u(\eta'))]\Omega(\eta')d\eta', \tag{A·6}$$

where $q(u)=u(3u^2+8u+6)/(1+u)^3$.

First we study the finiteness condition of δQ. Since $\mathrm{Tr}\, D_jN_{k(j)}$ damps exponentially for large k due to the factor $\exp(-H/T)$ (see Eqs. (2·18) and (2·43)), this condition is equivalent to

$$\int_0^\infty dk k^2 \mu_k |\beta_k|^2 < +\infty . \quad \text{(A·7)}$$

Here we have used the fact that the expression for δQ for the general case is given by replacing Ω by μ in Eq. (2·43). Noting that $\mathcal{B}=1+O(k^{-4})$ and $\dot{\mathcal{B}}=O(k^{-3})$ for large k (see Eqs. (4·20) and (4·21) in Ref. 10)), we obtain the following asymptotic estimate for $\mathcal{D}_i$:

$$\mathcal{D}_1=O(k^{-1}), \qquad \mathcal{D}_4=O(k^{-1}),$$

$$\mathcal{D}_2=\frac{2\Omega_0}{\mu_0+\mu_1}(\mu_1{}^2-\mu_0{}^2)-\frac{2\mu_0}{\Omega_0+\Omega_1}(\Omega_1{}^2-\Omega_0{}^2)+O(k^{-1}),$$

$$\mathcal{D}_3=2(\Omega_0{}^2-\mu_0{}^2)+\frac{2\mu_0}{\mu_0+\mu_1}(\mu_0{}^2-\mu_1{}^2)+\frac{2\Omega_0}{\Omega_0+\Omega_1}(\Omega_1{}^2-\Omega_0{}^2)+O(k^{-1}). \quad \text{(A·8)}$$

Since μ, $\Omega\sim O(k)$, the condition (A·7) is rewritten as $|\mathcal{D}|^2=o(1)$. Thus from Eq. (A·8) we obtain the following constraint on μ:

$$\mu^2=\Omega^2+o(1). \qquad (k\sim\infty) \quad \text{(A·9)}$$

Since $\mu^2=\Omega^2+\dot{A}$ from Eq. (2·31) (note we adopted $\lambda_k=R$), this constraint yields

$$\dot{A}=o(1). \qquad (k\sim\infty) \quad \text{(A·10)}$$

This means that it is most natural to put $\dot{A}=0$.

Next we prove that $|\beta_k|^2$ is given by Eq. (3·5) in the lowest order in $\delta t/\tau_{\mathrm{exp}}$. We put $\lambda=R$ and $\dot{A}=0$ in Eqs. (2·31) and (2·32), hence $\mu_k=\Omega_k$ from now on. Then from Eq. (A·6) we obtain

$$|\dot{\mathcal{B}}_1/\mathcal{B}_1|\lesssim\Omega^2\Lambda\delta\eta . \quad \text{(A·11)}$$

Rewriting $\Omega_1{}^2-\Omega_0{}^2$ as $(\Omega^2)^{\cdot}\delta\eta$ and using Eq. (A·11), we obtain the following estimates of $\mathcal{D}_i$:

$$\mathcal{D}_1=(\Omega^2)^{\cdot}/\Omega[1+O(\delta t/\tau_{\mathrm{exp}})], \quad \text{(A·12a)}$$

$$\mathcal{D}_2=4\Omega^2 u[1+O(\delta t/\tau_{\mathrm{exp}})], \quad \text{(A·12b)}$$

$$\mathcal{D}_3=\frac{1}{8}\frac{[(\Omega^2)^{\cdot}]^2}{\Omega^4}[1+O(\delta t/\tau_{\mathrm{exp}})], \quad \text{(A·12c)}$$

$$\mathcal{D}_4=\frac{1}{2}\left[\frac{(\Omega^2)^{\cdot\cdot}}{\Omega}-\frac{3}{2}\frac{[(\Omega^2)^{\cdot}]^2}{\Omega^3}\right]\delta\eta-2\Omega\dot{u}+O((\delta t/\tau_{\mathrm{exp}})^2), \quad \text{(A·12d)}$$

where $u=\mathcal{B}-1$. To proceed further we must consider the cases $\delta\eta\Omega\ll1$ and $\delta\eta\Omega\gg1$ separately.

(i) $\delta\eta\Omega\ll1$ case. In this case from Eq. (A·6) it follows that

$$u \simeq -\frac{1}{2}\Omega^2 \Lambda(\delta\eta)^2$$

$$= -\frac{1}{2}\left(\frac{\delta\eta}{R/\dot{R}}\right)^2\left[\frac{5}{4}\frac{m^4R^4}{\Omega^4}-\frac{1}{2}\frac{m^2R^2}{\Omega^2}\left(1+\frac{R\ddot{R}}{\dot{R}^2}\right)\right], \qquad (\text{A}\cdot 13)$$

$$\dot{u} \simeq 2u/\delta\eta\ . \qquad (\text{A}\cdot 14)$$

From these equations we find

$$\mathcal{D}_2 \cos \Phi/\mathcal{D}_1 \sin \Phi \simeq \frac{\Omega^3}{(\Omega^2)^{\cdot}}\frac{u}{\delta\eta\Omega} = O\left(\frac{\delta t}{\tau_{\text{exp}}}\right) \ll 1\,,$$

$$\mathcal{D}_3 \sin \Phi + \mathcal{D}_4 \cos \Phi = O\left(\left(\frac{\delta t}{\tau_{\text{exp}}}\right)^2\right) \ll 1\,. \qquad (\text{A}\cdot 15)$$

Estimate (A·15) means that $\mathcal{D} \simeq \mathcal{D}_1 \sin \Phi$ in the lowest-order approximation.
(ii) $\delta\eta\Omega \gg 1$ case. In this case, $R/\dot{R} \gg \delta\eta \gg 1/\Omega$ and

$$u \sim O(\Omega\Lambda\delta\eta). \qquad (\text{A}\cdot 16)$$

From these we obtain the following estimates:

$$\mathcal{D}_2/\mathcal{D}_1 \sim O\left(\frac{\Omega^3 u}{(\Omega^2)^{\cdot}}\right) \sim O\left(\frac{\delta t}{\tau_{\text{exp}}}\right) \ll 1\,,$$

$$\mathcal{D}_3/\mathcal{D}_1 \sim O\left(\frac{(\Omega^2)^{\cdot}}{\Omega^3}\right) \sim O\left(\frac{m^2R^2}{\Omega^2}\frac{\dot{R}}{R\Omega}\right) \ll 1\,,$$

$$\mathcal{D}_4/\mathcal{D}_1 \sim O\left(\frac{(\Omega^2)^{\cdot}}{\Omega^2}\delta\eta\right) + O\left(\frac{1}{\Omega\delta\eta}\frac{\mathcal{D}_2}{\mathcal{D}_1}\right)$$

$$\sim O\left(\frac{m^2R^2}{\Omega^2}\frac{\delta\eta}{R/\dot{R}}\right) + O\left(\frac{1}{\Omega\delta\eta}\frac{\mathcal{D}_2}{\mathcal{D}_1}\right) \ll 1\,. \qquad (\text{A}\cdot 17)$$

Thus again $\mathcal{D}_1$-term dominates in $\mathcal{D}$. (i) and (ii) show that $\mathcal{D} \simeq \mathcal{D}_1 \sin \Phi$ in the lowest-order in $\delta t/\tau_{\text{exp}}$, which leads to Eq. (3·5).

Appendix B

In this appendix we estimate the integrals $I_1 \sim I_5$ which appeared in § 3. These integrals are classified into three types:

$$J_n(\lambda) \equiv \int_1^\infty dx \frac{\sqrt{x^2-1}}{x^n}\sin^2(\lambda x), \quad (n \geqq 4) \qquad (\text{B}\cdot 1)$$

$$K_n(\lambda, \beta) \equiv \int_1^\infty dx \frac{\sqrt{x^2-1}}{x^n}\sin^2(\lambda x)(e^{\beta x}-1)^{-1}\,, \quad (n \geqq 4) \qquad (\text{B}\cdot 2)$$

$$L_0(\beta) \equiv \int_1^\infty dx \sqrt{x^2-1}\,(e^{\beta x}-1)^{-1}\,. \qquad (\text{B}\cdot 3)$$

First let us consider $J_n(\lambda)$. Since we cannot perform the integration exactly, we seek the asymptotic behavior of J_n for $\lambda\gg 1$ and $\lambda\ll 1$. For $\lambda\gg 1$ we can replace $\sin^2(\lambda x)$ by 1/2 and obtain

$$J_n\sim\frac{1}{2}\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^n}=\frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n+1}{2}\right)}. \quad (\lambda\gg 1) \tag{B·4}$$

For $\lambda\ll 1$, if $n\geqq 5$, we can apply Lebesgue's theorem and obtain

$$J_n\sim\lambda^2\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^{n-2}}=\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{n}{2}-2\right)}{\Gamma\left(\frac{n-1}{2}\right)}\lambda^2. \quad (\lambda\ll 1, n\geqq 5) \tag{B·5}$$

For $\lambda\ll 1$ and $n=4$, we must make a little delicate argument. We divide the integrand into two parts:

$$J_4=\int_1^\infty dx\frac{\sqrt{x^2-1}-x}{x^4}\sin^2(\lambda x)+\int_1^\infty\frac{dx}{x^3}\sin^2(\lambda x). \tag{B·6}$$

By Lebesgue's theorem the first term is estimated as

$$\int_1^\infty dx\frac{\sqrt{x^2-1}-x}{x^4}\sin^2(\lambda x)\sim\lambda^2\int_1^\infty dx\frac{\sqrt{x^2-1}-x}{x^2}=\lambda^2\ln\frac{2}{e}. \tag{B·7}$$

The second term is expressed by the cosine integral function $Ci(x)$ by partial integration. Using the asymptotic expansion formula for $Ci(x)$,[14] we obtain

$$\int_1^\infty dx\frac{\sin^2(\lambda x)}{x^3}=\frac{1}{2}\frac{\sin^2\lambda}{\lambda^2}+\frac{\sin 2\lambda}{2\lambda}-Ci(\lambda)$$

$$\sim\frac{3}{2}-\gamma-\ln(2\lambda), \qquad (\lambda\ll 1) \tag{B·8}$$

where γ is Euler constant. Equations (B·7) and (B·8) yield

$$J_4\sim\frac{3}{2}-\gamma-\ln(2\lambda). \qquad (\lambda\ll 1) \tag{B·9}$$

Next we estimate K_n and L_0. Let $L_n(\beta)$ be

$$L_n(\beta)\equiv\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^n}(e^{\beta x}-1)^{-1}. \tag{B·10}$$

Then Lebesque's theorem yields

$$K_n(\lambda,\beta)\sim\begin{cases}\frac{1}{2}L_n(\beta) & \text{for } \lambda\gg 1\,, & (\text{B}\cdot 11)\\ \lambda^2 L_{n-2}(\beta) & \text{for } \lambda\ll 1\,. & (\text{B}\cdot 12)\end{cases}$$

Thus we only have to examine the asymptotic behavior of $L_n(\beta)$. We are interested only in the case $\beta\ll 1$. For $n\geqq 2$ we can apply Lebesgue's theorem to Eq. (B·10) directly to obtain

$$L_n(\beta)\sim\beta^{-1}\int_1^\infty dx\frac{\sqrt{x^2-1}}{x^{n+1}}=\frac{\sqrt{\pi}}{8}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\beta^{-1}\,. \quad (\beta\ll 1,\ n\geqq 2) \qquad (\text{B}\cdot 13)$$

Equations (B·11)~(B·13) yield the required asymptotic estimate of $K_n(\lambda,\beta)$. For $n=0$ we change the integration variable from x to $t=\beta x$:

$$L_0(\beta)=\beta^{-2}\int_\beta^\infty dt\sqrt{t^2-\beta^2}\,(e^t-1)^{-1}\,. \qquad (\text{B}\cdot 14)$$

Then again by Lebesgue's theorem we obtain

$$L_0(\beta)\sim\beta^{-2}\int_0^\infty dt\ t(e^t-1)^{-1}=\frac{\pi^2}{6}\beta^{-1}\,. \quad (\beta\ll 1) \qquad (\text{B}\cdot 15)$$

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