Noname manuscript No.
(will be inserted by the editor)

Statistical estimation of Lévy-type stochastic volatility models

José E. Figueroa-López

Received: date / Accepted: date

Abstract Volatility clustering and leverage are two of the most prominent stylized features of the dynamics of asset prices. In order to incorporate these features as well as the typical fat-tails of the log return distributions, several types of exponential Lévy models driven by random clocks have been proposed in the literature. These models constitute a viable alternative to the classical stochastic volatility approach based on SDEs driven by Wiener processes. This paper has two main objectives. First, using threshold type estimators based on high-frequency discrete observations of the process, we consider the recovery problem of the underlying random clock of the process. We show consistency of our estimator in the mean-square sense, extending former results in the literature for more general Lévy processes and for irregular sampling schemes. Secondly, we illustrate empirically the estimation of the random clock, the Blumenthal-Geetor index of jump activity, and the spectral Lévy measure of the process using real intraday high-frequency data.

Keywords Time-changed Lévy models · stochastic volatility · random clocks · non-parametric estimation · parameter estimation based on high-frequency data

JEL Classification: C14 · C32 · C51.

1 Introduction

1.1 The model and some motivation

Accurate modeling of the *stylized features* of the stock prices has been a fundamental problem in mathematical finance for a long time. In addition to the well-known lep-

Department of Statistics
Purdue University
West Lafayette, IN 47906
Tel.: +1-765-494-6036
Fax: +1-765-494-0558
E-mail: figueroa@stat.purdue.edu

tokurtic properties of the distributions of log returns, *volatility clustering* and *leverage* are the other two most prominent features of the dynamics of asset prices (see, e.g., Cont (2001) for a review of these and other stylized features). Indeed, typical asset prices exhibit changes in the overall variability of the process through time. This phenomenon can be perceived more clearly when looking at the the time series of log returns, which often show high variability periods followed by low variability periods. Roughly speaking, "high-volatility" events exhibit a tendency to cluster in time. Leverage refers to the empirical observation of a volatility growth after a drop in prices, suggesting that volatility is negatively correlated with returns.

One of the traditional approaches to incorporate volatility leverage and clustering consists of treating the volatility parameter σ of the Black-Scholes model stochastic, resulting in a model of the form:

$$dS_t = S_t\left(bdt + \sigma_t dW_t\right), \tag{1}$$

where $\{\sigma_t\}_{t\geq 0}$ is an adapted stochastic process driven by another factor. For instance, in the traditional Heston model, $r(t) := \sigma_t^2$ is given by the Cox-Ingersoll-Ross (CIR) model

$$dr(t) = \alpha(m - r(t))dt + \gamma\sqrt{r(t)}dB_t, \tag{2}$$

where $E\left(dB_t \cdot dW_t\right) = \rho dt$ and $\alpha m/\gamma^2 > 1/2$. The reader is referred to, e.g., the monograph of Fouque *et al.* (2000) for the stochastic volatility approach using SDEs driven by Wiener processes.

In this paper, we adopt an alternative approach where volatility clustering is assumed to be a byproduct of changes in the business activity of the market: periods of higher (resp. lower) trading activity results in higher (resp. lower) volatility. This phenomenon can be incorporated into the model via a random clock

$$\tau(t) := \int_0^t r(u)du,$$

where $\{r(t)\}_{t\geq 0}$, called the speed or rate process of the random clock, is a non-negative process. Hence, in this paradigm, the stock price process is given by

$$S_t = S_0 \exp\left\{W_{\tau(t)} + b\tau(t)\right\}. \tag{3}$$

The application of random clocks in asset price modeling can be traced back to the work of Clark (1973), who proposed to link future prices of cotton to the variations in volume during different trading periods (see also Ané & Geman (2000)). In fact, the two models (1) and (3) are closely related in view of a fundamental result of Monroe (1978), stating that any semimartingale can be written as a time-changed Wiener process. For instance, $\int_0^t \sigma_u dW_u$ can be written as $B_{\tau(t)}$, where B is certain Wiener process and $\tau(t)$ has rate process $r(t) = \sigma_t^2$ (Karatzas & Shreve, 1988, Theorem 4.6).

The use of a Wiener process W in the model (3) is not essential. Indeed, during the last decade, several subclasses of Lévy processes have been shown to describe better the stylized features of log returns than the Wiener process. Among the better known models are the *double exponential model* of Kou & Wang (2004) (see also Ramezani & Zeng (2007) for its empirical performance), *variance Gamma model* of Carr *et al.* (1998), the *CGMY model* of Carr *et al.* (2002), and the *generalized hyperbolic motion* of Barndorff-Nielsen (1998) and Eberlein & Keller (1995) (see also Behr & Pötter (2009) for its empirical performance).While preserving the simple statistical properties of the increments of a Wiener process (namely, independent and stationary increments), Lévy processes $\{Z_t\}_{t\geq 0}$ can exhibit flexible marginal distributions with heavy tails, high-kurtosis, and asymmetry. Hence, exponential Lévy models of the form

$$S_t = S_0 \exp\{Z_t\}, \tag{4}$$

will yield time series of log returns with leptokurtic distributions when the Lévy process Z is suitably chosen. Furthermore, given that a Lévy process is not constrained to have continuous paths, the stock price dynamics under (4) can incorporate sudden price shifts or jumps, which can account for major information news affecting the market perspective about the company. At the same time, under certain conditions, the Lévy processes Z can display *infinite jump activity* (infinite many jumps on any finite time interval), which is a plausible model approximation to a random measurement whose value is the result of a large-number of small "shocks" occurring through time with high-frequency.

Summarizing, in this paper we combine the two previous approaches and consider an exponential time-changed Lévy model of the form

$$(i)\; S_t := S_0 \exp\{X_t\}, \quad (ii)\; X_t := Z_{\tau(t)}, \quad (iii)\; \tau(t) := \int_0^t r(u)du, \tag{5}$$

$$(iv)\; Z \text{ is Lévy with triplet } (b, \sigma^2, \nu), \quad (v)\; r(t) \geq 0, \text{ independent of } Z. \tag{6}$$

As explained above, (5-6) is a natural stochastic volatility model with jumps, where the speed process $r(t)$ dictates the volatility of the process in the sense that when $r(t)$ takes a high value (relative to its overall mean), the clock τ will run faster, which in turn results in a higher variability and more frequent jumps. Hence, a mean reverting processes $r(t)$ is an appealing model in order to incorporate the volatility clustering phenomenon of real stock price dynamics. This crucial observation was first noticed by Carr *et al.* (2003), who studied the performance of the model for option pricing when Z is normal inverse Gaussian or CGMY, and r is a CIR model. It is important to remark that the independence assumption between Z and r is an undesirable restriction from a financial point of view in light of the leverage phenomenon present in real stock prices. One could think of ad hoc treatments to incorporate certain degree of dependence such as common driving factors for r and Z, but we will not explore this direction in this work.

We finish this section with a digression about the connection between the stochastic volatility models (1) and the time-changed Lévy model (5). To illustrate this point, let us consider the particular Heston model (2) and its discrete-time approximations. Using Euler's method, the high-frequency log returns $R_i := \log\{S_{t_i}/S_{t_{i-1}}\}$ satisfy the following approximations (when the time span $dt := t_i - t_{i-1}$ is small):

$$R_i \approx \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \approx b\, dt + \sqrt{dt\, r(t_{i-1})}\, \varepsilon_i,$$

$$r(t_i) \approx r(t_{i-1}) + \alpha(m - r(t_{i-1}))dt + \gamma\sqrt{dt\, r(t_{i-1})}\, \varepsilon_i',$$

for $i=1,\ldots,n$, where $\varepsilon_i, \varepsilon_i'$ are i.i.d. standard normal variables (assuming for simplicity $\rho=0$). On the other hand, using the well-known fact that Z can be decomposed as

$$Z_t = bt + \sigma W_t + Y_t,$$

where $b \in \mathbb{R}$, $\sigma \geq 0$, and Y is a pure-jump Lévy process, the model (5) admits the following discrete-time approximations:

$$\begin{aligned} R_i &= b(\tau(t_i) - \tau(t_{i-1})) + W_{\tau(t_i)} - W_{\tau(t_{i-1})} + Y_{\tau(t_i)} - Y_{\tau(t_{i-1})} \\ &\approx b\, r(t_{i-1})\, dt + \sqrt{dt\, r(t_{i-1})}\, \varepsilon_i + \varepsilon_i'', \\ r(t_i) &\approx r(t_{i-1}) + \alpha(m - r(t_{i-1}))dt + \gamma\sqrt{dt\, r(t_{i-1})}\, \varepsilon_i', \end{aligned}$$

where the random variables $\varepsilon_1'', \ldots, \varepsilon_n''$ given $r(t_0), \ldots, r(t_{n-1})$ are independent with respective distributions $f_{r(t_0)dt}, \ldots, f_{r(t_{n-1})dt}$, and where f_t denotes the marginal distribution of Y_t. Two interesting points become apparent from these approximations. The essential difference between the Heston model and the corresponding time changed-Lévy model is that the later may have an extra non-Gaussian innovation ε_i''. Compared to the light-tail normal distributions of ε_i', the additional innovations ε_i'' will contribute weight to the tails of the log return distributions when the marginal distributions of the pure-jump component of Y exhibit heavy tails. Also, given that $\mathrm{Var}(\varepsilon_i'' | \{r(t)\}_t) = r(t_{i-1})\, dt\, \mathrm{Var}(Y_1)$ (when $\mathbb{E} Y_1^2 < \infty$), the rate process r will also introduce a volatility clustering effect to the time series $\varepsilon_1'', \ldots, \varepsilon_n''$.

1.2 Background on Lévy processes

Before introducing the statistical problems we consider in this paper and our results, let us briefly review a few well-known facts of Lévy processes (see, e.g., the monographs of Cont & Tankov (2004) and Sato (1999) or the review paper Figueroa-López (2010b) for the necessary background on Lévy processes). By definition, a Lévy process $Z = \{Z_t\}_{t\geq 0}$ is a process with independent and stationary increments, with right-continuous with left limits paths, and with no fixed jump times. The law of the process is determined uniquely by the distribution of Z_1. Thus, for instance, when Z_1 is Normally distributed, Z_t is necessarily a Brownian motion with drift, $\sigma W_t + bt$.

It is known that the distribution of a Lévy process is determined by three parameters (b, σ^2, ν): a non-negative real σ^2, a real b, and a measure ν on $\mathbb{R}\backslash\{0\}$ such that $\int (x^2 \wedge 1)\nu(dx) < \infty$. These parameters dictate the dynamics of the process according to the decomposition

$$Z_t = bt + \sigma W_t + Y_t, \tag{7}$$

where W is a Wiener process and Y is a pure-jump process such as a compound Poisson process. The measure ν controls the jump dynamics of the process Z in that for any $A \in \mathscr{B}(\mathbb{R})$,

$$N_A(t) := \sum_{s \leq t} \chi_A(\Delta Z(s)),$$

is a Poisson process with intensity $\nu(A)$, and for any disjoint B, the two Poisson processes N_A and N_B are independent (Sato, 1999, Section 19). In summary, $\nu(A)$

gives the average number of jumps (per unit time) whose magnitudes fall in the set A. A common assumption in Lévy-based financial models is that ν is determined by a function $s:\mathbb{R}\backslash\{0\}\to[0,\infty)$, called the *Lévy density*, as follows

$$\nu(A)=\int_A s(x)dx,\ \ \forall A\in\mathscr{B}(\mathbb{R}\backslash\{0\}).$$

Intuitively, the value of s at x_0 provides information on the frequency of jumps with sizes "close" to x_0. Note that the process Z will display *infinite jump activity* if and only if $\nu(\mathbb{R}\backslash\{0\})=\infty$.

Another very useful characterization of ν that we will heavily use here has to do with the short-term ergodic properties of the Lévy process Z_t. Concretely, ν is such that

$$\lim_{t\to 0}\frac{1}{t}\mathbb{P}(Z_t\geq a)=\nu(x\geq a), \tag{8}$$

for any point of continuity a of ν (e.g. Bertoin, 1996, Chapter 1).

1.3 Our results and related known results

Throughout this paper, we adopt the exponential time-changed Lévy model (5-6). Statistical inference for this model is not a simple matter, not even under parametric specifications of the model, due to the unobservable random clock τ. The likelihood function is in general intractable, requiring simulation-based methods for obtaining maximum likelihood estimates such as particle filters or Bayesian MCMC methods (see, e.g., Johannes & Polson (2003) and Li (2009)). Our goal here is to take a "non-parametric" approach, where we impose only "qualitative" constraints about the parameters of the model, and to perform parameter estimation based on high-frequency data. The following statistical problems are of particular interest here:

(1) Estimation of the Lévy density $s:\mathbb{R}\backslash\{0\}\to\mathbb{R}_+$.
(2) Recovery of the random clock $\tau(t)=\int_0^t r(u)du$;
(3) Estimation of the "jump-activity" intensity of the process (see below);

The estimation problem (1) above has been considered in some detail in Figueroa-López (2009, 2010a), where consistent estimators for the integral parameters $\varphi\to\int\varphi(x)s(x)dx$ are constructed and central limit theorems are also proved assuming that r is an ergodic diffusion satisfying certain moment conditions. By appropriately choosing the function φ (e.g. piece-polynomials or wavelet functions), one can construct non-parametric estimators for s such as sieve or kernel estimators.

In this paper, our main focus is on problems (2) and (3) above. Motivated by a procedure proposed by Winkel (2001), we use a threshold type estimator of the form

$$\hat{\tau}_n(T):=\frac{1}{\nu(|x|\geq a_n)}\sum_{k=1}^{n}\mathbf{1}_{\left\{|X_{t_k^n}-X_{t_{k-1}^n}|\geq a_n\right\}} \tag{9}$$

to recover the random clock τ based on high-frequency discrete observations of the log return process $X_t=\log(S_t/S_0)$. We show consistency of our estimator when the time mesh $\delta_n := \max_i(t_i^n - t_{i-1}^n)$ of our sampling times $0 \le t_1^n < \cdots < t_n^n = T$ converges to 0 fast enough compared to a_n, who is also made to converge to 0. We cover some of the same modeling framework of previous works such as Woerner (2007) and Aït-Sahalia & Jacod (2009), but we obtain stronger statements in three directions: the sampling times $\{t_i\}$ are allowed to be irregular, the convergence of the main estimator is in the L^2 sense, and the Lévy measure ν is less restrictive.

The key assumption for our results, common in many of the Lévy-based financial models proposed in the literature, is that the Lévy density $s(x) := \nu(dx)/dx$ satisfies the asymptotics

$$\lim_{x\to 0^{\pm}} |x|^{\beta+1} s(x) = c^{\pm}, \tag{10}$$

for a constant $\beta \in (0,2)$ and $c^-, c^+ \ge 0$ such that $c^- + c^+ > 0$. The parameter β, called the Blumenthal-Geetor index or the jump-activity index, models the intensity of small jumps present in the process. The larger β is, the faster the Lévy density s diverges at the origin, and the more frequent small jumps are to occur. The property (10) is a simplifying assumption that interestingly enough seems to be validated by our empirical study presented in this paper.

It is relevant to point out that another popular estimation method, pioneered by Barndorff-Nielsen and Shephard, is based on multipower variations such as

$$V_n^{(r,s)}(X) := \sum_{k=1}^{n-1} \left| X_{t_k^n} - X_{t_{k-1}^n} \right|^r \left| X_{t_{k+1}^n} - X_{t_k^n} \right|^s, \tag{11}$$

(see, e.g., Barndorff-Nielsen & Shephard (2004, 2006), Woerner (2006)). In particular, under a slightly stronger assumption than (10), Woerner (2007) shows that

$$\delta_n^{1-(r+s)/\beta} V_n^{(r,s)}(X) \xrightarrow{\mathbb{P}} \gamma_{\beta,r,s} \left(\frac{c_0}{\beta}\right)^{(r+s)/\beta} \int_0^T (r(u))^{(r+s)/\beta} du, \tag{12}$$

provided that $\max\{r,s\} < \beta$, for certain computable constant $\gamma_{\beta,r,s}$, and where it is assumed regular sampling times such that $\delta_n \equiv t_k^n - t_{k-1}^n \to 0$. In particular, when $r = s = \beta/2$, we recover $\tau(T) = \int_0^T r(u)du$, up to the constant c_0. Lévy processes are not the only models that benefit from statistical estimation based on high-frequency data. Recently, power variations have been also applied to estimate the "memory strenght" index H present in certain continuous-time long memory processes such as fractional Brownian motion (see, e.g., Nourdin *et al.* (2007), and Chronopoulou *et al.* (2010)).

The paper is organized as follows. In Section 2, we analyze the recovery problem using the threshold estimator (9). We show the mean-square convergence of (9) to $\tau(T)$, under certain moment conditions on the random clock. The proof of this result is deferred until Appendix A for the sake of clarity. Section 3 gives an application of the consistency of $\hat{\tau}_n$ to devise estimators for the jump-activity index β appearing in (10). In Section 4, the numerical performance of the estimation methodology is illustrated using simulated data and also real data. We consider intraday stock prices (at a 5-second frequency) of Intel (INTC) from January 2, 2003 to December 30, 2005. Our results complement the enlightening empirical study of Aït-Sahalia & Jacod (2009), where the same stock and frequency is considered for the year of 2006.

We conduct an analysis of the behavior of the estimator (27), with $a_n = \alpha\delta_n^{\omega}$ (as suggested in Aït-Sahalia & Jacod (2009)), on the parameters α and α', which shows some interesting features.

2 Recovery of the random clock

In this part, we analyze the problem of recovering the random clock τ, under the exponential time-changed Lévy model (5-6), based on discrete observations of the process at times $0 \leq t_0^n < \cdots < t_n^n = T < \infty$. This problem is of great importance in finance since τ plays the same role in this class of models as the integrated variance process $\bar{\sigma}_t := \int_0^t \sigma_u^2 du$ plays in stochastic volatility models driven by Wiener processes.

Let us start by noticing that we have an identifiability issue. Concretely, the clock

$$\tau(t) = \int_0^t r(u)du, \quad t \leq T,$$

cannot be recovered from observations of the process $X_t = Z_{\tau(t)}$. This is clear since, for any constant $k > 0$, one can always write

$$X_t = Z^{(k)}_{\tau^{(k)}(t)}, \tag{13}$$

with $\tau^{(k)}(t) := \tau(t)/k$ and $Z_t^{(k)} := Z_{kt}$, which is still a Lévy process but with Lévy triplet $(kb, k\sigma^2, ks(x)dx)$. Hence, one must impose additional restriction to the model in order to uniquely identify the random clock τ. For instance, any of the constraints

$$r(0) = 1, \quad \text{or} \quad \mathbb{E}\,\tau(T) = T, \tag{14}$$

will suffice to identify uniquely τ among the above parameterizations $Z^{(k)}_{\tau^{(k)}(t)}$. Also, under the assumption that

$$s(x) = |x|^{-\beta-1}\left(c_0 + o(1)\right), \quad \text{as} \quad x \to 0,$$

for some $c_0 > 0$, there exists a unique $Z^{(k)}$ process having Lévy density

$$\hat{s}(x) = x^{-\beta-1}\left(1 + o(1)\right), \quad \text{as} \quad x \to 0. \tag{15}$$

The problem of clock recovery can be traced back to the work of Winkel (2001) who, assuming that τ and Z are independent and that Z exhibits infinite-jump activity, shows that

$$\tau(T) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} M_k, \quad a.s., \tag{16}$$

where $M_k = \#\{t \leq T : |\Delta X_t| \in [a_k, a_{k-1})\}$, and $\{a_k\}_{k\geq 0}$ are such that $a_0 = \infty$ and $k = \nu(x : |x| \geq a_k)$, for $k \geq 1$. The result (16) follows directly from the strong law of large numbers since, conditional on $\{\tau(t)\}_{t\leq T}$, the variables $\{M_k\}_{k\geq 0}$ are independent Poisson distributed with a common mean

$$\lambda = \tau(T)\nu(|x| \in [a_k, a_{k-1})) = \tau(T)\left\{\nu(|x| \geq a_k) - \nu(|x| \geq a_{k-1})\right\} = \tau(T).$$

Winkel's procedure is the key motivation behind our main estimators, who are designed to overcome the two clear drawbacks of (16): inaccessibility of the Lévy measure ν, and inaccessibility of the jumps $\Delta X_t := X_t - X_{t^-}$ (at least based on discrete sampling). To solve the first issue we first note that the sequence $\{a_k\}_k$ is superfluous since

$$\frac{1}{n}\sum_{k=1}^{n} M_k = \frac{\#\{t \le T : |\Delta X_t| \ge a_n\}}{\nu(|x| \ge a_n)} = \frac{1}{\nu(|x| \ge a_n)} \sum_{t \le T} \mathbf{1}_{\{|\Delta X_t| \ge a_n\}},$$

suggesting the use of the statistics

$$\hat{\tau}^{(a)}(T) := \frac{1}{\nu(|x| \ge a)} \sum_{t \le T} \mathbf{1}_{\{|\Delta X_t| \ge a\}}, \tag{17}$$

to recover $\tau(T)$ by making $a \to 0$. Still, $\nu(|x| \ge a)$ is needed or, to be more precise, an assumption about the behavior of $\nu(|x| \ge a)$ as $a \to 0$ is needed. As it was explained in the introduction, one of such assumptions appearing commonly in the literature is to take a Lévy density $s(x) := \nu(dx)/dx$ such that

$$\lim_{x \to 0^{\pm}} |x|^{\beta+1} s(x) = c^{\pm}, \tag{18}$$

for constants $\beta \in (0,2)$ and $c^-, c^+ \ge 0$ such that $c^- + c^+ > 0$. In that case, it follows that

$$\lim_{a \downarrow 0} a^{\beta} \nu(x \ge a) = \frac{c_+}{\beta}, \quad \text{and} \quad \lim_{a \downarrow 0} a^{\beta} \nu(x \le -a) = \frac{c_-}{\beta}. \tag{19}$$

The previous limits motivate us to consider the statistics

$$\hat{\tau}'^{(a)}(T) := a^{\beta} \sum_{t \le T} \mathbf{1}_{\{|\Delta X_t| \ge a\}}, \tag{20}$$

who a priori should satisfy that

$$\lim_{a \to 0} \hat{\tau}'^{(a)}(T) = \frac{c^+ + c^-}{\beta} \tau(T). \tag{21}$$

For the issue of jump inaccessibility, we use the natural approach of replacing the jumps $\{\Delta X_t : t \le T\}$ by the increments of the process

$$\Delta_k^n X := X_{t_k^n} - X_{t_{k-1}^n}, \tag{22}$$

who are expected to be good proxies of the jumps under high-frequency sampling. Concretely, the following statistics are the natural discrete-time-based proxies of (17) and (20), respectively:

$$\hat{\tau}_n(T) := \frac{1}{\nu(|x| \ge a_n)} \sum_{k=1}^{n} \mathbf{1}_{\{|\Delta_k^n X| \ge a_n\}}, \quad \hat{\tau}'_n(T) := a_n^{\beta} \sum_{k=1}^{n} \mathbf{1}_{\{|\Delta_k^n X| \ge a_n\}}, \tag{23}$$

where again $0 = t_0^n < \cdots < t_n^n = T$ denote the sampling times and $\{a_n\}_{n \ge 1}$ denotes a sequence such that $a_n \to 0$. We proceed to investigate the asymptotics of (23) as $a_n \to 0$, and $\delta_n := \max_k \{t_k^n - t_{k-1}^n\} \to 0$. The following is our main result showing the convergence of (23) towards $\tau(T)$ in the mean square sense under certain mild structural constraint on s near the origin and under certain conditions about the asymptotic behavior of $s(x)$ as $x \to 0$.

Theorem 1 *Consider the model (5-6) and suppose that the Lévy density $s(x) = \nu(dx)/dx$ of Z exists and is differentiable decreasing on $(0,x_0)$, and differentiable increasing on $(-x_0,0)$, for some $x_0 > 0$. Also, assume that*

$$\limsup_{x\to 0} |x|^{\beta+1} s(x) < \infty, \qquad \liminf_{a\to 0} \nu(|x| \geq a) a^{\beta} > 0, \tag{24}$$

for some $\beta \in (0,2)$, and that $\mathbb{E}\left(\int_0^T r^6(u)du\right) < \infty$. Then, for any fixed $T > 0$, the estimator $\hat{\tau}_n(T)$ in (23) is such that

$$\lim_{n\to 0} \mathbb{E}\left(\hat{\tau}_n(T) - \tau(T)\right)^2 = 0, \tag{25}$$

whenever $\delta^n := \max\{t_k^n - t_{k-1}^n\} \to 0$ and $a_n \to 0$ such that $a_n^{\beta-4}\delta^n = o(1)$ as $n \to \infty$. If $\sigma = 0$, it suffices that $a_n^{-\beta}\delta^n = o(1)$ as $n \to \infty$.

Corollary 2 *Suppose that the conditions of Theorem 1 are satisfied replacing (24) by condition (18). Then, the estimator $\hat{\tau}'_n(T)$ in (23) is such that*

$$\lim_{n\to 0} \mathbb{E}\left(\hat{\tau}'_n(T) - \frac{c^+ + c^-}{\beta}\tau(T)\right)^2 = 0, \tag{26}$$

whenever δ^n and a_n converge to 0 as stated in Theorem 1.

For the proof of Theorem 1, we exploit heavily a new estimate for the rate of convergence in (8). In order to motivate the connection between (8) and (25), consider for simplicity the estimator $\hat{\tau}_n^+(T) := \sum_{k=1}^n \mathbf{1}_{\{\Delta_k^n X \geq a_n\}}/\nu(x \geq a_n)$. Conditioning on the random clock $\{\tau(t) : t \leq T\}$ and using the independence between τ and Z as well as the independence and stationarity of the increments of Z, one can check that

$$\begin{aligned}\mathbb{E}\left(\hat{\tau}_n^+(T) - \tau(T)\right)^2 &= \mathbb{E}\left(\frac{1}{\nu(x \geq a_n)}\sum_{k=1}^n \left\{\mathbb{P}(Z_t \geq a_n) - t\nu(x \geq a_n)\right\}\Big|_{t=\Delta_k^n\tau}\right)^2 \\ &\quad + \frac{1}{\nu(x \geq a_n)^2}\sum_{k=1}^n \mathbb{E}\left(\left\{\mathbb{P}(Z_t \geq a_n) - \mathbb{P}(Z_t \geq a_n)^2\right\}\Big|_{t=\Delta_k^n\tau}\right).\end{aligned}$$

From the above expression, it is now clear why it is necessary to have precise bounds on $|\mathbb{P}(Z_t \geq a) - t\nu(x \geq a)|$ in order to obtain (25). For the sake of clarity, we defer the details of the proof to the Appendix A.

Remark 3 *Aït-Sahalia & Jacod (2009) also considers the threshold estimator $\hat{\tau}'_n(T)$ fixing $a_n = \alpha\delta_n^{\omega}$, for some $\alpha > 0$ and $\omega \in (0,1/2)$. The motivation of their estimator seems to come from the desire to filter away the continuous component of the process X, while ours come from Winkel's estimator (16). Aït-Sahalia & Jacod (2009) are able to consider the most general setting of an Itô semimartingale $X = \{X_t\}_{t\geq 0}$, which means that its characteristic triplet (B,C,η) are of the form:*

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t \sigma_s ds, \quad \eta(dt,dx) = \nu_t(dx)dt.$$

It was also assumed in there that ν_t can be written as $\nu_t = \nu'_t + \nu''_t$ for ν'_t and ν''_t of the form

$$\nu'_t(dx) = \frac{1+|x|^{\gamma} f(t,x)}{|x|^{1+\beta}} \left(a^+_t \mathbf{1}_{\{0<z\le z^+_t\}} + a^-_t \mathbf{1}_{\{-z^-_t \le x<0\}} \right),$$
$$\nu''_t(dx) \perp \nu'_t(dx), \quad \text{and} \quad \int (|x|^{\beta'} \wedge 1)\nu''_t(dx) \le L_t.$$

Here, $L_t \ge 1$ is a locally bounded predictable process, $\beta \in (0,2)$, $\gamma > 0$, $\beta' \in [0,\beta)$, and $a^+_t, a^-_t, z^+_t, z^-_t$ and $f(\omega,t,x)$ are predictable such that

$$\frac{1}{L_t} \le z^{\pm}_t \le 1, \quad a^+_t + a^-_t \le L_t, \quad 1+|a| f(t,x) \ge 0, \quad |f(t,x)| \le L_t.$$

Under the above structural assumption and under regular sampling ($\delta^n := t^n_k - t^n_{k-1}$), it was shown that $\hat{\tau}'_n(T)$ converges in probability to

$$\bar{A}_t = \frac{\int_0^t (a^+_s + a^-_s) ds}{\beta},$$

when $a_n = \alpha \delta^{\omega}_n$ and $\alpha, \omega > 0$ satisfy certain relationship. Since our time-changed Lévy model (5) is an Itô semimartingale with $a^+_t = a^-_t = r(t)$, Aït-Sahalia & Jacod (2009)'s result can be applied. By specifically considering the time-change Lévy framework, we are able to obtain convergence in the mean-square sense, to consider non-regular sampling times, and weaken the constraints on the Lévy measure. Note that, when taking $a_n := \alpha(\delta^n)^{\bar{\omega}}$, the condition $a_n^{\beta-4}\delta^n = o(1)$ of Theorem 1 is satisfied if $1+\bar{\omega}(\beta-4) > 0$, which in turn is satisfied whenever $\bar{\omega} \le 1/4$ (regardless of $\beta \in (0,2)$). If $\sigma = 0$, then $a_n^{-\beta}\delta^n = o(1)$ always that $\omega \le 1/2$.

3 An application: estimation of the index of jump-intensity

Following closely ideas by Aït-Sahalia & Jacod (2009), we now apply our main result (Theorem 1) to devise an estimator for the parameter β appearing in (18). Estimating β has great relevance in the whole theory since, once an estimate $\hat{\beta}$ has been found, we can plug it in the estimators for the random clock of equations (23) (or also (12)). This approach will then allow us to estimate the random clock $\tau(t)$ at each fixed t up to a constant (c_0 in the case of (12) and $c_+ + c_-$ in the case of (23)).

The index β coincides with the Blumenthal-Geetor index defined by

$$\beta := \inf \left\{ \gamma > 0 : \int_{\{|x|\le 1\}} |x|^{\gamma} \nu(dx) < \infty \right\},$$

which is known to exist and being in $[0,2]$ since always $\int_{\{|x|\le 1\}} |x|^2 \nu(dx) < \infty$. The parameter β serves as a summary measurement of the jump activity of the process: the larger is β, the more frequent is the jump activity. Also, if $p > \beta$, the p^{th}−power realized variation of the process (namely, $V_n^{(p,0)}(X)$ using the notation (11)) converges to a non-trivial limit, while it does converge to 0 for any $p < \beta$ (see Hudson & Mason (1976)).

The main estimator for β is constructed by considering the ratio between $\hat{\tau}'_a(T)$ and $\hat{\tau}'_{\alpha' a}(T)$. Concretely, let

$$\hat{\beta}_{n,\alpha'} := \frac{1}{\ln \alpha'} \cdot \ln \left(\frac{\sum_{k=1}^n \mathbf{1}_{\{|\Delta_k^n X| \geq a_n\}}}{\sum_{k=1}^n \mathbf{1}_{\{|\Delta_k^n X| \geq \alpha' a_n\}}} \right), \tag{27}$$

where $\alpha' > 0$ represent a fixed arbitrary parameter of the estimator. In light of (21),

$$\hat{\beta}_{n,\alpha'} \xrightarrow{\mathbb{P}} \beta,$$

under the conditions of Theorem 1, regardless the value of α' (though we shall see that the performance of the estimator for finite sample sizes varies greatly with the choice of α'). We remark that one does not need to assume the same degree of jump activity for the positive and negative jumps. For instance, assuming that

$$\lim_{x \to 0^+} x^{\beta_+ + 1} s(x) = c_+, \tag{28}$$

for some $c_+ > 0$ and $\beta_+ \in (0, 2)$, then it is expected that

$$\hat{\beta}^+_{n,\alpha'} := \frac{1}{\ln \alpha'} \cdot \ln \left(\frac{\sum_{k=1}^n \mathbf{1}_{\{\Delta_k^n X \geq a_n\}}}{\sum_{k=1}^n \mathbf{1}_{\{\Delta_k^n X \geq \alpha' a_n\}}} \right) \to \beta^+, \tag{29}$$

with a similar result holding for the estimate $\hat{\beta}^-_{n,\alpha'}$.

4 Numerical and empirical performance

Our goal in this part is to study the performance of the estimator (23) for the random clock, the estimator (27) for the Blumenthal-Geetor index β, and the estimators for the spectral Lévy function $a \to \nu(|x| \geq a)$ developed in Figueroa-López (2009). We divide our study in two part: the finite-sample performance using simulations of popular jump-diffusion models and the empirical study using 5-second log returns of intel for the period of 2003 and 2005. Following Aït-Sahalia & Jacod (2009) we fix $a_n = \alpha \delta_n^{\omega}$ in the case of the estimator (27). We are particularly interested in the behavior of the later estimator as a function of the control parameter α and α'.

Throughout this section, X_t stands for the log return process $\log\{S_t/S_0\}$, $\{W_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ stand for correlated Wiener processes with correlation $\rho = \mathbb{E}\,(dW_t \cdot dB_t)$, and $\{Z_t\}_{t\geq 0}$ stands for a pure-jump Lévy process independent of W and B. All the figures are presented in Appendix B.

4.1 Finite-sample performance using simulated data

In order to get a bettet idea of the performance of the estimator (27) for small sample size, we consider several popular jump-diffusion models for the log return process. We then compute the estimator (27) for the sample frequency and time horizon of our empirical study presented below.

4.1.1 A stochastic volatility model with stable jump in returns

The following model was studied in Aït-Sahalia & Jacod (2009):

$$dX_t = \sigma_t dW_t + \theta dZ_t,$$
$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \gamma\sigma_t dB_t,$$

where Z is a symmetric β-stable process with $\beta = 1.5$ and scale parameter 1. The parameter setting is as follows:

$$\beta = 1.5; \quad \eta^{1/2} = .25; \quad \gamma = .5; \quad \kappa = 5; \quad \rho = -.5; \quad \theta = .1.$$

As we can see, the overall volatility of the continuous component is 25%, and there is a leverage effect. The stable process is consider to exhibit very heavy-tailed distributions with not even a finite second moment. The simulated 5-sec log returns of process are shown in Figure 1, together with the graphs of the estimators $\alpha \to \hat{\beta}_{\alpha\delta_n^{\omega},\alpha'}$, for different value of α'. The graph shows a relatively high variability of the estimate $\hat{\beta}$ as function of α in a hump-liked shape. The estimate seems to settle down about the true parameter of β for values of α around .06, but eventually the estimate start to increase as α becomes larger. The value .06 seems to correspond to the average value of σ_t^2 which is $\eta = .0625$.

4.1.2 Time-changed Normal Inverse Gaussian Lévy models

Normal inverse Gaussian (NIG) processes is one the most popular Lévy process for modeling financial prices (see, e.g., Cont & Tankov (2004) for more information). This process is defined as

$$Z(t) = \sigma_Z W_Z(U(t)) + \theta_Z U(t) + bt, \tag{30}$$

where σ_Z and θ_Z are constants, W_Z is a Wiener process, and U is an Inverse Gaussian subordinator such that $\mathbb{E}U(t) = \nu t$, for $\nu > 0$. This model is an infinite-jump activity process with jumps activity $\beta = 1$.

The increments of Z can be simulated by a rejection-type method (see Cont & Tankov (2004), pp. 183). Figure 2 shows the 5-second log returns of the time-changed Normal Inverse Gaussian (TCNIG) process:

$$X_t = Z(\tau(t)) + bt, \quad \tau(t) = \int_0^t r(u)du, \tag{31}$$

$$dr(t) = \alpha(m - r(t))dt + \sigma_r\sqrt{r(t)}\,dB_r(t). \tag{32}$$

The following settings were used:

$$\theta_Z = -.080, \quad \sigma_Z = .5, \quad \nu = .210, \quad \mathbb{E}(dW_Z(t)dB_r(t)) = 0$$
$$b = .143, \quad \alpha = 1.763, \quad m = 1, \quad \sigma_r = 0.563.$$

The second display in Figure 2 shows the typical graphs of $\alpha \to \hat{\beta}_{\alpha\delta_n^{\omega},\alpha'}$, for different value of α'. We again observe relatively high variability of the estimate $\hat{\beta}$ about the true value $\beta = 1$, but in sharp contrast with the previous model, we don't perceived the hump-liked shape, and hence, we can conjecture that the continuous component of the previous model is the reason of the hump.

4.1.3 Time-changed of a Wiener process plus a NIG process

This model is similar to (31-32), but we assume that the Lévy process has a continuous component. Concretely, the model takes the form

$$X_t = Z(\tau(t)) + \sigma_X W(\tau(t)) + bt, \tag{33}$$

where Z is the NIG process (30), τ is given by (31-32), and W is independent of Z and τ. Here, we take the same parameter setting for Z and τ as above, and also $\sigma_X = .25$. Figure 3 shows the 5-second log returns of the model (33), together with the graphs of $\alpha \to \hat{\beta}_{\alpha\delta_n^{\omega},\alpha'}$, for different value of α'. We again perceive the hump-liked shape, confirming at least empirically our previous conjecture. We note that the estimate of the β is much more stable than in the case of the stochastic volatility model with stable jumps.

4.2 Empirical case study: INTC

In this part we apply our estimation methodology to the intraday stock prices of Intel (INTC) from January 2, 2003 to December 30, 2005, which were obtained via the NASDAQ TAQ database. Once all the transactions from 9:30 am until 4:00 pm have been collected, we sampled the prices of the stock at intervals of 5 seconds and subsequently computed the log returns, filtering out the overnight log returns. Figures 4 show the prices and log returns during the mentioned three year period. One can clearly perceived the volatility clustering effect, in this case, higher variability during the first year as opposed to the relatively low volatility at the end of the period.

4.2.1 Index of jump activity β

Figures 5 shows the estimated index of jump activity based in frequencies of 5 and 15 seconds. There are small differences between the two displays, but both of them suggest an index of jump activity of about 1.5. Interestingly, this value is also consistent with the findings in Aït-Sahalia & Jacod (2009) for the year of 2006, suggesting certain persistency of the underlying jump activity of the process. It is also worth pointing out the hump-like shape of the graph of $\hat{\beta}_{n,\alpha'}$, suggesting the presence of a continuous component in the dynamics of the price process. We also apply the one sided versions of $\hat{\beta}$ (see (29)) to determine if there is a significant difference between the indexes of positive and negative jump activity. We found that they are almost identical. The estimators were also applied to 1-min. log returns, but the results were not good since the estimates happen to be greater than 2 almost all the time. This shows how crucial high-frequency is for a reliable estimation of index of jump activity β.

4.2.2 Estimates for the rate process r of the random clock

Once an estimate for the jump activity index β has been found, it is natural to plug this value in the estimators for the random clock of equations (23) and (12). This approach will yield estimates for the random clock $\tau(t)$ at each fixed t up to a constant (c_0 in the case of (12) and $c_+ + c_-$ in the case of (23)). The recovery of the daily rate process $r(t_i)$ for $t_i = i\delta$, where $\delta = 1/252$ years and $1 \le i \le T/\delta$, is interesting. We recover these values using the natural approximation

$$r(t_i) \approx \frac{1}{\delta}\int_{t_i}^{t_{i+1}} r(u)du = \frac{1}{\delta}\left(\tau(t_{i+1}) - \tau(t_i)\right). \tag{34}$$

In order to estimate $\tau(t_i)$ on the right-hand side of the approximation (34), we used our estimator $\hat{\tau}_n(t_i)$ of (23), which in light of Theorem 1, converges in L^2 to $\tau(t_i)$ for each $1 \le i \le T/\delta$. Due to the fact that this estimator is the discrete version of Winkel's estimator (17), we called it discrete-time Winkel estimator. For comparison purposes, we also apply Woerner's estimator appearing in (12). Figures 6 shows the results of both methods taking $a = .1\delta_n^w$. Even though there are some differences (mainly in terms of the level), the essential features are very similar. In particular, both estimators point out to a high volatility period during the first 6 months of 2003, followed by a decrease in volatility until the end of 2005. This is consistent with the overall appearance of the log return time series of Figure 4.

4.2.3 Estimates for the spectral Lévy density $\nu(|x| \ge a)$

The last estimation problem (3) listed at the beginning of Section 1.3, namely the non-parametric estimation of the Lévy density s, has been considered in detail in Figueroa-López (2009, 2010a). Let us first note that this problem cannot be solved on a finite-time horizon $T < \infty$. Indeed, the only information about the sample path $t \in [0,T] \to X_t$ that is relevant to estimate the Lévy density s, on some region $[c,d] \subset \mathbb{R}\backslash\{0\}$, are the jumps of X with size between $[c,d]$. However, for any finite time horizon $[0,T]$, there will be finitely-many of such jumps and consistency would not be achievable no matter how frequently the process is sampled.

In Figueroa-López (2009), we assume that the rate process r is ergodic, and, by combining its long-run ergodic properties with the short-term ergodic properties of Z, we attain consistent estimation of the integral parameters $\beta(\varphi) := \int \varphi(x)s(x)dx$. These integral parameters, with φ judiciously chosen, can subsequently be combined with a numerical approximation method for the function s to yield a non-parametric estimator for s. For instance, indicator functions φ will lead to histogram or kernel type estimators for s, piece-wise polynomials φ will lead to spline type estimators, etc. The estimators for $\beta(\varphi)$ are given by the so-called *realized φ-variations of X per unit time*:

$$\hat{\beta}_n(\varphi) := \frac{1}{t_n^n}\sum_{k=1}^{n} \varphi\left(X_{t_k^n} - X_{t_{k-1}^n}\right). \tag{35}$$

Under certain moment conditions on r, $\hat{\beta}_n(\varphi)$ converges to $\bar{\zeta}\,\beta(\varphi)$ when $t_n^n \to \infty$ and $\delta_n = \max\{t_k^n - t_{k-1}^n\} \to 0$ such that $t_n^n\delta_n^2 \to 0$. Here, $\bar{\zeta}$ is defined as

$$\bar{\zeta} := \lim_{T\to\infty} \frac{1}{T}\int_0^T r(u)du \in (0,\infty), \tag{36}$$

whose existence is assumed.

We briefly analyze the performance of the estimators (35) with respect to the time horizon T. Our goal is to illustrate the convergence of $\hat{\beta}_n(\mathbf{1}_{\{|x|\geq a\}})$ as $T_n := t_n^n \to \infty$. Figure 7 shows the graphs of $\hat{\beta}_n(\mathbf{1}_{\{|x|\geq a\}})$ as a function of t_n^n for different values of a. As expected, the convergence is much faster when a is large, while for value of a close to the origin, we would require a longer time horizon for reaching stability.

A Proof of the main Theorem 1

In order to obtain (25), we rely heavily on a new bound for $|\mathbb{P}(Z_t \geq a)/t - \nu(x \geq a)|$. As it is often the case with Lévy processes, our estimates use the well-known decomposition of the pure-jump component of Z into a compound Poisson process and a process with small jump sizes. Concretely, suppose that Z has Lévy-Itô decomposition

$$Z_t = bt + \sigma W_t + \int_0^t \int_{|x|\leq 1} x\,\bar{\mu}(dx,ds) + \int_0^t \int_{|x|>1} x\,\mu(dx,ds), \tag{37}$$

where W is a standard Brownian motion and μ is an independent Poisson measure on $\mathbb{R}\backslash\{0\}\times\mathbb{R}_+$ with mean measure $\nu(dx)dt$, and $\bar{\mu}(dx,dt) := \mu(dx,dt) - \nu(dx)dt$. Next, for a given fixed $0<\varepsilon<1$, we set

$$\widetilde{Z}_t^{\varepsilon} := \int_0^t \int_{\mathbb{R}} x\mathbf{1}_{\{|x|\geq\varepsilon\}}\mu(dx,ds), \quad \text{and} \quad Z_t^{\varepsilon} := Z_t - \widetilde{Z}_t^{\varepsilon}; \tag{38}$$

hence, $\widetilde{Z}^{\varepsilon}$ is a compound Poisson process with intensity $\lambda_\varepsilon := \nu(|x|\geq\varepsilon)$, and jumps $\{\xi_i^{\varepsilon}\}_i$ with common distribution $\mathbf{1}_{|x|\geq\varepsilon}\nu(dx)/\lambda_\varepsilon$, while the remaining process Z^{ε} is a Lévy process with triplet $(\sigma^2, b_\varepsilon, \mathbf{1}_{\{|x|\leq\varepsilon\}}\nu(dx))$, where

$$b_\varepsilon := b - \int_{|x|\leq 1} x\mathbf{1}_{\{|x|\geq\varepsilon\}}\nu(dx).$$

The following estimate for the tails of Z_t^{ε} will be also useful in the sequel.

Lemma 4 *For any $a>0$, $0<\varepsilon<1$, and $t>0$ such that $t(b-\int_{\varepsilon<|x|\leq 1} x\nu(dx)) \leq a/2$,*

$$\mathbb{P}(Z_t^{\varepsilon}\geq a) \leq \left(\frac{4eV_\varepsilon^2}{\varepsilon a}\right)^{a/4\varepsilon} t^{a/4\varepsilon} + \frac{4\sigma t^{1/2}}{a\sqrt{2\pi}} e^{-\frac{a^2}{16\sigma t}}, \tag{39}$$

where $V_\varepsilon^2 := \int_{\{|x|\leq\varepsilon\}} x^2\nu(dx)$.

Proof Note that $\mathbb{E}Z_t^{\varepsilon} = t(b-\int_{\varepsilon<|x|\leq 1} x\nu(dx))$. Thus, if $\mathbb{E}Z_t^{\varepsilon} < a/2$,

$$\mathbb{P}(Z_t^{\varepsilon}\geq a) = \mathbb{P}(Z_t^{\varepsilon} - \mathbb{E}Z_t^{\varepsilon} \geq a - \mathbb{E}Z_t^{\varepsilon}) \leq \mathbb{P}(Z_t^{\varepsilon} - \mathbb{E}Z_t^{\varepsilon} \geq a/2).$$

Also, denoting $\hat{Z}_t^{\varepsilon} := Z_t^{\varepsilon} - \sigma W_t$, we have that

$$\mathbb{P}(Z_t^{\varepsilon}\geq a) \leq \mathbb{P}\left(\hat{Z}_t^{\varepsilon} - \mathbb{E}\hat{Z}_t^{\varepsilon} \geq a/4\right) + \mathbb{P}(\sigma W_t \geq a/4).$$

The estimate (39) will then follow from the standard tail estimate for Gaussian random variables Z (namely, $\mathbb{P}(Z\geq x) \leq \exp(-x^2/2)/(x\sqrt{2\pi})$) and a generic concentration inequality such as

$$\mathbb{P}(\hat{Z}_t^{\varepsilon} - \mathbb{E}\hat{Z}_t^{\varepsilon} \geq x) \leq e^{\frac{x}{\varepsilon} - \left(\frac{x}{\varepsilon} + \frac{tV_\varepsilon^2}{\varepsilon^2}\right)\log\left(1+\frac{\varepsilon x}{tV_\varepsilon^2}\right)} \leq \left(\frac{eV_\varepsilon^2}{\varepsilon x}\right)^{\frac{x}{\varepsilon}} t^{\frac{x}{\varepsilon}},$$

see e.g. (Houdré, 2002, Corollary 1).

Lemma 5 *Let $H_t^+(a) := \mathbb{P}(Z_t \geq a) - t\nu([a,\infty))$ and assume that ν admits a Lévy density $s : \mathbb{R}\backslash\{0\} \to [0,\infty)$ which is non-decreasing and differentiable on $(0,x_0)$ for some $x_0 > 0$. Then, for any $0 < a < x_0/2$, $t > 0$, and $0 < \varepsilon < a$,*

$$\begin{aligned}\left|H_t^+(a)\right| \leq \mathbb{P}(Z_t^\varepsilon \geq a) + 2\lambda_\varepsilon t\left((\lambda_\varepsilon t)\wedge 1\right) \\ + ts(a)|\mathbb{E}Z_t^\varepsilon| + t\frac{m(\varepsilon,a)+1}{2}\mathbb{E}\left\{(Z_t^\varepsilon)^2\right\} \\ + t\mathbb{P}(|Z_t^\varepsilon| \geq a-\varepsilon)(\lambda_\varepsilon + a - \varepsilon + 1),\end{aligned} \tag{40}$$

where

$$m(\varepsilon,a) := \sup_{\varepsilon\leq x<a, a<x\leq 2a-\varepsilon}\frac{s(x)-s(a)}{a-x} \in (0,\infty). \tag{41}$$

Proof In terms of the decomposition $Z := Z^\varepsilon + \widetilde{Z}^\varepsilon$ in (38), by conditioning on the number of jumps of $\widetilde{Z}^\varepsilon$ during the interval $[0,t]$, we get

$$H_t^+(a) = \lambda_\varepsilon t\mathbb{P}(Z_t^\varepsilon + \xi_1 \geq a) - t\nu([a,\infty)) \tag{42}$$

$$+ e^{-\lambda_\varepsilon t}\mathbb{P}(Z_t^\varepsilon \geq a) + \left(e^{-\lambda_\varepsilon t} - 1\right)\lambda_\varepsilon t\mathbb{P}(Z_t^\varepsilon + \xi_1 \geq a) \tag{43}$$

$$+ e^{-\lambda_\varepsilon t}\sum_{k=2}^{\infty}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{P}(Z_t^\varepsilon + \sum_{i=1}^{k}\xi_i \geq a). \tag{44}$$

The last two terms on the right-hand side of the previous equation can be bounded as follows:

$$0 \leq \left(1 - e^{-\lambda_\varepsilon t}\right)\lambda_\varepsilon t\mathbb{P}(Z_t^\varepsilon + \xi_1 \geq a) \leq \lambda_\varepsilon t\left((\lambda_\varepsilon t)\wedge 1\right), \tag{45}$$

$$0 \leq e^{-\lambda_\varepsilon t}\sum_{k=2}^{\infty}\frac{(\lambda_\varepsilon t)^k}{k!}\mathbb{P}(Z_t^\varepsilon + \sum_{i=1}^{k}\xi_i \geq a) \leq (\lambda_\varepsilon t)^2 \wedge 1. \tag{46}$$

Let us consider the first two terms on the right-hand side of (42). Conditioning on ξ_1 and using that $a \geq \varepsilon$, the expression

$$A_t := \lambda_\varepsilon\mathbb{P}(Z_t^\varepsilon + \xi_1 \geq a) - \nu([a,\infty))$$

can be written as follows:

$$\int_{|x|\geq\varepsilon}\left(\mathbb{P}(Z_t^\varepsilon \geq a-x) - \mathbf{1}_{x\geq a}\right)s(x)dx = \int_{-\infty}^{-\varepsilon}\mathbb{P}(Z_t^\varepsilon \geq a-x)s(x)dx$$

$$- \int_a^\infty \mathbb{P}(Z_t^\varepsilon < a-x)s(x)dx + \int_\varepsilon^a \mathbb{P}(Z_t^\varepsilon \geq a-x)s(x)dx$$

$$= \int_{-\infty}^{-\varepsilon}\mathbb{P}(Z_t^\varepsilon \geq a-x)s(x)dx - \int_{2a-\varepsilon}^{\infty}\mathbb{P}(Z_t^\varepsilon < a-x)s(x)dx \tag{47}$$

$$- \int_{-(a-\varepsilon)}^{0}\mathbb{P}(Z_t^\varepsilon < u)s(a-u)du + \int_0^{a-\varepsilon}\mathbb{P}(Z_t^\varepsilon \geq u)s(a-u)du, \tag{48}$$

The two terms in line (47) can be bounded in absolute value by

$$\mathbb{P}(|Z_t^\varepsilon| \geq a-\varepsilon)\int_{|x|\geq\varepsilon}s(x)dx.$$

To dealt with the two term in line (48), note first that, since s is assumed to be continuous and decreasing on $(0,x_0)$, the supremum in (41) exists and also,

$$\begin{aligned}&0 \leq s(a-u) - s(a) \leq m(\varepsilon,a)u, \quad \text{for} \quad 0 < u \leq a-\varepsilon,\\ &0 \leq -(s(a-u) - s(a)) \leq -m(\varepsilon,a)u, \quad \text{for} \quad -(a-\varepsilon) \leq u < 0.\end{aligned}$$

Also, note that

$$
\begin{aligned}
A_t^1 &:= \int_0^{a-\varepsilon} \mathbb{P}(Z_t^\varepsilon \geq u)\,du - \int_{-(a-\varepsilon)}^0 \mathbb{P}(Z_t^\varepsilon < u)\,du \\
&= \mathbb{E}\{Z_t^\varepsilon\} - \mathbb{E}\left\{Z_t^\varepsilon \mathbf{1}_{\{|Z_t^\varepsilon| \geq a-\varepsilon\}}\right\} \\
&\quad + (a-\varepsilon)\{\mathbb{P}(Z_t^\varepsilon \geq (a-\varepsilon)) - \mathbb{P}(Z_t^\varepsilon < -(a-\varepsilon))\}, \\
A_t^2 &:= \int_0^{a-\varepsilon} \mathbb{P}(Z_t^\varepsilon \geq u)\,u du - \int_{-(a-\varepsilon)}^0 \mathbb{P}(Z_t^\varepsilon < u)\,u du \leq \frac{1}{2}\mathbb{E}\left\{(Z_t^\varepsilon)^2\right\}.
\end{aligned}
$$

In that case,

$$
0 \leq \int_0^{a-\varepsilon} \mathbb{P}(Z_t^\varepsilon \geq u)\,s(a-u)du - \int_{-(a-\varepsilon)}^0 \mathbb{P}(Z_t^\varepsilon < u)\,s(a-u)du - s(a)A_t^1 \leq m(\varepsilon,a)A_t^2.
$$

Therefore,

$$
\begin{aligned}
|A_t| &\leq \mathbb{P}(|Z_t^\varepsilon| \geq a-\varepsilon)\,\nu(|x| \geq \varepsilon) + s(a)|A_t^1| + m(\varepsilon,a)A_t^2 \\
&\leq s(a)|\mathbb{E}Z_t^\varepsilon| + \frac{m(\varepsilon,a)+1}{2}\mathbb{E}\left\{(Z_t^\varepsilon)^2\right\} \\
&\quad + \mathbb{P}(|Z_t^\varepsilon| \geq a-\varepsilon)\,(\nu(|x| \geq \varepsilon) + a - \varepsilon + 1),
\end{aligned}
$$

where we used that

$$
\left|\mathbb{E}\left\{Z_t^\varepsilon \mathbf{1}_{\{|Z_t^\varepsilon| \geq a-\varepsilon\}}\right\}\right| \leq \frac{1}{2}\left(\mathbb{E}\left\{(Z_t^\varepsilon)^2\right\} + \mathbb{P}(|Z_t^\varepsilon| \geq a-\varepsilon)\right).
$$

Applying the above bound for A_t and (45-46) to (42), we obtain (40).

We are ready to show the consistency of the estimators (23).

Proof (Theorem 1) Let $\bar{F}_t^+(a) := \mathbb{P}(Z_t \geq a)$ and $\bar{F}_t^-(a) := \mathbb{P}(Z_t \leq -a)$. Define also

$$
E_n^{\pm} := \frac{1}{\nu(|x| \geq a_n)^2}\mathbb{E}\left\{\left(\sum_{k=1}^n \mathbf{1}_{\{\pm\Delta_k^n X \geq a_n\}} - \nu(\pm x \geq a_n)\tau(T)\right)^2\right\}.
$$

It is easy to check that

$$
\mathbb{E}\left(\hat{\tau}_n(T) - \tau(T)\right)^2 \leq 2E_n^+ + 2E_n^-,
$$

hence, in order to obtain (25), it suffices to show that $E_n^+ \to 0$ and $E_n^- \to 0$. Let us prove this for E_n^+ (the case E_n^- can be done analogously). By conditioning on $\mathscr{F}_T^\tau := \sigma(\tau(t) : t \leq T)$ and using the independence between τ and Z, it follows that

$$
\begin{aligned}
E_n^+ &= \mathbb{E}\left(\frac{1}{\nu(|x| \geq a_n)}\sum_{k=1}^n \left(\bar{F}_t^+(a_n) - t\nu(x \geq a_n)\right)\Big|_{t=\Delta_k^n\tau}\right)^2 \\
&\quad + \frac{1}{\nu(|x| \geq a_n)^2}\sum_{k=1}^n \mathbb{E}\left(\left(\bar{F}_t^+(a_n) - (\bar{F}_t^+(a_n))^2\right)\Big|_{t=\Delta_k^n\tau}\right).
\end{aligned}
$$

Since $\nu(|x| \geq a_n) \to \infty$ as $n \to \infty$ and $\mathbb{E}\,\tau(T) < \infty$, the first term on the right-hand side above dominates the second one, and thus, we only have to show that

$$
\lim_{n\to\infty}\mathbb{E}\left(\frac{1}{\nu(|x| \geq a_n)}\sum_{k=1}^n \left(\bar{F}_t^+(a_n) - t\nu(x \geq a_n)\right)\Big|_{t=\Delta_k^n\tau}\right)^2 = 0. \tag{49}
$$

Using (40) with $\varepsilon = a_n/\kappa$, for some $\kappa \geq 2$ to be determined below, and since $a_n \to 0$, the following limits will suffice for (49):

$$A_n := \lim_{n\to\infty} \mathbb{E}\left(\frac{1}{\nu(|x|\geq a_n)}\sum_{k=1}^{n} \mathbb{P}(Z_t^{\varepsilon}\geq a_n)|_{t=\Delta_k^n\tau}\right)^2 = 0, \tag{50}$$

$$B_n := \lim_{n\to\infty} \mathbb{E}\left(\frac{\nu(|x|\geq \frac{a_n}{\kappa})}{\nu(|x|\geq a_n)}\sum_{k=1}^{n} \Delta_k^n\tau((\lambda_\varepsilon\Delta_k^n\tau)\wedge 1)\right)^2 = 0, \tag{51}$$

$$C_n := \lim_{n\to\infty} \mathbb{E}\left(\frac{s(a_n)}{\nu(|x|\geq a_n)}\sum_{k=1}^{n} \Delta_k^n\tau|\mathbb{E}Z_t^{\varepsilon}|_{t=\Delta_k^n\tau}\right)^2 = 0, \tag{52}$$

$$D_n := \lim_{n\to\infty} \mathbb{E}\left(\frac{m(\frac{a_n}{\kappa},a_n)}{\nu(|x|\geq a_n)}\sum_{k=1}^{n} \Delta_k^n\tau\;\mathbb{E}\left\{(Z_t^{\varepsilon})^2\right\}\Big|_{t=\Delta_k^n\tau}\right)^2 = 0, \tag{53}$$

$$E_n := \lim_{n\to\infty} \mathbb{E}\left(\frac{\nu(|x|\geq \frac{a_n}{\kappa})}{\nu(|x|\geq a_n)}\sum_{k=1}^{n} \Delta_k^n\tau\;\mathbb{P}\left(|Z_t^{\varepsilon}|\geq \frac{a_n}{2}\right)\Big|_{t=\Delta_k^n\tau}\right)^2 = 0. \tag{54}$$

In light of (39), (50) will follow from the limits:

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{\left(\int_{\{|x|\leq \frac{a_n}{\kappa}\}} x^2\nu(dx)\right)^{\kappa/4}}{\nu(|x|\geq a_n)a_n^{\kappa/2}}\sum_{k=1}^{n}(\Delta_k^n\tau)^{\kappa/4}\right)^2 = 0, \tag{55}$$

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{\nu(|x|\geq a_n)a_n^4}\sum_{k=1}^{n}(\Delta_k^n\tau)^2\right)^2 = 0, \tag{56}$$

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{1}{\nu(|x|\geq a_n)}\sum_{k=1}^{n}\mathbf{1}_{\left\{\Delta_k^n\tau\left(b-\int_{\frac{a_n}{\kappa}<|x|\leq 1}x\nu(dx)\right)>\frac{a_n}{2}\right\}}\right)^2 = 0, \tag{57}$$

where we used that $e^{-x}\leq Cx^{-3/2}$, for any $x>0$ and some $C<\infty$. Fixing $\kappa=8$, (56) will imply (55). Using that $\left(\int_{t_{k-1}^n}^{t_k^n} r(u)du\right)^2 \leq \delta^n\int_{t_{k-1}^n}^{t_k^n} r^2(u)du$ and (19),

$$\mathbb{E}\left(\frac{1}{\nu(|x|\geq a_n)a_n^4}\sum_{k=1}^{n}(\Delta_k^n\tau)^2\right)^2 \leq \left(\frac{\delta_n a_n^{\beta-4}}{\nu(|x|\geq a_n)a_n^{\beta}}\right)^2\mathbb{E}\left(\int_0^T r^2(u)du\right)^2;$$

hence, (56) will follows provided that

$$\liminf_{a\to 0}\nu(|x|\geq a)a^{\beta}>0, \quad \text{and} \quad \delta^n a_n^{\beta-4}=o(1), \quad \text{as } n\to\infty. \tag{58}$$

For (57), it suffices that

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{1}{\nu(|x|\geq a_n)}\sum_{k=1}^{n}\mathbf{1}_{\{b\Delta_k^n\tau>\frac{a_n}{4}\}}\right)^2=0, \quad \lim_{n\to\infty}\mathbb{E}\left(\frac{1}{\nu(|x|\geq a_n)}\sum_{k=1}^{n}\mathbf{1}_{\left\{-\int_{\frac{a_n}{\kappa}<|x|\leq 1}x\nu(dx)\Delta_k^n\tau>\frac{a_n}{4}\right\}}\right)^2=0.$$

To verify the second limit above (the first can be proved in a similar manner), we observe that the quantity after the limit can be bounded, up to a constant, as follows:

$$\mathbb{E}\left(\frac{\left(\int_{\frac{a_n}{\kappa}<|x|\leq 1}x\nu(dx)\right)^2}{\nu(|x|\geq a_n)a_n^2}\sum_{k=1}^{n}(\Delta_k^n\tau)^2\right)^2 \leq \left(\frac{\left(\int_{|x|\leq 1}x^2\nu(dx)\right)^2\delta_n}{\nu(|x|\geq a_n)a_n^4}\right)^2\mathbb{E}\left(\int_0^T r^2(u)du\right)^2; \tag{59}$$

hence, this quantity vanishes as $n\to\infty$ provided that (58) holds true. Note also that if one assumes that

$$\limsup_{x\to 0}|x|^{\beta+1}s(x)<\infty, \tag{60}$$

there exists an $0 < x_0 < 1$ small enough and n large enough such that $\int_{\frac{a_n}{\kappa}<|x|\le 1} |x|\nu(dx) \le K a_n^{-\beta+1}$, for some constant $0 < K < \infty$. Thus, up to a constant,

$$\mathbb{E}\left(\frac{\left(\int_{\frac{a_n}{\kappa}<|x|\le 1} x\nu(dx)\right)^2}{\nu(|x|\ge a_n)a_n^2}\sum_{k=1}^{n}(\Delta_k^n\tau)^2\right)^2 \le \left(\frac{a_n^{-\beta}\delta_n}{\nu(|x|\ge a_n)a_n^{\beta}}\right)^2 \mathbb{E}\left(\int_0^T r^2(u)du\right)^2, \tag{61}$$

which converges to 0 provided that $a_n^{-\beta}\delta_n = o(1)$, as $n\to\infty$. To show (54), we use the same arguments as for (50).

Next, we use that $\limsup_{n\to\infty} \nu(|x|\ge a_n/\kappa)/\nu(|x|\ge a_n) < \infty$, that $(x\wedge 1)\le\sqrt{x}$, for $x>0$, and that $(\Delta_k^n\tau)^{3/2} \le (\delta^n)^{1/2}\int_{t_{k-1}^n}^{t_k^n} r^{3/2}(u)du$ to conclude that $\delta^n a_n^{-\beta} = o(1)$, as $n\to\infty$ suffices for (51).

Using the formula $\mathbb{E}Z_t^{\varepsilon} = t\left(b - \int_{\frac{a_n}{\kappa}<|x|\le 1} x\nu(dx)\right)$ and that $\limsup_{n\to\infty} a_n s(a_n)/\nu(|x|\ge a_n) < \infty$, the limit (52) will follows if $a_n^{-\beta}\delta^n = o(1)$, as $n\to\infty$. Similarly, using the identity

$$\mathbb{E}(Z_t^{\varepsilon})^2 = t\int_{|x|\le\frac{a_n}{\kappa}} x^2\nu(dx) + t\sigma^2 + t^2\left(b - \int_{\frac{a_n}{\kappa}<|x|\le 1} x\nu(dx)\right)^2,$$

and that $\limsup_{n\to\infty} a_n^2|m(a_n/\kappa,a_n)|/\nu(|x|\ge a_n) < \infty$, the limit (53) will hold if $\mathbb{E}(\int_0^T r^3(u)du)^2 < \infty$ and $a_n^{-\beta}\delta^n = o(1)$, when $\sigma = 0$, or if $a_n^{-2}\delta^n = o(1)$, if $\sigma\ne 0$. Note that $a_n^{-2}\delta^n = o(1)$ is implied by $a_n^{\beta-4}\delta^n = o(1)$.

B Figures

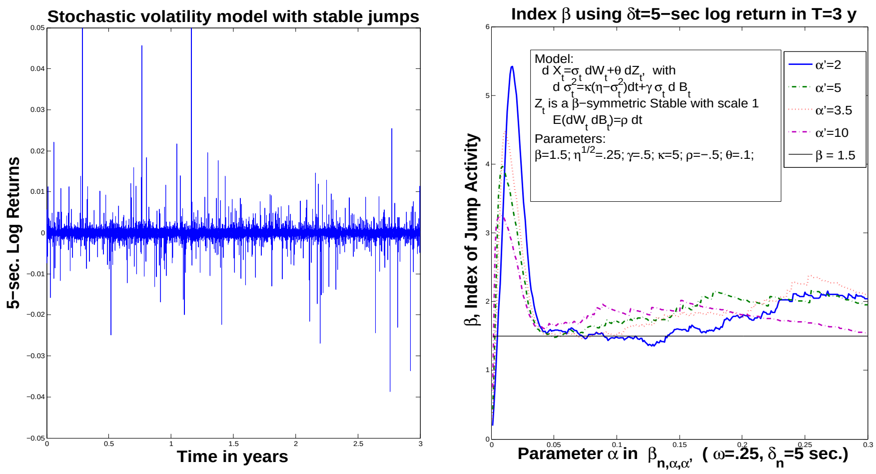

Fig. 1 5-sec returns and estimation of β for a stochastic volatility model with stable jumps.

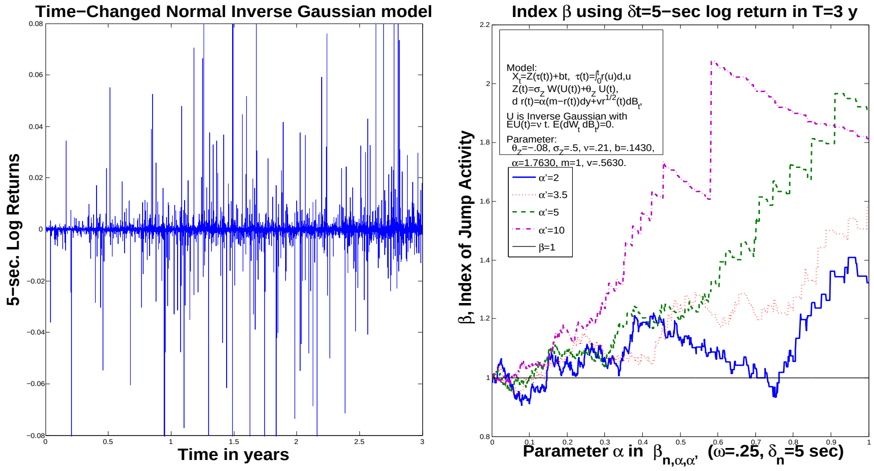

Fig. 2 5-sec returns and estimation of β for the time-changed Normal Inverse Gaussian process.

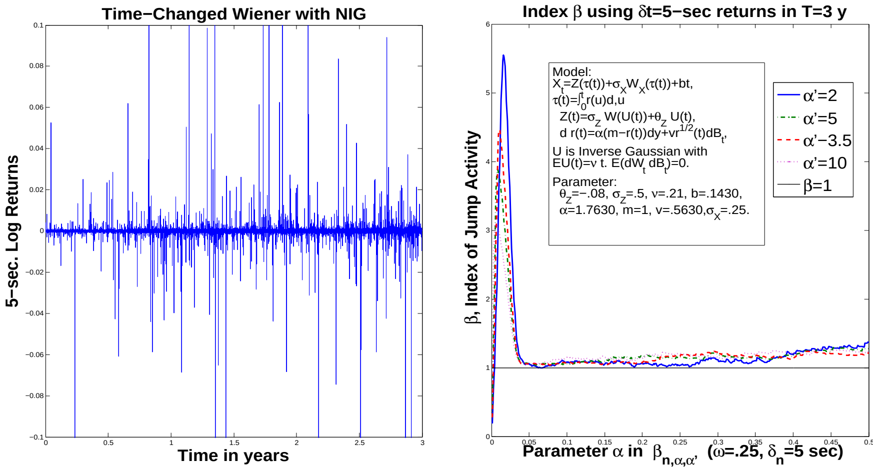

Fig. 3 5-sec returns of the time-changed Wiener process plus a Normal Inverse Gaussian process.

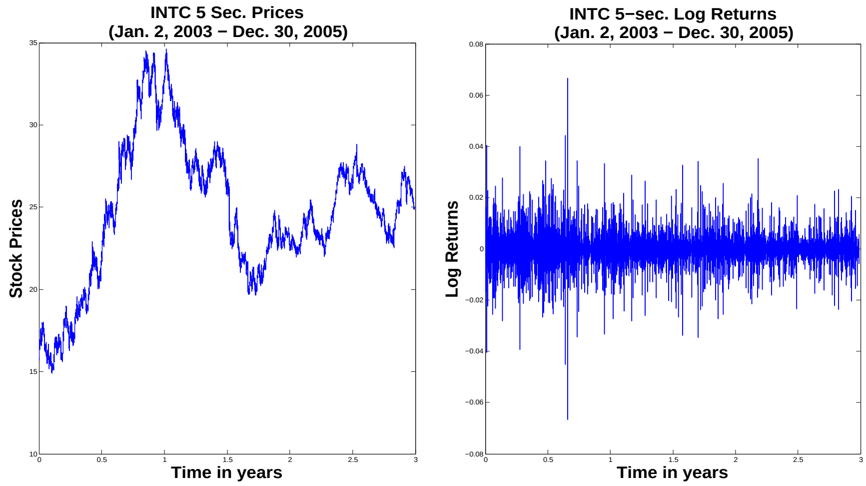

Fig. 4 INTC stock prices and log returns in 5-sec frequencies from 9:30 a.m. to 4:00 p.m.

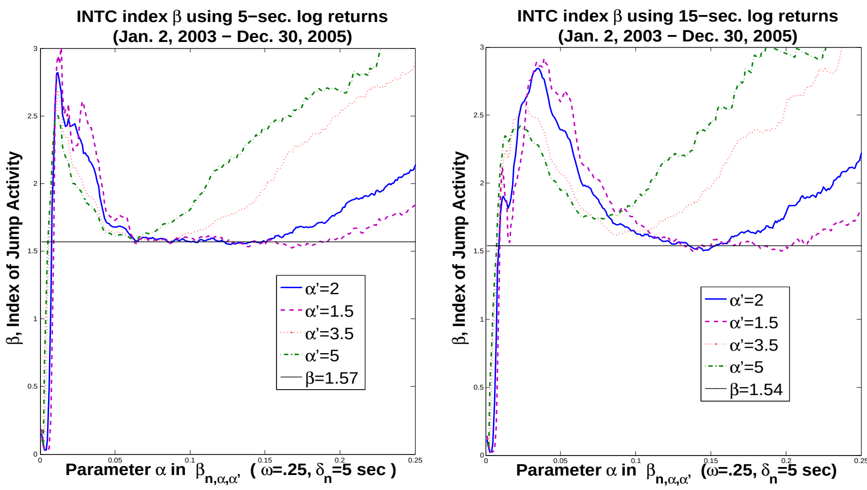

Fig. 5 INTC index of jump activity based on 5 sec. and 15 sec. log returns

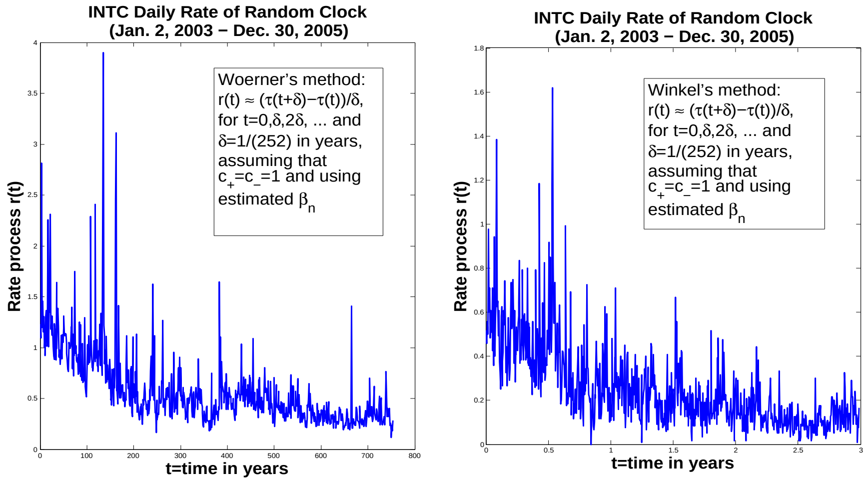

Fig. 6 Daily rate process $r(t)$ of the random clock using Woerner's and Winkel's method.

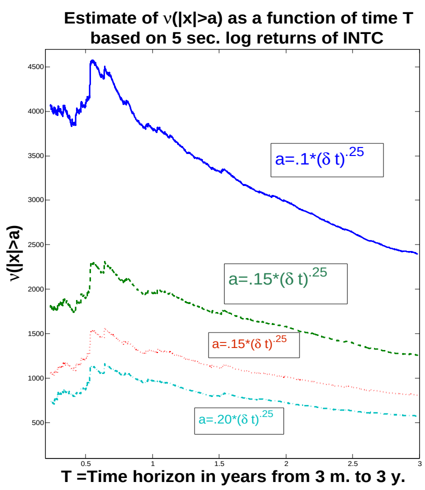

Fig. 7 Convergence of the estimator for $\nu(|x| \geq a)$ in function of the time horizon.

Acknowledgements Research partially supported by the NSF grant #DMS 0906919. The author is indebted to the associate editor and referee for their careful reading of paper and for their many suggestions that improve it considerably. It is also a pleasure to thank Yue Wu for assistance gathering and preprocessing the high-frequency data used for the empirical part of the paper.

References

AÏT-SAHALIA, Y. & JACOD, J. (2009). Estimating the degree of activity of jumps in high-frequency data. *Annals of Statistics* **37**(5A), 2202–2244.

ANÉ, T. & GEMAN, H. (2000). Order Flow, Transaction Clock and Normality of Asset Returns. *Journal of Finance* .

BARNDORFF-NIELSEN, O. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics* **2**, 41–68.

BARNDORFF-NIELSEN, O. & SHEPHARD, N. (2004). Power and bipower variation with stochastic volatility and jumps. *Journal of Financial Econometrics* **2**, 1–48.

BARNDORFF-NIELSEN, O. & SHEPHARD, N. (2006). Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics* **4**(1), 1–30.

BEHR, A. & PÖTTER, U. (2009). Alternatives to the normal model of stock returns: Gaussian mixture, generalised logF and generalised hyperbolic models. *Annals of Finance* **5**, 49–68.

BERTOIN, J. (1996). *Lévy processes*. Cambridge University Press.

CARR, P., GEMAN, H., MADAN, D. & YOR, M. (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business* , 305–332.

CARR, P., GEMAN, H., MADAN, D. & YOR, M. (2003). Stochastic volatility for Lévy processes. *Mathematical Finance* **13**, 345–382.

CARR, P., MADAN, D. & CHANG, E. (1998). The variance Gamma process and option pricing. *European Finance Review* **2**, 79–105.

CHRONOPOULOU, A., VIENS, F. & TUDOR, C. (2010). Variations and Hurst index estimation for a Rosenblatt process using longer filters. *To appear in Electronic Journal of Statistics* .

CLARK, P. (1973). A Subordinated Stochastic Process with Finite Variance for Speculative Prices. *Econometrica* **41**, 135–155.

CONT, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance* **1**, 223–236.

CONT, R. & TANKOV, P. (2004). *Financial modelling with Jump Processes*. Chapman & Hall.

EBERLEIN, E. & KELLER, U. (1995). Hyperbolic Distribution in Finance. *Bernoulli* **1**, 281–299.

FIGUEROA-LÓPEZ, J. (2009). Nonparametric estimation of time-changed Lévy models under high-frequency data. *Advances in Applied Probability* **41**(4).

FIGUEROA-LÓPEZ, J. (2010a). Central limit theorems for the non-parametric estimation of time-changed Lévy models. Tech. Rep. 10-01, Department of Statistics, Purdue University.

FIGUEROA-LÓPEZ, J. (2010b). *Jump-diffusion models driven by Lévy processes*. Springer. To appear in Handbook of Computational Finance. Jin-Chuan Duan, James E. Gentle, and Wolfgang Hardle (eds.).

FOUQUE, J., PAPANICOLAOU, G. & SIRCAR, K. (2000). *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press.

HOUDRÉ, C. (2002). Remarks on deviation inequalities for functions of infinitely divisible random vectors. *The Annals of Probabilty* **30**(3), 1223–1237.

HUDSON, W. & MASON, J. (1976). Variational sums for additive processes. *Proceedings of the American Mathematical Society* **55**, 395–399.

JOHANNES, M. & POLSON, N. (2003). *MCMC Methods for Continuous-Time Financial Econometrics*. Elsevier: Amsterdam. In L.P. Hansen and Y. Ait-Sahalia (eds), Handbook of Financial Econometrics.

KARATZAS, I. & SHREVE, S. (1988). *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York.

KOU, S. & WANG, H. (2004). OptionPricingUnderaDoubleExponentialJumpDiffusionModel. *Management Science* **50**, 1178–1192.

LI, J. (2009). A Bayesian Estimation of Time-Changed Infinite Activity Lévy Models: Explaining Return-Volatility Relations. Tech. rep., Bocconi University.

MONROE, I. (1978). Processes that can be embedded in Brownian motion. *The Annals of Probability* **6**, 42–56.

NOURDIN, I., NUALART, D. & TUDOR, C. (2007). Central and Non-Central Limit Theorems for weighted power variations of the fractional Brownian motion. Tech. rep., Preprint.

RAMEZANI, C. & ZENG, Y. (2007). Maximum likelihood estimation of the double exponential jump-diffusion process. *Annals of Finance* **3**, 487–507.

SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

WINKEL, M. (2001). The recovery problem for time-changed lévy processes. Tech. Rep. 2001-37, MaPhySto. Available also at www.stats.ox.ac.uk/∼winkel/indexe.html.

WOERNER, J. (2006). Power and multipower variations: inference for high frequency data. *In Stochastic Finance, A.N. Shiryaev, M. do Rosário Grosshino, P. Oliviera, M. Esquivel, eds.* , 343–354.

WOERNER, J. (2007). Inference in Lévy-type stochastic volatility models. *Adv. Appl. Prob.* **39**, 531–549.