BIOMATHEMATICS TRAINING PROGRAM

STATISTICAL INFERENCE FOR CENSORED BIVARIATE NORMAL DISTRIBUTIONS BASED ON INDUCED ORDER STATISTICS*

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SUMMARY

In life testing studies, where failure times are usually recorded only for the first k (out of n) units that fail, one may wish to study the relationship between the failure time and other concomitant variates. The present investigation concerns this study for the case when these concomitant variates are observable only for the units pertaining to the actual failures, i.e., for the induced order statistics. For a bivariate normal distribution, estimation of parameters based on these induced order statistics is considered and a related test for independence is also studied. Empirical power studies are made for this test.

Key Words and Phrases: Bivariate normal distribution, censoring, independence, induced order statistics, order statistics, modified maximum likelihood estimator and likelihood ratio test.

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1. INTRODUCTION

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent and identically distributed random vectors (i.i.d.r.v) and let $Y_{n,1} \le \cdots \le Y_{n,n}$ be the *order statistics* corresponding to $Y_1, \ldots, Y_n$. Also, assume that the conditional distribution function (d.f.) of Y_i given X_i is of the form $F_{\beta X_i}(y)$ with a probability density function (p.d.f.) $f_{\beta X_i}(y)$, for $i = 1, \ldots, n$; here, the X_i may be $p(\ge 1)$ - vectors and β is also an unknown vector.

In a life testing problem, treating the Y_i as the *failure times*, if an experiment is run until the first k failures (out of n) occur, the *likelihood function* of the *censored sample* (and given the X_i) is

$$\tilde{L}_{n,k} = \left\{\prod_{i=1}^{k} f_{\beta X_{n[i]}}(Y_{n,i})\right\}\left\{\prod_{i=k+1}^{n} [1 - F_{\beta X_{n[i]}}(Y_{n,k})]\right\} \tag{1.1}$$

where

$$X_{n[i]} = X_j \quad \text{if} \quad Y_{n,i} = Y_j \quad \text{for} \quad i, j = 1, \ldots, n\ . \tag{1.2}$$

Bhattacharya (1974) has termed the $X_{n[i]}$ as the *induced order statistics* while David and Galambos (1974) have named these as the *concomitants of order statistics*.

Various authors have used (1.1) to draw statistical inference [on β and other parameters associated with p.d.f. $f_{\beta x}(y)$]; we may refer to David and Moeschberger (1978) for some of these procedures. The conditional likelihood approach of Cox (1972) is similar to this, but uses fewer parametric assumptions. These procedures, however, assume that though $Y_{n,1}, \ldots, Y_{n,k}$ are only observable, the entire set $(X_1, \ldots, X_n)$ [or equivalently, $X_{n[1]}, \ldots, X_{n[n]}$] is given prior to experimentation, so that (1.1) is properly defined. In many other life testing problems, especially, arising in clinical trials, $X_{n[i]}$ is observable only when $Y_{n,i}$ is so (for $i \le n$), so that the second factor on the right hand side (rhs) of (1.1) is no longer deterministic. This is particularly true if recording of X_i necessitates the failure of the ith unit, so that for the surviving $[n-k]$ units, the X_i are not observable. In toxicological experimentation this is a common feature.

The necessity of observing concomitant data for all individuals no longer exists if we make the assumption that (X_i, Y_i) has a bivariate normal d.f. and denote its density by $\phi_{\underset{\sim}{\theta}}(x,y)$ where $\underset{\sim}{\theta} = (\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$, μ_X, μ_Y stand for the means, σ_X^2, σ_Y^2 for the variances and ρ for the correlation coefficient of X and Y. Watterson (1959) found linear estimators of μ_X and $\rho\sigma_X$ using $X_{n[1]}, \ldots, X_{n[k]}$, while the estimation of μ_Y and σ_Y using the order statistics $Y_{n,1} < \cdots < Y_{n,k}$ can be made as in Sarhan and Greenberg (1962). However, the estimation of σ_X^2 (and ρ) poses certain difficulties and constitutes the main objective of the current study.

In Section 3, an estimator of σ_X^2 is developed. Likelihood equations are incorporated in Section 4 for deriving Cramèr-Rao lower bounds for variances of unbiased estimators based on censored data. Estimation of μ_X, σ_X^2, and ρ using both order and induced order statistics is formulated in Section 5. In Section 6, the desirable properties of the classical Pearsonian correlation coefficient as a test statistic for testing H_0: $\rho = 0$ are considered and some power calculations are presented for various combinations of (k, n). Foundations of many of these results are laid down in Section 2.

2. CONDITIONAL DISTRIBUTIONS AND MOMENTS OF INDUCED ORDER STATISTICS

Since $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d.r.v. having the p.d.f. $\phi_{\underset{\sim}{\theta}}(x,y)$, by an appeal to Lemma 1 of Bhattacharya (1974), we conclude that given $Y_{n,1}, \ldots, Y_{n,k}$, $X_{n[1]}, \ldots, X_{n[k]}$ are (conditionally) independently distributed and the conditional p.d.f. of $X_{n[j]}$ given the $Y_{n,i}$, $1 \le i \le k$ is univariate normal with mean $\mu_X + \rho\sigma_X(Y_{n,j} - \mu_Y)/\sigma_Y$ and variance $\sigma_X^2(1-\rho^2)$, for $j = 1, \ldots, k$ and every $n \ge k \ge 1$. Let then $\underset{\sim}{X} = (X_{n[1]}, \ldots, X_{n[k]})'$, $\underset{\sim}{Y} = (Y_{n,1}, \ldots, Y_{n,k})'$ and $\underset{\sim}{Z} = \sigma_Y^{-1}(\underset{\sim}{Y} - \underset{\sim}{1}\mu_Y)$. The joint (conditional) p.d.f. of $\underset{\sim}{X}$ given $\underset{\sim}{Y}$ is then

$$[\sigma_X^2(1-\rho^2)2\pi]^{-k/2} \exp\left\{-\frac{1}{2\sigma_X^2(1-\rho^2)} [\underset{\sim}{X} - \mu_X\underset{\sim}{1} - \rho\sigma_X\underset{\sim}{Z}]'[X - \mu_X\underset{\sim}{1} - \rho\sigma_X\underset{\sim}{Z}]\right\}. \quad (2.1)$$

It follows from (2.1) that

$$E(\underset{\sim}{X}|\underset{\sim}{Y}) = \mu_X \underset{\sim}{1} + \rho\sigma_X \underset{\sim}{Z} \quad \text{and} \quad E(\underset{\sim}{X}\underset{\sim}{X}'|\underset{\sim}{Y}) = \sigma_X^2(1-\rho^2)\underset{\sim}{I}_k + [E(\underset{\sim}{X}|\underset{\sim}{Y})][E(\underset{\sim}{X}|\underset{\sim}{Y})]' . \quad (2.2)$$

Thus, if we let

$$E\underset{\sim}{Z} = \underset{\sim}{u} = (u_{n,1}, \ldots, u_{n,k}) \quad \text{and} \quad E(\underset{\sim}{Z}-E\underset{\sim}{Z})(\underset{\sim}{Z}-E\underset{\sim}{Z})' = \underset{\sim}{V} = ((v_{n,ij})), \quad (2.3)$$

then, from (2.2) and (2.3), we obtain that

$$E(\underset{\sim}{X}) = \mu_X \underset{\sim}{1} + \rho\sigma_X \underset{\sim}{u} \quad \text{and} \quad E(\underset{\sim}{X}-E\underset{\sim}{X})(\underset{\sim}{X}-E\underset{\sim}{X})' = \sigma_X^2(1-\rho^2)\underset{\sim}{I}_k + \rho^2\sigma_X^2 \underset{\sim}{V} . \quad (2.4)$$

Note that for $k \le n \le 20$, $\underset{\sim}{u}$ and $\underset{\sim}{V}$ are tabulated in Sarhan and Greenberg (1962, pp. 193-205) and approximations for the same for higher values of n are considered in David (1970, pp. 65-7). These are quite useful in subsequent sections.

3. ESTIMATION OF σ_X^2 FROM THE INDUCED ORDER STATISTICS

Watterson (1959) used (2.4) to find estimators of μ_X and $\rho\sigma_X$ which are linear functions of $X_{n[1]}, \ldots, X_{n[k]}$. His estimator of μ_X is

$$\tilde{\mu}_X = \underset{\sim}{\tilde{\alpha}}' \underset{\sim}{X} = \sum_{i=1}^{k} \tilde{\alpha}_i X_{n[i]} \quad ; \quad (3.1)$$

$$\tilde{\alpha}_i = k^{-1} - \bar{u}_k(u_{n,i} - \bar{u}_k)/\sum_{m=1}^{k}(u_{n,m} - \bar{u}_k)^2, \quad 1 \le i \le k, \quad (3.2)$$

and $\bar{u}_k = k^{-1}\sum_{i=1}^{k} u_{n,i}$. $\tilde{\mu}_X$ is the minimum variance unbiased (MVU) estimator of μ_X when $\rho = 0$. $\tilde{\mu}_X$ losses its MVU property when $\rho \ne 0$ and it is not in general possible to find a MVU estimator since ρ is not known. However, he observed **that** for a certain range of ρ $\tilde{\mu}_X$ in (3.1) has a smaller variance than that of the MVU estimator, assuming $\rho = 1$. The $\tilde{\alpha}_i$ in (3.2) are the same as in Gupta (1952) where the estimation of μ_Y based on $Y_{n,1}, \ldots, Y_{n,k}$ is treated.

It follows from (2.4) that there is no estimator of σ_X^2, linear in $\underset{\sim}{X}$ and valid for all ρ. It is therefore natural to try a quadratic estimator of the form $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}$. Again, the variance of such an estimator would depend on the unknown ρ. Following the Watterson approach, we might minimize the variance of $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}$ when $\rho = 0$, which amounts to minimizing $\mathrm{tr}\underset{\sim}{A}^2$ subject to $E(\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}) = \sigma_X^2$. The authors have found (by extensive simulations) the use of this estimator, which requires the solution of k simultaneous equations, has little to gain over a much simpler estimator which we present below. Let

$$\tilde{\sigma}_X^2 = \sum_{i=1}^{k} c_i (X_{n[i]} - \tilde{\mu}_X)^2 , \tag{3.3}$$

where $\tilde{\mu}_X$ is defined by (3.1) and the c_i are real constants. Using (2.4), (3.1), (3.2), and (3.3), it follows by some some standard steps that

$$E\tilde{\sigma}_X^2 = \sigma_X^2\, \underset{\sim}{a}'\underset{\sim}{c} + \rho\sigma_X^2\, \underset{\sim}{b}'\underset{\sim}{c} , \tag{3.4}$$

where $\underset{\sim}{a} = (a_1, \ldots, a_k)'$, $\underset{\sim}{b} = (b_1, \ldots, b_k)'$, $\underset{\sim}{c} = (c_1, \ldots, c_k)'$ with

$$a_i = (k-1)/k + \bar{u}_k\{1+2(u_{n,i} - \bar{u}_k)\}/\sum_{m=1}^{k} (u_{n,m} - \bar{u}_k)^2 \tag{3.5}$$

$$b_i = -a_i - 2\sum_{j=1}^{k} \tilde{\alpha}_j \nu_{n,ji} + \sum_{r=1}^{k}\sum_{s=1}^{k} \tilde{\alpha}_r\tilde{\alpha}_s \nu_{n,rs} + \nu_{n,ii} + u_{n,i}^2 , \tag{3.6}$$

for $i=1, \ldots, k$. Thus, for an unbiased estimator, we require that $\underset{\sim}{a}'\underset{\sim}{c}=1$ and $\underset{\sim}{b}'\underset{\sim}{c}=0$ and, motivated by Watterson, we minimize $\underset{\sim}{c}'\underset{\sim}{c}$ which minimizes $V(\tilde{\sigma}_X^2)$ when $\rho=0$, considering $\tilde{\mu}_X$ given. The result is

$$c_i = \{b_i\underset{\sim}{a}'\underset{\sim}{b} - a_i\underset{\sim}{b}'\underset{\sim}{b}\}/\{(\underset{\sim}{a}'\underset{\sim}{b})^2 - (\underset{\sim}{a}'\underset{\sim}{a})(\underset{\sim}{b}'\underset{\sim}{b})\}, \quad 1 \le i \le k. \tag{3.7}$$

An estimator of ρ can also be obtained by considering the Watterson estimator of $\rho\sigma_X$ (based on $\underset{\sim}{X}$ alone) and dividing the same by the square root of $\tilde{\sigma}_X^2$ in (3.3) and (3.7). However, such an estimator does not depend on $\underset{\sim}{Y}$, and hence, when the latter is given, may not be a very efficient one. For this reason, in the next Section, we proceed to study the efficiency of estimators of $\underset{\sim}{\theta} = (\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$ based on $(\underset{\sim}{X}, \underset{\sim}{Y})$.

4. INFORMATION LIMITS TO THE DISPERSION OF UNBIASED ESTIMATORS OF $\underset{\sim}{\theta}$

Note that the joint p.d.f. of $\underset{\sim}{Y}$ is

$$n^{[k]} \prod_{i=1}^{k} \phi_1((y_{n,i} - \mu_Y)/\sigma_Y)[1 - \Phi_1((y_{n,k} - \mu_Y)/\sigma_Y)]^{n-k} \sigma_Y^{-k}, \tag{4.1}$$

where $n^{[k]} = n \cdots (n-k+1)$, Φ_1 is the standard univariate normal d.f. and ϕ_1 is its p.d.f. Multiplying (2.1) and (4.1), we obtain that the joint p.d.f. of $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ is equal to

$$\left\{n^{[k]} \prod_{i=1}^{k} \phi_{\underset{\sim}{\theta}}(x_{n[i]}, y_{n,i})\right\}\left\{[1 - \Phi_1((y_{n,k} - \mu_Y)/\sigma_Y)]^{n-k}\right\}, \tag{4.2}$$

$$y_{n,1} \le y_{n,2} \cdots \le y_{n,k}$$

where $\phi_{\underset{\sim}{\theta}}(x,y)$ is the bivariate normal p.d.f. with the parameter $\underset{\sim}{\theta}$. With the notations in (2.3), we let $\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_k)$ where $\omega_i = \nu_{n,ii} + u^2_{n,i}$, $1 \le i \le k$. Also, we denote by $L_{n,k}$ the logarithm of (4.2) and define $\underset{\sim}{X}$, $\underset{\sim}{Y}$, and $\underset{\sim}{Z}$ as in before (2.1). Then, we have (on letting $\underset{\sim}{X}_0 = \underset{\sim}{X} - \mu_X \underset{\sim}{1}$)

$$\left.\begin{aligned}
(\partial/\partial\mu_X)L_{n,k} &= \{(\underset{\sim}{X}_0 - \rho\sigma_X\underset{\sim}{Z})'\underset{\sim}{1}\}/\sigma_X^2(1-\rho^2),\\
(\partial/\partial\sigma_X)L_{n,k} &= -k/\sigma_X + [\underset{\sim}{X}_0'\underset{\sim}{X}_0 - \rho\sigma_X\underset{\sim}{X}_0'\underset{\sim}{Z}]/\sigma_X^3(1-\rho^2),\\
(\partial/\partial\rho)L_{n,k} &= (1-\rho^2)^{-2}[k\rho(1-\rho^2) - \rho\sigma_X^{-2}\underset{\sim}{X}_0'\underset{\sim}{X}_0 - \rho\underset{\sim}{Z}'\underset{\sim}{Z} + \sigma_X^{-1}(1+\rho^2)\underset{\sim}{X}_0'\underset{\sim}{Z}],\\
(\partial/\partial\mu_Y)L_{n,k} &= (n-k)\sigma_Y^{-1}\phi_1(Z_{n,k})/[1 - \Phi_1(Z_{n,k})]\\
&\quad - (1-\rho^2)^{-1}\sigma_Y^{-1}[\rho\sigma_X^{-1}\underset{\sim}{X}_0'\underset{\sim}{1} - \underset{\sim}{Z}'1],\\
(\partial/\partial\sigma_Y)L_{n,k} &= (n-k)\sigma_Y^{-1}Z_{n,k}\phi_1(Z_{n,k})/[1 - \Phi_1(Z_{n,k})]\\
&\quad - \sigma_Y^{-1}(1-\rho^2)^{-1}\{\rho\sigma_X^{-1}\underset{\sim}{X}_0'\underset{\sim}{Z} - \underset{\sim}{Z}'\underset{\sim}{Z}\} - k\sigma_Y^{-1},
\end{aligned}\right\} \quad (4.3)$$

where $Z_{n,k} = (Y_{n,k} - \mu_Y)/\sigma_Y$. Let

$$\psi(n,k;\ a,b,c,d) = E\{[\phi_1(z_{n,k})]^a[\Phi_1(z_{n,k})]^{-b}[1 - \Phi_1(z_{n,k})]^{-c}z^d_{n,k}\} \quad (4.4)$$

where a, b, c, d are non-negative numbers and the p.d.f. of $Z_{n,k}$ is

$$[n^{[k]}/(k-1)!][\Phi_1(z)]^{k-1}[1 - \Phi_1(z)]^{n-k}\phi_1(z), \quad -\infty < z < \infty, \quad (4.5)$$

for $k = 1, \ldots, n$. Differentiating each side of each equation in (4.3) with respect to μ_X, μ_Y, σ_X, σ_Y, and ρ and taking expectations, we obtain by some standard steps that

$$I_{11} = E\{-(\partial^2/\partial\mu_X^2)L_{n,k}\} = k[\sigma_X^2(1-\rho^2)]^{-1}, \tag{4.6}$$

$$I_{13} = E\{-(\partial^2/\partial\mu_X\partial\sigma_X)L_{n,k}\} = (\underset{\sim}{u}'\underset{\sim}{1})\rho[\sigma_X^2(1-\rho^2)]^{-1}, \tag{4.7}$$

$$I_{15} = E\{-(\partial^2/\partial\mu_X\partial\rho)L_{n,k}\} = (\underset{\sim}{u}'\underset{\sim}{1})[\sigma_X(1-\rho^2)]^{-1}, \tag{4.8}$$

$$I_{12} = E\{-(\partial^2/\partial\mu_X\partial\mu_Y)L_{n,k}\} = -k\rho[\sigma_X\sigma_Y(1-\rho^2)]^{-1}, \tag{4.9}$$

$$I_{14} = E\{-(\partial^2/\partial\mu_X\partial\sigma_Y)L_{n,k}\} = -\rho(\underset{\sim}{u}'\underset{\sim}{1})[\sigma_X\sigma_Y(1-\rho^2)]^{-1}, \tag{4.10}$$

$$I_{33} = E\{-(\partial^2/\partial\sigma_X^2)L_{n,k}\} = \{2k(1-\rho^2) + \rho^2\underset{\sim}{\omega}'\underset{\sim}{1}\}/[\sigma_X^2(1-\rho^2)], \tag{4.11}$$

$$I_{23} = E\{-(\partial^2/\partial\sigma_X\partial\mu_Y)L_{n,k}\} = -\rho^2(\underset{\sim}{u}'\underset{\sim}{1})[\sigma_X\sigma_Y(1-\rho^2)]^{-1}, \tag{4.12}$$

$$I_{35} = E\{-(\partial^2/\partial\sigma_X\partial\rho)L_{n,k}\} = \rho(\underset{\sim}{\omega}'\underset{\sim}{1}-2k)[\sigma_X(1-\rho^2)]^{-1}, \tag{4.13}$$

$$I_{34} = E\{-(\partial^2/\partial\sigma_X\partial\sigma_Y)L_{n,k}\} = -\rho^2(\underset{\sim}{\omega}'\underset{\sim}{1})[\sigma_X\sigma_Y(1-\rho^2)]^{-1}, \tag{4.14}$$

$$I_{55} = E\{-(\partial^2/\partial\rho^2)L_{n,k}\} = \{2k\rho^2 + (1-\rho^2)\underset{\sim}{\omega}'\underset{\sim}{1}\}(1-\rho^2)^{-2}, \tag{4.15}$$

$$I_{52} = E\{-(\partial^2/\partial\rho\partial\mu_Y)L_{n,k}\} = -\rho(\underset{\sim}{u}'\underset{\sim}{1})[\sigma_Y(1-\rho^2)]^{-1}, \tag{4.16}$$

$$I_{54} = E\{-(\partial^2/\partial\rho\partial\sigma_Y)L_{n,k}\} = -\rho(\underset{\sim}{\omega}'\underset{\sim}{1})[\sigma_Y(1-\rho^2)]^{-1}, \tag{4.17}$$

$$\begin{aligned} I_{22} = E\{-(\partial^2/\partial\mu_Y^2)L_{n,k}\} = \{&(n-k)[\psi(n,k;\ 2,0,2,0) - \psi(n,k;\ 1,0,1,1)] \\ &+ k(1-\rho^2)^{-1}\}/\sigma_Y^2, \end{aligned} \tag{4.18}$$

$$\begin{aligned} I_{24} = E\{-(\partial^2/\partial\mu_Y\partial\sigma_Y)L_{n,k}\} &= \{(n-k)[\psi(n,k;\ 1,0,1,0) - \psi(n,k;\ 1,0,1,2) \\ &+ \psi(n,k;\ 2,0,2,1] + \underset{\sim}{u}'\underset{\sim}{1}(2-\rho^2)/(1-\rho^2)\}/\sigma_Y^2, \end{aligned} \tag{4.19}$$

$$\begin{aligned} I_{44} = E\{-(\partial^2/\partial\sigma_Y^2)L_{n,k}\} &= \{(n-k)[2\psi(n,k;\ 1,0,1,1) - \psi(n,k;\ 1,0,1,3) \\ &+ \psi(n,k;\ 2,0,2,2)] - k - (1-\rho^2)^{-1}(2\rho^2-3)\underset{\sim}{\omega}'\underset{\sim}{1}\}/\sigma_Y^2, \end{aligned} \tag{4.20}$$

and where $I_{j\ell} = I_{\ell j}$ for every $j, \ell = 1, \ldots, 5$.

[In passing, we may remark that Des Raj (1953) in the context of maximum likelihood estimators (MLE) of $\underset{\sim}{\theta}$, when Y is truncated instead of censored, obtained equations similar to those in (4.3). However our (4.6) - (4.20) are different from his.]

To evaluate $\underset{\sim}{I} = ((I_{j\ell}))_{j,\ell=1,\ldots,5}$, the information matrix, we need to evaluate $\underset{\sim}{u}'\underset{\sim}{1}$ and $\underset{\sim}{\omega}'\underset{\sim}{1}$. For this, as in Saw (1958), we note that given $Z_{n,k}$, the joint distribution of $(Z_{n,1},\ldots,Z_{n,k-1})$ is reducible to that of $k-1$ independent rv's from a univariate normal d.f., truncated from above by $Z_{n,k}$; we denote the latter variables by $S_1,\ldots,S_{k-1}$, so that for $1 \le i \le k-1$,

$$E(S_i|Z_{n,k}) = -\phi_1(Z_{n,k})/\Phi_1(Z_{n,k}), \quad E(S_i^2|Z_{n,k}) = 1-Z_{n,k}\phi(Z_{n,k})/\Phi_1(Z_{n,k}) , \tag{4.21}$$

and, in general, for $r \ge 0$,

$$E(S_i^{r+2}|Z_{n,k}) = (r+1)E(S_i^r|Z_{n,k}) - Z_{n,k}^{r+1}\phi_1(Z_{n,k})/\Phi_1(Z_{n,k}) . \tag{4.22}$$

Hence,

$$\begin{aligned}\underset{\sim}{u}'\underset{\sim}{1} &= \sum_{i=1}^{k} E(Z_{n,i}) = E(Z_{n,k}) + \textstyle\sum_{i=1}^{k-1}E[S_i] \\ &= E(Z_{n,k}) + \textstyle\sum_{i=1}^{k-1}E\{E[S_i|Z_{n,k}]\} \\ &= \psi(n,k;\ 0,0,0,1) - (k-1)\psi(n,k;\ 1,1,0,0);\end{aligned} \tag{4.23}$$

$$\begin{aligned}\underset{\sim}{\omega}'\underset{\sim}{1} &= \textstyle\sum_{i=1}^{k}E(Z_{n,i}^2) = E(Z_{n,k}^2) + \sum_{i=1}^{k-1}E\{E(S_i^2|Z_{n,k})\} \\ &= \psi(n,k;\ 0,0,0,2) + (k-1)[1-\psi(n,k;\ 1,1,0,1)].\end{aligned} \tag{4.24}$$

[Note that, in Saw's notation, $\psi(n,k;\ a,a,0,b) = \psi(k/(n+1), n\colon a,b)$ and these can be easily calculated for given (k,n) by numerical integration.]

For the calculations which follow, σ_X and σ_Y are both taken as unity. For three selected sample sizes and five values of ρ the elements of $\underset{\sim}{I}$ were calculated. The Cramèr-Rao lower bounds for variances of unbiased estimators, which are the diagonal elements of $\underset{\sim}{I}^{-1}$, are presented in the following table. These bounds also represent asymptotic variances of the maximum likelihood estimators using (4.3), $\hat{\mu}_X$, $\hat{\mu}_Y$, $\hat{\sigma}_X$, $\hat{\sigma}_Y$, and $\hat{\rho}$.

TABLE I

Cramèr-Rao lower bounds for variances

n	k	ρ	$V[\hat{\mu}_X]$	$V[\hat{\mu}_Y]$	$V[\hat{\sigma}_X]$	$V[\hat{\sigma}_Y]$	$V[\hat{\rho}]$
20	10	0	.2433	.0761	.0500	.0601	.2433
		.1	.2417	.0761	.0514	.0601	.2372
		.3	.2283	.0761	.0618	.0601	.1916
		.5	.2015	.0761	.0775	.0601	.1181
		.9	.1078	.0761	.0787	.0601	.0049
100	20	0	.4598	.0567	.0250	.0345	.2134
		.1	.4557	.0567	.0266	.0345	.2077
		.3	.4235	.0567	.0385	.0345	.1653
		.5	.3590	.0567	.0562	.0345	.0984
		.9	.1333	.0567	.0564	.0345	.0032
1000	50	0	.6170	.0508	.0100	.0155	.1410
		.1	.6114	.0508	.0112	.0155	.1370
		.3	.5661	.0508	.0200	.0155	.1081
		.5	.4755	.0508	.0330	.0155	.0630
		.9	.1584	.0508	.0322	.0155	.0017

It is interesting to note that for a large censoring proportion (small k/n) μ_X cannot be estimated well at all from induced order statistics yet σ_X can be estimated with more accuracy than σ_Y for small ρ. Also, lower bounds for variance of estimators of μ_Y and σ_Y are independent of ρ as one would expect; μ_Y and σ_Y can be estimated efficiently with a small number of order statistics regardless whether or not induced order statistics are observable.

The difficulty in estimating μ_X stems from ρ being unknown, therefore bringing an unknown location bias to the sample of induced order statistics due to the selection process on Y. For example, if ρ were known to be zero μ_X would be estimated by the sample mean with variance σ_X^2/k.

5. MODIFIED MAXIMUM LIKELIHOOD ESTIMATORS OF $\underset{\sim}{\theta}$

An iterative method may be used to estimate $\underset{\sim}{\theta}$ by using (4.3). Des Raj (1953) set up the implicit solutions to the likelihood equations for the case of double truncation on Y from a bivariate normal sample which is similar to our case relating to censoring. Assuming the theorem outlined by Halperin (1952) to be valid for our bivariate case†, the ML-estimator will be consistent, asymptotically normally distributed and asymptotically efficient too. However, the

†Since $Y_{n,1}, \ldots, Y_{n,k}$ are not independent, study of the asymptotic properties of MLE may demand additional regularity conditions [viz. Sen (1976)]. Nevertheless, for bivariate normal d.f.'s, these hold under quite general setups.

complexity of the solutions in (4.3) makes the study of the properties of the estimator difficult for any finite sample size. Also, if the results obtained by Saw (1961) for estimating μ_Y and σ_Y by the MLE from censored data can be extrapolated to our bivariate case, we can expect the MLE to have undersirable properties for small k or n. Particularly, the MLE may have considerable bias (relative to its standard error). The linearization of the log-likelihood function described in Chan (1967), which results in simple and efficient estimators of μ_Y and σ_Y, does not work here as, in this case, we will face 4 equations in 5 unknown parameters. Therefore, a second order Taylor-series expansion of $L_{n,k}$ is needed and this will be as complicated as using the original equations in (4.3) and does not lead to unbiased estimators.

There are other good estimators of μ_Y and σ_Y besides the one considered by Chan (1967). For instance, Saw (1959) developed an unbiased estimator of μ_Y which is a linear combination of $Y_{n,k}$ and $(k-1)^{-1}\sum_{i=1}^{k-1} Y_{n,i}$ and has asymptotic efficiency (a function of $k/(n+1)) \geq 94\%$ for $k/(n+1)$ as low as 0.25. He has also considered a linear combination of $\sum_{i=1}^{k-1}(Y_{n,i} - Y_{n,k})^2$ and $\{\sum_{i=1}^{k-1}(Y_{n,i} - Y_{n,k})\}^2$ as an unbiased estimator of σ_Y^2 which is of 100% asymptotic efficiency for all $k/(n+1)$.

Since the lower bounds for $V(\hat{\mu}_X)$ and $V(\hat{\sigma}_X)$ are independent of p (and hence, equal to the bounds when $p = 0$), for the estimation of (μ_Y, σ_Y), the induced order statistics may not contribute any information. In view of this, it seems natural to estimate (μ_Y, σ_Y) by some (efficient) procedure [based only on $\underset{\sim}{Y} = (Y_{n,1}, \ldots, Y_{n,k})$] [denote the estimators by (μ_Y^*, σ_Y^*)] and then to consider only the first three equations in (4.3) wherein μ_Y, σ_Y are replaced by μ_Y^*, σ_Y^* and equating these to 0, we get three equations in three unknown parameters (μ_X, σ_X, ρ) and denote these solutions $(\mu_X^*, \sigma_X^*, \rho^*)$ as the modified MLE. For this purpose, we denote by

$$\left.\begin{aligned} &\bar{X} = k^{-1}\textstyle\sum_{i=1}^{k} X_{n[i]}, \quad \bar{Y} = k^{-1}\sum_{i=1}^{k} Y_{n,i}, \quad S_X^2 = k^{-1}\sum_{i=1}(X_{n[i]} - \bar{X})^2 , \\ &S_Y^2 = k^{-1}\textstyle\sum_{i=1}^{k}(Y_{n,i} - \bar{Y})^2 \quad \text{and} \quad S_{XY} = k^{-1}\sum_{i=1}^{k}(X_{n[i]} - \bar{X})(Y_{n,i} - \bar{Y}). \end{aligned}\right\} \tag{5.1}$$

Then, for the modified MLE, we have

$$(\bar{X} - \mu_X^*)/\sigma_X^* = \rho^*(\bar{Y} - \mu_Y^*)/\sigma_Y^* , \tag{5.2}$$

$$\sigma_X^{*2} = S_X^2 + \sigma_Y^{*2}(\bar{X} - \mu_X^*)^2/(\bar{Y} - \mu_Y^*)^2 - S_{XY}(\bar{X} - \mu_X^*)/(\bar{Y} - \mu_Y^*), \tag{5.3}$$

$$[\text{or} \quad (\rho^*)^2 - \rho^* S_{XY}/\sigma_X^*\sigma_Y^* = 1 - S_X^2/\sigma_X^{*2}] \tag{5.4}$$

and the third equation in (4.3) along with (5.2) leads us to

$$\rho^*(1-\rho^{*2}) - \rho^*(S_X^2/\sigma_X^{*2} + S_Y^2/\sigma_Y^{*2}) + (1+\rho^{*2})S_{XY}/\sigma_X^*\sigma_Y^* = 0, \tag{5.5}$$

so that by (5.4) and (5.5), we have

$$\begin{aligned} \rho^* &= \sigma_Y^* S_{XY}/\sigma_X^* S_Y^2 \\ &= (S_{XY}/S_X S_Y)[(\sigma_Y^*/S_Y)(S_X/\sigma_X^*)]. \end{aligned} \tag{5.7}$$

From (5.4) and (5.7), we obtain that

$$\sigma_X^{*2} = S_X^2 + (S_{XY}^2/S_Y^2)(\sigma_Y^{*2}/S_Y^2 - 1) \tag{5.8}$$

and from (5.2), (5.7) and (5.8), we have

$$\mu_X^* = \bar{X} + (S_{XY}/S_Y^2)(\mu_Y^* - \bar{Y}) \ . \tag{5.9}$$

Thus, having estimated μ_Y, σ_Y by μ_Y^*, σ_Y^*, we may estimate μ_X, σ_X, and p by (5.9), (5.8), and (5.7), respectively. One nice feature of these estimators is that as $k/n \to 1$ (i.e., the amount of censoring decreases), the estimators approach the ordinary MLE, which are known to be efficient.

Noting that given $\underset{\sim}{Y}$, μ_X^* is linear in $\underset{\sim}{X}$, we obtain by (2.2), (5.1), and (5.9) that

$$E[\mu_X^* | \underset{\sim}{Y}] = \mu_X + p\sigma_X(\mu_Y^* - \mu_Y)/\sigma_Y, \tag{5.10}$$

and hence, μ_X^* is unbiased for μ_X if μ_Y^* is unbiased for μ_Y. It can also be shown that

$$V(\mu_X^*) = \sigma_X^2(1-p^2)[1 + E\{(\mu_Y^* - \bar{Y})^2/S_Y^2\}]/k + p^2\sigma_X^2\sigma_Y^{-2}V(\mu_Y^*), \tag{5.11}$$

$$E(\sigma_X^{*2}) = \sigma_X^2 + p^2\sigma_X^2[\sigma_Y^{-2}E(\sigma_Y^{*2}) - 1] + \sigma_X^2(1-p^2)k^{-1}[E\{\sigma_Y^{*2}/S_Y{}^2\} - 2]. \tag{5.12}$$

In order to estimate the expected values of σ_X^* and p^* as well as the variances of μ_X^*, σ_X^*, and p^*, for each of three different combinations of (n, k) and five different values of p, 500 random sets of $(\underset{\sim}{X}, \underset{\sim}{Y})$ were generated and the statistics were evaluated.

μ_Y^*, σ_Y^* were taken as the estimators due to Gupta (1952). Also, for comparison $V(\tilde{\mu}_X)$ (where $\tilde{\mu}_X$ is defined by (3.1)-(3.2)) was evaluated and for the latter, the $u_{n,i}$ in (3.2) were approximated by the method due to Harter (1961). For simplicity, we have taken here $\sigma_X = \sigma_Y = 1$. We have also estimated the trace efficiency of the modified MLE, which is the ratio of the sum of the five Cramèr-Rao lower bounds from Table I to the sum of the estimated variances of the five modified MLE.

TABLE II

Estimated moments of modified maximum likelihood estimators

n	k	p	$V[\mu_X^*]$	$V[\tilde{\mu}_X]$	$V[\mu_Y^*]$	$E[\sigma_X^*]$	$V[\sigma_X^*]$	$V[\sigma_Y^*]$	$E[p^*]$	$V[p^*]$	Trace Efficiency
20	10	0	.281	.297	.074	1.021	.077	.078	-.020	.236	.902
		.1	.287	.293	.092	.998	.074	.088	.060	.213	.884
		.3	.281	.293	.082	1.043	.079	.079	.275	.196	.862
		.5	.265	.276	.087	.998	.102	.086	.449	.172	.749
		.9	.123	.126	.093	1.014	.106	.083	.889	.024	.764
100	20	0	.495	.508	.076	1.038	.042	.045	-.044	.169	.955
		.1	.548	.564	.087	1.059	.047	.051	.080	.176	.859
		.3	.487	.507	.090	1.054	.050	.049	.256	.152	.868
		.5	.400	.420	.081	1.020	.057	.046	.442	.115	.865
		.9	.202	.209	.087	1.014	.080	.048	.885	.009	.667
1000	50	0	.663	.685	.078	1.045	.019	.022	-.016	.114	.931
		.1	.601	.616	.091	1.044	.018	.025	.092	.105	.983
		.3	.624	.648	.094	1.048	.024	.024	.274	.103	.875
		.5	.477	.508	.091	1.030	.032	.025	.462	.066	.923
		.9	.187	.188	.082	.987	.038	.023	.887	.003	.777

From Table II we see that μ_X^* is consistently slightly better than $\tilde{\mu}_X$ although a small part of this may have to do with the approximation used for the $u_{n,i}$. The simulated efficiency of μ_X^* is between 66% and 102% according to the bounds in Table I. As expected, μ_X^* has large variance for large censoring proportion. On the other hand, σ_X is estimated as well as σ_Y in almost every case and has lower variance in some cases. Also, σ_X^* appears to be not badly biased and is 53-103% efficient. p^* has efficiency between 69% and 130% for the cases studied except when $p = .9$. The bias in p^* is very low except for moderate p ($p = .3$ or .5 resulted in maximum estimated bias of .06).

6. TESTS FOR INDEPENDENCE OF (X,Y)

Since μ_Y^*, σ_Y^* are efficient estimators of μ_Y, σ_Y for all p, the modified MLE in Section 5 may be incorporated to prescribe a modified likelihood ratio test for H_0: $p = 0$. As before, μ_Y^*, σ_Y^*; μ_X^*, σ_X^*, p^* are the estimators considered in Section 5. Over the parameter space restricted by $p = 0$, the parallel estimators are μ_Y^*, σ_Y^*; $\tilde{\mu}_X^*$, $\tilde{\sigma}_X^*$, 0 where

$$\tilde{\mu}_X^* = \bar{X} \quad \text{and} \quad \tilde{\sigma}_X^{*2} = S_X^2 \quad \text{are defined by (5.1).} \tag{6.1}$$

Since the second factor of (4.2) involves only (μ_Y, σ_Y) and we have the same estimator (μ_Y^*, σ_Y^*) in both the null and non-null cases, substituting the two sets of estimates in (4.2), taking the ratio and proceeding by some standard steps, we obtain that -2 times the logarithm of the (modified) likelihood ratio statistic is equal to

$$-k\log(1-r^{*2}) \quad \textbf{where} \quad r^* = S_{XY}/S_X S_Y \tag{6.2}$$

and the S_X, S_Y, S_{XY} are defined by (5.1). Thus, r^* is the ordinary product moment correlation of the set $(Y_{n,i}, X_{n[i]})$, $i = 1, \ldots, k$. Thus, as a test statistic (for testing $H_0: \rho = 0$), we may use r^*. We define

$$T^* = (k-2)r^{*2}/(1-r^{*2}) = S_{XY}^2/\{(k-2)^{-1}[S_X^2 S_Y^2 - S_{XY}^2]\}, \tag{6.3}$$

$$U^* = kS_Y^2/\sigma_Y^2 = \sigma_Y^{-2}\sum_{i=1}^{k}(Y_{n,i} - \bar{Y})^2. \tag{6.4}$$

Let $H_\Delta(t;\ a,b)$ be the non-central F d.f. with degrees of freedom (DF)(a,b) and non-centrality parameter Δ and $G_{n,k}(u) = P\{U^* \le u\}$ for $0 \le u < \infty$. Finally, let

$$H_\rho^*(t;\ a,b) = \int_0^\infty H_{\rho^2 u/(1-\rho^2)}(t;\ a,b)\,dG_{n,k}(u), \quad 0 \le t < \infty. \tag{6.5}$$

Then, we have the following

<u>Theorem 6.1</u>

For every $t \ge 0$ *and given* $\underset{\sim}{U}^*$

$$P\{T^* \le t \mid \underset{\sim}{U}^*\} = H_{\rho^2(1-\rho^2)^{-1}U^*}(t;\ 1, k-2). \tag{6.6}$$

Hence, $P\{T^* \le t\} = H_\rho^*(t;\ 1, k-2)$, $\forall\ t \in [0,\infty)$. *Under* $H_0: \rho = 0$, $H_\rho^*(t;\ 1, k-2) = H_0(t;\ 1, k-2)$ *is the central* F *d.f. with DF* $(1, k-2)$.

Proof:

As in Section 2, given $\underset{\sim}{Y}$, $X_{n[1]}, \ldots, X_{n[k]}$ are (conditionally) independently normally distributed with means $\mu_X + \rho\sigma_X\sigma_Y^{-1}(Y_{n,i} - \mu_Y)$ $1 \le i \le k$ and a common variance $\sigma_X^2(1-\rho^2)$. Hence, given Y, kS_{XY} is conditionally normally distributed with mean $(\underset{\sim}{Y} - \bar{Y}\underset{\sim}{1})'\{\mu_X\underset{\sim}{1} + \rho\sigma_X\sigma_Y^{-1}(\underset{\sim}{Y} - \mu_Y\underset{\sim}{1})\} = k\rho\sigma_X S_Y^2/\sigma_Y = \rho\sigma_X\sigma_Y U^*$ and variance $\sigma_X^2(1-\rho^2)(\underset{\sim}{Y} - \bar{Y}\underset{\sim}{1})'\underset{\sim}{I}_k(\underset{\sim}{Y} - \bar{Y}\underset{\sim}{1}) = \sigma_X^2\sigma_Y^2(1-\rho^2)U^*$. Therefore, given $\underset{\sim}{Y}$,

$$k^2 S_{XY}^2/\{\sigma_X^2\sigma_Y^2(1-\rho^2)U^*\} \sim \chi_1^2(\rho^2 U^*(1-\rho^2)^{-1}), \tag{6.7}$$

where $\chi_p^2(\Delta)$ stands for the non-central chi-square d.f. with p DF and non-centrality parameter Δ. Further,

$$k^2(S_X^2 S_Y^2 - S_{XY}^2)/\{\sigma_X^2\sigma_Y^2(1-\rho^2)U^*\} = \underset{\sim}{U}'\underset{\sim}{A}\underset{\sim}{U} \tag{6.8}$$

where

$$\left.\begin{aligned} \underset{\sim}{A} &= \sigma_Y^2 U^*(\underset{\sim}{I}_k - k^{-1}\underset{\sim}{1}\underset{\sim}{1}') - (\underset{\sim}{Y} - \bar{Y}\underset{\sim}{1})(\underset{\sim}{Y} - \bar{Y}\underset{\sim}{1})' \\ \underset{\sim}{U} &= \{\sigma_X^2\sigma_Y^2(1-\rho^2)U^*\}^{-1/2}\underset{\sim}{X} \end{aligned}\right\} \tag{6.9}$$

and given $\underset{\sim}{Y}, \underset{\sim}{U}$ is conditionally normally distributed with mean vector

$$\underset{\sim}{\gamma} = \{\mu_X\underset{\sim}{1} + \rho\sigma_X\sigma_Y^{-1}(\underset{\sim}{Y} - \mu_Y\underset{\sim}{1})\}\{\sigma_X^2\sigma_Y^2(1-\rho^2)U^*\}^{-1/2} \tag{6.10}$$

and dispersion matrix $k^{-1}S_Y^{-2}\underset{\sim}{I}_k$. Further, it can be shown that $\underset{\sim}{A}k^{-1}S_y^{-2}$ is an idempotent matrix and rank of $\underset{\sim}{A} = k-2$. Also, $\underset{\sim}{\gamma}'\underset{\sim}{A}\underset{\sim}{\gamma} = 0$. Thus, given $\underset{\sim}{Y}$,

$$k^2(S_X S_Y^2 - S_{XY}^2)/\{\sigma_X^2\sigma_Y^2(1-\rho^2)U^*\} \sim \chi_{k-2}^2(0). \tag{6.11}$$

Finally, noting that $(\underset{\sim}{Y} - \bar{Y}\underset{\sim}{1})'\underset{\sim}{A} = \underset{\sim}{0}$, we obtain by (5.1), (6.8), and

(6.9) that given $\underset{\sim}{Y}$, S_{XY} and $\{S_X^2 S_Y^2 - S_{XY}^2\}$ are conditionally independent, and hence, (6.6) follows from (6.3), (6.7), (6.11) and the fact that this conditional df depends on $\underset{\sim}{Y}$ through U^* alone. Finally, by (6.5) and (6.6),

$P(T^* \le t) = E\{P(T^* \le t \mid \underset{\sim}{U}^*)\} = H_p^*(t;\ 1, k-2)$. Under $H_0: p = 0$, (6.6) is equal to $H_0(t;\ 1, k-2)$ for all U^*, and hence, $H_p^* = H_0$. Q.E.D.

If follows from Theorem 6.1 that under $H_0: p = 0$, T^* has the classical variance-ratio distribution with DF (1, k -2) and the test can be made without any difficulty. Let F_α^* be the upper $100\alpha\%$ point of $H_0(\ ;\ 1, k-2)$ i.e., $H_0(F_\alpha^*;\ 1, k-2) = 1 - \alpha$. Then, the power of the test based on T^* (or r^*) and corresponding to the level of significance $\alpha (0 < \alpha < 1)$ is given by

$$1 - H_p^*(F_\alpha^*;\ 1, k-2)\ . \tag{6.12}$$

In general, for $k < n$, $G_{n,k}$, the d.f. of U^*, is quite complicated, and hence, analytical solutions for (6.12) are difficult to obtain. However, one can obtain one or two moment approximations for it, using the following formulae[†] due to Saw (1958):

[†]This paper contains additional errors not reported in its Corrigenda. Let $\psi_{ab} = \psi(n,k;\ a,a,o,b)$, defined by (4.4). Then, noting that $U^* = (\underset{\sim}{Z} - \bar{Z}\underset{\sim}{1})'(\underset{\sim}{Z} - \bar{Z}\underset{\sim}{1})$ where $\underset{\sim}{Z} = (Z_{n,1}, \ldots, Z_{n,k})'$ with $Z_{n,i} = (Y_{n,i} - \mu_Y)/\sigma_Y$, $1 \le i \le k$ relate to the standard normal df, we find after rederiving Saw's formulae that

$$EU^* = k^{-1}(k-1)[(k-1) - (k-3)\psi_{11} + \psi_{02} - (k-2)\psi_{20}] = ES_Y^2/\sigma_Y^2$$

and the coefficient of ψ_{13} in equation (3.6) of Saw (1958), relating to ES_Y^4, should be $-(3k^2 - 14k + 15)$ instead of $(5k^2 - 10k + 1)$. Dr. Saw has confirmed these findings in a personal communication. Also, we limit ourselves to $k > 4$ (or > 6) for these approximations.

$$E(T^*) = \{1+\zeta EU^*\}(k-2)/(k-4), \quad \zeta = p^2/(1-p^2), \tag{6.13}$$

$$E(T^{*2}) = \{3+6\zeta EU^* + \zeta^2 EU^{*2}\}(k-2)^2/(k-4)(k-6). \tag{6.14}$$

For the one moment approximation, we replace the distribution of T^* by $F_{1,k-2}(\lambda)$ where $\lambda = \zeta EU^*$ and $F_{a,b}(\Delta)$ has the d.f. $H_\Delta(t; a,b)$, defined after (6.4). For the two-moment approximation, we replace the distribution of T^* by $aF_{b,k-2}(0)$ and equating its moments to (6.13) - (6.14), we obtain that

$$a = 1+\zeta EU^* \quad \text{and} \quad b = 2a^2/\{\zeta^2 V(U^*) + 4\zeta EU^* + 2\}\ . \tag{6.15}$$

Note that under $p = 0$, $a = 1$, $b = 1$ and $\lambda = 0$, so that both the approximations give the correct null d.f. For the two-moment approximations, (a, b) values for different n, k, p combinations are as follows.

n	k	p	EU^*	$V(U^*)$	a	b
20	10	0	3.43246	3.87797	1	1
		0.1			1.035	1.001
		0.3			1.339	1.057
		0.5			2.144	1.312
		0.9			15.633	3.731
100	20	0	4.22029	3.60135	1	1
		0.1			1.043	1.002
		0.3			1.417	1.085
		0.5			2.407	1.443
		0.9			18.992	5.174
1000	50	0	6.79174	4.41258	1	1
		0.1			1.069	1.004
		0.3			1.672	1.182
		0.5			3.264	1.845
		0.9			29.954	9.063

For each n, k, and p, the power of the r^* test under censoring at the 5% level was estimated by generating 2000 $k \times 2$ random vectors. The exact upper 5% critical values were used and the power was calculated by counting the number of r^*'s having absolute value greater than the true critical value. For purposes of comparison the simulated critical values are also given. In order to compare the power of the test using censored data to that using a complete bivariate sample, the complete sample size c required to achieve the same simulated power using the Fisher-Yates Z-test is given.

TABLE III

Estimated power of 5% r-test for censored data

n	k	p	Estimated Critical Value of $\lvert r^*\rvert$	True Critical Value of $\lvert r^*\rvert$	Power from One Moment Approximation	Power from Two Moment Approximation	Simulated Power	c
20	10	0	.624	.632	.0500	.0500	.0485	
		.1			.0531	.0531	.0615	13
		.3			.0810	.0802	.0810	6
		.5			.1570	.1522	.1645	7
		.9			.9160	.8331	.8080	7
100	20	0	.433	.444	.0500	.0500	.0445	
		.1			.0544	.0544	.0555	8
		.3			.0939	.0921	.0915	7
		.5			.2025	.1929	.1920	7
		.9			.9796	.9468	.9305	9
1000	50	0	.280	.276	.0500	.0500	.0535	
		.1			.0576	.0575	.0595	12
		.3			.1266	.1219	.1130	9
		.5			.3140	.2967	.3225	11
		.9			.9995	.9985	.9980	14

Note that a complete sample of about one-fourth the size of the censored sample would give the same power for $n = 1000$, $k = 50$. The two power approximations are very similar but the two-moment approximation appears to be slightly better. However, because of the simplicity of the one-moment approximation, it might be preferred.

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