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STATISTICAL ESTIMATION FOR MULTIPLICATIVE CASCADES[1]

BY MINA OSSIANDER AND EDWARD C. WAYMIRE

Oregon State University

The probability distribution of the cascade generators in a random multiplicative cascade represents a hidden parameter which is reflected in the fine scale limiting behavior of the scaling exponents (sample moments) of a single sample cascade realization as a.s. constants. We identify a large class of cascade generators uniquely determined by these scaling exponents. For this class we provide both asymptotic consistency and confidence intervals for two different estimators of the cumulant generating function (log Laplace transform) of the cascade generator distribution. These results are derived from investigation of the convergence properties of the fine scale sample moments of a single cascade realization.

1. Introduction and preliminaries. Early versions of multiplicative cascades were introduced by Kolmogorov (1941, 1962) in the statistical theory of turbulence for use in modeling the redistribution of energy under a rapid stirring motion as a repeated random splitting of energy into finer scale eddies. Refinements were developed further by Yaglom (1966) and by Mandelbrot (1974), and steps toward a rigorous mathematical foundation were initiated by Kahane and Peyrière (1976), with earlier work done by Joffe, LeCam, and Neveu (1973); also see Frisch (1995) in this context. Other highly variable and intermittent phenomena whose statistics appear to be well represented by such models are spatial rainfall, internet traffic, financial markets, etc. For example, see Gupta and Waymire (1993), Gilbert, Willinger and Feldman (1999), Mandelbrot (1998), respectively. Our goal is to extend some of the existing statistical theory required for accurate parameter estimation and rigorous tests of hypotheses within this framework.

In such physical contexts as noted above, the data structure is a distribution of some random quantities $R(\Delta)$, for example, rain, energy, message packets, etc., measured over some region S of space or time, and binned into some b^N pixels Δ at the scale of resolution $\delta = |\Delta| = b^{-N}$. Here $R(\cdot)$ can be thought of as a random measure observed on a partition of S into sets of diameter b^{-N}; that is, the pixels of resolution b^{-N}. Typically $R(\cdot)$ does not have independent increments, but instead exhibits a random splitting phenomena as the diameter of the sets Δ decreases, or equivalently, the pixel resolution is increased. An example of this type is seen in Gupta and Waymire (1993), with the observed data, $R(\Delta)$, comprising spatial rainfall measurements converted

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from observed radar scans over a square region S, 256-km on a side, binned into 4096 4-km $\times$ 4-km pixels at the finest resolution. Alternatively, Figure 2(a) provides an example of a single simulated scan from a multiplicative cascade over an interval $S = [0, 1]$ binned into pixels of length 2^{-13}. The *multiscaling exponents* $\tau(h)$ arise in this context through log–log plots of empirical moments of various orders h versus length scale δ; that is,

$$\log \sum_{\Delta: |\Delta| = \delta} R^h(\Delta) = \tau(h) \log \delta + C_h. \tag{1}$$

In particular, in such data one observes (i) log–log linearity in δ for fixed h, and (ii) nonlinear slopes $\tau(h)$ as a function of h. This naturally leads one to seek a theoretical framework which will accomodate such empirically observed structure.

Notice that linear multiscaling exponents (simple scaling) $\tau(h) = \theta h$ can be accommodated by the theoretical framework of statistical self-similarity via the relation $R(\Delta) \overset{\text{dist}}{=} |\Delta|^\theta R(\mathbf{1})$. In this setting a rather extensive theory exists. On the other hand, to illustrate a model which exhibits true multiscaling structure of the form (1) consider $R(\Delta) \overset{\text{dist}}{=} R_0 \exp\{B_{\log(1/\delta)}\}$, where $\{B_t \colon t \geq 0\}$ is standard Brownian motion independent of R_0. Then a simple computation of the Gaussian moment generating function $ER^h(\Delta) = ER_0^h \; Ee^{hB_{\log(1/\delta)}}$ furnishes a log–log linear relation with nonlinear (quadratic) exponent $\tau(h)$ and intercept $C_h = \log ER_0^h$. Since the Brownian motion exponent process is additively comprised of stationary independent increments, this framework suggests the following multiplicative spatial extrapolation. Address each pixel $\Delta \equiv \Delta_j(v)$ at the resolution b^{-j} by a sequence $v = (v_1, \ldots, v_j)$ of digits $v_i \in \{0, 1, 2, \ldots, b - 1\}$, and write

$$R(\Delta_N(v)) = R(\Delta_0(v)) \frac{R(\Delta_1(v))}{R(\Delta_0(v))} \cdots \frac{R(\Delta_N(v))}{R(\Delta_{N-1}(v))}. \tag{2}$$

If it is assumed that random splitting occurs independently at each level of resolution, then the ratios $R(\Delta_j(v))/R(\Delta_{j-1}(v))$ will be independent random variables. On the other hand, for a nearby pixel $\Delta_N(v')$ with $v' = (v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_N)$ we have correlation since

$$R(\Delta_N(v')) = R(\Delta_0(v)) \frac{R(\Delta_1(v))}{R(\Delta_0(v))} \cdots \frac{R(\Delta_{k+1}(v'))}{R(\Delta_k(v))} \cdots \frac{R(\Delta_N(v'))}{R(\Delta_{N-1}(v'))}. \tag{3}$$

Due to the additive nature of the $R(\cdot)$ process,

$$R(\Delta_k(v)) = \sum_{v'} R(\Delta_N(v')), \tag{4}$$

where the sum is taken over all $v' = (v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_N)$ as described above. The following model incorporates both recursive random splitting and the additivity of random measures in modeling $R(\cdot)$.

Let $\{W_v\}$ be an i.i.d. family of nonnegative mean 1 random variables indexed by the pixel addresses v at different fine scales b^{-n} for $n \geq 1$ and write

$$R(\Delta(v)) = \lambda_\infty(\Delta(v)) = Z_\infty(v) \times \prod_{j=1}^{N} W_{(v_1,\dots,v_j)} b^{-N}, \tag{5}$$

where $\lambda_\infty(dx)$ is the underlying random distribution at the fine scale limit ($\delta \to 0$), which is being sampled at the given resolution of b^{-N}. In view of the recursive structure which defines the fine scale limit distribution, $Z_\infty(v)$ is distributed as the total quantity $\lambda_\infty(S)$. Though purely phenomenological, this is precisely the mathematical structure underlying Kolmogorov's multiplicative cascade model of energy redistribution cited above. This provides a framework for interpreting and analyzing data structures of the type (1); however for this paper the main focus is on statistical inference on the distribution of cascade generators W_v from measurements of the form $\{\lambda_\infty(\Delta)\colon \Delta \subset S\}$ at some given resolution of the pixels Δ. That is, how does one choose the distribution of the W's? In this paper we regard the parameter b as a given parameter which is imposed on the data by the binning of observations.

This is an area of physical science in which the connection between the physics of the phenomena and the model being analyzed is at best primitive. Thus the role of statistics is to provide a framework for interpreting and analyzing the observed data structures in ways that might enhance the physical understanding. In the context of the somewhat more well-developed physical science of turbulence, Kolmogorov (1962) hypothesised a log-Normal distribution for the cascade generators. However recent results on the multiscaling exponents by She and Levesque (1994) suggest a log-Poisson distribution for the cascade generators; see Dubrulle (1994), She and Waymire (1995). Thus the mathematically precise results of the type given in the present paper are motivated by a need for accurate statistical hypothesis testing methodology.

In order to describe our results, let us now turn to a more precise mathematical specification of the multiplicative cascade. For simplicity, first consider repeated splittings of $\mathbf{T}$, a unit cube in $\mathbf{R^d}$ for some $d \geq 1$, into a number b^n, $b \geq 2$, of subcubes (pixels) of volume $b^{-n}, n = 1, 2, \dots,$ such that the nth-stage mass of the pixel $\Delta_n(t_1, \dots, t_n)$, $t_k \in \{0, 1, \dots, b-1\}$ is given by the random measure $\lambda_n(\Delta_n(t_1, \dots, t_n)) = b^{-n} \prod_{k=1}^{n} W_{(t_1 \cdots t_k)}$, where the W_v's are as described above. The cascade measure λ_∞ is obtained as the a.s. vague limit of the sequence λ_n as $n \to \infty$. See Figure 1 for a sketch of the histograms of λ_1, λ_2, and λ_∞ with $b = 4$ and $\mathbf{T}$ taken to be the unit square in $\mathbf{R^2}$. The main problem for applications noted above is to infer the distribution of the random factors W_v from data on the random masses $\lambda_\infty(\Delta_n(t_1, \dots, t_n))$, at some prescribed fine scale b^{-n}. This problem is the focus of this paper.

Our results can be briefly described as follows. First, the distribution of the W_v's is parameterized by the *structure function*

$$\chi_b(h) = \log_b \mathrm{E}[W_v^h \mathbf{1}[W > 0]] - (h-1), \tag{6}$$

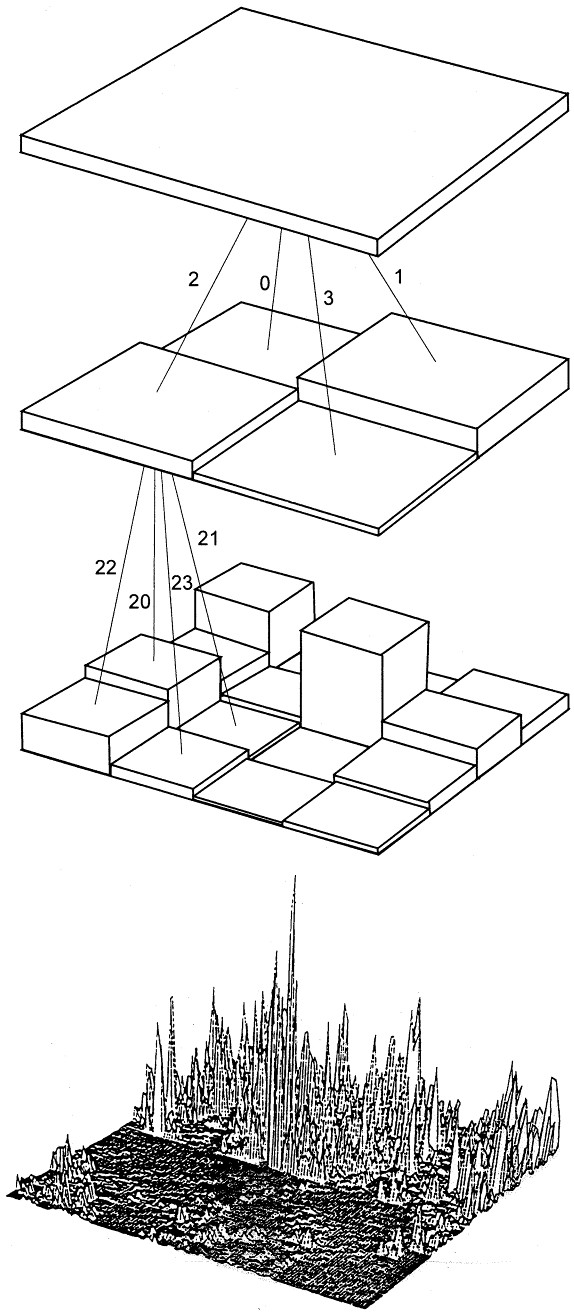

FIG. 1. *Ingredients of a multiplicative cascade distribution on the unit square with $b = 4$; the histograms of λ_1, λ_2 and λ_∞ are indicated.*

the modified cumulant generating function for $\log W_v$. It is shown that the estimators

$$\hat{\tau}_n(h) = n^{-1} \log_b \sum_{\Delta_n} \lambda_\infty^h(\Delta_n) \tag{7}$$

and

$$\tilde{\tau}_n(h) = \log_b \left(\sum_{\Delta_{n+1}} \lambda_\infty^h(\Delta_{n+1}) \Big/ \sum_{\Delta_n} \lambda_\infty^h(\Delta_n) \right) \tag{8}$$

converge a.s. to $\chi_b(h)$ as $n \to \infty$ for all h within a critical interval. Consequently, if the moment generating function of $\log W_v$ is defined in a neighborhood of the origin, the collection $\{\lambda_\infty(\Delta_n)\colon n \geq 0\}$ uniquely determines the distribution of the cascade generators, $\{W_v\}$. Theorem 2.2 in the following section gives a more precise statement of this result. The convergence of the estimators $\hat{\tau}_n(h)$ and $\tilde{\tau}_n(h)$ themselves is addressed in Theorem 3.2 and Corollary 3.4 of Section 3. In particular it is shown that $\hat{\tau}_n(h)$ converges a.s. to a functional of $\chi_b(h)$ for all h. [Notice that $\hat{\tau}_n(h)$ corresponds asymptotically to the $\tau(h)$ appearing in (1) if δ is taken to be b^{-n}.] On the other hand, it is shown that $\tilde{\tau}_n(h)$ converges a.s. to $\chi_b(h)$ for h within a critical interval specified by the structure function itself. Central limit theorems for suitably normalized versions of both $\hat{\tau}_n(h)$ and $\tilde{\tau}_n(h)$ are given in Section 4. In particular, Corollary 4.7 at the end of Section 4 gives a central limit theorem that allows computation of asymptotically exact confidence intervals for $\chi_b(h)$ by using the fine-scale sample moments of a single sample realization of a multiplicative cascade.

The overall organization of this paper is as follows. Fine-scale consistency is explored via the use of size-bias methods in Section 2. Results here include Theorem 2.2 mentioned above, with its proof appearing as a corollary to a somewhat more general theorem (Theorem 2.4) on the rate of growth of the sample moments. This latter result generalizes previous results of Holley and Waymire (1992) and Collet and Koukiou (1992) to a much larger class of generators. Versions of this generalization utilizing "convergence in distribution" were obtained by Franchi (1995) and Molchan (1996). Also, by employing a Legendre transform formalism for expected values, such results were anticipated in the physics literature by Lovejoy and Schertzer (1991). These results all follow from our more general theory (Theorems 2.4, 3.1). Although we restrict this paper to the case of i.i.d. generators, an advantage of the approach via size-bias methods is that the tools are also amenable to the analysis of dependent cascades; see Waymire and Williams (1996). While size-biasing provides convergence in probability and in L^1, an almost sure version of these results is obtained in Section 3. The methods used here are more classical in nature and depend on a delicate truncation argument. Related methods are also required for our treatment of the fluctuation laws (central limit theorems) given in Section 4. These latter results extend the central limit theorems of Troutman and Vecchia (1999) to a correspondingly larger class of generators.

A presentation of simulation results illustrating convergence and fluctuations of the estimator $\tilde{\tau}_n(h)$ of $\chi_b(h)$ is made in Section 5. These simulations point to some interesting phenomena with regard to the behavior of the scale and/or distribution of the estimators outside a critical range for the parameter h. The paper concludes with some remarks on unresolved problems and possible extensions of this work.

2. Fine scale consistency. It is mathematically convenient to exploit the b-ary tree structure underlying the multiplicative cascade as follows. Let $b \geq 2$ be a natural number and let $\mathbf{T}$ denote the product space

$$\mathbf{T} = \{0, 1, 2, \ldots, b-1\}^{\mathbf{N}} \tag{9}$$

equipped with the metric $\rho(s,t) = b^{-|s\wedge t|}$, $s, t \in \mathbf{T}$, where $\mathbf{N}$ denotes the set of natural numbers and $|s \wedge t| = \inf\{n \geq 0\colon s_{n+1} \neq t_{n+1}\}$, $s = (s_1, s_2, \ldots)$, $t = (t_1, t_2, \ldots) \in \mathbf{T}$. $\mathscr{B}(\mathbf{T})$ will denote the corresponding Borel sigma field on $\mathbf{T}$ for this metric. For $t = (t_1, t_2, \ldots) \in \mathbf{T}$ let $t|n = (t_1, t_2, \ldots, t_n)$. If points $t \in \mathbf{T}$ are viewed as paths through a b-ary tree then $v = t|n$ denotes the nth generation vertex along t.

For $s \in \mathbf{T}$, $n \in \mathbf{N}$, let

$$\Delta_n(s) \equiv \Delta_n(s|n) = B_{b^{-n}}(s) = \{t \in \mathbf{T}\colon t_i = s_i, i \leq n\} \tag{10}$$

denote the closed ball of radius $r = b^{-n}$ centered at $s \in \mathbf{T}$. The normalized Haar measure λ on $\mathbf{T}$, viewed as a countable product of cyclic groups of order b, is specified by

$$\lambda(\Delta_n(s)) = b^{-n}, \ s \in \mathbf{T}, n \geq 1. \tag{11}$$

Now let $\{W_v\colon v \in \{0, 1, \ldots, b-1\}^n, n \geq 1\}$ be a denumerable family of i.i.d. nonnegative mean 1 random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Also let $\mathscr{F}_n$, $n \geq 1$, denote the filtration defined by

$$\mathscr{F}_n = \sigma\{W_v\colon |v| \leq n\}, \qquad n \geq 1, \tag{12}$$

where for $v = (v_1, v_2, \ldots, v_n)$, $v_i \in \{0, 1, \ldots, b-1\}$, $n \geq 1$, $|v| = n$. The random variables W_v are referred to as the *cascade generators* and, as such, define a sequence of random measures λ_n on $(\mathbf{T}, \mathscr{B}(\mathbf{T}))$, $n \geq 1$, via

$$\frac{d\lambda_n}{d\lambda}(t) = Q_n(t) = \prod_{i=0}^{n} W_{t|i} = W_\varnothing \prod_{i=1}^{n} W_{t|i}, \qquad t \in \mathbf{T}, \tag{13}$$

where $W_\varnothing$, referred to as the *cascade initiator*, is an a.s. positive random variable independent of $\mathscr{F}_n$, $n \geq 1$.

It is well known [e.g., see Kahane and Peyrière (1976)] that there is a random measure λ_∞ on $(\mathbf{T}, \mathscr{B}(\mathbf{T}))$ such that

$$P(\lambda_n \Rightarrow \lambda_\infty \text{ as } n \to \infty) = 1, \tag{14}$$

where, throughout, $\Rightarrow$ denotes vague convergence. In fact, for any countable dense family Φ of bounded Borel measurable functions [cf. Kahane (1989)],

$$P\left(\lim_{n\to\infty}\int_{\mathbf{T}} f(t)\lambda_n(dt) = \int_{\mathbf{T}} f(t)\lambda_\infty(dt),\ f \in \Phi\right) = 1. \tag{15}$$

The random measure λ_∞ defines the *multiplicative cascade*. The following basic structure theorem for λ_∞ is also well known. First let

$$\chi_b(h) = \log_b \mathrm{E}[W^h \mathbf{1}[W > 0]] - (h-1), \tag{16}$$

where W is a generic cascade generator distributed as W_v for $v \neq \emptyset$. The structure function $\chi_b(h)$ is defined for all real numbers h but may be infinite, with the conventions that $0^0 = 0, 0 \cdot \infty = 0$. The use of the indicator function $\mathbf{1}[W > 0]$ allows incorporation of the case $h < 0$ into the general theory.

THEOREM 2.1 [Kahane and Peyrière (1976)]. (i) $\mathbf{E}\lambda_\infty(\mathbf{T}) > 0$ *iff* $\chi_b'(1-) < 0$.

(ii) $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$ *for* $0 \le h \le 1$, *and, if* $h_c := \sup\{h \ge 1: \chi_b(h) \le 0\} > 1$, *then* $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$ *for* $1 < h < h_c$.

(iii) $E\lambda_\infty(\mathbf{T}) = 1$ *iff* $E\lambda_\infty(\mathbf{T}) > 0$.

The following example of a multiplicative random cascade is explicitly related to the Galton–Watson branching process. Let W take values p^{-1} and 0 with probability p and $1-p$, respectively; that is, W is nonzero–zero with mean 1. χ_b is then linear with slope $\log_b(1/bp)$. Then the random measure λ_n has total mass $\lambda_n(\mathbf{T})$ obtained as a sum of b^n terms which are each of the form $\prod_{i=1}^n W_{v|i} b^{-n}$, which takes on either value 0 or value $(bp)^{-n}$. The number of nonzero terms in the sum is the number X_n in the nth generation of a Galton–Watson branching process with a Binomial(b, p) offspring distribution, that is,

$$\lambda_n(\mathbf{T}) = \frac{X_n}{(bp)^n}. \tag{17}$$

$\lambda_n(\mathbf{T})$ is the nonnegative mean one martingale associated with X_n, with $\lambda_\infty(\mathbf{T})$ being its a.s. limit. Part (i) above tells us that $P(\lambda_\infty(\mathbf{T}) > 0) > 0$ iff the slope of χ_b is less than 0; that is, $p > b^{-1}$. This coincides exactly with the well-known condition guaranteeing that the mean-normalized Galton–Watson branching process $(bp)^{-n}X_n$ lives with positive probability.

The following theorem gives a precise statement of the main results of this paper.

THEOREM 2.2. *Assume that* $\chi_b'(1-) < 0$. *If* $\mathbf{E}[W^h \mathbf{1}[W > 0]]$ *exists and is finite for* h *belonging to some neighborhood of* 0, *then* $\{\lambda_\infty(\Delta_n(v)): v \in \{0, 1, \dots, b-1\}^n, n \ge 0\}$ *uniquely determines the distribution of the cascade generator* W.

Throughout the remainder of the paper we will restrict our consideration to cascade generators for which

$$\chi_b'(1-) < 0, \tag{18}$$

so that $\mathbf{E}\lambda_\infty(\mathbf{T}) > 0$; compare Theorem 2.1. A basic family of statistics which we consider are the *nth-scale sample moments* defined by

$$M_n(h) = \sum_{|v|=n} \lambda_\infty^h(\Delta_n(v)), \qquad h \in \mathbf{R}. \tag{19}$$

The estimators $\hat{\tau}_n(h)$ and $\tilde{\tau}_n(h)$ are defined in terms of the $M_n(h)$'s as follows:

$$\hat{\tau}_n(h) = n^{-1} \log_b M_n(h) \tag{20}$$

and

$$\tilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h)) \tag{21}$$

A particularly useful tool for our considerations are the *h-cascades*, denoted by the random measures $\lambda_\infty(h; dt)$, $h \in \mathbf{R}$, which we define via the *h-cascade generators*

$$W_v(h) = \frac{W_v^h}{\mathbf{E}W_v^h}, \qquad h \in \mathbf{R}. \tag{22}$$

With this one may easily check that

$$\frac{\lambda_n^h(\Delta_n(v))}{b^{n\chi_b(h)}} = \lambda_n(h; \Delta_n(v)), \tag{23}$$

where

$$\frac{d\lambda_n(h;\cdot)}{d\lambda}(t) \equiv Q_n(h;t) = \prod_{i=0}^{n} W_{t|i}(h), \qquad t \in \mathbf{T}, \tag{24}$$

is the sequence of *nth level h-cascades*, $n = 1, 2, \ldots$.

PROPOSITION 2.1. *For $h \in \mathbf{R}$, $n \geq 1$, one has*

$$\frac{M_n(h)}{b^{n\chi_b(h)}} = \sum_{|v|=n} Z_\infty^h(v)\lambda_n(h; \Delta_n(v)) = \int_{\mathbf{T}} Z_\infty^h(t|n)\lambda_n(h; dt),$$

where a.s.,

$$Z_\infty(v) = \lim_{N\to\infty} \sum_{|u|=N-n} \prod_{i=1}^{N-n} W_{v*(u_1\cdots u_i)} b^{-(N-n)},$$

and $*$ denotes the concatenation

$$(v_1, \ldots, v_n) * (u_1, \ldots, u_N) = (v_1, \ldots, v_n, u_1, \ldots, u_N).$$

PROOF. First note that for $N \geq n+1$, $n = 1, 2, \ldots$ one has

$$\lambda_N(\Delta_n(v)) = Z_N^{(n)}(v) \cdot \lambda_n(\Delta_n(v)), \tag{25}$$

where

$$Z_N^{(n)}(v) = \sum_{|u|=N-n} \prod_{i=1}^{N-n} W_{v*(u_1\cdots u_i)} b^{-(N-n)}. \tag{26}$$

For each $n \geq 1$ and $|v| = n$, the sequence $\{Z_N^{(n)}(v), N = n+1, n+2, \ldots\}$ is a nonnegative martingale and therefore $\lim_{N\to\infty} Z_N^{(n)}(v) = Z_\infty^{(n)}(v)$ exits a.s. and is independent of $\mathscr{F}_n$. Moreover, using (25),

$$\lambda_\infty(\Delta_n(v)) = Z_\infty^{(n)}(v) \cdot \lambda_n(\Delta_n(v)). \tag{27}$$

Thus, combining this with (23),

$$\begin{aligned} \frac{\lambda_\infty^h(\Delta_n(v))}{b^{n\chi_b(h)}} &= Z_\infty^{(n)}(v) \cdot \frac{\lambda_n^h(\Delta_n(v))}{b^{n\chi_b(h)}} \\ &= Z_\infty^{(n)h}(v)\lambda_n(h; \Delta_n(v)). \end{aligned} \tag{28}$$

This completes the proof. □

The following proposition delineates the critical interval of h-values which is central to the convergence results of this and following sections.

PROPOSITION 2.2. *Assume that* $\chi_b'(1-) < 0$ *and let*

$$H_c^+ = \sup\{h \geq 1: h\chi_b'(h) - \chi_b(h) < 0\}$$

and

$$H_c^- = \inf\{h \leq 0: h\chi_b'(h) - \chi_b(h) < 0\}.$$

Then $H_c^- \leq 0 < 1 \leq H_c^+$, *with* $h\chi_b'(h) - \chi_b(h) < 0$ *for all* $H_c^- < h < H_c^+$. *Furthermore, for* $h \in [0, 1] \cup (H_c^-, H_c^+)$, $\lambda_n(h; \mathbf{T}) \to \lambda_\infty(h; \mathbf{T}) P$*-a.s., where* $\mathbf{E}\lambda_\infty(h; \mathbf{T}) = 1$.

PROOF. Define

$$\chi_{b,h}(r) = \log_b \mathbf{E}\{W(h)^r \mathbf{1}[W(h) > 0]\} - (r-1) = \chi_b(hr) - r\chi_b(h). \tag{29}$$

Note that $\chi_{b,0}(r) = (1-r)\chi_b(0)$ so $\chi_{b,0}'(1) = -\chi_b(0) < 0$ since $\chi_b(h)$ is convex, $\chi_b(1) = 0$, and $\chi_b'(1-) < 0$. Also $\chi_b'(1-) = 1\chi_b'(1-) - \chi_b(1) = \chi_b'(1-) < 0$. Then simply observe that

$$\chi_{b,h}'(1-) = \begin{cases} h\chi_b'(h-) - \chi_b(h), & \text{if } h > 0, \\ h\chi_b'(h+) - \chi_b(h), & \text{if } h < 0. \end{cases}$$

Taking a derivative with respect to h, and using the convexity of $\chi_b(h)$ it is easy to see that $\chi_{b,h}'(1-) < 0$ if $h \in [0, 1] \cup (H_c^-, H_c^+)$. Now apply (29) and Theorem 2.1. □

Define a reverse filtration by

$$\widetilde{\mathscr{F}}_n = \sigma\{W_v\colon |v| > n\}, \qquad n \geq 0. \tag{30}$$

Then for each n, $Z_\infty^{(n)}(v)$ is $\widetilde{\mathscr{F}}_n$-measurable, $|v| = n$. In fact we will see below that $\{Z_\infty^{(n)h}(v), \widetilde{\mathscr{F}}_n\}$, viewed as a sequence of random variables on $\Omega \times \mathbf{T}$, comprises a reverse martingale under an appropriate size-bias change of measure. More precisely, define

$$\widetilde{Z}_n(\omega, t) = Z_\infty^{(n)}(t|n)(\omega), \qquad \omega \in \Omega,\ t \in \mathbf{T}. \tag{31}$$

Also define a probability measure $\mathscr{Q}(h; d\omega \times dt)$ on $(\Omega, \mathscr{F} \times \mathscr{B}(\mathbf{T}))$ by the disintegration formula,

$$\mathscr{Q}(h; d\omega \times dt) = P_t(h; d\omega)\lambda(dt), \tag{32}$$

where $P_t(h; d\omega) \ll P(d\omega)$ on $\mathscr{F}_n$ with

$$\frac{dP_t(h;\cdot)}{dP}\bigg|_{\mathscr{F}_n}(\omega) = \prod_{i=0}^{n} W_{t|i}(h)(\omega). \tag{33}$$

Equivalently, for each bounded measurable function f on $\Omega \times \mathbf{T}$ one has

$$\mathbf{E}\Big[\int_{\Omega\times\mathbf{T}} f(\omega, t)\mathscr{Q}(h; d\omega \times dt)|\mathscr{F}_n\Big] = \mathbf{E}\Big[\int_{\mathbf{T}} f(\omega, t)\lambda_n(h; dt)\Big]. \tag{34}$$

PROPOSITION 2.3. *One has*

$$\mathbf{E}_{\mathscr{Q}}(h;\cdot)\widetilde{Z}_n^h = \mathbf{E}_P Z_\infty^h.$$

PROOF. One has using (27) that

$$\begin{aligned}
\mathbf{E}_{\mathscr{Q}(h;\cdot)}\widetilde{Z}_n^h &= \mathbf{E}_P\Big[\int_{\mathbf{T}} \widetilde{Z}_n^h(\omega, t)\lambda_\infty(h; dt)\Big] \\
&= \mathbf{E}_P\Big[\mathbf{E}_P\Big\{\int_T \widetilde{Z}_\infty^h(t|n)\lambda_\infty(h; dt)\Big|\mathscr{F}_n\Big\}\Big] \qquad (35)\\
&= \mathbf{E}_P\Big[\int_{\mathbf{T}} \mathbf{E}_P\widetilde{Z}_\infty^h(t|n)\lambda_n(h; dt)\Big] \\
&= \mathbf{E}_P Z_\infty^h \mathbf{E}_P\lambda_n(h; \mathbf{T}) = \mathbf{E}_P Z_\infty^h, \qquad (36)
\end{aligned}$$

where we used independence of $Z_\infty^{(n)h}$ with $\mathscr{F}_n$. This completes the proof. □

COROLLARY 2.1. *For each $\widetilde{\mathscr{F}}_n \times \mathscr{B}(\mathbf{T})$-measurable, bounded function $\widetilde{G}$ one has*

$$\mathbf{E}_{\mathscr{Q}(h;\cdot)}[\widetilde{G}\widetilde{Z}_n^h] = \mathbf{E}_{\mathscr{Q}(h;\cdot)}[\widetilde{G}\widetilde{Z}_{n+k}^h], \qquad k = 1, 2, \ldots.$$

THEOREM 2.3. *The sequence of random variables $\{\widetilde{Z}_n^h\colon n = 0, 1, 2, \ldots\}$ on $\Omega \times \mathbf{T}$ is a reverse martingale with respect to $\widetilde{\mathscr{F}}_n \times \mathscr{B}(\mathbf{T})$.*

PROOF. First note that

$$\widetilde{Z}_{n+1}(t,\omega) = Z_\infty(t|n+1) = \lim_{N\to\infty} \sum_{|u|=N} \prod_{i=1}^{N} W_{\langle t|n+1\rangle * \langle u_1,\dots,u_i\rangle} b^{-N}$$

is $\widetilde{\mathscr{F}}_{n+1} \times \mathscr{B}$-measurable. We must check that

$$E_{\mathscr{Q}(h;\cdot)}[\widetilde{Z}_n^h | \widetilde{\mathscr{F}}_{n+1} \times \mathscr{B}(\mathbf{T})] = \widetilde{Z}_{n+1}^h.$$

For this let $\widetilde{G}$ be an arbitrary bounded $\widetilde{\mathscr{F}}_{n+1} \times \mathscr{B}$-measurable function. Then

$$\begin{aligned}
\mathbf{E}_{\mathscr{Q}(h;\cdot)} \widetilde{G}\widetilde{Z}_n^h &= \mathbf{E}_P \int_{\mathbf{T}} \widetilde{G}(\omega,t)\widetilde{Z}_n^h(\omega,t)\lambda_\infty(h;dt) \\
&= \mathbf{E}_P\left[\mathbf{E}_P\left\{\int_{\mathbf{T}} \widetilde{G}(\omega,t)\widetilde{Z}_n^h(\omega,t)\lambda_\infty(h;dt)\middle|\widetilde{\mathscr{F}}_n\right\}\right] \\
&= \mathbf{E}_P\left[\int_{\mathbf{T}} \mathbf{E}_P[\widetilde{G}\widetilde{Z}_n^h]\lambda_n(h;dt)\right] \qquad (37) \\
&= \mathbf{E}_P[\widetilde{G}\widetilde{Z}_n^h]\mathbf{E}_P\lambda_n(h;\mathbf{T}) \\
&= \mathbf{E}_P\widetilde{G}\widetilde{Z}_n^h = \mathbf{E}_P\widetilde{G}\widetilde{Z}_{n+1}^h,
\end{aligned}$$

where we apply Corollary 2.1 in the last line. This completes the proof. □

COROLLARY 2.2. *One has*

$$\widetilde{Z}_n^h \to \mathbf{E}_p Z_\infty^h$$

$\mathscr{Q}(h;\cdot)$-a.s. and in $L^1(\mathscr{Q}(h;\cdot))$.

PROOF. In view of Theorem 2.3 there is a random variable Y such that $\widetilde{Z}_n^h \to Y \mathscr{Q}(h;\cdot)$-a.s. and in $L^1(\mathscr{Q}(h;\cdot))$. Moreover,

$$Y = \mathbf{E}[\widetilde{Z}_1^h | \widetilde{\mathscr{F}}_\infty \times \mathscr{B}(\mathbf{T})].$$

However, $\widetilde{\mathscr{F}}_\infty \times \mathscr{B}(\mathbf{T})$ is trivial with respect to $\mathscr{Q}(h;\cdot)$. Since $\{W_v: v \in \mathbf{T}\}$ are independent under $\mathscr{Q}(h;\cdot)$; see Waymire and Williams (1996), it follows that Y is $\mathscr{Q}(h;\cdot)$-a.s. constant. Thus

$$\widetilde{Z}_n^h \to \mathbf{E}_{\mathscr{Q}(h;\cdot)}\widetilde{Z}_1^h = \mathbf{E}_P Z_\infty^h, \qquad n \to \infty,$$

$\mathscr{Q}(h;\cdot)$-a.s. and in $L^1(\mathscr{Q}(h;\cdot))$. □

THEOREM 2.4. *Assume that $\chi_b'(1-) < 0$ and $\chi_b(h) < 0$. Take*

$$h \in [0,1] \cup (H_c^-, H_c^+).$$

Then

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to \lambda_\infty(h,\mathbf{T})\mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]\mathbf{E}Z_\infty^h$$

in probability as $n \to \infty$. Moreover the limit is finite and positive with positive probability.

PROOF. First note that

$$\left|\frac{M_n(h)}{b^{n\chi_b}(h)} - \lambda_\infty(h,\mathbf{T})\mathbf{E}Z_\infty^h\right| \le \left|\frac{M_n(h)}{b^{n\chi_b}(h)} - \lambda_n(h,\mathbf{T})\mathbf{E}Z_\infty^h\right| + \left|\lambda_n(h,\mathbf{T}) - \lambda_\infty(h,\mathbf{T})\right|\mathbf{E}Z_\infty^h.$$

The second term is P-a.s. $o(1)$ by the martingale convergence theorem, and thus converges to 0 in probability as well. We will show that the first term is $o(1)$ in $L^1(P)$ and thus in P-probability. In view of Proposition 2.1, one has

$$\begin{aligned}\left|\frac{M_n(h)}{b^{n\chi_b(h)}} - \lambda_\infty(h;\mathbf{T})\mathbf{E}_P Z_\infty^h\right| &\le \mathbf{E}_P \int_{\mathbf{T}} \left|\widetilde{Z}_n^h - \mathbf{E}_P Z_\infty^h\right| \lambda_n(h;dt) \\ &= \int_{\mathbf{T}} \mathbf{E}_P\left|\widetilde{Z}_n^h - \mathbf{E}_P Z_\infty^h\right| \mathbf{E}_P\lambda_n(h;dt) \\ &= \int_{\mathbf{T}} \mathbf{E}_P\left|\widetilde{Z}_n^h - \mathbf{E}_P Z_\infty^h\right| \mathbf{E}_P\lambda_\infty(h;dt) \\ &= \mathbf{E}_P \int_{\mathbf{T}} \left|\widetilde{Z}_n^h - \mathbf{E}_P Z_\infty^h\right| \lambda_\infty(h;dt) \\ &= \mathbf{E}_{\mathscr{Q}(h;\cdot)}\left|\widetilde{Z}_n^h - \mathbf{E}_P Z_\infty^h\right|\end{aligned} \tag{38}$$

Now simply use the L^1 convergence property of reverse martingales. □

REMARK. One may note that the conditions are weaker than previously required in Holley and Waymire.

Theorem 2.2 may now be obtained by an application of the above results as follows.

PROOF OF THEOREM 2.2. First note that

$$\frac{\log_b M_n(h)}{\log_b(b^n)} = \frac{1}{n}\log\left(\frac{M_n(h)}{b^{n\chi_b(h)}}\right) + \chi_b(h).$$

Thus, in view of Theorem 2.4, one may obtain $\chi_b(h)$ as a limit on a set of positive probability for a countable dense set of $h \in [0,1] \cup (H_c^-, H_c^+)$. Since $\chi_b(h)$ exists for h in a neighborhood of the origin one may check that $H_c^- < 0$ and the distribution of $\log W$ is uniquely determined by its moment generating function in a neighborhood of the origin [cf. Billingsley (1986), page 408]. □

3. Laws of large numbers. The main result of this section is the following Theorem 3.1, which provides a strengthening of Theorem 2.4 to almost sure convergence. It is the key ingredient in the derivation of the a.s. convergence of both $\hat{\tau}_n(h)$ and $\tilde{\tau}_n(h)$.

THEOREM 3.1. *For* $h \in [0,1] \cup (H_c^-, H_c^+)$,

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to \lambda_\infty(h, \mathbf{T})\mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]\mathbf{E}\lambda_\infty^h(\mathbf{T})$$

P-a.s. as $n \to \infty$.

The proof of Theorem 3.1 is deferred until later in this section. Convergence of the estimators $\hat{\tau}_n(h)$ and $\tilde{\tau}_n(h)$ to the structure function $\chi_b(h)$ for h inside the critical interval (H_c^-, H_c^+) is provided in Corollaries 3.2, 3.3 and 3.4, respectively.

COROLLARY 3.1. *For any* $h \in \{0\} \cup (H_c^-, H_c^+)$ *with* $h \neq 1$,

$$P([\lambda_\infty(\mathbf{T}) > 0]\Delta[\lambda_\infty(h; \mathbf{T}) > 0]) = 0.$$

COROLLARY 3.2. *For any* $h \in [0,1] \cup (H_c^-, H_c^+)$, *the following hold P-a.s. as $n \to \infty$ on the set* $[\lambda_\infty(\mathbf{T}) > 0]$:

(i) $$(\log_b M_n(h) - n\chi_b(h)) \to \log_b \lambda_\infty(h, \mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T})$$

and

(ii) $$\hat{\tau}_n(h) \to \chi_b(h).$$

COROLLARY 3.3. *On the set* $[\lambda_\infty(\mathbf{T}) > 0]$,

$$\{\hat{\tau}_n(h): h \in [0,1] \cup (H_c^-, H_c^+)\} \to \{\chi_b(h): h \in [0,1] \cup (H_c^-, H_c^+)\}$$

P-a.s. as $n \to \infty$.

PROOF. The function $\hat{\tau}_n(h) = n^{-1} \log_b M_n(h)$ is a convex function of h, which converges a.s. to the continuous convex function $\chi_b(h)$ on any countable set of $h \in (H_c^-, H_c^+)$. □

COROLLARY 3.4. *On the set* $[\lambda_\infty(\mathbf{T}) > 0]$ *one has P-a.s. that*

$$\{\tilde{\tau}_n(h): h \in [0,1] \cup (H_c^-, H_c^+)\} \to \{\chi_b(h): h \in [0,1] \cup (H_c^-, H_c^+)\}$$

as $n \to \infty$.

PROOF. On the set $[\lambda_\infty(\mathbf{T}) > 0]$,

$$b^{\tilde{\tau}_n(h)} = \frac{M_{n+1}(h)}{M_n(h)} = b^{\chi_b(h)} \frac{M_{n+1}(h)}{b^{(n+1)\chi_b(h)}} \left(\frac{M_n(h)}{b^{n\chi_b(h)}}\right)^{-1} \to b^{\chi_b(h)},$$

P-a.s. for $h \in [0,1] \cup (H_c^-, H_c^+)$. □

The following theorem provides a strengthening of Theorem 2.4, both probabilistically and in scope. It shows that the form of the limiting behavior of $\hat{\tau}_n(h) = n^{-1} \log_b M_n(h)$, viewed as a function of h, is different outside the set $[0,1] \cup (H_c^-, H_c^+)$; that is, when the low frequency h-cascade $\lambda_n(h, \mathbf{T})$ dies out a.s. with respect to P. It also makes clear that the interval (H_c^-, H_c^+) is a bona fide critical interval for estimation purposes. Weaker versions of this appear in Lovejoy and Schertzer (1991), Holley and Waymire (1992), Collect and Koukiou (1992), Franchi (1995) and Molchan (1996).

THEOREM 3.2. *Let*

$$\overline{\chi}_b(h) = \begin{cases} h\chi_b'(H_c^-), & \text{if } h \le H_c^- < 0, \\ \chi_b(h), & \text{if } h \in (H_c^-, H_c^+) \cup [0, 1], \\ h\chi_b'(H_c^+), & \text{if } h \ge H_c^-. \end{cases}$$

If $H_c^+ < h_c$ *and* $H_c^- < 0$, *with* $\mathbf{E}W^h\mathbf{1}[W > 0] < \infty$ *for some* $h < H_c^-$, *then on* $A = [\lambda_\infty(\mathbf{T}) > 0]$,

$$\{\hat{\tau}_n(h): h \in \mathbf{R}\} \to \{\overline{\chi}_b(h): h \in \mathbf{R}\}$$

P-a.s. as $n \to \infty$. *If* $H_c^- = 0$ *and* $H_c^+ < h_c$, *then on* A,

$$\{\hat{\tau}_n(h): h \ge 0\} \to \{\overline{\chi}_b(h): h \ge 0\}$$

P-a.s. as $n \to \infty$.

REMARK. The simulation results given in Section 5 strongly suggest that the estimator $\tilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h))$ converges to neither $\chi_b(h)$ nor $\overline{\chi}_b(h)$ for $h > H_c^+$ and $h < H_c^-$. This is partially explained by the following Theorem 3.3, which gives a result complementary to those of Theorems 3.1 and 3.2 for $h > H_c^+$; analogous statements hold, of course, for h in an open interval bounded above by H_c^-. The proof of Theorem 3.3 is deferred until the end of this section.

THEOREM 3.3. *For* $h \in (H_c^+, h_c)$, *P-a.s. as* $n \to \infty$,

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to 0.$$

We now proceed to the proof of Theorem 3.1. It relies on a truncation argument and careful use of Chebyshev's inequality. Similar methods are used in Ossiander (2000) to show that the support sets (in $\mathbf{T}$) of the collection of h-cascades are disjoint a.s. P.

PROOF OF THEOREM 3.1. For $h = 1$, the result holds trivially. For $h = 0$,

$$\begin{aligned} &\frac{M_n(0)}{b^{n\chi_b(0)}} - \lambda_n(0; \mathbf{T})P(\lambda_\infty(\mathbf{T}) > 0) \\ &\qquad = \sum_{|v|=n} \{\mathbf{1}[Z_\infty(v) > 0] - P(\lambda_\infty(\mathbf{T}) > 0)\}\lambda_n(0; \Delta_n(v)) \end{aligned} \tag{39}$$

It is easy to check that the variance of the sum is $P(\lambda_\infty(\mathbf{T}) > 0)P(\lambda_\infty(\mathbf{T}) = 0)b^{-n\chi_b(0)}$. Since $\chi_b(0) > 0$, the sum of these variances is finite. The result follows from Proposition 2.2 in this case.

For the remaining values of h the first step involves a truncation as follows. Fix $h \in (H_c^-, H_c^+)$ and write

$$\frac{M_n(h)}{b^{n\chi_b(h)}} = \sum_{|v|=n} Z_\infty^h(v)\lambda_n(h; \Delta_n(v)) \tag{40}$$

as given by Proposition 2.1. Since $\chi'_{b,h}(1) < 0$, we can choose $\varepsilon > 0$ small enough to have both $\chi_{b,h}(1+\varepsilon) < 0$ and $h(1+\varepsilon) \in (H_c^-, H_c^+)$. Set $\alpha = b^{\chi_{b,h}(1+\varepsilon)} < 1$. Fix $n > 1$, and for $|v| = n$ let

$$\widetilde{Z}(v) = Z_\infty(v)\mathbf{1}\big[Z_\infty^h(v)\lambda_n(h; \Delta_n(v)) < \alpha^{n/2(1+\varepsilon)}\big]. \tag{41}$$

We use this truncation and a conditional centering to decompose $M_n(h)/b^{n\chi_b(h)}$. Write

$$\begin{aligned}
&\frac{M_n(h)}{b^{n\chi_b(h)}} - \mathbf{E}\lambda_\infty^h(\mathbf{T}) \cdot \lambda_n(h; \mathbf{T}) \\
&\quad = \sum_{|v|=n} \big(Z_\infty^h(v) - \widetilde{Z}^h(v)\big)\lambda_n(h; \Delta_n(v)) \\
&\qquad + \sum_{|v|=n} (\widetilde{Z}^h(v) - \mathbf{E}[\widetilde{Z}^h(v)|\mathscr{F}_n])\lambda_n(h; \Delta_n(v)) \\
&\qquad - \bigg(\mathbf{E}\lambda_\infty^h(\mathbf{T}) \cdot \lambda_n(h; \mathbf{T}) - \sum_{|v|=n} \mathbf{E}[\widetilde{Z}^h(v)|\mathscr{F}_n]\lambda_n(h; \Delta_n(v))\bigg).
\end{aligned} \tag{42}$$

We will show that each of the three terms on the right side of (42) converges to 0 P-a.s. as $n \to \infty$. This will then give

$$\frac{M_n(h)}{b^{n\chi_b(h)}} - \mathbf{E}\lambda_\infty^h(\mathbf{T}) \cdot \lambda_n(h; \mathbf{T}) \to 0$$

P-a.s. as $n \to \infty$. Since, by Proposition 2.2, $\lambda_n(h; \mathbf{T}) \to \lambda_\infty(h; \mathbf{T})$ P-a.s. as $n \to \infty$, the asserted result is obtained. Let

$$A_n = \bigcup_{|v|=n} [Z_\infty(v) \neq \widetilde{Z}(v)]. \tag{43}$$

The first term is treated as follows. Using subadditivity followed by the well-known bound for nonnegative random variables, $\mathbf{E}(X\mathbf{1}[X > a]) \le \mathbf{E}X^{1+\delta}a^{-\delta}$ for $a, \delta > 0$, one obtains

$$\begin{aligned}
P(A_n) &\le \sum_{|v|=n} P(Z_\infty(v) \neq \widetilde{Z}(v)) \\
&= b^n P\Big(Z_\infty^h(v_0)\lambda_n(h; \Delta_n(v_0)) > \alpha^{n/2(1+\varepsilon)}\Big) \\
&\le b^n \mathbf{E}\big[Z_\infty^{h(1+\varepsilon)}(v_0)\lambda_n^{1+\varepsilon}(h; \Delta_n(v_0))\alpha^{-n/2}\big] \\
&= b^n \mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T}) b^{n(\chi_{b,h}(1+\varepsilon)-1)}\alpha^{-n/2} \\
&= \alpha^{n/2}\mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T}).
\end{aligned} \tag{44}$$

Theorem 2.1 gives $\mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T}) < \infty$. Since $\alpha < 1$, $\sum_n P(A_n) < \infty$, so that $P(A_n \text{ i.o.}) = 0$. Thus, as $n \to \infty$, one has, P-a.s.,

$$\sum_{|v|=n} (Z_\infty^h(v) - \widetilde{Z}^h(v))\lambda_n(h;\Delta_n(v)) \to 0. \tag{45}$$

The third term in (42) is

$$\begin{aligned}
&\mathbf{E}\lambda_\infty^h(\mathbf{T}) \cdot \lambda_n(h;\mathbf{T}) - \sum_{|v|=n} \mathbf{E}[\widetilde{Z}^h(v)|\mathscr{F}_n]\lambda_n(h;\Delta_n(v)) \\
&\quad = \sum_{|v|=n} (\mathbf{E}Z_\infty^h(v) - \mathbf{E}[\widetilde{Z}^h(v)|\mathscr{F}_n])\lambda_n(h;\Delta_n(v)) \\
&\quad = \sum_{|v|=n} \mathbf{E}\big(Z_\infty^h(v)\mathbf{1}\big[Z_\infty^h(v)\lambda_n(h;\Delta_n(v)) > \alpha^{n/2(1+\varepsilon)}\big]\big|\mathscr{F}_n\big)\lambda_n(h;\Delta_n(v)).
\end{aligned} \tag{46}$$

This difference can now be seen to be nonnegative. Using the conditional version of the bound used previously in (44), (46) above is bounded as follows:

$$\begin{aligned}
&\sum_{|v|=n} \mathbf{E}\big(Z_\infty^{h(1+\varepsilon)}(v)\lambda_n^{1+\varepsilon}(h;\Delta_n(v))|\mathscr{F}_n\big)\alpha^{-n\varepsilon/2(1+\varepsilon)} \\
&\quad = \mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T})\alpha^{-n\varepsilon/(2(1+\varepsilon))} \sum_{|v|=n} \lambda_n^{1+\varepsilon}(h;\Delta_n(v)) \\
&\quad = \mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T})\alpha^{n(2+\varepsilon)/2(1+\varepsilon)} \sum_{|v|=n} \lambda_n(h(1+\varepsilon);\Delta_n(v)) \\
&\quad = \mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T})\alpha^{n(2+\varepsilon)/2(1+\varepsilon)}\lambda_n(h(1+\varepsilon);\mathbf{T}).
\end{aligned} \tag{47}$$

Since $\mathbf{E}\lambda_\infty^{h(1+\varepsilon)}(\mathbf{T}) < \infty$, $\lambda_n(h(1+\varepsilon);\mathbf{T}) \to \lambda_\infty(h(1+\varepsilon);\mathbf{T})$ P-a.s. as $n \to \infty$, and $\alpha < 1$, this sum converges to 0 P-a.s. as $n \to \infty$.

Finally, let us consider the middle term of (42), namely the sum

$$S_n := \sum_{|v|=n} \big(\widetilde{Z}^h(v) - \mathbf{E}\big[\widetilde{Z}^h(v)\big|\mathscr{F}_n\big]\big)\lambda_n(h;\Delta_n(v)). \tag{48}$$

Notice that for $|u| = |v| = n$, with $u \neq v$, one has that $\widetilde{Z}(u)$ and $\widetilde{Z}(v)$ are conditionally independent and thus conditionally uncorrelated given $\mathscr{F}_n$. Thus

$$\begin{aligned}
\operatorname{Var} S_n &= \mathbf{E}S_n^2 = \mathbf{E}[\mathbf{E}(S_n^2|\mathscr{F}_n)] \\
&= \mathbf{E} \sum_{|v|=n} \mathbf{E}[(\widetilde{Z}^h(v) - \mathbf{E}(\widetilde{Z}^h(v)|\mathscr{F}_n))^2|\mathscr{F}_n]\lambda_n^2(h;\Delta_n(v)) \\
&\le \mathbf{E} \sum_{|v|=n} \mathbf{E}[\widetilde{Z}^{2h}(v)\lambda_n^2(h;\Delta_n(v))|\mathscr{F}_n] \\
&\le \alpha^{n/2(1+\varepsilon)} \sum_{|v|=n} \mathbf{E}[\widetilde{Z}^h(v)\lambda_n(h;\Delta_n(v))] \\
&\le \alpha^{n/2(1+\varepsilon)}\mathbf{E}\lambda_\infty^h(\mathbf{T}).
\end{aligned} \tag{49}$$

Since $\alpha < 1$ and $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$, it follows that $\sum_n \operatorname{Var} S_n < \infty$, and hence $S_n \to 0$ P-a.s. as $n \to \infty$. □

The following propositions are utilized in the proof of Theorem 3.2.

PROPOSITION 3.1. *If $H_c^+ < \infty$, then for $h \ge H_c^+$, on $A = [\lambda_\infty(\mathbf{T}) > 0]$,*

$$\limsup_n \hat{\tau}_n(h) \le \frac{h}{H_c^+}\chi_b(H_c^+-) \quad P\text{-}a.s. \tag{50}$$

and

$$\limsup_n \hat{\tau}_n(h) \ge h\chi_b'(H_c^+-) \quad P\text{-}a.s. \tag{51}$$

PROOF. Fix $h \ge H_c^+$ and take $r \in (0, H_c^+)$. For $a, b > 0$ and $p \ge 1$, $a^p + b^p \le (a+b)^p$. Thus, taking $h/r = p > 1$,

$$M_n(h) = \sum_{|v|=n} Z_\infty^h(v)\lambda_n^h(\Delta_n(v)) \le M_n^{h/r}(r). \tag{52}$$

From Corollary 3.2. one has P-a.s.,

$$\limsup_{n\to\infty} \hat{\tau}_n(h) \le \frac{h}{r}\chi_b(r). \tag{53}$$

Let $r \uparrow H_c^+$ to see that P-a.s.,

$$\limsup_{n\to\infty} \hat{\tau}_n(h) \le \frac{h}{H_c^+}\chi_b(H_c^+-). \tag{54}$$

To derive the lower bound again take $r \in (0, H_c^+)$. From Jensen's inequality, for any $\varepsilon > 0$,

$$\begin{aligned}\frac{M_n(h)}{M_n(r)} &= \frac{\sum_{|v|=n}(Z_\infty(v)\lambda_n(\Delta_n(v)))^{h-r}\lambda_\infty^r(\Delta_n(v))}{M_n(r)}\\ &\ge \left(\frac{\sum_{|v|=n}(Z_\infty(v)\lambda_n(\Delta_n(v)))^{(h-r)/(1+\varepsilon)}\lambda_\infty^r(\Delta_n(v))}{M_n(r)}\right)^{1+\varepsilon}\\ &= \left(\frac{M_n(r+(h-r)/(1+\varepsilon))}{M_n(r)}\right)^{1+\varepsilon}\end{aligned} \tag{55}$$

If we require $\varepsilon > (h - H_c^+)/(H_c^+ - r)$, so that $r + (h-r)/(1+\varepsilon) = (h+\varepsilon r)/(1+\varepsilon) < H_c^+$, then we can apply Corollary 3.2 to see that P-a.s.,

$$\liminf_{n\to\infty} \hat{\tau}_n(h) \ge (1+\varepsilon)\chi_b\left(\frac{h+\varepsilon r}{1+\varepsilon}\right) - \varepsilon\chi_b(r). \tag{56}$$

Letting $\varepsilon \downarrow (h - H_c^+)/(H_c^+ - r)$, the right-hand side becomes $\chi_b(H_c^+-) + (h - H_c^+)/(H_c^+ - r)(\chi_b(H_c^+-) - \chi_b(r))$. Now let $r \uparrow H_c^+$ and use the definition of H_c^+ to obtain

$$\liminf_{n\to\infty} \hat{\tau}_n(h) \geq \chi_b(H_c^+-) + (h - H_c^+)\chi_b'(H_c^+-) \geq h\chi_b'(H_c^+-). \quad \square \tag{57}$$

PROPOSITION 3.2. *If* $-\infty < H_c^- < 0$, *then for* $h \leq H_c^-$,

$$\limsup_n \hat{\tau}_n(h) \leq \frac{h}{H_c^-}\chi_b(H_c^-+) \tag{58}$$

and

$$\liminf_n \hat{\tau}_n(h) \geq h\chi_b'(H_c^-+). \tag{59}$$

PROOF. The proof follows exactly the same pattern as the proof of Proposition 3.1. $\square$

PROOF OF THEOREM 3.2. If $H_c^+ < h_c$, then both $\chi_b(H_c^+)$ and $\chi_b'(H_c^+)$ are defined and finite with $0 = \chi'_{b,\,H_c^+}(1) = H_c^+\chi_b'(H_c^+) - \chi_b(H_c^+)$. Thus the upper and lower bounds of Proposition 3.1 are the same. If $\mathbf{E}W^h\mathbf{1}[W > 0] < \infty$ for some $h < H_c^-$, then both $\chi_b(H_c^-)$ and $\chi_b'(H_c^-)$ are defined and finite with $0 = \chi'_{b,\,H_c^-}(1) = H_c^-\chi_b'(H_c^-) - \chi_b(H_c^-)$. Then the upper and lower bounds of Propositions 3.1 and 3.2 are identical. $\square$

It is easy to see that the above Theorem 3.1 may be generalized as follows.

THEOREM 3.4. *Suppose that* $\{X(v)\colon |v| = n, n \geq 1\}$ *is a collection of identically distributed nonnegative random variables defined on* $(\Omega, \mathscr{F}, P)$ *with* $\mathbf{E}X^{1+\varepsilon}(v_0) < \infty$ *for some* $\varepsilon > 0$ *and, for each* $n \geq 1$, $\{X(v)\colon |v| = n\}$ *is a collection of independent random variables which is also independent of* $\mathscr{F}_n$. *Then for* $h \in [0,1] \cup (H_c^-, H_c^+)$,

$$\sum_{|v|=n} X(v)\lambda_n(h; \Delta_n(v)) \to \lambda_\infty(h; \mathbf{T})\mathbf{E}X(v_0)$$

P-*a.s. as* $n \to \infty$.

PROOF. In the proof of Theorem 3.1 replace $Z_\infty^h(v)$ by $X(v)$ and $\widetilde{Z}^h(v)$ with

$$\widetilde{X}(v) = X(v)\mathbf{1}\big[X(v)\lambda_n(h; \Delta_n(v)) \leq \alpha^{n/2(1+\varepsilon)}\big] \tag{60}$$

for suitably small $\varepsilon > 0$. The proof is then identical. $\square$

The proof of Theorem 3.3 relies on the following proposition.

PROPOSITION 3.3. *Assume* $\chi_b'(1) > 0$ *and let* $\{X(v)\colon |v| = n, n \geq 1\}$ *be a collection of random variables defined on* $(\Omega, \mathscr{F}, P)$ *with* $\sup_n \max_{|v|=n} \mathbf{E}|X(v)| = C < \infty$ *and, for each* $n \geq 1$, $\{X(v)\colon |v| = n\}$ *independent of* $\mathscr{F}_n$. *Then*

$$\sum_{|v|=n} X(v)\lambda_n(\Delta_n(v)) \to 0, \quad P\text{-a.s. as } n \to \infty.$$

PROOF. Fix $r \in (0,1)$ so that $\chi_b(r) < 0$. Then

$$\begin{aligned} E\Big|\sum_{|v|=n} X(v)\lambda_n(\Delta_n(v))\Big|^r &\le \mathbf{E}\sum_{|v|=n} \big|X(v)\lambda_n(\Delta_n(v))\big|^r \\ &\le \max_{|v|=n} \mathbf{E}\big|X(v)\big|^r b^n \mathbf{E}\lambda_n^r(\Delta_n(v_0)) \\ &\le C^r b^{n(1-r)} \mathbf{E}\prod_1^n W_{v_0|i}^r \\ &= C^r b^{n\chi_b(r)}. \end{aligned}$$

Since $\chi_b(r) < 0$ the sum over n of this bound is finite. Applying Chebyshev's inequality and the Borel–Cantelli lemma, we see that for any $\varepsilon > 0$, $P\big(\big|\sum_{|v|=n} X(v)\lambda_n(\Delta_n(v))\big| > \varepsilon \text{ i.o.}\big) = 0$, and thus

$$\sum_{|v|=n} X(v)\lambda_n(\Delta_n(v)) \to 0,$$

P-a.s. □

PROOF OF THEOREM 3.3. For $h \in (H_c^+, h_c)$ write

$$\frac{M_n(h)}{b^{n\chi_b(h)}} = \sum_{|v|=n} Z_\infty^h(v)\lambda_n(h;\Delta_n(v)).$$

For $h \in (H_c^+, h_c)$, $\chi'_{b,h}(1) > 0$ and $\{Z_\infty^h(v)\colon |v| = n\}$ forms a collection of i.i.d. nonnegative random variables independent of $\mathscr{F}_n$ with $\mathbf{E}Z_\infty^h(v) < \infty$. From Proposition 3.3 this converges to 0 P-a.s. □

4. Central limit theorems. Here we develop asymptotic error distributions for the estimators $\hat{\tau}_n(h)$ and $\tilde{\tau}_n(h)$ for h within the scaled critical interval $(H_c^-/2, H_c^+/2)$. The central limit theorem for a normalized $\hat{\tau}_n(h)$ appears in Corollary 4.4. The central limit theorem for the estimator $\tilde{\tau}_n(h)$ is given in Corollary 4.7.

For each $n \ge 1$, let $\{X_n(v)\colon |v| = n\}$ be a collection of independent random variables which are also independent of $\mathscr{F}_n$. Define

$$S_n(h) = \sum_{|v|=n} X_n(v)\lambda_n(h;\Delta_n(v)). \tag{61}$$

Also let

$$R_n(h) = \frac{S_n(h)}{(\sum_{|v|=n} \lambda_n^2(h;\Delta_n(v)))^{1/2}}. \tag{62}$$

PROPOSITION 4.1. *If* $\mathbf{E}X_n^2(v) = 1$ *and* $\mathbf{E}X_n(v) = 0$ *for each* v, *and if*

$$\sup_n \sup_{|v|=n} \mathbf{E}|X_n(v)|^{2(1+\delta)} < \infty$$

for some $\delta > 0$, *then for* $h \in \left(H_c^-/2, H_c^+/2\right)$,

$$\lim_{n\to\infty} \mathbf{E}[e^{izR_n(h)}|\mathscr{F}_n] = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0] + e^{-(1/2)z^2}\mathbf{1}[\lambda_\infty(\mathbf{T}) > 0], \tag{63}$$

with the convention that $R_n(h) = 0$ *if* $\lambda_n(\mathbf{T}) = 0$.

PROOF. Fix $h \in \left(H_c^-/2, H_c^+/2\right)$, and choose $\delta > 0$ sufficiently small that $h(1+\delta) \in \left(H_c^-/2, H_c^+/2\right)$, $\sup_n \sup_{|v|=n} \mathbf{E}|X_n(v)|^{2(1+\delta)} < \infty$, and $\chi'_{b,2h}(1+\delta) < 0$. Together with the convexity of the $\chi_{b,2h}$, this last implies that $\chi_b(2h(1+\delta)) - (1+\delta)\chi_b(2h) < 0$. For $|v| = n$, let

$$Y_n(v) = \frac{X_n(v)\lambda_n(h;\Delta_n(v))}{(\sum_{|u|=n}\lambda_n^2(h;\Delta_n(u)))^{1/2}}. \tag{64}$$

Again we take $Y_n(v) = 0$ if $\lambda_n(\mathrm{T}) = 0$. Conditionally on $\mathscr{F}_n$, the $Y_n(v)$'s are independent, mean zero random variables. Furthermore,

$$\mathbf{E}\left[\left(\sum_{|v|=n} Y_n(v)\right)^2 |\mathscr{F}_n\right] = \sum_{|v|=n}\mathbf{E}[Y_n^2(v)|\mathscr{F}_n] = \mathbf{1}[\lambda_n(\mathbf{T} > 0)]. \tag{65}$$

and $A_n := [\lambda_n(\mathbf{T}) > 0] \downarrow A = [\lambda_\infty(\mathbf{T}) > 0]$ a.s. P. We will show that Lindeberg's condition holds conditionally. The following is useful in this regard:

$$\begin{aligned}\sum_{|v|=n}\lambda_n^r(h;\Delta_n(v)) &= \sum_{|v|=n}\lambda_n^{rh}(\Delta_n(v))\mathbf{E}^{-rn}W^h b^{nr(h-1)}\\ &= \sum_{|v|=n}\lambda_n(rh;\Delta_n(v))\mathbf{E}^{-rn}W^h\mathbf{E}^n W^{rh} b^{n(1-r)}\\ &= \lambda_n(rh;\mathbf{T})b^{n(\chi_b(rh)-r\chi_b(h))}\\ &= \lambda_n(rh;\mathbf{T})b^{n\chi_{b,h}(r)}.\end{aligned} \tag{66}$$

Now

$$\begin{aligned}&\mathbf{E}\left[\sum_{|v|=n}Y_n^2(v)\mathbf{1}[|Y_n(v)| > \varepsilon]|\mathscr{F}_n\right]\\ &\quad\le \sum_{|v|=n}\mathbf{E}\big[Y_n^2(v)\mathbf{1}\big[|Y_n(v)| > \varepsilon\big]\big|\mathscr{F}_n\big]\mathbf{1}[A] + \mathbf{1}[A_n - A]\\ &\quad\le \varepsilon^{-2\delta}\sum_{|v|=n}\mathbf{E}\big[|Y_n(v)|^{2(1+\delta)}\big|\mathscr{F}_n\big]\mathbf{1}[A] + \mathbf{1}[A_n - A]\\ &\quad\le \varepsilon^{-2\delta}\sup_{|v|=n}\mathbf{E}|X_n(v)|^{2(1+\delta)}\frac{\sum_{|v|=n}\lambda_n^{2(1+\delta)}(h;\Delta_n(v))}{(\sum_{|v|=n}\lambda^2(h;\Delta_n(v)))^{1+\delta}}\mathbf{1}[A] + \mathbf{1}[A_n - A]\\ &\quad\le C\varepsilon^{-2\delta}\frac{\lambda_n(2h(1+\delta);\mathbf{T})}{\lambda_n^{1+\delta}(2h;\mathbf{T})}b^{n(\chi_b(2h(1+\delta))-(1+\delta)\chi_b(2h))}\mathbf{1}[A] + \mathbf{1}[A_n - A].\end{aligned} \tag{67}$$

Both $\lambda_n(2h;\mathbf{T}) \to \lambda_\infty(2h;\mathbf{T})$ and $\lambda_n(2h;\mathbf{T}) > 0$ P-a.s. on the set A. Since $\mathbf{1}[A_n - A] \to 0$ P-a.s. and $\chi_b(2h(1+\delta)) - (1+\delta)\chi_b(2h) < 0$, the right-hand side above converges to 0 P-a.s. as $n \to \infty$. This gives P-a.s.,

$$\lim_n \sum_{|v|=n} \mathbf{E}(Y_n^2(v)\mathbf{1}[|Y_n(v)| > \varepsilon]|\mathscr{F}_n) = 0. \tag{68}$$

If we set $\sigma_n^2(v) = \mathbf{E}(Y_n^2(v)|\mathscr{F}_n) = \mathbf{E}(Y_n^2(v)\mathbf{1}[A]|\mathscr{F}_n)$, then we can follow the usual proof of Lindeberg's theorem [cf. Billingsley (1986), page 369] from here to see that

$$\lim_n \mathbf{E}[e^{izR_n(h)}|\mathscr{F}_n] = \mathbf{1}[A^c] + e^{-(1/2)z^2}\mathbf{1}[A]. \tag{69}$$ □

COROLLARY 4.1. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\lim_n \mathbf{E}e^{izR_n(h)} = P(\lambda_\infty(\mathbf{T}) = 0) + e^{-(1/2)z^2}P(\lambda_\infty(\mathbf{T}) > 0). \tag{70}$$

For the proof, apply the dominated convergence theorem to the result given by Proposition 4.1.

COROLLARY 4.2. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$R_n(h) \xrightarrow{d} \eta N_h, \tag{71}$$

where η *and* N_h *are independent with* $\eta =^d \mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]$ *and* N_h *has a standard normal distribution.*

PROOF. Note that $\mathbf{E}e^{iz\eta N_h} = P(\lambda_\infty(\mathbf{T}) = 0) + e^{-(1/2)z^2}P(\lambda_\infty(\mathbf{T}) > 0)$. This follows immediately from (70) using the continuity theorem for characteristic functions. □

REMARK. One may check that, as a stochastic process indexed by h, $\{N_h\}$ is a Gaussian white noise. Moreover, Proposition 4.1 may be used to show that the convergence in Corollary 4.2 is *mixing* in the sense of Rootzen (1976).

COROLLARY 4.3. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\frac{((M_n(h)/b^{n\chi_b(h)}) - \mathbf{E}\lambda_\infty^h(\mathbf{T}) \cdot \lambda_n(h;\mathbf{T}))}{(\operatorname{Var} \lambda_\infty^h(\mathbf{T}))^{1/2}(\sum \lambda_n^2(h;\Delta_n(v)))^{1/2}} \xrightarrow{d} \eta N_h, \tag{72}$$

where η *and* N_h *are independent with* $\eta =^d \mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]$ *and* N_h *has a standard normal distribution.*

PROOF. Taking $X_n(v) = (Z_\infty^h(v) - \mathbf{E}\lambda_\infty^h(\mathbf{T}))(\operatorname{Var} \lambda_\infty^h(\mathbf{T}))^{-1/2}$ in (61), this follows from the previous corollary. □

COROLLARY 4.4. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\frac{nM_n(h)}{M_n^{1/2}(2h)}\left(\hat{\tau}_n(h) - \chi_b(h) - n^{-1}\left(\log_b \mathbf{E}\lambda_\infty^h(\mathbf{T}) + \log_b \lambda_n(h, \mathbf{T})\right)\right) \xrightarrow{d} \hat{c}_h \eta N_h, \tag{73}$$

where η *and* N_h *are independent with* $\eta =^d \mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]$, N_h *has a standard normal distribution, and* $\hat{c}_h^2 = \mathrm{Var}\ \lambda_\infty^h(\mathbf{T})/(\mathbf{E}\lambda_\infty^h(\mathbf{T}) \log^2 b)$.

REMARK. Notice that this central limit theorem for $\hat{\tau}_n(h)$ has an unobservable centering term: $n^{-1}(\log_b \mathbf{E}\lambda_\infty^h(\mathbf{T}) + \log_b \lambda_n(h, \mathbf{T}))$.

PROOF. Start with Corollary 4.3, take logarithms and use a Taylor's expansion together with Theorem 3.1 [see Serfling (1980) for details] to see

$$\lambda_\infty(h; \mathbf{T})\mathbf{E}\lambda_\infty^h(\mathbf{T}) \times \left(\frac{\log_b M_n(h) - n\chi_b(h) - \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T}) - \log_b \lambda_n(h; \mathbf{T})}{(\mathrm{Var}\ \lambda_\infty^h(\mathbf{T}))^{1/2}(\sum \lambda_n^2(h; \Delta_n(v)))^{1/2}}\right) \tag{74}$$

converges in distribution to $\eta N_h/\log b$. Using the change of variables for $\lambda_n(h; \cdot)$ given in (66), along with Proposition 2.2 and Theorem 3.1, the corollary follows. □

Investigating the distributional limit of $\tilde{\tau}_n(h)$ leads to a more satisfactory denouement.

COROLLARY 4.5. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\frac{(M_n(h)/b^{n\chi_b(h)}) - (M_{n+1}(h)/b^{(n+1)\chi_b(h)})}{(\sum \lambda_n^2(h; \Delta_n(v)))^{1/2}} \xrightarrow{d} c_h \eta N_h, \tag{75}$$

where η *and* N_h *are independent with* $\eta =^d \mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]$, N_h *has a standard normal distribution, and* $c_h = (\mathrm{Var}\ (\lambda_\infty^h(\mathbf{T}) - b^{-\chi_b(h)} M_{\mathbf{1}}(h)))^{1/2}$.

PROOF. The numerator can be written as $\sum_{|v|=n} X_n(v)\lambda_n(h; \Delta_n(v))$, where

$$X_n(v) = Z_\infty^h(v) - \sum_{i=0}^{b-1} \frac{W_{v*i}^h}{b\mathbf{E}W^h} Z_\infty^h(v * i). \tag{76}$$

For each n, these are i.i.d. random variables and independent of $\mathscr{F}_n$ with $\mathbf{E}X_n(v) = 0$ and $\mathrm{Var}\ X_n(v) = c_h^2$. □

The estimator $\tilde{\tau}_n(h)$ of $\chi_b(h)$ is obtained by differencing the logarithms of the hth sample moments at scales of resolution $n+1$ and n; namely $\tilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h))$. In view of Corollary 3.4 we have asymptotic consistency of this estimator for $h \in (H_c^-, H_c^+)$. In the following two corollaries we develop an observable normalization of this estimator which yields, in turn, a computable estimate of its variance and, thus, a central limit theorem for computing error bars.

Define

$$D_n^2(h) = \sum_{|v|=n} \left(\frac{\lambda_\infty^h(\Delta_n(v))}{M_n(h)} - \frac{1}{M_{n+1}(h)} \sum_{i=0}^{b-1} \lambda_\infty^h(\Delta_{n+1}(v*i)) \right)^2. \tag{77}$$

REMARK. Although we do not make use of it here, it should be mentioned that this statistic $D_n^2(h)$ has been shown to be an ordinary least squares variance estimator for sufficiently small h; see Troutman and Vecchia (1999).

COROLLARY 4.6. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\frac{(M_{n+1}(h)/M_n(h))b^{-\chi_b(h)} - 1}{D_n(h)} \xrightarrow{d} \eta N_h, \tag{78}$$

where η *and* N_h *are independent with* $\eta =^d \mathbf{1}[\lambda_\infty(\mathbf{T}) > 0]$ *and* N_h *has a standard normal distribution.*

PROOF. Simply note that P-a.s.,

$$\sum_{|v|=n} \lambda_n^2(h; \Delta_n(v)) b^{-n\chi_{b,h}(2)} = \lambda_n(2h; \mathbf{T}) \to \lambda_\infty(2h, \mathbf{T}) \tag{79}$$

and, letting

$$\widetilde{X}_n(v) = Z_\infty^h(v) - \frac{M_n(h)}{M_{n+1}(h)} \sum_{i=0}^{b-1} b^{-h} W_{v*i}^h Z_\infty(v*i),$$

one has upon expanding the square and repeatedly applying Theorem 3.1 that

$$\begin{aligned} M_n^2(h) D_n^2(h) b^{-n\chi_b(2h)} &= \sum_{|v|=n} \widetilde{X}_n^2(v) \lambda_n(2h; \delta_n(v)) \\ &\to c_h^2 \lambda_\infty(2h; \mathbf{T}), \end{aligned} \tag{80}$$

so that

$$\frac{M_n^2(h) D_n^2(h)}{b^{2n\chi_b(h)} \sum_{|v|=n} \lambda_n^2(h; \Delta_n(v))} \to c_h^2 \mathbf{1}[\lambda_\infty(\mathbf{T}) > 0].$$

Now rewrite

$$\begin{aligned} &\frac{(M_n(h)/b^{n\chi_b(h)}) - (M_{n+1}(h)/b^{(n+1)\chi_b(h)})}{c_h \sqrt{\sum_{|v|=n} \lambda_n^2(h; \Delta_n(v))}} \\ &\quad = -\frac{M_n(h) D_n(h)}{b^{n\chi_b(h)} c_h \sqrt{\sum_{|v|=n} \lambda_n^2(h; \Delta_n(v))}} \left(\frac{(M_{n+1}(h)/M_n(h)) b^{-\chi_b(h)} - 1}{D_n(h)} \right) \end{aligned}$$

to get the asserted result. □

The following corollary then gives a central limit theorem for a completely observable statistic whose asymptotic distribution does not depend on the distributions of the unobservable generator variables W or the unknown distribution of the cascade itself, $\lambda_\infty(\mathbf{T})$. The independence in h of the N_h's noted above indicates that the errors in this estimator of $\chi_b(h)$, namely $\tilde{\tau}(h)$, are asymptotically independent.

COROLLARY 4.7. *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\frac{\tilde{\tau}(h) - \chi_b(h)}{D_n(h)} \xrightarrow{d} (\log b)^{-1} \eta N_h. \tag{81}$$

The proof follows by taking logarithms in the previous corollary and making a Taylor approximation to get the distributional limit; for example, see Serfling (1980).

5. Simulations and numerical illustrations. In this section we will provide some MATLAB simulations to illustrate the estimation theory developed in the previous sections and its limitations.

For the first example we consider generators W which are uniformly distributed on $[0, 2]$ with binary branching number $b = 2$. In this case it is simple to check using recursions that the total mass $Z_\infty := \lambda_\infty(\mathbf{T})$ has the Gamma distribution

$$P(Z_\infty \in dx) = 4xe^{-2x}\,dx, \qquad x \geq 0. \tag{82}$$

Thus an exact simulation of the limit cascade may be achieved using this and Proposition 2.1. Figure 2(a) provides a sample realization of $\lambda_\infty(\Delta_n) = \lambda_n(\Delta_n)Z_\infty$ at the resolution $n = 13$, that is, $2^{13} = 8192$ pixels Δ_n. Also, in this case one has

$$\chi_2(h) = 1 - \log_2(1 + h), \qquad h > -1. \tag{83}$$

In particular, $H_c^- \approx -0.6266$, $\chi_2'(H_c^-) \approx -2.4210$, $H_c^+ \approx 3.3111$, $\chi_2'(H_c^+) \approx -0.3346$, $h_c = \infty$. A plot of $\bar{\chi}_2(h)$ is given in Figure 2(b). Sample values of the estimator $\tilde{\tau}_{12}(h) = \log_2(M_{13}(h)/M_{12}(h))$ for $H_c^-/2 = -0.31133 < h < 1.6555 = H_c^+/2$ are given by an overlay of $\times$'s with error bars denoting a 95% confidence interval. In Table 1 we have included the values marked by italics for discussion below; however, these are not error bars predicted by theory and are not plotted in Figure 2(b). There is a cutoff effect from the print code which should be ignored outside the theoretically applicable range.

Now let us turn to the nature of the errors outside of this range. While asymptotic consistency has been shown to hold outside this range of h-values [i.e., for all $h \in (H_c^-, H_c^+)$], the asymptotic distribution of the fluctuations is unknown there. One may note from Table 1 that $\bar{\chi}_2(h)$ is in the interval $(\tilde{\tau}_{12}(h) - D_{12}(h)\zeta_{0.025}/\log 2, \tilde{\tau}_{12}(h) + D_{12}(h)\zeta_{0.025}/\log 2)$ for some values of h in the range not covered by the present central limit theory; namely $h = -0.5$ and $h = 2.5, 3.5$. However $\bar{\chi}_2(-0.75)$ is outside the suggested interval. In

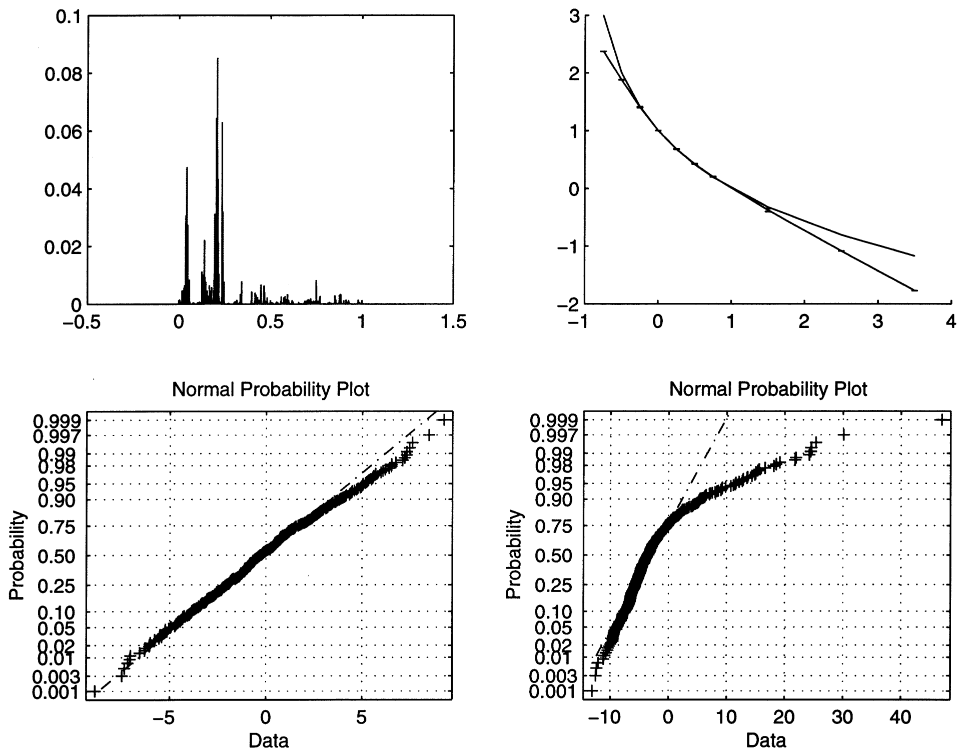

FIG. 2. *Binary cascade simulation with generators uniformly distributed on* [0, 2]. (a) *Graph of a single simulation of* $\lambda_\infty(\Delta)$ *with* $|\Delta| = 2^{-13}$; (b) *Graph of* $\bar{\chi}_2$ *and* $\tilde{\tau}_{12}$ *with error bars marked*; (c) *Probability plot of the normalized values of* $\tilde{\tau}_{12}(2.5)$; (d) *Probability plot of the normalized values of* $\tilde{\tau}_{12}(-0.75)$.

the former cases the tails of the fluctuation law appear to be comparable to Gaussian. However, deviations from normality are illustrated in the normal probability plot based on the sample of 500 realizations of the statistic $(\tilde{\tau}_{12}(h) - \chi_2(h)) \log 2/D_{12}(h)$ for $h = 2.5, h = -0.75$ in Figure 2(c) and 2(d), respectively.

6. Conclusions. The results presented here provide a rigorous basis for statistical inference based on the scaling exponents associated with intermittent and highly variable phenomena modeled by random cascades. The values H_c^- and H_c^+ as defined are *critical points* for the survival of the h-cascades; compare Proposition 2.2 and Theorems 2.4, 3.1 and 3.3. Criticality in this context is analogous to the quenched versus annealed transitions in the study of spin glass models [cf. Koukiou (1997)].

The results presented here permit a comparison of the two estimators of $\chi_b(h)$, $\hat{\tau}(h) = n^{-1} \log_b M_n(h)$ and $\tilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h))$. Within the critical interval $[0, 1] \cup (H_c^-, H_c^+)$ these are both strongly consistent estima-

TABLE 1
Estimation at resolution $n = 13$

h	$\tilde{\tau}_{12}(h)$	$\tilde{\tau}_{12}(h) \pm D_{12}(h)\zeta_{0.025}/\log 2$	$\bar{\chi}_2(h)$
−0.75	*2.6047*	*(2.3746, 2.8343)*	2.8978
−0.50	1.9614	*(1.8831, 2.0397)*	2.000
−0.25	1.1414	(1.4008, 1.4274)	1.4150
0.25	0.6780	(0.6735, 0.6825)	0.6781
0.50	0.4149	(0.4078, 0.4220)	0.4150
0.75	0.1924	(0.1856, 0.1992)	0.1926
1.50	−0.3227	(−0.3625, −0.2829)	−0.3219
2.50	−0.8293	*(−1.0311, −0.6275)*	−0.8074
3.50	*−1.2317*	*(−1.63753, −0.7881)*	−1.1221

tors. Outside this range $\hat{\tau}_n(h)$ estimates $\bar{\chi}_b(h)$. Simulation results suggest that outside this critical interval the estimator $\tilde{\tau}_n(h)$ is converging to some unknown function of h. Perhaps in practice a comparison of the two estimators can be used to delineate the critical points H_c^- and H_c^+. However the limiting behavior of $\tilde{\tau}_n(h)$ for h outside the critical interval will need to be better understood to make this a useful approach.

The fluctuations of $\tilde{\tau}_n(h)$, appropriately scaled, are approximately Gaussian within the scaled critical zone $(H_c^-/2, H_c^+/2)$. Outside this interval our probability plots suggest that either a nonnormal limit law holds, or that a different scaling is appropriate. This is also indicated by Theorem 3.3. While this extends the range of speculation based on simulation offered by Troutman and Vecchia (1999), we have also shown that this range is critical in a certain important sense. Also, one may oberve from Jensen's inequality for $h > H_c^+ \geq 1$ that

$$\begin{aligned} M_{n+1}(h) &= \frac{1}{b^{h-1}} \sum_{|v|=n} \sum_{i=0}^{b-1} \frac{Z_{v*i}^h W_{v*i}^h \lambda_n^h(\Delta_n v)}{b} \\ &\geq \frac{1}{b^{h-1}} \sum_{|v|=n} Z_v^h \lambda_n^h(\Delta_n v) = b^{1-h} M_n(h). \end{aligned}$$

Thus, in particular,

$$\tilde{\tau}_n(h) \geq 1 - h,$$

with a similar estimate for $h < H_c^-$. This bound on the estimator clearly indicates that the growth of this statistic is constrained to be finite outside the critical range as $n \to \infty$.

Building on this approach to make further inferences ragarding the distribution of the underlying cascade generators relies on Laplace transform inversion in one form or another. By restricting the class of generators to those with a density whose Fourier transform is a simple analytic continuation of the Laplace transform one may be able to obtain explicit density estimates.

This is an interesting problem which does not seem to have been treated in the literature.

The extension of the methods used here to the case of i.i.d. nonnegative random vectors $(W_0, W_1, \ldots, W_{b-1})$ of generators with $\mathbf{E}(1/b)\sum_{i=0}^{b-1} W_i = 1$ appears to be rather straightforward by the methods of this paper, although we have not checked all of the details. Our calculations show that for such vector generators one should expect the same results in this correlated case with $\chi_b(h)$ replaced by the transform $\xi_b(h) = \log \mathbf{E}\widetilde{W}^h - (h-1)$, where $\widetilde{W} = W_i$ with probability b^{-1}. In particular, the statistical estimation is restricted to the distribution of $\widetilde{W}$. Thus, in the absence of further symmetry assumptions, statistics additional to the multiscaling exponents will be required for inference on the joint distribution of the entries in the random vector $(W_0, W_1, \ldots, W_{b-1})$. An alternative approach based on wavelets is given in a recent preprint by Resnick, Gilbert, and Willinger (1999) for the special perfectly correlated case in which $b = 2$, and $W_0 = 2 - W_1$ and W_0 is symmetrically distributed about 1.

Another estimation problem that comes up in this context is that of estimating the support dimension of the random measure λ_∞ from the pixel observations $\{\lambda_\infty(\Delta_n(v))\colon v \in \{0, 1, \ldots, b-1\}^n,\ n \geq 0\}$. This problem can also be addressed using extensions of the methods of this paper; see Ossiander and Waymire (2000). Simultaneous estimation of these various parameters of the cascade and its generators requires functional analogues of Theorem 3.4 and Proposition 4.1. This suggests development of an understanding of the convergence of multiplicative cascades along the lines seen in empirical process theory; compare Dudley (1999).

Related problems arise when attempting to estimate the distribution of the cascade generators when random binning is present; that is, the branching or binning number b is a random parameter, selected independently at each splitting. The foundations for a cascade model with random binning are given in Burd and Waymire (2000).

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DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY
CORVALLIS, OREGON 97331
E-MAIL: ossiand@math.orst.edu
waymire@math.orst.edu