STATISTICAL INFERENCE FOR ROUGH VOLATILITY: CENTRAL LIMIT THEOREMS

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In recent years, there has been substantive empirical evidence that stochastic volatility is rough. In other words, the local behavior of stochastic volatility is much more irregular than semimartingales and resembles that of a fractional Brownian motion with Hurst parameter $H < 0.5$. In this paper, we derive a consistent and asymptotically mixed normal estimator of H based on high-frequency price observations. In contrast to previous works, we work in a semiparametric setting and do not assume any a priori relationship between volatility estimators and true volatility. Furthermore, our estimator attains a rate of convergence that is known to be optimal in a minimax sense in parametric rough volatility models.

1. Introduction. For many years, continuous-time stochastic volatility models were predominantly based on stochastic differential equations driven by Brownian motion or Lévy processes. But more recently, [21] found empirical evidence that stochastic volatility is actually much rougher than semimartingales, in the sense that it locally resembles a fractional Brownian motion with Hurst index $H < 0.5$, a statement that was further supported by other empirical work based on both return data [9, 20, 22] and options data [8, 19, 34].

The data-driven approach of [21] to uncover rough volatility starts by considering high-frequency log-price data $\{x_{i\delta_n} : i = 0, \dots, [T/\delta_n]\}$, where for example $\delta_n = 5\,\text{min}$ and $T = 1\,\text{year}$. In a next step, daily realized variance estimates are calculated from the formula

$$RV_j = \sum_{i=1}^{k_n} (\delta^n_{(j-1)k_n+i} x)^2, \qquad j = 1, \dots, [T/(k_n\delta_n)], \tag{1.1}$$

where $\delta_i^n x = x_{i\delta_n} - x_{(i-1)\delta_n}$ and $k_n = 78$ is the number of $5\,\text{min}$ increments during one trading day. On a one-year horizon, RV_j can be viewed as daily spot volatility estimates. In a next step, realized power variations of $\log RV_j$, that is,

$$m(q, \Delta) = \frac{1}{[T/\Delta]} \sum_{j=1}^{[T/\Delta]} |\log RV_{j\Delta} - \log RV_{(j-1)\Delta}|^q$$

are computed for different values of $q > 0$ and $\Delta \in \{1\,\text{day}, 2\,\text{days}, \dots\}$. If $\log RV_j$ were discrete observations of a continuous Itô semimartingale, then one would expect that $m(q, \Delta)$ scales as $\Delta^{q/2}$, implying that the slope ζ_q in a regression of $\log m(q, \Delta)$ on $\log \Delta$ satisfies

$$\zeta_q / q \approx \frac{1}{2}.$$

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However, for large set of high-frequency data, [21] consistently found values of $\zeta_q/q < \frac{1}{2}$, indicating that stochastic volatility locally behaves as a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$.

As was pointed out by [9, 20], the above approach rests on the assumption that realized variances have the same scaling behavior as the true unobserved volatility. At the same time, it is well known (see e.g., [3, Chapter 8]) that in the absence of jumps and if volatility is a semimartingale, spot volatility estimators of the type (1.1) converge to true volatility plus a small modulated white noise. In a first attempt to take estimation errors for spot volatility into account, [9, 20] assume that

$$\log RV_j = \log \text{true volatility}_j + \varepsilon_j, \tag{1.2}$$

where ε_j is a zero-mean iid sequence that is independent of everything else. Under assumption (1.2), [9, 20] derive consistent estimators of the roughness parameter H in parametric rough volatility models and uphold the conclusion of [21] that volatility is rough in a large set of financial time series. We also refer to [10], where the authors assume (1.2) with slightly different assumptions on $(\log RV_j, \varepsilon_j)$, and to [38], where a central limit theorem (CLT) for H is established under (1.2) (see also [42]).

This paper aims to substantially generalize the aforementioned results in two directions: first, we establish consistent and asymptotically mixed normal estimators of H in a semiparametric setting, where except for H all other model ingredients are fully nonparametric; and second, we shall do so without assuming any relationship (such as (1.2)) between volatility proxies and true volatility. The rate of convergence of our best estimator is

$$\delta_n^{-1/(4H+2)}, \tag{1.3}$$

which as our companion paper [13] shows is optimal in a minimax sense in parametric rough volatility models. In follow-up work, we will discuss the finite-sample performance of our estimators and leverage the results of this paper into real data applications. Also, the inclusion of price jumps [2, 28] and the separation of volatility jumps from volatility roughness [14] are left to future research.

The remaining paper is structured as follows: in Section 2, after introducing the model assumptions, we state the main technical result of this paper, Theorem 2.1, a CLT for *volatility of volatility (VoV)* estimators in a rough volatility framework. The proof will be given in Section 3, with certain technical details postponed to Appendices A–C. Section 4 discusses how we turn Theorem 2.1 into rate-optimal and feasible estimators of H. In addition to a usual application of the delta method, the rough volatility setting requires us overcome two distinct challenges:

- eliminating a nonnegligible asymptotic bias term for which we do not have a sufficiently fast estimator;
- constructing an optimal sequence k_n for spot volatility estimation that depends on the unknown parameter H without losing a marginal bit of convergence rate.

Our final estimator H_n for H is given in Equation (4.34). As Theorems 4.3 and 4.5 show, H_n is a feasible and rate-optimal estimator of H if $H \in (0, \frac{1}{2})$ and is equal to $\frac{1}{2}$ with high probability if volatility is a continuous Itô semimartingale.

In what follows, we write $A \lesssim B$ if there is a constant $C \in (0, \infty)$ that does not depend on any important parameter such that $A \leq CB$. Furthermore, if $A_n(t)$ and $B_n(t)$ are stochastic processes, we write $A_n \approx B_n$ if $\mathbb{E}[\sup_{t\in[0,T]}|A_n(t) - B_n(t)|] \to 0$ as $n \to \infty$. For two sequences a_n and b_n we write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$. If $x \in \mathbb{R}^n$, we denote its Euclidean norm by $|x|$. For any $\alpha \in \mathbb{R}$, we write $x_+^\alpha = x^\alpha$ if $x > 0$ and $x_+^\alpha = 0$ otherwise. We also use the notation $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

2. Model and CLT for VoV estimators. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, we assume that the log-price x of an asset is given by a continuous Itô semimartingale of the form

$$x_t = x_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \qquad t \geq 0. \tag{2.1}$$

We assume that the squared volatility process $c = \sigma^2$ satisfies

$$c_t = c_0 + \int_0^t a_s ds + \int_0^t \tilde{g}(t-s)\tilde{\eta}_s ds + \int_0^t g(t-s)(\eta_s dW_s + \hat{\eta}_s d\hat{W}_s), \tag{2.2}$$

where

$$\begin{aligned} \eta_t^2 &= \eta_0^2 + \int_0^t a_s^\eta ds + \int_0^t \tilde{g}^\eta(t-s)\tilde{\theta}_s ds + \int_0^t g^\eta(t-s)\boldsymbol{\theta}_s d\bar{\boldsymbol{W}}_s, \\ \hat{\eta}_t^2 &= \hat{\eta}_0 + \int_0^t a_s^{\hat{\eta}} ds + \int_0^t \tilde{g}^{\hat{\eta}}(t-s)\tilde{\vartheta}_s ds + \int_0^t g^{\hat{\eta}}(t-s)\boldsymbol{\vartheta}_s d\bar{\boldsymbol{W}}_s. \end{aligned} \tag{2.3}$$

The ingredients of (2.1)–(2.3) are assumed to satisfy the following conditions.

ASSUMPTION CLT. *Suppose that the log-price process x is given by (2.1) with the following specifications:*

1. *There is $H \in (0, \frac{1}{2}]$ such that the squared volatility process $c_t = \sigma_t^2$ satisfies (2.2) with η and $\hat{\eta}$ given by (2.3). The variables x_0, c_0, η_0^2 and $\hat{\eta}_0^2$ are $\mathcal{F}_0$-measurable.*
2. *The processes a, b, a^η and $a^{\hat{\eta}}$ (resp., $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$) are adapted and locally bounded real-valued (resp., $\mathbb{R}^{1\times 4}$-dimensional) processes. Moreover, for all $T > 0$, we assume that*

$$\lim_{h\to 0} \sup_{s,t\in[0,T], |s-t|\leq h} \big\{\mathbb{E}[1 \wedge |b_t - b_s|] + \mathbb{E}[1 \wedge |a_t - a_s|]\big\} = 0. \tag{2.4}$$

3. *The processes $\tilde{\eta}$, $\tilde{\theta}$ and $\tilde{\vartheta}$ are adapted, locally bounded and for all $T > 0$, there is $K_T \in (0, \infty)$ such that*

$$\sup_{s,t\in[0,T]} \big\{\mathbb{E}[1 \wedge |\tilde{\eta}_t - \tilde{\eta}_s|] + \mathbb{E}[1 \wedge |\tilde{\theta}_t - \tilde{\theta}_s|] + \mathbb{E}[1 \wedge |\tilde{\vartheta}_t - \tilde{\vartheta}_s|]\big\} \leq K_T |t-s|^H. \tag{2.5}$$

4. *The processes W and $\hat{W}$ are independent standard $\mathbb{F}$-Brownian motions and $\bar{\boldsymbol{W}}$ is a four-dimensional $\mathbb{F}$-Brownian motion that is jointly Gaussian with $(W, \hat{W})$. The components of $\bar{\boldsymbol{W}}$ may depend on each other and on $(W, \hat{W})$.*
5. *We have*

$$\begin{aligned} g(t) &= g_H(t) + g_0(t), \quad g^\eta(t) = g_{H_\eta}(t) + g_0^\eta(t), \quad g^{\hat{\eta}}(t) = g_{H_{\hat{\eta}}}(t) + g_0^{\hat{\eta}}(t), \\ \tilde{g}(t) &= g_{\tilde{H}}(t) + \tilde{g}_0(t), \quad \tilde{g}^\eta(t) = g_{\tilde{H}_\eta}(t) + \tilde{g}_0^\eta(t), \quad \tilde{g}^{\hat{\eta}}(t) = g_{\tilde{H}_{\hat{\eta}}}(t) + \tilde{g}_0^{\hat{\eta}}(t), \end{aligned} \tag{2.6}$$

where

$$g_H(t) = K_H^{-1} t_+^{H-1/2}, \qquad K_H = \frac{\Gamma(H + \frac{1}{2})}{\sqrt{\sin(\pi H)\Gamma(2H+1)}}, \tag{2.7}$$

and $H_\eta, H_{\hat{\eta}} \in (0, \frac{1}{2}]$, $\tilde{H}, \tilde{H}_\eta, \tilde{H}_{\hat{\eta}} \in [H, \frac{1}{2}]$ and $g_0, g_0^\eta, g_0^{\hat{\eta}}, \tilde{g}_0, \tilde{g}_0^\eta, \tilde{g}_0^{\hat{\eta}} \in C^1([0,\infty))$ are functions vanishing at $t = 0$.

Let us comment on the conditions imposed in Assumption CLT. Except for the parameter H, the assumptions on x, c, η and $\hat{\eta}$ are fully nonparametric and designed in such a way that it contains the rough Heston model [17, 18] as an example, which is a particular important one as it is founded in the microstructure of financial markets [16, 29]. Note that we allow c, η and $\hat{\eta}$ to have both a usual (differentiable) and a rough (non-differentiable) drift. Moreover, by considering W, $\hat{W}$ and $\bar{\boldsymbol{W}}$, we allow for the most general dependence between the Brownian motions driving x, c, η, $\hat{\eta}$. Also note that H_η and $H_{\hat{\eta}}$ are not coupled with H, so the VoV processes η and $\hat{\eta}$ can be much rougher than the volatility process c itself.

We should also mention that, because of the various g_0-functions in (2.6), the kernels in (2.2) and (2.3) are only specified around $t=0$. In particular, H, H_η and $H_{\hat{\eta}}$ are parameters of roughness and are not related to long-range dependence / long-memory / persistence. This distinction is important as [9, 36, 37] point out.

If c was directly observable, a classical way to feasibly estimate H would be to prove a joint CLT for *realized autocovariances* $\delta_n^{1-2H}\sum_{i=1}^{[T/\delta_n]-\ell}\delta_i^n c\,\delta_{i+\ell}^n c$ with different values of $\ell\in\mathbb{N}_0$ and then to obtain an estimator of H from the ratio of two such functionals; see [6, 11, 12, 15, 24, 26, 33]. Since we do not observe c, we first consider spot volatility estimators

$$(2.8)\qquad \hat{c}^n_{t,s}=\frac{1}{k_n\delta_n}\hat{C}^n_{t,s},\quad \hat{C}^n_{t,s}=\sum_{i=[t/\delta_n]}^{[(t+s)/\delta_n]-1}(\delta_i^n x)^2,\quad \delta_i^n x=x_{i\delta_n}-x_{(i-1)\delta_n},$$

where $k_n\in\mathbb{N}$ and $k_n\sim\theta\delta_n^{-\kappa}$ for some $\kappa,\theta>0$. Then we form realized autocovariances of these spot volatility estimators by defining

$$(2.9)\qquad \begin{aligned}\tilde{V}_t^{n,\ell,k_n}=(k_n\delta_n)^{1-2H}\frac{1}{k_n}&\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\left(\hat{c}^n_{(i+k_n)\delta_n,k_n\delta_n}-\hat{c}^n_{i\delta_n,k_n\delta_n}\right)\\&\times\left(\hat{c}^n_{(i+(\ell+1)k_n)\delta_n,k_n\delta_n}-\hat{c}^n_{(i+\ell k_n)\delta_n,k_n\delta_n}\right)\end{aligned}$$

for $\ell\geq 0$. Note that we write $[x]$ and $\{x\}$ for the integer and fractional part of x, respectively. The normalization in the last line is chosen in such a way that $\tilde{V}_t^{n,\ell,k_n}$ converges in probability. In the semimartingale context (with $H=\frac{1}{2}$ and $\ell=0$), the functional $\tilde{V}_t^{n,0,k_n}$ was used in [40] to estimate the integrated VoV process $\int_0^t(\eta_s^2+\hat{\eta}_s^2)ds$ (see also [23, 32]). Still in the semimartingale framework, functionals similar to (2.9) have also been investigated in the literature to estimate the leverage effect; see [1, 4, 5, 30, 39, 41].

To state a CLT for $\tilde{V}^{n,\ell,k_n}$ for $H<\frac{1}{2}$, we have to introduce some additional notation: for $n\in\mathbb{N}$, $h>0$ and a function $f:\mathbb{R}\to\mathbb{R}$, we define the forward and central difference operators by

$$\Delta_h^n f(t)=\sum_{i=0}^n(-1)^{n-i}\binom{n}{i}f(t+ih),\qquad \delta_h^n f(t)=\sum_{i=0}^n(-1)^i\binom{n}{i}f(x+(\tfrac{n}{2}-i)h),$$

respectively. For $n=1$, we simply write $\Delta_h f(t)=\Delta_h^1 f(t)=f(t+h)-f(t)$ and $\delta_h f(t)=\delta_h^1 f(t)=f(t+\frac{h}{2})-f(t-\frac{h}{2})$. Moreover, given $\alpha\in\mathbb{R}$, we use the shorthand notation $\Delta_h^n t_+^\alpha$ or $\Delta_h^n|t|^\alpha$ for $\Delta_h^n f(t)$ where $f(t)=t_+^\alpha$ or $f(t)=|t|^\alpha$ ($\delta_h^n t_+^\alpha$ and $\delta_h^n|t|^\alpha$ are used similarly). Finally, for any $d\in\mathbb{N}$, we use $\stackrel{st}{\Longrightarrow}$ to denote functional stable convergence in law in the space of càdlàg functions $[0,\infty)\to\mathbb{R}^d$ equipped with the local uniform topology. The following CLT is the main technical result of this paper.

THEOREM 2.1. *Let $d \in \mathbb{N}$ and $\ell_1, \dots, \ell_d \geq 2$ be integers. Furthermore, consider deterministic integer sequences $(k_n^{(1)})_{n\in\mathbb{N}}, \dots, (k_n^{(d)})_{n\in\mathbb{N}}$ such that for some $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$ and $\theta_1, \dots, \theta_d \in (0, \infty)$ we have $k_n^{(j)} \sim \theta_j \delta_n^{-\kappa}$ for all $j = 1, \dots, d$. For each $j = 1, \dots, d$, let*

$$\mathcal{Z}_t^{n,j} = \delta_n^{-(1-\kappa)/2}(\tilde{V}_t^{n,\ell_j,k_n^{(j)}} - V_t^{\ell_j} - \mathcal{A}_t^{n,\ell_j,k_n^{(j)}}), \tag{2.10}$$

where for $\ell \geq 2$, we define

$$V_t^\ell = \Phi_\ell^H \int_0^t (\eta_s^2 + \hat{\eta}_s^2) ds \tag{2.11}$$

with

$$\begin{aligned} \Phi_\ell^H &= \frac{\delta_1^4 |\ell|^{2H+2}}{2(2H+1)(2H+2)} \\ &= \frac{(\ell+2)^{2H+2} - 4(\ell+1)^{2H+2} + 6\ell^{2H+2} - 4(\ell-1)^{2H+2} + (\ell-2)^{2H+2}}{2(2H+1)(2H+2)} \end{aligned} \tag{2.12}$$

and for a general integer sequence k_n,

$$\begin{aligned} \mathcal{A}_t^{n,\ell,k_n} = -\frac{2K_H^{-1}}{H+\frac{1}{2}}(k_n\delta_n)^{-1/2-H} \int_0^t \frac{1}{k_n} \sum_{i=0}^{k_n-1} \Delta_1^3(\ell - 1 - \tfrac{i+\{u/\delta_n\}}{k_n})_+^{H+1/2} \\ \times \int_{[u/\delta_n]\delta_n}^u \sigma_v dW_v (\sigma_u \eta_u - \sigma_{[u/\delta_n]\delta_n} \eta_{[u/\delta_n]\delta_n}) du. \end{aligned} \tag{2.13}$$

Under Assumption CLT, the process $\mathcal{Z}_t^n = (\mathcal{Z}_t^{n,1}, \dots, \mathcal{Z}_t^{n,d})^T$ satisfies the joint CLT

$$\mathcal{Z}^n \stackrel{st}{\Longrightarrow} \mathcal{Z}, \tag{2.14}$$

where $\mathcal{Z} = ((\mathcal{Z}_t^1, \dots, \mathcal{Z}_t^d)^T)_{t\geq 0}$ is a continuous $\mathbb{R}^d$-valued process that is defined on a very good filtered extension $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}, \bar{\mathbb{P}})$ of the original probability space (see e.g. [27, Chapter 2.1.4]) and conditionally on $\mathcal{F}$ is a centered Gaussian process with independent increments and $\mathcal{F}$-conditional covariance function

$$\mathcal{C}_t^{jj'} = \bar{\mathbb{E}}[\mathcal{Z}_t^j \mathcal{Z}_t^{j'} \mid \mathcal{F}] = \sum_{\nu=1}^3 \gamma_\nu^{\ell_j,\theta_j,\ell_{j'},\theta_{j'}}(H)\Gamma_\nu(t). \tag{2.15}$$

In the last line,

$$\Gamma_1(t) = \int_0^t \sigma_s^8 ds, \quad \Gamma_2(t) = \int_0^t (\eta_s^2 + \hat{\eta}_s^2)^2 ds, \quad \Gamma_3(t) = \int_0^t \sigma_s^4(\eta_s^2 + \hat{\eta}_s^2) ds \tag{2.16}$$

and for arbitrary $\ell, \ell' \geq 2$ and $\theta, \theta' \in (0, \infty)$,

$$\begin{aligned} \gamma_1^{\ell,\theta,\ell',\theta'}(H) &= \frac{\delta_\theta^4 \delta_{\theta'}^4 |\ell\theta - \ell'\theta'|^3}{3(\theta\theta')^{2H+2}} \mathbf{1}\left\{\kappa = \frac{2H}{2H+1}\right\}, \\ \gamma_2^{\ell,\theta,\ell',\theta'}(H) &= \frac{\Gamma(1+2H)^2(1 - 1/\cos(2\pi H))}{4\Gamma(6+4H)(\theta\theta')^{2H+2}} \\ &\qquad \times \delta_\theta^4 \delta_{\theta'}^4 \left[|\ell\theta - \ell'\theta'|^{4H+5} + |\ell\theta + \ell'\theta'|^{4H+5}\right], \\ \gamma_3^{\ell,\theta,\ell',\theta'}(H) &= -\frac{\delta_\theta^4 \delta_{\theta'}^4 [|\ell\theta + \ell'\theta'|^{2H+4} + |\ell\theta - \ell'\theta'|^{2H+4}]}{8(H+\frac{1}{2})(H+1)(H+\frac{3}{2})(H+2)(\theta\theta')^{2H+2}} \mathbf{1}\left\{\kappa = \frac{2H}{2H+1}\right\}. \end{aligned} \tag{2.17}$$

If $H = \frac{1}{4}$, $\gamma_2^{\ell,\theta,\ell',\theta'}(H)$ *is defined via continuous extension by*

$$\gamma_2^{\ell,\theta,\ell',\theta'}(\tfrac{1}{4}) = \frac{\delta_\theta^4 \delta_{\theta'}^4 \left[|\ell\theta - \ell'\theta'|^6 \log|\ell\theta - \ell'\theta'| + |\ell\theta + \ell'\theta'|^6 \log|\ell\theta + \ell'\theta'|\right]}{5760(\theta'\theta')^{5/2}}. \tag{2.18}$$

A few remarks are in order. Theorem 2.1 is a joint functional stable CLT for d realized autocovariances of the form (2.9) with potentially different lags ℓ_j and sequences $k_n^{(j)}$ that are of the same asymptotic order $\delta_n^{-\kappa}$ but potentially with different constants θ_j. We include this multivariate CLT not for the sole purpose of pursing utmost generality but really because we need it in Section 4, when we construct a rate-optimal estimator of H. For technical reasons, we need $\ell \geq 3$ in Section 4, which is why we only consider $\ell \geq 2$ in Theorem 2.1. (If $\ell = 0, 1$, an additional dominating bias term appears; since we do not use this result, we refrain from stating it.) The upper bound on κ could be relaxed to some extent (we will not need this), but the lower bound cannot. Since taking the lower bound $\kappa = \frac{2H}{2H+1}$ yields the optimal rate of convergence given in (1.3), one might be tempted to take a shortcut by just proving Theorem 2.1 for that value of κ, but unfortunately, we will need Theorem 2.1 with a general κ in Section 4. An informal argument why (1.3) is the optimal rate is given after Equation (3.10) below; a formal proof is the subject of our companion paper [13].

For any value of $H \in (0, \frac{1}{2}]$, the functional $\tilde{V}_t^{n,\ell,k_n}$ converges in probability to the law of large numbers (LLN) limit V_t^ℓ given in (2.11), which is given by integrated VoV times a constant Φ_ℓ^H (given in (2.12)). It is the dependence of this constant on H that allows us to construct an estimator of H from $\tilde{V}_t^{n,\ell,k_n}$. If $\kappa > \frac{1}{2H+2}$, the proof below (more precisely, Lemma B.3) shows that $\mathcal{A}_t^{n,\ell,k_n} = o_{\mathbb{P}}(\delta_n^{(1-\kappa)/2})$, so $\tilde{V}_t^{n,\ell,k_n}$ satisfies a CLT with rate $\delta_n^{-(1-\kappa)/2}$ by (2.14). For the optimal $\kappa = \frac{2H}{2H+1}$, this is true if and only if $H > \frac{1}{4}(\sqrt{5} - 1) \approx 0.3090$. For other values of κ (in particular, for small H if we take the optimal κ), the bias term $\mathcal{A}_t^{n,\ell,k_n}$ does not converge to 0 fast enough. Even worse, we were not able to find a debiasing statistic that converges to $\mathcal{A}_t^{n,\ell,k_n}$ sufficiently fast. This is why we have to resort to a nonstandard debiasing procedure in Section 4. The distribution of the limit $\mathcal{Z}$ is mixed normal, with a fully explicit covariance function $\mathcal{C}_t$. To make this CLT feasible, we exhibit consistent estimators of $\mathcal{C}_t$ in Proposition 4.4.

The next section is devoted to the main ideas in the proof of Theorem 2.1. The reader who wishes to first understand how this limit theory can be applied to feasible estimation of H can first jump to Section 4.

3. Proof of Theorem 2.1. The proof of Theorem 2.1 essentially consists of two parts: an approximation step (see Section 3.1), where we isolate terms that contribute to the limit $\mathcal{Z}$ in (2.14), and a CLT step (see Section 3.2), where we actually prove their stable convergence in law to $\mathcal{Z}$.

Let us start with a remark about drifts: by the stochastic and ordinary Fubini theorem,

$$\int_0^t g_0(t-s)\eta_s dW_s = \int_0^t \int_s^t g_0'(r-s) dr \eta_s dW_s = \int_0^t \int_0^r g_0'(r-s)\eta_s dW_s dr,$$

$$\int_0^t \tilde{g}_0(t-s)\tilde{\eta}_s ds = \int_0^t \int_s^t \tilde{g}_0'(r-s) dr \tilde{\eta}_s ds = \int_0^t \int_0^r \tilde{g}_0'(r-s)\tilde{\eta}_s ds dr,$$

and similarly for other integrals, so we can rewrite (2.2) and (2.3) as

$$\begin{aligned} &c_t = c_0 + A_t + \int_0^t g_H(t-s)\boldsymbol{\eta}_s d\boldsymbol{W}_s, \qquad \eta_t^2 = \eta_0^2 + A_t^\eta + \int_0^t g_{H_\eta}(t-s)\boldsymbol{\theta}_s d\bar{\boldsymbol{W}}_s, \\ &\hat{\eta}_t^2 = \hat{\eta}_0^2 + A_t^{\hat{\eta}} + \int_0^t g_{H_{\hat{\eta}}}(t-s)\boldsymbol{\vartheta}_s d\bar{\boldsymbol{W}}_s, \end{aligned} \tag{3.1}$$

where $\boldsymbol{\eta}_t=(\eta_t,\hat\eta_t)$, $\boldsymbol{W}_t=(W_t,\hat W_t)^T$ and

(3.2)
$$A_t=\int_0^t\left(a_s+\int_0^s g_0'(s-r)\boldsymbol{\eta}_r d\boldsymbol{W}_r+\int_0^s \tilde g_0'(s-r)\tilde\eta_r dr\right)ds+\int_0^t g_{\tilde H}(t-s)\tilde\eta_s ds,$$
$$A_t^\eta=\int_0^t\left(a_s^\eta+\int_0^s (g_0^\eta)'(s-r)\boldsymbol{\theta}_r d\bar{\boldsymbol{W}}_r+\int_0^s (\tilde g_0^\eta)'(s-r)\tilde\theta_r dr\right)ds+\int_0^t g_{\tilde H_\eta}(t-s)\tilde\theta_s ds,$$
$$A_t^{\hat\eta}=\int_0^t\left(a_s^{\hat\eta}+\int_0^s (g_0^{\hat\eta})'(s-r)\boldsymbol{\vartheta}_r d\bar{\boldsymbol{W}}_r+\int_0^s (\tilde g_0^{\hat\eta})'(s-r)\tilde\vartheta_r dr\right)ds+\int_0^t g_{\tilde H_{\hat\eta}}(t-s)\tilde\vartheta_s ds.$$

In the last display, the processes in parentheses are all locally bounded, and so are $\tilde\eta$, $\tilde\theta$ and $\tilde\vartheta$. Therefore, there is no loss of generality to assume

$$g_0\equiv\tilde g_0\equiv g_0^\eta\equiv\tilde g_0^\eta\equiv g_0^{\hat\eta}\equiv\tilde g_0^{\hat\eta}\equiv 0,\qquad \begin{cases}\tilde H,\tilde H_\eta,\tilde H_{\hat\eta}\in(0,\frac12) & \text{if } H\in(0,\frac12),\\ \tilde\eta\equiv\tilde\theta\equiv\tilde\vartheta\equiv 0 & \text{if } H=\frac12.\end{cases}\tag{3.3}$$

In addition, as it is usual when infill asymptotics are considered, Assumption CLT can be localized (cf. [27, Lemma 4.4.9]). Therefore, there is no loss of generality if we assume the following strengthened hypotheses.

ASSUMPTION CLT'. *In addition to Assumption CLT, we have* (3.3) *and there is a deterministic constant* $K\in(0,\infty)$ *such that*

$$\sup_{t\in[0,\infty)}\left\{|a_t|+|a_t^\eta|+|a_t^{\hat\eta}|+|b_t|+|\tilde\eta_t|+|\tilde\theta_t|+|\tilde\vartheta_t|+|\boldsymbol{\theta}_t|+|\boldsymbol{\vartheta}_t|\right\}<K\quad a.s.\tag{3.4}$$

In particular, all processes appearing in (2.1), (2.2) *and* (2.3) *have uniformly bounded moments of all orders. In addition, for all* $p>0$, *there is a constant* $K_p\in(0,\infty)$ *such that*

$$\lim_{h\to 0}\sup_{s,t\in[0,\infty):|s-t|\le h}\left\{\mathbb{E}[|a_t-a_s|^p]+\mathbb{E}[|b_t-b_s|^p]\right\}=0\tag{3.5}$$

and

$$\sup_{s,t\in[0,\infty)}\left\{\mathbb{E}[|\tilde\eta_t-\tilde\eta_s|^p]^{1/p}+\mathbb{E}[|\tilde\theta_t-\tilde\theta_s|^p]^{1/p}+\mathbb{E}[|\tilde\vartheta_t-\tilde\vartheta_s|^p]^{1/p}\right\}\le K_p|t-s|^H.\tag{3.6}$$

3.1. *Main decomposition and approximations.* Since the arguments can be applied component by component, there is no loss of generality to assume $d=1$ in this subsection. For brevity, we also write $\ell=\ell_1$, $k_n=k_n^{(1)}$, $\theta=\theta^{(1)}$ and $\tilde V_t^{n,\ell}=\tilde V_t^{n,\ell,k_n}$. In a first step, write

$$\hat c^n_{i\delta_n,k_n\delta_n}=J^n_{1,i}+J^n_{2,i},\tag{3.7}$$

where

$$J^n_{1,i}=\frac{1}{k_n\delta_n}\sum_{j=0}^{k_n-1}\left((\delta^n_{i+j}x)^2-\int_{(i+j-1)\delta_n}^{(i+j)\delta_n}c_s ds\right),$$
$$J^n_{2,i}=\frac{1}{k_n\delta_n}\sum_{j=0}^{k_n-1}\int_{(i+j-1)\delta_n}^{(i+j)\delta_n}c_s ds=\frac{1}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i-1+k_n)\delta_n}c_s ds=\frac{C_{(i-1+k_n)\delta_n}-C_{(i-1)\delta_n}}{k_n\delta_n},$$

and $C_t=\int_0^t c_s ds=\int_0^t\sigma_s^2 ds$ is the integrated volatility. The decomposition (3.7) shows that the spot volatility estimator $\hat c^n_{i\delta_n,k_n\delta_n}$ is first and foremost an estimator of $J^n_{2,i}$, a local average of spot volatility (with $J^n_{1,i}$ being the estimation error). With this decomposition, we have

$$
\begin{aligned}
&\big(\hat{c}^n_{(i+k_n)\delta_n,k_n\delta_n} - \hat{c}^n_{i\delta_n,k_n\delta_n}\big)\big(\hat{c}^n_{(i+(\ell+1)k_n)\delta_n,k_n\delta_n} - \hat{c}^n_{(i+\ell k_n)\delta_n,k_n\delta_n}\big) \\
&\quad = Z^{n,\ell}_{1,i} + Z^{n,\ell}_{2,i} + Z^{n,\ell}_{3,i} + Z'^{n,\ell}_{3,i},
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
Z^{n,\ell}_{1,i} &= (J^n_{1,i+k_n} - J^n_{1,i})(J^n_{1,i+(\ell+1)k_n} - J^n_{1,i+\ell k_n}), \\
Z^{n,\ell}_{2,i} &= (J^n_{2,i+k_n} - J^n_{2,i})(J^n_{2,i+(\ell+1)k_n} - J^n_{2,i+\ell k_n}), \\
Z^{n,\ell}_{3,i} &= (J^n_{1,i+k_n} - J^n_{1,i})(J^n_{2,i+(\ell+1)k_n} - J^n_{2,i+\ell k_n}), \\
Z'^{n,\ell}_{3,i} &= (J^n_{2,i+k_n} - J^n_{2,i})(J^n_{1,i+(\ell+1)k_n} - J^n_{1,i+\ell k_n}).
\end{aligned}
$$

Correspondingly, we obtain the decomposition

$$
\tilde{V}^{n,\ell}_t = Z^{n,\ell}_1(t) + Z^{n,\ell}_2(t) + Z^{n,\ell}_3(t) + Z'^{n,\ell}_3(t), \tag{3.9}
$$

where

(3.10)

$$
\begin{aligned}
Z^{n,\ell}_{1|2}(t) &= \frac{(k_n\delta_n)^{1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} (J^n_{1|2,i+k_n} - J^n_{1|2,i})(J^n_{1|2,i+(\ell+1)k_n} - J^n_{1|2,i+\ell k_n}), \\
Z^{n,\ell}_3(t) &= \frac{(k_n\delta_n)^{1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} (J^n_{1,i+k_n} - J^n_{1,i})(J^n_{2,i+(\ell+1)k_n} - J^n_{2,i+\ell k_n}), \\
Z'^{n,\ell}_3(t) &= \frac{(k_n\delta_n)^{1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} (J^n_{2,i+k_n} - J^n_{2,i})(J^n_{1,i+(\ell+1)k_n} - J^n_{1,i+\ell k_n}),
\end{aligned}
$$

and $1 \mid 2$ means that we can take either 1 or 2 (consistently for the whole line).

We can now give an informal argument why $\kappa = \frac{2H}{2H+1}$ is the optimal window size in our estimation procedure. Note that $(k_n\delta_n)^{-H}(J^n_{2,i+k_n} - J^n_{2,i})$ is a normalized second-order increment of C over an interval of length $k_n\delta_n$. Therefore, $Z^{n,\ell}_2(t)$ is nothing else but the normalized second-order quadratic variation of C (computed with a lag ℓ). By definition, C is the integral of a fractional process. It is well known from [6, 7] that the normalized higher-order quadratic variation of a fractional process, computed over a step size of $k_n\delta_n$, converges to a limit at rate $(k_n\delta_n)^{-1/2}$, for all $H \in (0, \frac{1}{2}]$. Of course, C is not a fractional process but rather its integral. Our analysis of $Z^{n,\ell}_2(t)$ below shows that the rate of convergence remains unchanged. In other words, if we were able to observe C_t directly, we would have chosen $k_n = 1$, and the optimal rate of convergence would be $\delta_n^{-1/2}$.

But we do not observe C_t directly, which means that we have an estimation error of C_t in the form of $J^n_{1,i}$. By the integration by parts formula for semimartingales,

$$
\begin{aligned}
J^n_{1,i} &= \frac{2}{k_n\delta_n} \sum_{j=0}^{k_n-1} \int_{(i+j-1)\delta_n}^{(i+j)\delta_n} (x_s - x_{(i+j-1)\delta_n})dx_s \\
&= \frac{2}{k_n\delta_n} \int_{(i-1)\delta_n}^{(i+k_n-1)\delta_n} (x_s - x_{[s/\delta_n]\delta_n})dx_s,
\end{aligned} \tag{3.11}
$$

which shows that $J^n_{1,i}$ is a term of order $k_n^{-1/2}$, uniformly in i. Moreover, if we neglect the drift in x (which, of course, is fine as we will see below), then $J^n_{1,i}$ is a martingale increment with step size $k_n\delta_n$, so $J^n_{1,i+k_n} - J^n_{1,i}$ will be a martingale increment, too, just with step size $2k_n\delta_n$. Because $\ell \geq 2$, if we take the product with $J^n_{1,i+(\ell+1)k_n} - J^n_{1,i+\ell k_n}$ in $Z^{n,\ell}_1(t)$ and apply integration by parts one more time, we only get martingale increments (with step size $O(k_n\delta_n)$) but no quadratic variation / drift part. Therefore, the sum over i in $Z^{n,\ell}_1(t)$ will be of order $O_{\mathbb{P}}((k_n\delta_n)^{-1/2})$ and $Z^{n,\ell}_1(t)$ will be of order $O_{\mathbb{P}}((k_n\delta_n)^{1/2-2H}k_n^{-1})$. Thus, contrary to $Z^n_2(t)$, the error term $Z^n_1(t)$ is small if k_n is large. Of course, this is expected, for the larger the window size k_n is, the better integrated volatility is approximated by realized variance. Nonchalantly ignoring $Z^{n,\ell}_3$ and $Z'^{n,\ell}_3$ for the moment, we obtain the optimal convergence rate if κ is chosen such that $(k_n\delta_n)^{1/2}$ and $(k_n\delta_n)^{1/2-2H}k_n^{-1}$ are of the same order. This precisely gives $\kappa = \frac{2H}{2H+1}$ and the optimal rate of convergence of $\delta_n^{-1/(4H+2)}$.

This informal argument clearly does not *prove* that $\delta_n^{-1/(4H+2)}$ is the best possible rate. But in our companion paper [13], we actually show that this is the case in parametric rough volatility models. For now, let us make two more remarks before we return to the main line of the proof. First, in the above argument, we have neglected $Z^{n,\ell}_3$ and $Z'^{n,\ell}_3$, which are mixed terms and ought to be no worse than $Z^{n,\ell}_1$ and $Z^{n,\ell}_2$ in terms of rate. This is indeed true for the fluctuations, but with the caveat that with the optimal κ, $Z^{n,\ell}_3$ comes with an asymptotic bias term (given by (2.13)) that dominates the CLT if (and only if) $H \leq \frac{1}{4}(\sqrt{5}-1) \approx 0.3090$.

Second, notice that the optimal rate of convergence of $\tilde{V}^{n,\ell,k_n}_t$ (after removing the bias) is $\delta_n^{-1/(4H+2)}$, which approaches $\delta_n^{-1/2}$ as $H \downarrow 0$ and $\delta_n^{-1/4}$ as $H \uparrow \frac{1}{2}$ and is monotone in between. The fact that the convergence rate is faster for small H compared to the semi-martingale case $H = \frac{1}{2}$ (see [40]) seems counter-intuitive since the spot volatility estimator $\hat{c}^n_{t,s}$ should be less precise if c is rough. This is true, and if one decided to first estimate c_t and then extract H from variations of c_t, the resulting estimator would definitely perform poorly for small H. But there is no need to estimate c_t: recall from the discussion above that $\hat{c}^n_{i\delta_n,k_n\delta_n}$ is mainly an estimator of $J^n_{2,i} = (k_n\delta_n)^{-1}(C_{(i-1+k_n)\delta_n} - C_{(i-1)\delta_n})$, an increment of *integrated* volatility, which as we shall see below contains as much information about H as an increment of c_t. So the faster convergence rate of $\tilde{V}^{n,\ell,k_n}_t$ for small H is really due to the fact that the total error $Z^{n,\ell}_1(t)$ in pre-estimating $Z^{n,\ell}_2(t)$ is $(k_n\delta_n)^{1/2-2H}k_n^{-1}$ and hence *smaller* for small H for any given window size k_n.

The following three propositions determine the main parts of $Z^{n,\ell}_1(t)$, $Z^{n,\ell}_2(t)$, $Z^{n,\ell}_3(t)$ and $Z'^{n,\ell}_3(t)$ that contribute to the CLT.

PROPOSITION 3.1. *Let the assumptions be as in Theorem 2.1. Then for all* $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$ *and integer sequences* $k_n \sim \theta\delta_n^{-\kappa}$ *with* $\theta > 0$*, the following convergence holds:*

$$(k_n\delta_n)^{-1/2}(Z^{n,\ell}_1(t) - M^{n,\ell}_1(t)) \overset{L^1}{\Longrightarrow} 0, \tag{3.12}$$

where using the notations $y_t = \int_0^t \sigma_s dW_s$ *and* $\chi(t) = -1$ *for* $t \in [0, \frac{1}{2}]$ *and* $\chi(t) = 1$ *for* $t \in (\frac{1}{2}, 1]$*, we define*

$$M^{n,\ell}_1(t) = \frac{4(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})dy_s$$
$$\times \int_{(i+\ell k_n-1)\delta_n}^{(i-1+(\ell+2)k_n)\delta_n} \chi(\tfrac{[s/\delta_n]-i-\ell k_n+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})dy_s. \tag{3.13}$$

If $\kappa \in (\frac{2H}{2H+1}, \frac{1}{2}]$*, we further have*

$$(k_n\delta_n)^{-1/2} M_1^{n,\ell}(t) \overset{L^1}{\Longrightarrow} 0. \tag{3.14}$$

PROPOSITION 3.2. *Under the assumptions of Theorem 2.1, we have for all* $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$ *and integer sequences* $k_n \sim \theta\delta_n^{-\kappa}$ *with* $\theta > 0$ *that*

$$(k_n\delta_n)^{-1/2}(Z_2^{n,\ell}(t) - V_t^\ell - M_2^{n,\ell}(t)) \overset{L^1}{\Longrightarrow} 0,$$

where

$$\begin{aligned} M_2^{n,\ell}(t) = \frac{(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_0^{(i-1+(\ell+2)k_n)\delta_n} \int_0^r \Big\{ \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - r) \\ \times \Delta^2_{k_n\delta_n} G_H((i+\ell k_n - 1)\delta_n - u) + \Delta^2_{k_n\delta_n} G_H((i+\ell k_n - 1)\delta_n - r) \\ \times \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - u) \Big\} \boldsymbol{\eta}_u d\boldsymbol{W}_u \boldsymbol{\eta}_r d\boldsymbol{W}_r \end{aligned} \tag{3.15}$$

and

$$G_H(t) = \frac{K_H^{-1}}{H + \frac{1}{2}} t_+^{H+1/2}. \tag{3.16}$$

PROPOSITION 3.3. *Under the assumptions of Theorem 2.1, we have for all* $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$ *and integer sequences* $k_n \sim \theta\delta_n^{-\kappa}$ *with* $\theta > 0$ *that*

$$(k_n\delta_n)^{-1/2}(Z_3^{n,\ell}(t) - \mathcal{A}_t^{n,\ell} - M_{31}^{n,\ell}(t) - M_{32}^{n,\ell}(t)) \overset{L^1}{\Longrightarrow} 0,$$

$$(k_n\delta_n)^{-1/2}(Z_3'^{n,\ell}(t) - M_3'^{n,\ell}(t)) \overset{L^1}{\Longrightarrow} 0,$$

where $\mathcal{A}_t^{n,\ell} = \mathcal{A}_t^{n,\ell,k_n}$ *and*

$$\begin{aligned} M_{31}^{n,\ell}(t) = \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n}) \\ \times \int_0^s \Delta^2_{k_n\delta_n} G_H((i-1+\ell k_n)\delta_n - r) \boldsymbol{\eta}_r d\boldsymbol{W}_r dy_s, \end{aligned}$$

$$\begin{aligned} M_{32}^{n,\ell}(t) = \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_0^{(i-1+(\ell+2)k_n)\delta_n} \Delta^2_{k_n\delta_n} G_H((i-1+\ell k_n)\delta_n - r) \\ \times \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n \wedge r} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n}) dy_s \boldsymbol{\eta}_r d\boldsymbol{W}_r \end{aligned}$$

and

$$\begin{aligned} M_3'^{n,\ell}(t) = \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_{(i+\ell k_n-1)\delta_n}^{(i-1+(\ell+2)k_n)\delta_n} \chi(\tfrac{[s/\delta_n]-i-\ell k_n+1}{2k_n-1}) \\ \times (y_s - y_{[s/\delta_n]\delta_n}) dy_s \int_0^{(i-1+2k_n)\delta_n} \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - r) \boldsymbol{\eta}_r d\boldsymbol{W}_r. \end{aligned} \tag{3.17}$$

If $\kappa \in (\frac{2H}{2H+1}, \frac{1}{2}]$*, we also have* $(k_n\delta_n)^{-1/2}(M_{31}^{n,\ell}(t) + M_{32}^{n,\ell}(t) + M_3'^{n,\ell}(t)) \overset{L^1}{\Longrightarrow} 0.$

We now give an overview of the proof of Proposition 3.2, with details delegated to Section A. Note that $Z_2^{n,\ell}(t)$ is the only term that contributes to the LLN and, furthermore, is the only term that contributes to the CLT for any $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$. The other three terms $Z_1^{n,\ell}(t)$, $Z_3^{n,\ell}(t)$ and $Z_3'^{n,\ell}(t)$ never contribute to the LLN and, unless $\kappa = \frac{2H}{2H+1}$, do not contribute to the CLT, either. Also, the approximations we need to make for them are mostly similar to those for $Z_2^{n,\ell}(t)$. This is why we postpone the whole proof of Propositions 3.1 and 3.3 to Section B.

By (3.1), we have for any $t \geq 0$ that

$$\int_t^{t+k_n\delta_n} c_s ds = c_0 k_n \delta_n + \int_t^{t+k_n\delta_n} A_s ds + \int_0^{t+k_n\delta_n} \Delta_{k_n\delta_n} G_H(t-r)\boldsymbol{\eta}_r d\boldsymbol{W}_r.$$

Consequently,

$$J^n_{2,i+k_n\delta_n} - J^n_{2,i} = \frac{1}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i-1+k_n)\delta_n} (c_{s+k_n\delta_n} - c_s) ds = \frac{1}{k_n\delta_n}(D^n_{1,i} + D^n_{2,i}), \tag{3.18}$$

where

$$\begin{aligned} D^n_{1,i} &= \int_0^{(i-1+2k_n)\delta_n} \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - r)\boldsymbol{\eta}_r d\boldsymbol{W}_r, \\ D^n_{2,i} &= \int_{(i-1)\delta_n}^{(i-1+k_n)\delta_n} (A_{s+k_n\delta_n} - A_s) ds. \end{aligned} \tag{3.19}$$

We can safely remove the drift part $D^n_{2,i}$:

LEMMA 3.4. *Under Assumption CLT', we have* $(k_n\delta_n)^{-1/2}(Z_2^{n,\ell}(t) - \tilde{Z}_2^{n,\ell}(t)) \overset{L^1}{\Longrightarrow} 0$, *where*

$$\tilde{Z}_2^{n,\ell}(t) = (k_n\delta_n)^{-1-2H}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D^n_{1,i} D^n_{1,i+\ell k_n}. \tag{3.20}$$

Next, an application of the integration by parts formula shows that

$$\tilde{Z}_2^{n,\ell}(t) = M_{21}^{n,\ell}(t) + M_{22}^{n,\ell}(t) + Q_2^{n,\ell}(t) = M_2^{n,\ell}(t) + Q_2^{n,\ell}(t), \tag{3.21}$$

where

$$\begin{aligned} M_{21}^{n,\ell}(t) = \frac{(k_n\delta_n)^{-1-2H}}{k_n} &\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_0^{(i-1+2k_n)\delta_n} \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - r) \\ &\times \int_0^r \Delta^2_{k_n\delta_n} G_H((i+\ell k_n - 1)\delta_n - u)\boldsymbol{\eta}_u d\boldsymbol{W}_u \boldsymbol{\eta}_r d\boldsymbol{W}_r, \\ M_{22}^{n,\ell}(t) = \frac{(k_n\delta_n)^{-1-2H}}{k_n} &\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_0^{(i-1+(\ell+2)k_n)\delta_n} \Delta^2_{k_n\delta_n} G_H((i+\ell k_n - 1)\delta_n - r) \\ &\times \int_0^r \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - u)\boldsymbol{\eta}_u d\boldsymbol{W}_u \boldsymbol{\eta}_r d\boldsymbol{W}_r, \\ Q_2^{n,\ell}(t) = \frac{(k_n\delta_n)^{-1-2H}}{k_n} &\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} \int_0^{(i-1+2k_n)\delta_n} \Delta^2_{k_n\delta_n} G_H((i-1)\delta_n - r) \\ &\times \Delta^2_{k_n\delta_n} G_H((i+\ell k_n - 1)\delta_n - r)|\boldsymbol{\eta}_r|^2 dr. \end{aligned}$$

For the proof of Proposition 3.2, we only have to further consider $Q_2^{n,\ell}(t)$.

Interchanging summation and integration and factoring $k_n\delta_n$ out of $\Delta^2_{k_n\delta_n}G_H$, we obtain

$$Q_2^{n,\ell}(t) = \frac{1}{k_n}\int_0^{([t/\delta_n]-\ell k_n)\delta_n} \sum_{i=([r/\delta_n]-2k_n+2)\vee 1}^{[t/\delta_n]-(\ell+2)k_n+1} \Delta_1^2 G_H(\tfrac{i-1-r/\delta_n}{k_n}) \times \Delta_1^2 G_H(\tfrac{i-1+\ell k_n - r/\delta_n}{k_n})|\boldsymbol{\eta}_r|^2 dr. \tag{3.22}$$

Writing $r/\delta_n = [r/\delta_n] + \{r/\delta_n\}$ as the sum of its integer and fractional part and changing the index $i-1-[r/\delta_n]$ to i result in

$$Q_2^{n,\ell}(t) = \frac{1}{k_n}\int_0^{([t/\delta_n]-\ell k_n)\delta_n} \sum_{i=(1-2k_n)\vee(-[r/\delta_n])}^{[t/\delta_n]-[r/\delta_n]-(\ell+2)k_n} \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n}) \times \Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n})|\boldsymbol{\eta}_r|^2 dr. \tag{3.23}$$

The next lemma shows that we can replace the lower bound in the summation by $1-2k_n$ and the upper bound by $+\infty$.

LEMMA 3.5. *Under Assumption CLT', we have* $(k_n\delta_n)^{-1/2}(Q_2^{n,\ell}(t) - \hat{Q}_2^{n,\ell}(t)) \stackrel{L^1}{\Longrightarrow} 0$, *where*

$$\hat{Q}_2^{n,\ell}(t) = \int_0^{([t/\delta_n]-\ell k_n)\delta_n} \frac{1}{k_n}\sum_{i=1-2k_n}^{\infty} \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n})|\boldsymbol{\eta}_r|^2 dr. \tag{3.24}$$

Since $\{r/\delta_n\} \in [0,1)$, the sum over i is a Riemann sum that converges as $k_n \to \infty$ to the limit $\int_{-2}^{\infty} \Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\ell)dv$. This integral is nothing else but Φ_ℓ^H defined in (2.12).

LEMMA 3.6. *For any* $H \in (0, \frac{1}{2})$ *and* $\ell \geq 2$,

$$\Phi_\ell^H = \int_{-2}^{\infty} \Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\ell)dv. \tag{3.25}$$

As an immediate consequence, we obtain $\hat{Q}_2^{n,\ell}(t) \stackrel{L^1}{\Longrightarrow} \Phi_\ell^H \int_0^t |\boldsymbol{\eta}_r|^2 dr = V_t^\ell$, the desired LLN limit. There is only one problem: the convergence rate. Even for a smooth function (which $\Delta_1^2 G_H(v)$ is not), a Riemann sum converges to its limit only with rate k_n, which for small H and small κ (including the optimal $\kappa = \frac{2H}{2H+1}$) is much slower than the needed $(k_n\delta_n)^{-1/2}$. Nevertheless, we shall prove

LEMMA 3.7. *Under Assumption CLT', we have* $(k_n\delta_n)^{-1/2}(\hat{Q}_2^{n,\ell}(t) - V_t^\ell) \stackrel{L^1}{\Longrightarrow} 0$.

This unexpected gain in convergence rate is only possible because we have a very special Riemann sum *and* a very special process $\boldsymbol{\eta}$ in (3.24). To understand what is so particular about the former, let us exploit the periodicity of the mapping $u \mapsto \{u\}$ and change variables a few times to rewrite

$$\begin{aligned}\Phi^H_\ell &= \sum_{i=1-2k_n}^{\infty} \int_{\frac{i-1}{k_n}}^{\frac{i}{k_n}} \Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\ell)dv \\ &= \frac{1}{k_n}\sum_{i=1-2k_n}^{\infty} \int_0^1 \Delta_1^2 G_H(\tfrac{i-v}{k_n})\Delta_1^2 G_H(\tfrac{i-v}{k_n}+\ell)dv \qquad (3.26) \\ &= \frac{1}{k_n}\sum_{i=1-2k_n}^{\infty} \delta_n^{-1}\int_{([r/\delta_n])\delta_n}^{([r/\delta_n]+1)\delta_n} \Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n}+\ell)du,\end{aligned}$$

which is valid for any $r>0$ and $n\in\mathbb{N}$. Comparing the last line of the previous display with (3.24), we realize that there is no need to study how fast the sum over i approaches its limit since

$$\begin{aligned}&\hat{Q}_2^{n,\ell}(t)-V_t^\ell = \hat{Q}^{n,\ell}(t) - \Phi_\ell^H \int_0^{([t/\delta_n]-\ell k_n)\delta_n} |\boldsymbol{\eta}_u|^2du + O_{\mathbb{P}}(k_n\delta_n) \\ &= \frac{1}{k_n}\sum_{i=1-2k_n}^{\infty}\int_0^{([t/\delta_n]-\ell k_n)\delta_n}\Bigg(\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n}) \\ &\quad - \delta_n^{-1}\int_{([r/\delta_n])\delta_n}^{([r/\delta_n]+1)\delta_n}\Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n}+\ell)du\Bigg)|\boldsymbol{\eta}_r|^2dr + O_{\mathbb{P}}(k_n\delta_n).\end{aligned}$$

What matters is therefore how fast the difference in parentheses goes to 0 (as long as we obtain a bound that is an integrable function of $\frac{i}{k_n}$). With this in mind, we rewrite the last line in the previous display as

$$\begin{aligned}&\sum_{j=1}^{[t/\delta_n]-\ell k_n}\frac{1}{k_n\delta_n}\sum_{i=1-2k_n}^{\infty}\int_{(j-1)\delta_n}^{j\delta_n}\int_{(j-1)\delta_n}^{j\delta_n}\Bigg\{\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n}) \\ &\qquad - \Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{u/\delta_n\}}{k_n})\Bigg\}du|\boldsymbol{\eta}_r|^2dr + O_{\mathbb{P}}(k_n\delta_n).\end{aligned}$$

The $dudr$-double integral on the right-hand side can be split into an $\int_{(j-1)}^{j\delta_n}\int_{(j-1)\delta_n}^{r}$-part and an $\int_{(j-1)}^{j\delta_n}\int_{r}^{j\delta_n}$-part. By symmetry, the latter is equal to

$$\begin{aligned}&-\int_{(j-1)\delta_n}^{j\delta_n}\int_{(j-1)\delta_n}^{r}\Bigg\{\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n}) \\ &\qquad - \Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{u/\delta_n\}}{k_n})\Bigg\}|\boldsymbol{\eta}_u|^2dudr,\end{aligned}$$

which implies that

$$\begin{aligned}&\hat{Q}_2^{n,\ell}(t)-V_t^\ell \\ &= \sum_{j=1}^{[t/\delta_n]-\ell k_n}\frac{1}{k_n\delta_n}\sum_{i=1-2k_n}^{\infty}\int_{(j-1)\delta_n}^{j\delta_n}\int_{(j-1)\delta_n}^{r}\Bigg\{\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n}) \qquad (3.27) \\ &\qquad - \Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{u/\delta_n\}}{k_n})\Bigg\}(\eta_r^2-\eta_u^2+\hat{\eta}_r^2-\hat{\eta}_u^2)dudr + O_{\mathbb{P}}(k_n\delta_n).\end{aligned}$$

In the last line, the regularity of η^2 and $\hat{\eta}^2$ starts to play a role. If η^2 and $\hat{\eta}^2$ were just any H-Hölder regular function, the best bound we can hope for is δ_n^H, which is clearly not enough if H is small. However, this bound can be significantly improved if we have some structure on η^2 and $\tilde{\eta}^2$. This is therefore the first (and only) place in this paper where the assumption (2.3) is used. Leveraging (2.3) into a sufficiently good bound in (3.27) is still nontrivial, so we complete the proof of Lemma 3.7 in Section A.

PROOF OF PROPOSITION 3.2. Proposition 3.2 follows by combining Lemma 3.4, Equation (3.21), Lemma 3.5 and Lemma 3.7. □

3.2. *Multivariate stable convergence in law.* In a first step, we carry out a few approximations of $M_1^{n,\ell,k_n}(t)$, $M_2^{n,\ell,k_n}(t)$, $M_{31}^{n,\ell,k_n}(t)$, $M_{32}^{n,\ell,k_n}(t)$ and $M_3'^{n,\ell,k_n}(t)$. The proof can be found in Section C.

PROPOSITION 3.8. *Under the conditions of Theorem 2.1, we have*

$$\begin{aligned}
&\delta_n^{-(1-\kappa)/2} M_1^{n,\ell,k_n}(t) \approx \sum_{j=1}^{[t/\delta_n]} \zeta_1^{n,j,\ell,k_n}, \qquad \delta_n^{-(1-\kappa)/2} M_2^{n,\ell,k_n}(t) \approx \sum_{j=1}^{[t/\delta_n]} \zeta_2^{n,j,\ell,k_n},\\
&\delta_n^{-(1-\kappa)/2}(M_{31}^{n,\ell,k_n}(t) + M_{32}^{n,\ell,k_n}(t) + M_3'^{n,\ell,k_n}(t)) \approx \sum_{j=1}^{[t/\delta_n]} \zeta_3^{n,j,\ell,k_n},
\end{aligned} \tag{3.28}$$

where, with the notation $\xi(t) = ((1-3|t|) \vee (|t|-1))\mathbf{1}_{[-1,1](t)}$,

$$\begin{aligned}
\zeta_1^{n,j,\ell,k_n} = 8\delta_n^{-(1-\kappa)/2}(k_n\delta_n)^{-1-2H} \int_{(j-1)\delta_n}^{j\delta_n} \sigma^4_{(j-1)\delta_n} \int_{([s/\delta_n]-(\ell+2)k_n+1)\delta_n}^{([s/\delta_n]-(\ell-2)k_n)\delta_n}\\
\xi(\tfrac{[r/\delta_n]-[s/\delta_n]+\ell k_n}{2k_n})(W_r - W_{[r/\delta_n]\delta_n})dW_r(W_s - W_{[s/\delta_n]\delta_n})dW_s
\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}
\zeta_2^{n,j,\ell,k_n} = \delta_n^{-(1-\kappa)/2} \int_{(j-1)\delta_n}^{j\delta_n} \int_{r-k_n\delta_n^{1-\varepsilon}}^{r} \int_{-2}^{\infty} \Big\{ \Delta_1^2 G_H(v)\Delta_1^2 G_H(v + \tfrac{r-u}{k_n\delta_n} + \ell)\\
+ \Delta_1^2 G_H(v)\Delta_1^2 G_H(v + \tfrac{r-u}{k_n\delta_n} - \ell)\Big\} dv \boldsymbol{\eta}_{(j-1)\delta_n} d\boldsymbol{W}_u \boldsymbol{\eta}_{(j-1)\delta_n} d\boldsymbol{W}_r
\end{aligned} \tag{3.30}$$

and $\zeta_3^{n,j,\ell,k_n} = \zeta_{31}^{n,j,\ell,k_n} + \zeta_{32}^{n,j,\ell,k_n}$ *with*

$$\begin{aligned}
\zeta_{31}^{n,j,\ell,k_n} &= -2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2} \int_{(j-1)\delta_n}^{j\delta_n} \sigma^2_{(j-1)\delta_n}\\
&\quad\times \int_{s-k_n\delta_n^{1-\varepsilon}}^{(j-1)\delta_n} \int_0^1 \Big\{ \Delta_1^3 G_H(\tfrac{s-r}{k_n\delta_n} + \ell - u - 1)\\
&\qquad + \Delta_1^3 G_H(\tfrac{s-r}{k_n\delta_n} - \ell - u - 1)\Big\} du \boldsymbol{\eta}_{(j-1)\delta_n} d\boldsymbol{W}_r (W_s - W_{(j-1)\delta_n})dW_s,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
\zeta_{32}^{n,j,\ell,k_n} = &-2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2} \int_{(j-1)\delta_n}^{j\delta_n} \sigma^2_{(j-1)\delta_n} \int_{r-(\ell+2)k_n\delta_n}^{(j-1)\delta_n}\\
&\int_0^1 \Delta_1^3 G_H(\ell - \tfrac{r-s}{k_n\delta_n} - u - 1) du (W_s - W_{[s/\delta_n]\delta_n}) dW_s \boldsymbol{\eta}_{(j-1)\delta_n} d\boldsymbol{W}_r.
\end{aligned}$$

Now let us define

$$\zeta_t^n = \begin{pmatrix} \zeta_t^{n,\ell_1,k_n^{(1)}} \\ \vdots \\ \zeta_t^{n,\ell_d,k_n^{(d)}} \end{pmatrix} = \sum_{j=1}^{[t/\delta_n]} \begin{pmatrix} \zeta_1^{n,j,\ell_1,k_n^{(1)}} & \zeta_2^{n,j,\ell_1,k_n^{(1)}} & \zeta_3^{n,j,\ell_1,k_n^{(1)}} \\ \vdots & \vdots & \vdots \\ \zeta_1^{n,j,\ell_d,k_n^{(d)}} & \zeta_2^{n,j,\ell_d,k_n^{(d)}} & \zeta_3^{n,j,\ell_d,k_n^{(d)}} \end{pmatrix}. \tag{3.32}$$

By (3.29), (3.30) and (3.31), we see that the jth matrix on the right-hand side of (3.32) is $\mathcal{F}_{j\delta_n}$-measurable with a zero $\mathcal{F}_{(j-1)\delta_n}$-conditional expectation. In conjunction with the fact that

$$\mathcal{Z}^n \approx \zeta^n \mathbf{1}, \qquad \mathbf{1} = (1,1,1)^T,$$

which follows from Propositions 3.1, 3.2, 3.3 and 3.8, we can complete the proof of Theorem 2.1 using a stable CLT for martingale arrays (see [27, Theorem 2.2.15]) upon showing the following:

1. For any $t > 0$, $m, m' \in \{1, \dots, d\}$ and $\nu, \nu' \in \{1,2,3\}$ such that $\nu \neq \nu'$, we have

$$\sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_\nu^{n,j,\ell_m,k_n^{(m)}} \zeta_\nu^{n,j,\ell_{m'},k_n^{(m')}} \mid \mathcal{F}_{(j-1)\delta_n}] \xrightarrow{\mathbb{P}} \gamma_\nu^{\ell_m,\theta_m,\ell_{m'},\theta_{m'}}(H)\Gamma_\nu(t), \tag{3.33}$$

$$\sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_\nu^{n,j,\ell_m,k_n^{(m)}} \zeta_{\nu'}^{n,j,\ell_{m'},k_n^{(m')}} \mid \mathcal{F}_{(j-1)\delta_n}] \xrightarrow{\mathbb{P}} 0. \tag{3.34}$$

2. For any $m \in \{1, \dots, d\}$, $\nu \in \{1,2,3\}$ and $t > 0$, we have

$$\sum_{j=1}^{[t/\delta_n]} \mathbb{E}[(\zeta_\nu^{n,j,\ell_m,k_n^{(m)}})^4 \mid \mathcal{F}_{(j-1)\delta_n}] \xrightarrow{\mathbb{P}} 0. \tag{3.35}$$

3. If $N \in \{W, \hat{W}\}$ or N is a bounded martingale on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that is orthogonal in the martingale sense to both W and $\hat{W}$, then

$$\sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_\nu^{n,j,\ell_m,k_n^{(m)}}(N_{j\delta_n} - N_{(j-1)\delta_n}) \mid \mathcal{F}_{(j-1)\delta_n}] \xrightarrow{\mathbb{P}} 0. \tag{3.36}$$

The proof of these three properties will be given in Section C. This completes the proof of Theorem 2.1. □

4. Debiasing and rate-optimal inference for *H*. There are two main challenges in deriving a rate-optimal estimator of H on the basis of Theorem 2.1: first, if H is small, $\tilde{V}_t^{n,k_n,\ell}$ has a nonnegligible bias that dominates the CLT fluctuations; and second, the optimal rate to be achieved is $\delta_n^{-1/(4H+2)}$ and therefore depends on the unknown roughness parameter H itself.

In order to account for the asymptotic bias, our strategy is to consider multiple window sizes k_n and combine the resulting $\tilde{V}_t^{n,\ell,k_n}$'s in a very specific way that cancels the bias terms up to a negligible contribution. For $M \in \mathbb{N}$, let us introduce the Vandermonde matrix

$$V_M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^{-1} & 3^{-1} & \cdots & M^{-1} \\ 1 & 2^{-2} & 3^{-2} & \cdots & M^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{-(M-1)} & 3^{-(M-1)} & \cdots & M^{-(M-1)} \end{pmatrix}, \tag{4.1}$$

which has an inverse V_M^{-1} by a standard result from linear algebra. Thus, we can define

$$\tilde{w}(M) = V_M^{-1} e_M, \quad e_M = (0, \dots, 0, 1)^T, \quad w(M) = \frac{\tilde{w}(M)}{|\tilde{w}(M)|}, \tag{4.2}$$

so that $w(M)$ is the normalized last column of V_M^{-1}. The following proposition shows that a very specific linear combination of $\tilde{V}_t^{n,\ell,mk_n}$ for different m's removes the dominating part of the bias. While Theorem 2.1 only requires $\ell \geq 2$, we have to impose $\ell \geq 3$ from now on.

PROPOSITION 4.1. *Suppose that the conditions of Theorem 2.1 are satisfied with $H \in (0, \frac{1}{2}]$ and that $k_n \sim \theta\delta_n^{-\kappa}$ for some $\theta > 0$ and $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$. Furthermore, assume that $\ell \geq 3$. Defining*

$$M = M(H) = [\tfrac{1}{2} - H + \tfrac{1}{4H}] + 1, \tag{4.3}$$

we have that

$$\sum_{m=1}^{M} w(M)_m m^{1/2+H} \tilde{V}_t^{n,\ell,mk_n} - V_t^{\ell} \sum_{m=1}^{M} w(M)_m m^{1/2+H} = O_{\mathbb{P}}((k_n\delta_n)^{1/2}). \tag{4.4}$$

Of course, the left-hand side of (4.4) multiplied by $(k_n\delta_n)^{-1/2}$ satisfies a CLT, but since we do not need this in the following, we only prove the simpler version (4.4).

PROOF OF PROPOSITION 4.1. By Theorem 2.1, it suffices to show that

$$\sum_{m=1}^{M} w(M)_m m^{1/2+H} \mathcal{A}_t^{n,\ell,mk_n} = o_{\mathbb{P}}((k_n\delta_n)^{1/2}), \tag{4.5}$$

where $\mathcal{A}_t^{n,\ell,k_n}$ is defined in (2.13). For $\ell \geq 3$, the function $v \mapsto \Delta_1^3 G_H(\ell - 1 - \frac{i}{k_n} + v)$ is smooth on $[-\frac{1}{2}, 0]$,with derivatives $(\Delta_1^3 G_H)^{(j)}(\ell - 1 - \frac{i}{k_n} + v)$ that are uniformly bounded in $v \in [-\frac{1}{2}, 0]$, i, k_n and $j = 0, \dots, M$. Thus, by (2.13) and Taylor's theorem,

$$\begin{aligned}\mathcal{A}_t^{n,\ell,k_n} &= -2(k_n\delta_n)^{-1/2-H} \sum_{j=0}^{M-1} \frac{1}{j!} \int_0^t \frac{1}{k_n} \sum_{i=0}^{k_n-1} (\Delta_1^3 G_H)^{(j)}(\ell - 1 - \tfrac{i}{k_n})\big(-\tfrac{\{u/\delta_n\}}{k_n}\big)^j \\ &\qquad \times (y_u - y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u - \sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du + O_{\mathbb{P}}(k_n^{-1/2-H-M}).\end{aligned} \tag{4.6}$$

As the reader can verify, by our definition of M in (4.3), we have that $k_n^{-1/2-H-M} = o((k_n\delta_n)^{1/2})$.

Next, we recognize that the sum over i is a Riemann sum approximation of the integral $\int_0^1 (\Delta_1^3 G_H)^{(j)}(\ell - 1 - v)dv$. By the Euler–Maclaurin formula (see e.g., [31, Theorem 1]), there are finite numbers $\xi_{j,j'}^{\ell}$ such that

$$\frac{1}{k_n} \sum_{i=0}^{k_n-1} (\Delta_1^3 G_H)^{(j)}(\ell - 1 - \tfrac{i}{k_n}) = \sum_{j'=0}^{M-1} \xi_{j,j'}^{\ell} k_n^{-j'} + O(k_n^{-M}).$$

Inserting this back into (4.6), we can ignore the $O(k_n^{-M})$-term as before. In fact, we only have to keep those terms for which $j + j' \leq M - 1$. Thus, letting

$$\begin{aligned}\Xi_p^{n,\ell}(t) &= -2\delta_n^{-1/2-H} \sum_{j,j'=0}^{M-1} \mathbf{1}_{\{j+j'=p\}} \frac{\xi_{j,j'}^{\ell}}{j!} \\ &\qquad \times \int_0^t (-\{u/\delta_n\})^j (y_u - y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u - \sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du,\end{aligned}$$

we have that

$$\mathcal{A}_t^{n,\ell,k_n} = k_n^{-1/2-H} \sum_{p=0}^{M-1} \Xi_p^{n,\ell}(t) k_n^{-p} + o_{\mathbb{P}}((k_n\delta_n)^{1/2}).$$

Note that $\Xi_p^{n,\ell}(t)$ depends on δ_n but not on k_n. Therefore, applying the previous identity to mk_n for $m = 1, \ldots, M$, we arrive at the following systems of equations:

$$m^{1/2+H} \mathcal{A}_t^{n,\ell,mk_n} = \sum_{p=0}^{M-1} m^{-p} \Xi_p^{n,\ell}(t) k_n^{-1/2-H-p} + o_{\mathbb{P}}((k_n\delta_n)^{1/2}), \quad m = 1, \ldots, M. \tag{4.7}$$

Thus, introducing

$$\underline{\mathcal{A}}_t^{n,\ell,k_n} = (1^{1/2+H} \mathcal{A}_t^{n,\ell,k_n}, \ldots, M^{1/2+H} \mathcal{A}_t^{n,\ell,Mk_n})^T,$$
$$\underline{\Xi}^{n,\ell}(t) = (\Xi_0^{n,\ell}(t) k_n^{-1/2-H-0}, \ldots, \Xi_{M-1}^{n,\ell}(t) k_n^{-1/2-H-(M-1)})^T,$$

we can rewrite (4.7) as

$$\underline{\mathcal{A}}_t^{n,\ell,k_n} = V_M^T \underline{\Xi}^{n,\ell}(t) + o_{\mathbb{P}}((k_n\delta_n)^{1/2}),$$

where V_M is the Vandermonde matrix (4.1). Thus, by the definition of $w(M)$ (see (4.2)),

$$\begin{aligned}
\sum_{m=1}^{M} w(M)_m m^{1/2+H} \mathcal{A}_t^{n,\ell,mk_n} &= w(M)^T \underline{\mathcal{A}}_t^{n,\ell,k_n} \\
&= |\tilde{w}(M)|^{-1} e_M^T (V_M^{-1})^T V_M^T \underline{\Xi}^{n,\ell}(t) + o_{\mathbb{P}}((k_n\delta_n)^{1/2}) \\
&= |\tilde{w}(M)|^{-1} \Xi_{M-1}^{n,\ell}(t) k_n^{-1/2-H-(M-1)} + o_{\mathbb{P}}((k_n\delta_n)^{1/2}).
\end{aligned}$$

Since $\Xi_{M-1}^{n,\ell}(t) = O_{\mathbb{P}}(1)$, (4.5) follows from our choice of M. □

We now explain how to implement this debiasing procedure in practice. For the remaining part of this section, we assume that

$$\int_0^t (\eta_s^2 + \hat{\eta}_s^2) ds > 0 \quad \text{a.s.} \tag{4.8}$$

(or, equivalently, all forthcoming statements are valid without (4.8) but in restriction to the set $\{\int_0^t (\eta_s^2 + \hat{\eta}_s^2) ds > 0\}$). Define

$$\begin{aligned}
\hat{V}_t^{n,\ell,k_n} = \delta_n \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} & (\hat{c}^n_{(i+k_n)\delta_n,k_n\delta_n} - \hat{c}^n_{i\delta_n,k_n\delta_n}) \\
& \times (\hat{c}^n_{(i+(\ell+1)k_n)\delta_n,k_n\delta_n} - \hat{c}^n_{(i+\ell k_n)\delta_n,k_n\delta_n})
\end{aligned} \tag{4.9}$$

for $\ell \geq 3$ and $k_n \in \mathbb{N}$, which clearly satisfies $\hat{V}_t^{n,\ell,k_n} = (k_n\delta_n)^{2H} \tilde{V}_t^{n,\ell,k_n}$ but in contrast to $\tilde{V}_t^{n,\ell,k_n}$ is actually a statistic since it does not depend on the unknown H. We construct a first pilot estimator of H by fixing two lags $\ell_1, \ell_2 \geq 3$ and then defining

$$\tilde{H}_n = \varphi^{-1}\left(\frac{\hat{V}_t^{n,\ell_1,\tilde{k}_n}}{\hat{V}_t^{n,\ell_2,\tilde{k}_n}}\right), \tag{4.10}$$

where $\varphi : H \mapsto \Phi_{\ell_1}^H / \Phi_{\ell_2}^H$ is assumed to be a diffeomorphism and

$$\tilde{k}_n = [\delta_n^{-1/2}]. \tag{4.11}$$

This choice of $\tilde{k}_n$ has the advantage that it makes $\tilde{H}_n$ a consistent estimator of H, which furthermore satisfies a bias-free central limit theorem regardless of the value of $H \in (0, \frac{1}{2})$. On the downside, its rate is poor if H is small. In the following, we therefore propose an iterative approach to improve the rate, which at the same time retains the bias-free property of the resulting estimators. To this end, let

$$\begin{aligned}\mathcal{H} &= \{H^{(j)} = \tfrac{1}{4}(\sqrt{4j^2-4j+5}-2j+1) : j \in \mathbb{N}\}\\ &= \{0.3090, 0.1514, 0.0963, 0.0700, \dots\},\end{aligned} \tag{4.12}$$

which is precisely the set of values of H for which $\frac{1}{2} - H - \frac{1}{4H}$ (as it appears in (4.3)) is an integer. Therefore, if $H^{(j)} < H \leq H^{(j-1)}$ for $j \in \mathbb{N}$ (where $H^{(0)} = \frac{1}{2}$), then $M = j$. Using the pilot estimator $\tilde{H}_n$, we now define

$$\hat{M}_n = \left[\tfrac{1}{2} - \tilde{H}_n + \tfrac{1}{4\tilde{H}_n} + \delta_n^{1/4}\log\delta_n^{-1}\right] + 1 \tag{4.13}$$

as an estimator of the number M from (4.3). Since $\tilde{H}_n$ is a consistent estimator of H and $\delta_n^{1/4}\log\delta_n^{-1} \to 0$, if $H \notin \mathcal{H}$, we have

$$\lim_{n\to\infty} \mathbb{P}(\hat{M}_n = M) = 1. \tag{4.14}$$

If $H \in \mathcal{H}$, then we still have $\frac{1}{2} - \tilde{H}_n + \frac{1}{4\tilde{H}_n} \to \frac{1}{2} - H + \frac{1}{4H}$ in probability, but since the limit is an integer, after rounding, $[\frac{1}{2} - \tilde{H}_n + \frac{1}{4\tilde{H}_n}]$ will typically jump between two consecutive integers as n increases. To avoid that, we have included $\delta_n^{1/4}\log\delta_n^{-1}$ in the definition of $\hat{M}_n$, which is asymptotically bigger than the $\delta_n^{1/4}$-fluctuations of $\frac{1}{2} - \tilde{H}_n + \frac{1}{4\tilde{H}_n}$ and therefore guarantees that we have $\mathbb{P}(\hat{M}_n = M) \to 1$ for $H \in \mathcal{H}$ as well.

Having defined $\hat{M}_n$, we now set $\bar{H}_n^{(0)} = \tilde{H}_n$ and define consecutively

$$\bar{H}_n^{(j)} = \varphi^{-1}\left(\frac{\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}} \hat{V}_t^{n,\ell_1,m\bar{k}_n^{(j)}}}{\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}} \hat{V}_t^{n,\ell_2,m\bar{k}_n^{(j)}}}\right), \quad \bar{k}_n^{(j)} = [\delta_n^{-2H^{(j)}/(2H^{(j)}+1)}], \tag{4.15}$$

for $j = 1, \dots, \hat{M}_n - 1$ and let

$$\bar{H}_n = \bar{H}_n^{(\hat{M}_n - 1)}. \tag{4.16}$$

PROPOSITION 4.2. *Suppose that the conditions of Theorem 2.1 are satisfied with $H \in (0, \frac{1}{2})$ and assume (4.8). Further fix two lags $\ell_1, \ell_2 \geq 3$ such that the function $\varphi : H \mapsto \Phi_{\ell_1}^H / \Phi_{\ell_2}^H$, where Φ_ℓ^H is defined in (2.12), is a diffeomorphism on $(0, \frac{1}{2})$. For any $j \in \mathbb{N}_0$, if $H \leq H^{(j)}$, then*

$$\bar{H}_n^{(j)} - H = O_{\mathbb{P}}((\bar{k}_n^{(j)}\delta_n)^{1/2}). \tag{4.17}$$

PROOF. We prove the claim by induction, and since the base case $j = 0$ corresponds to the CLT of $\tilde{H}_n$, we can consider $j \geq 1$ and assume that (4.17) is true for $j - 1$. We rewrite

$$\begin{aligned}\bar{H}_n^{(j)} &= \varphi^{-1}\left(\frac{\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} (m\bar{k}_n^{(j)}\delta_n)^{-2H} \hat{V}_t^{n,\ell_1,m\bar{k}_n^{(j)}}}{\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} (m\bar{k}_n^{(j)}\delta_n)^{-2H} \hat{V}_t^{n,\ell_2,m\bar{k}_n^{(j)}}}\right)\\ &= \varphi^{-1}\left(\frac{\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} \tilde{V}_t^{n,\ell_1,m\bar{k}_n^{(j)}}}{\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} \tilde{V}_t^{n,\ell_2,m\bar{k}_n^{(j)}}}\right)\end{aligned} \tag{4.18}$$

and recall (2.11), (2.13) and that φ is a diffeomorphism. Therefore, defining $\psi(x_1,x_2) = \varphi^{-1}(x_1/x_2)$, we can use the mean-value theorem to find $(\xi_t^{n,1},\xi_t^{n,2})$ satisfying

$$\xi_t^{n,\iota} \xrightarrow{\mathbb{P}} \sum_{m=1}^{j} w(j)_m m^{1/2+H}\Phi_{\ell_\iota}^H$$

for $\iota = 1,2$ such that

$$\begin{aligned} \bar{H}_n^{(j)} - H = {} & \sum_{\iota=1,2} \partial_{x_\iota}\psi(\xi_t^{n,1},\xi_t^{n,2}) \sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} \mathcal{A}_t^{n,\ell_\iota,m\bar{k}_n^{(j)}} \\ & + \sum_{\iota=1,2} \partial_{x_\iota}\psi(\xi_t^{n,1},\xi_t^{n,2}) \sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} \\ & \times (\tilde{V}_t^{n,\ell_\iota,m\bar{k}_n^{(j)}} - \mathcal{A}_t^{n,\ell_\iota,m\bar{k}_n^{(j)}} - V_t^{\ell_\iota}). \end{aligned} \tag{4.19}$$

By Theorem 2.1, $\tilde{V}_t^{n,\ell_\iota,m\bar{k}_n^{(j)}} - \mathcal{A}_t^{n,\ell_\iota,m\bar{k}_n^{(j)}} - V_t^{\ell_\iota} = O_{\mathbb{P}}((\bar{k}_n^{(j)}\delta_n)^{1/2})$. It remains to show that the first term on the right-hand side of (4.19) is $O_{\mathbb{P}}((\bar{k}_n^{(j)}\delta_n)^{1/2})$. Let us fix ι. Since $\partial_{x_\iota}\psi(\xi_t^{n,1},\xi_t^{n,2})$ converges in probability, we only have to show that for any $\ell \geq 3$,

$$\sum_{m=1}^{j} w(j)_m m^{1/2-\bar{H}_n^{(j-1)}+2H} \mathcal{A}_t^{n,\ell,m\bar{k}_n^{(j)}} = O_{\mathbb{P}}((\bar{k}_n^{(j)}\delta_n)^{1/2}). \tag{4.20}$$

By Lemma B.3, $\mathcal{A}_t^{n,\ell,m\bar{k}_n^{(j)}} = O_{\mathbb{P}}((\bar{k}_n^{(j)})^{-1/2-H}) = o_{\mathbb{P}}(\delta_n^{H^{(j)}/(1+2H^{(j)})})$. At the same time, for any $m = 1,\dots,j$, we have that $m^{1/2-\bar{H}_n^{(j-1)}+2H} - m^{1/2+H} = O_{\mathbb{P}}((\bar{k}_n^{(j-1)}\delta_n)^{1/2}) = O_{\mathbb{P}}(\delta_n^{1/(4H^{(j-1)}+2)})$ by the induction hypothesis. So if we replace $m^{1/2-\bar{H}_n^{(j-1)}+2H}$ by $m^{1/2+H}$ in (4.20), the overall error is $o_{\mathbb{P}}(\delta_n^{H^{(j)}/(1+2H^{(j)})+1/(4H^{(j-1)}+2)})$, which can be shown to be $o_{\mathbb{P}}(\delta_n^{1/(4H^{(j)}+2)})$ by using the explicit formula for $H^{(j)}$ from (4.12). Now once we have replaced $m^{1/2-\bar{H}_n^{(j-1)}+2H}$ by $m^{1/2+H}$, (4.20) follows from Proposition 4.1 (or, more directly, from (4.5)). □

By (4.14) and the previous proposition, $\bar{H}_n$ is our best estimator so far: it is bias-free and satisfies a CLT with rate $\delta_n^{1/(4H^{(j-1)}+2)}$, where $j \in \mathbb{N}$ is such that $H^{(j)} < H \leq H^{(j-1)}$. Unless $H \in \mathcal{H}$, this rate is close but still not equal to the optimal one, which is $\delta_n^{1/(4H+2)}$. As alluded to before, the remaining obstacle to rate efficiency is the fact that the optimal window size k_n should be of order $\delta_n^{-2H/(2H+1)}$, which depends on the parameter H to be estimated. While $\bar{H}_n$ is not rate-optimal in general, it is nevertheless consistent for H, so one might be tempted to use $\check{k}_n = [\delta_n^{-2\bar{H}_n/(2\bar{H}_n+1)}]$ as a new window size and to construct a new estimator similarly to (4.15) with $\check{k}_n$ substituted for $\tilde{k}_n$ and $\hat{M}_n$ substituted for j. While this is a natural approach, there is a pitfall inherent in any such plug-in estimator: the sequence $\check{k}_n$ is *random* as it depends on the data through $\bar{H}_n$. As Theorem 2.1 was shown with a deterministic window size, it cannot be applied with $\check{k}_n$.

In order tackle this problem, we use the randomization approach of [38] that relies on the following—seemingly paradoxical—idea: Add more randomness to $\check{k}_n$ in order to reduce its randomness! To see what this means and why it works, consider an auxiliary probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ equipped with a uniform random variable U. As usual, we form the product space

$$\hat{\Omega} = \Omega \times \Omega', \quad \hat{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}', \quad \hat{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$$

and extend all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ to the new space in the canonical fashion. To simplify the notation, we keep writing $\mathbb{P}$ in the following, but whenever U appears, of course, it stands for $\hat{\mathbb{P}}$. In addition, we choose two sequences $q_n \sim q/\log \delta_n^{-1}$ for some $q > 0$ and $r_n \to \infty$ such that $\delta_n^{-1/4}/r_n \to \infty$ and $\log \delta_n^{-1}/r_n \to 0$. We then define the *oracle sequence*

$$\hat{k}_n = [\delta_n^{-2\bar{H}_n^U/(2\bar{H}_n^U+1)}], \tag{4.21}$$

where

$$\bar{H}_n^U = \frac{[r_n(\bar{H}_n + q_n) + U] + 1}{r_n} \tag{4.22}$$

is a randomized version of $\bar{H}_n$. Note that $\bar{H}_n^U$ depends both on the data (through $\bar{H}_n$) and on U, which is what we mean by "adding randomness." The success of the randomization approach pivots on the following *oracle property*, proved in [38, Lemma 9]:

$$\lim_{n\to\infty} \mathbb{P}(\hat{k}_n = k_n^U) = 1, \tag{4.23}$$

where

$$k_n^U = [\delta_n^{-2H_n^U/(2H_n^U+1)}], \qquad H_n^U = \frac{[r_n(H + q_n) + U] + 1}{r_n}. \tag{4.24}$$

Note that k_n^U only depends on U but *no longer on* $\mathcal{F}$, in particular, no longer on the data. This is what we mean by "reducing randomness." In conclusion, what the randomization approach really does it to *exchange* data-dependent randomness for data-independent randomness in the sequence $\hat{k}_n$. And this clearly pays off: conditionally on U, the sequence k_n^U is deterministic, to which we can apply all limit theorems obtained so far. Thus, our rate-optimal estimator of H is

$$\hat{H}_n = \varphi^{-1}\left(\frac{\sum_{m=1}^{\hat{M}_n} w(\hat{M}_n)_m m^{1/2-\bar{H}_n} \hat{V}_t^{n,\ell_1,m\hat{k}_n}}{\sum_{m=1}^{\hat{M}_n} w(\hat{M}_n)_m m^{1/2-\bar{H}_n} \hat{V}_t^{n,\ell_2,m\hat{k}_n}}\right), \tag{4.25}$$

whose asymptotic behavior is given in the following theorem, our main result.

THEOREM 4.3. *Grant Assumption CLT and suppose that* $q_n \sim q/\log \delta_n^{-1}$ *for some* $q > 0$ *and* r_n *is an increasing sequence such that* $\delta_n^{-1/4}/r_n \to \infty$ *and* $\log \delta_n^{-1}/r_n \to 0$. *Moreover, fix two lags* $\ell_1, \ell_2 \geq 3$ *such that the function* $\varphi : H \mapsto \Phi_{\ell_1}^H/\Phi_{\ell_2}^H$, *where* Φ_ℓ^H *is defined in* (2.12), *is a diffeomorphism on* $(0, \frac{1}{2})$. *Assuming* (4.8) *and using the notations*

$$\beta(H) = e^{2q/(2H+1)^2} \tag{4.26}$$

and

$$\begin{aligned} w(M,H) &= (w(M,H)_1, \ldots, w(M,H)_M)^T, \\ w(M,H)_m &= \frac{w(M)_m m^{1/2+H}}{\sum_{m=1}^M w(M)_m m^{1/2+H}} \end{aligned} \tag{4.27}$$

and

$$\gamma_\nu^{\ell,\ell'}(H) = (\gamma_\nu^{\ell,\beta(H)m,\ell',\beta(H)m'}(H))_{m,m'=1}^M \in \mathbb{R}^{M\times M}, \quad \nu = 1,2,3, \quad \ell,\ell' \geq 3, \tag{4.28}$$

we have for any $H \in (0, \frac{1}{2})$ *that*

$$\begin{aligned} &\delta_n^{-1/(4H+2)}(\hat{H}_n - H) \\ &\xrightarrow{st} N\left(0, \left(\frac{\varphi(H)}{\varphi'(H)}\right)^2 \sum_{\iota,\iota'=1,2} \frac{(-1)^{\iota+\iota'}}{\Phi_{\ell_\iota}^H \Phi_{\ell_{\iota'}}^H} \sum_{\nu=1}^3 [w(M,H)^T \gamma_\nu^{\ell_\iota,\ell_{\iota'}}(H) w(M,H)] \Gamma_\nu(t)\right), \end{aligned} \tag{4.29}$$

where $M = M(H)$ is the number from (4.3) and the limit in the previous line is independent of $\mathcal{F}'$.

PROOF. By (4.14) and (4.23), it suffices to prove (4.29) for

$$H_n' = \varphi^{-1}\left(\frac{\sum_{m=1}^M w(M)_m m^{1/2-\bar{H}_n}\hat{V}_t^{n,\ell_1,mk_n^U}}{\sum_{m=1}^M w(M)_m m^{1/2-\bar{H}_n}\hat{V}_t^{n,\ell_2,mk_n^U}}\right) \tag{4.30}$$

instead of $\hat{H}_n$. And by the definition of stable convergence in law, it suffices to do so conditionally on U because U does not appear in the limit. Next, similarly to (4.18) and (4.19), we can write

$$H_n' = \varphi^{-1}\left(\frac{\sum_{m=1}^M w(M)_m m^{1/2-\bar{H}_n+2H}\tilde{V}_t^{n,\ell_1,mk_n^U}}{\sum_{m=1}^M w(M)_m m^{1/2-\bar{H}_n+2H}\tilde{V}_t^{n,\ell_2,mk_n^U}}\right)$$

and find $(\zeta_t^{n,1},\zeta_t^{n,2})$ such that

$$\begin{aligned}
&\delta_n^{-1/(4H+2)}(H_n'-H)\\
&\quad= \delta_n^{-1/(4H+2)}\sum_{\iota=1,2}\partial_{x_\iota}\psi(\zeta_t^{n,1},\zeta_t^{n,2})\sum_{m=1}^M w(M)_m m^{1/2-\bar{H}_n+2H}\mathcal{A}_t^{n,\ell_\iota,mk_n^U}\\
&\qquad+\sum_{\iota=1,2}\partial_{x_\iota}\psi(\zeta_t^{n,1},\zeta_t^{n,2})\sum_{m=1}^M w(M)_m m^{1/2-\bar{H}_n+2H}\\
&\qquad\qquad\times\delta_n^{-1/(4H+2)}(\tilde{V}_t^{n,\ell_\iota,mk_n^U}-\mathcal{A}_t^{n,\ell_\iota,mk_n^U}-V_t^{\ell_\iota}).
\end{aligned} \tag{4.31}$$

Conditionally on U, the sequence k_n^U is deterministic. Furthermore, since $q_n \sim q\log\delta_n^{-1}$ and $\log\delta_n^{-1}/r_n \to 0$, we have $k_n^U/\delta_n^{-2H/(2H+1)} \to e^{2q/(2H+1)^2}$. By Theorem 2.1, we know that $(\delta_n^{-1/(4H+2)}(\tilde{V}_t^{n,\ell_\iota,mk_n^U}-\mathcal{A}_t^{n,\ell_\iota,mk_n^U}-V_t^{\ell_\iota}))_{\iota=1,2,m=1,\dots,M}$ satisfies a joint CLT, so a tedious but entirely straightforward variance computation shows that the second line of the previous display converges stably to the right-hand side of (4.29). Analogously to how we proved (4.20), we can first use Proposition 4.2 to replace $m^{1/2-\bar{H}_n+2H}$ by $m^{1/2+H}$ and then apply Proposition 4.1 to show that the first term on the right-hand side of (4.31) is $o_{\mathbb{P}}(\delta_n^{1/(4H+2)})$, completing the proof. □

In order to make Theorems 2.1 and 4.3 feasible, we need to find consistent estimators of $\Gamma_1(t)$, $\Gamma_2(t)$ and $\Gamma_3(t)$ from (2.16). The following estimators are adapted from [3, Theorem 8.12].

PROPOSITION 4.4. *Let $\hat{K}_n = \hat{k}_n[\delta_n^{-\lambda}]$, where $\hat{k}_n$ is defined in (4.21) and $\lambda \in (0,\frac{1}{2})$. Moreover, define*

$$\begin{aligned}
\delta_i'^n X &= X_{i\hat{K}_n\delta_n} - X_{(i-1)\hat{K}_n\delta_n},\\
\delta_i'^n \hat{c} &= \frac{1}{\hat{k}_n\delta_n}(\hat{C}^n_{(1+i\hat{K}_n)\delta_n,\hat{k}_n\delta_n} - \hat{C}^n_{(1+(i-1)\hat{K}_n)\delta_n,\hat{k}_n\delta_n})
\end{aligned} \tag{4.32}$$

and

$$\hat\Gamma^n_1(t)=\frac{1}{9\delta_n^3}\sum_{i=1}^{[t/(\hat K_n\delta_n)]-1}(\delta'^n_i X)^4(\delta'^n_{i+1}X)^4,$$

$$\hat\Gamma^n_2(t)=\frac{(\hat K_n\delta_n)^{1-4\hat H_n}}{3}\sum_{i=1}^{[t/(\hat K_n\delta_n)]-2}(\delta'^n_i\hat c)^4, \tag{4.33}$$

$$\hat\Gamma^n_3(t)=\frac{(\hat K_n\delta_n)^{-1-2\hat H_n}}{3}\sum_{i=1}^{[t/(\hat K_n\delta_n)]-2}(\delta'^n_i\hat c)^2(\delta'^n_{i+1}X)^4.$$

Then under the assumptions of Theorem 4.3, we have $\hat\Gamma^n_\nu \stackrel{\mathbb{P}}{\Longrightarrow} \Gamma_\nu$ *for each* $\nu=1,2,3$.

PROOF. Let $\delta'^n_i\tilde c$ and $\tilde\Gamma^n_\nu(t)$, $\nu=1,2,3$, be defined in the same way as the corresponding quantities in (4.32) and (4.33) except that $\hat k_n$ and $\hat K_n$ are replaced by some deterministic sequences $k_n\sim\theta\delta_n^{-2H/(2H+1)}$ and $K_n\sim\Theta\delta_n^{-2H/(2H+1)-\lambda}$ with $\theta,\Theta>0$. Similarly to (4.23), we have $\mathbb{P}(\hat K_n=K_n^U)=1$, where $K_n^U\sim\Theta'\delta_n^{-2H/(2H+1)-\lambda}$ for some $\Theta'>0$ (and almost all realizations of U). Thus, it suffices to show

$$\tilde\Gamma^n_1\stackrel{L^1}{\Longrightarrow}\Gamma_1,\quad \tilde\Gamma^n_2\stackrel{L^1}{\Longrightarrow}\Gamma_2,\quad \tilde\Gamma^n_3\stackrel{L^1}{\Longrightarrow}\Gamma_3,$$

assuming Assumption CLT'. The first convergence is a consequence of [27, Theorem 8.4.1].

For the remaining two, we make the following observation: by (3.7), we have that

$$\delta'^n_i\tilde c-\delta'^n_i c=J^n_{1,1+iK_n}-J^n_{1,1+(i-1)K_n}+\frac{1}{k_n\delta_n}\int_{iK_n\delta_n}^{(iK_n+k_n)\delta_n}(c_s-c_{iK_n\delta_n})ds$$
$$+\frac{1}{k_n\delta_n}\int_{(i-1)K_n\delta_n}^{((i-1)K_n+k_n)\delta_n}(c_s-c_{(i-1)K_n\delta_n})ds.$$

It is not hard to see from the definition that J^n_i is of size $k_n^{-1/2}$, uniformly in i. Moreover, the last two terms on the right-hand side of the previous display are of size $(k_n\delta_n)^H$, uniformly in i. Therefore, if we define $\Gamma^n_2(t)$ and $\Gamma^n_3(t)$ in the same way as $\tilde\Gamma^n_2(t)$ and $\tilde\Gamma^n_3(t)$ but with $\delta'^n_i\tilde c$ replaced by $\delta'^n_i c$, then

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]}\big\{|\tilde\Gamma^n_2(t)-\Gamma^n_2(t)|+|\tilde\Gamma^n_3(t)-\Gamma^n_3(t)|\big\}\Bigg]\lesssim(\hat K_n\delta_n)^{-H}(k_n^{-1/2}+(k_n\delta_n)^H)\to 0$$

as $n\to\infty$. Consequently, it remains to show $\Gamma^n_2\stackrel{L^1}{\Longrightarrow}\Gamma_2$ and $\Gamma^n_3\stackrel{L^1}{\Longrightarrow}\Gamma_3$. The first convergence was shown in [6, Theorem 3], while the second is easily obtained from standard techniques of high-frequency statistics (involving drift removal, localization of σ, η and $\hat\eta$, and a LLN in the case where X is a Brownian motion and c is a fractional Brownian motion) and the fact that $\mathbb{E}[(W^H_1)^2(W_2-W_1)^4]=\mathbb{E}[(\hat W^H_1)^2(W_2-W_1)^4]=3$. $\square$

This could have been the end of our construction of a rate-optimal and feasible estimator of H if it was not for a crucial detail that we have overlooked so far. It turns out that *all* estimators considered in this section (including $\hat H_n$) break down if $H=\frac12$, that is, if volatility is not rough but just a semimartingale. This is because $\Phi^{1/2}_\ell=0$ for any $\ell\geq 3$, which implies by Theorem 2.1 that $(\tilde V^{n,\ell_1,\tilde k_n}_t,\tilde V^{n,\ell_2,\tilde k_n}_t)$ for $\ell_1,\ell_2\geq 3$ converges in law to a bivariate mixed normal distribution. In particular, the ratio $\tilde V_t^{n,\ell_1,\tilde k_n}/\tilde V_t^{n,\ell_2,\tilde k_n}$ and thus the estimator $\tilde H_n$ converges in distribution (not in probability) to a random variable with a density. In other words, naïvely applying $\tilde H_n$ if $H=\frac12$ can output any value in the interval $(0,\frac12)$ just by chance!

There at least two ways of remedying this problem. One possibility is to choose $\ell_2\in\{0,1\}$ in (4.10), which ensures that $\Phi^{1/2}_{\ell_2}\neq 0$. However, in this case, $Z_1^{n,\ell_2}(t)$ will have a dominating bias term that has to be removed (even for the LLN; see [3, Section 8.3]). But more importantly, the latent bias term $\mathcal{A}_t^{n,\ell,k_n}$ from (2.13) (which will also have a slightly different form) involves the function $v\mapsto \Delta_1^3(\ell_2-1-v)$, which is no longer smooth in $v\in[0,\frac32]$. This has the consequence that the debiasing procedure from Proposition 4.1 has to be modified. We propose a different, quicker, solution. Loosely speaking, we first use the limit theory of $\tilde V_t^{n,\ell_2,\tilde k_n}$ to test whether $H=\frac12$ and only use $\hat H_n$ if $H=\frac12$ is rejected. More precisely, we define

$$H_n=\hat H_n \mathbf{1}_{\mathcal{R}_n}(\hat V_t^{n,\ell_2,\tilde k_n})+\frac12\mathbf{1}_{\mathbb{R}\setminus\mathcal{R}_n}(\hat V_t^{n,\ell_2,\tilde k_n}), \tag{4.34}$$

where

$$\mathcal{R}_n=\left\{x\in\mathbb{R}:|x|>\delta_n^{3/4}\log\delta_n^{-1}\left(\sum_{\nu=1}^3\gamma_\nu^{\ell_2,1,\ell_2,1}(\tfrac12)\hat\Gamma^n_\nu(t)\right)^{1/2}\right\} \tag{4.35}$$

and $\hat\Gamma_\nu^n(t)$ is defined in (4.33).

THEOREM 4.5. *Under the assumptions of Theorem 4.3 and Proposition 4.4, we have*

$$\begin{cases}\lim\limits_{n\to\infty}\mathbb{P}(H_n=\hat H_n)=1 & \text{if } H\in(0,\frac12),\\ \lim\limits_{n\to\infty}\mathbb{P}(H_n=\frac12)=1 & \text{if } H=\frac12.\end{cases}$$

In particular, if $H\in(0,\frac12)$, (4.29) continues to hold with H_n instead of $\hat H_n$.

PROOF. If $H=\frac12$, note that by Theorem 2.1, $\delta_n^{-3/4}\hat V_t^{n,\ell_2,\tilde k_n}$ converges stably in law to a centered normal with conditional variance $\sum_{\nu=1}^3\gamma_\nu^{\ell_2,1,\ell_2,1}(\frac12)\Gamma_\nu(t)$. Thus, by Proposition 4.4, $\mathcal{V}_n=\delta_n^{-3/4}\hat V_t^{n,\ell_2,\tilde k_n}/(\sum_{\nu=1}^3\gamma_\nu^{\ell_2,1,\ell_2,1}(\frac12)\hat\Gamma_\nu(t))^{1/2}\xrightarrow{d}N(0,1)$, so $\mathbb{P}(H_n=\frac12)=\mathbb{P}(|\mathcal{V}_n|\leq\log\delta_n^{-1})\to1$. Similarly, if $H\in(0,\frac12)$, we know from Theorem 2.1 that

$$\mathcal{V}_n'=\delta_n^{1/2-H}\mathcal{V}_n\frac{\left(\sum_{\nu=1}^3\gamma_\nu^{\ell_2,1,\ell_2,1}(\frac12)\hat\Gamma_\nu(t)\right)^{1/2}}{(\gamma_2^{\ell_2,1,\ell_2,1}(H)\hat\Gamma_2(t))^{1/2}}\xrightarrow{d}N(0,1),$$

which shows that

$$\mathbb{P}(H_n=\hat H_n)=\mathbb{P}\left(|\mathcal{V}_n'|>\delta_n^{1/2-H}\log\delta_n^{-1}\frac{\left(\sum_{\nu=1}^3\gamma_\nu^{\ell_2,1,\ell_2,1}(\frac12)\hat\Gamma_\nu(t)\right)^{1/2}}{(\gamma_2^{\ell_2,1,\ell_2,1}(H)\hat\Gamma_2(t))^{1/2}}\right)\to1. \qquad \square$$

APPENDIX A: DETAILS OF THE PROOF OF PROPOSITION 3.2

We start with a lemma on the regularity of the process A from (3.2).

LEMMA A.1. *For any $T>0$, we have that*

$$\begin{aligned}&\mathbb{E}[(A_{t+h}-A_t)^2]^{1/2}+\mathbb{E}[(A^\eta_{t+h}-A^\eta_t)^2]^{1/2}+\mathbb{E}[(A^{\hat\eta}_{t+h}-A^{\hat\eta}_t)^2]^{1/2}\\ &\quad\lesssim\left[(1+t^{-H})h^{(2H+1/2)\wedge1}\right]\wedge h^H,\end{aligned}$$

with a constant that is uniform for $t \in [0,T]$, $h > 0$ and $i = 0, \dots, L$. If $H = \frac{1}{2}$, the previous bound can be improved to

$$\mathbb{E}[(A_{t+h} - A_t)^2]^{1/2} + \mathbb{E}[(A^{\eta}_{t+h} - A^{\eta}_t)^2]^{1/2} + \mathbb{E}[(A^{\hat{\eta}}_{t+h} - A^{\hat{\eta}}_t)^2]^{1/2} \lesssim h.$$

PROOF. The statement is obvious for $H = \frac{1}{2}$, so we assume $H \in (0, \frac{1}{2})$ in the following. We only consider increments of A_t; the bounds for A^{η} and $A^{\hat{\eta}}$ can be derived in the same way. Since the first term in the definition of A_t is differentiable almost surely with L^2-bounded derivative, we only need to consider the second term. To avoid introducing additional notation, we assume that $g_0' \equiv \tilde{g}_0' \equiv 0$ such that $A_t = (g_{\tilde{H}} * \tilde{\eta})(t)$, where $(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds$ denotes the convolution of two integrable functions. Note that we used the convention $\tilde{\eta}^{(i)}_s = 0$ for $s < 0$. Since

$$\begin{aligned}\mathbb{E}[(A_{t+h} - A_t)^2]^{1/2} &\le \int_0^t \mathbb{E}[(\tilde{\eta}_{t+h-s} - \tilde{\eta}_{t-s})^2]^{1/2} g_{\tilde{H}}(s)ds + \int_t^{t+h} \mathbb{E}[(\tilde{\eta}_{t+h-s})^2]^{1/2} g_{\tilde{H}}(s)ds \\ &\lesssim h^H + [(t+h)^{\tilde{H}+1/2} - t^{\tilde{H}+1/2}] \lesssim h^H + h^{\tilde{H}+1/2} \lesssim h^H,\end{aligned}$$

we have shown the second upper bound. To get the first one, observe that

$$\begin{aligned}\mathbb{E}[(\Delta^2_h A_t)^2]^{1/2} &= \mathbb{E}[(\Delta^2_h (g_{\tilde{H}} * \tilde{\eta})(t))^2]^{1/2} = \mathbb{E}[(\Delta_h g_{\tilde{H}} * \Delta_h \tilde{\eta})(t))^2]^{1/2} \\ &\le \int_{-h}^{t+h} |g_{\tilde{H}}(s+h) - g_{\tilde{H}}(s)| |\mathbb{E}[(\tilde{\eta}_{t+h-s} - \tilde{\eta}_{t-s})^2]^{1/2} ds.\end{aligned}$$

The last integral splits into three parts, according to whether $s \in (-h, 0)$, $s \in (t, t+h)$ or $s \in (0,t)$. Bounding them by

$$\int_{-h}^0 g_{\tilde{H}}(s+h) \mathbb{E}[(\tilde{\eta}_{t+h-s} - \tilde{\eta}_{t-s})^2]^{1/2} ds \lesssim h^H \int_{-h}^0 g_{\tilde{H}}(s+h) ds \lesssim h^{H+\tilde{H}+1/2},$$

$$\begin{aligned}\int_t^{t+h} |g_{\tilde{H}}(s+h) - g_{\tilde{H}}(s)| \mathbb{E}[(\tilde{\eta}_{t+h-s})^2]^{1/2} ds &\lesssim \int_t^{t+h} |g_{\tilde{H}}(s+h) - g_{\tilde{H}}(s)| ds \\ &\lesssim h^{\tilde{H}+1/2} \wedge t^{\tilde{H}-3/2} h^2 \le t^{-H} h^{H+\tilde{H}+1/2},\end{aligned}$$

$$\begin{aligned}\int_0^t |g_{\tilde{H}}(s+h) - g_{\tilde{H}}(s)| \mathbb{E}[(\tilde{\eta}_{t+h-s} - \tilde{\eta}_{t-s})^2]^{1/2} ds &\le h^H \int_0^t |g_{\tilde{H}}(s+h) - g_{\tilde{H}}(s)| ds \\ &\lesssim h^{H+\tilde{H}+1/2},\end{aligned}$$

we obtain the assertion of the lemma from [35, Proposition 2]. □

PROOF OF LEMMA 3.4. We start with $H \in (0, \frac{1}{2})$. Consider the first $[(k_n/\delta_n)^{1/2} \delta_n^{\varepsilon}]$ terms in the sum over i in the definition of $Z_2^{n,\ell}(t)$ in (3.10). Since $\mathbb{E}[|c_{s+k_n\delta_n} - c_s|^2]^{1/2} \lesssim (k_n\delta_n)^H$, their contribution to $Z_2^{n,\ell}(t)$, multiplied by the rate $(k_n\delta_n)^{-1/2}$, is of order at most $(k_n\delta_n)^{1/2-2H} k_n^{-1} (k_n/\delta_n)^{1/2} \delta_n^{\varepsilon} (k_n\delta_n)^{2H} = \delta_n^{\varepsilon}$ and hence asymptotically negligible. Similarly, because

$$\Delta_1^2 G_H(u) \lesssim u^{H-3/2} \wedge 1, \tag{A.1}$$

we have that

$$\begin{aligned}\mathbb{E}[|D^n_{1,i}|^2]^{1/2} &\lesssim (k_n\delta_n)^{1/2+H} \left(\int_0^{(i-1+2k_n)\delta_n} (\Delta_1^2 G_H(\tfrac{i-1-r/\delta_n}{k_n}))^2 dr \right)^{1/2} \\ &\lesssim (k_n\delta_n)^{1+H} \left(\int_{-2}^{\frac{i-1}{k_n}} (\Delta_1^2 G_H(u))^2 du \right)^{1/2} \lesssim (k_n\delta_n)^{1+H},\end{aligned} \tag{A.2}$$

uniformly in i, so the first $[(k_n/\delta_n)^{1/2}\delta_n^{\varepsilon}]$ terms in the sum over i in (3.20) are negligible as well. Furthermore, by Lemma A.1,

$$\begin{aligned}\mathbb{E}[|D_{2,i}^n|^2]^{1/2} &\lesssim (k_n\delta_n)^{(2H+1/2)\wedge 1}\int_{(i-1)\delta_n}^{(i-1+k_n)\delta_n} s^{-H}ds \\ &\lesssim (k_n\delta_n)^{1+(2H+1/2)\wedge 1}(k_n\delta_n)^{-H}\end{aligned}\tag{A.3}$$

uniformly for $i \geq [(k_n/\delta_n)^{1/2}\delta_n^{\varepsilon}]$, it follows from the mean-value theorem and the Cauchy–Schwarz inequality that the difference $Z_2^{n,\ell}(t) - \tilde{Z}_2^{n,\ell}(t)$, with summation restricted to $i \geq [(k_n/\delta_n)^{1/2}\delta_n^{\varepsilon}]$, is of order $(k_n\delta_n)^{-1-2H}(k_n\delta_n)^{-1}(k_n\delta_n)^{1+H}(k_n\delta_n)^{1-H+(2H+1/2)\wedge 1}$, which is $o((k_n\delta_n)^{1/2})$ for $H \in (0,\frac{1}{2})$ if $\varepsilon > 0$ is small enough.

If $H = \frac{1}{2}$, we note that

$$D_{1,i}^n = \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \Delta_{k_n\delta_n}^2 G_{1/2}((i-1)\delta_n - r)\boldsymbol{\eta}_r d\boldsymbol{W}_r = O_{\mathbb{P}}((k_n\delta_n)^{3/2})$$

and $D_{2,i}^n = O_{\mathbb{P}}((k_n\delta_n)^2)$. Thus, decomposing

$$\begin{aligned}(k_n\delta_n)^{-1/2}(Z_2^{n,\ell}(t) - \tilde{Z}_2^{n,\ell}(t)) = (k_n\delta_n)^{-2}\frac{1}{k_n}&\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{1,i}^n D_{2,i+\ell k_n}^n \\ &+ (k_n\delta_n)^{-2}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{2,i}^n D_{1,i+\ell k_n}^n \\ &+ (k_n\delta_n)^{-2}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{2,i}^n D_{2,i+\ell k_n}^n,\end{aligned}$$

we easily notice that the last term is $O_{\mathbb{P}}(k_n\delta_n)$ and therefore negligible. Let us consider the first expression on the right-hand side; the second one can be treated similarly. Bounding term by term, we notice that it is of order $O_{\mathbb{P}}(1)$. This means two things: to show convergence to zero, we need to find a better way of bounding this expression. But at the same time, we are allowed to make any modification that leads to an $o_{\mathbb{P}}(1)$ error. In particular, thanks to (3.5), we may replace $D_{2,i+\ell k_n}^n$ by (recall that we may assume $A_t = \int_0^t a_s ds$)

$$\tilde{D}_{2,i+\ell k_n}^n = \int_{(i-1+\ell k_n)\delta_n}^{(i-1+(\ell+1)k_n)\delta_n}\int_s^{s+k_n\delta_n} a_{(i-1)\delta_n} dr ds,$$

which has the advantage that it is $\mathcal{F}_{(i-1)\delta_n}$-measurable. Therefore, the product $D_{1,i}^n D_{2,i+\ell k_n}^n$ is $\mathcal{F}_{(i-1+2k_n)\delta_n}$-measurable with zero $\mathcal{F}_{(i-1)\delta_n}$-conditional expectation. By a martingale size estimate (see [11, Appendix A]), it follows that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left|(k_n\delta_n)^{-2}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{1,i}^n \tilde{D}_{2,i+\ell k_n}^n\right|\right] \\ &\qquad \lesssim (k_n\delta_n)^{-2}\frac{1}{k_n}(k_n/\delta_n)^{1/2}(k_n\delta_n)^{3/2}(k_n\delta_n)^2 = k_n\delta_n \to 0. \qquad \square\end{aligned}$$

PROOF OF LEMMA 3.5. We first remove $\vee(-[r/\delta_n])$ from the lower bound of i. Since this is only relevant for $r \leq 2k_n\delta_n$ and the two $\Delta_1^2 G_H$-terms are uniformly bounded for $i \in \{1-2k_n,\ldots,0\}$, this removal only incurs an error of order $k_n^{-1}(k_n\delta_n)k_n = k_n\delta_n$, which is smaller than the desired convergence rate of $(k_n\delta_n)^{1/2}$.

It remains to replace the upper bound of the sum by $+\infty$. In order to justify this, observe that

(A.4) $$|\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})| \lesssim \big(\tfrac{i}{k_n}\big)^{H-3/2} \wedge 1 \qquad (\text{and } \lesssim \mathbf{1}_{\{i\leq 2k_n+1\}} \text{ if } H=\tfrac{1}{2}),$$

uniformly in n, i and r. If $H \in (0,\frac{1}{2})$, we now choose some $p > 1 + (1-\kappa)/(4\kappa(1-H))$. For any $\kappa \in [\frac{2H}{2H+1},\frac{1}{2}]$, if p is sufficiently close to the lower bound, we still have $k_n^p\delta_n \to 0$. So if we consider the two cases $t-r \geq k_n^p\delta_n$ and $t-r\leq k_n^p\delta_n$ separately, we observe in the former case that

(A.5)
$$\mathbb{E}\Bigg[\Bigg|\frac{1}{k_n}\int_0^{t-k_n^p\delta_n} \sum_{i=[t/\delta_n]-[r/\delta_n]-(\ell+2)k_n+1}^{\infty} \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n})|\boldsymbol{\eta}_r|^2 dr\Bigg|\Bigg]$$
$$\lesssim \frac{1}{k_n}\sum_{i=k_n^p/2}^{\infty}\left(\frac{i}{k_n}\right)^{2H-3}\int_0^{t-k_n^p\delta_n}\mathbb{E}[|\boldsymbol{\eta}_r|^2]dr \lesssim k_n^{(2-2H)(1-p)} = o((k_n\delta_n)^{1/2})$$

by our choice of p. If $t-r\leq k_n^p\delta_n$, we pick some $\varepsilon>0$ to be specified later and, for the moment, small enough such that we have the bound

$$|\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n})| \leq \big(\tfrac{i}{k_n}\big)^{2H-3}\wedge 1 \leq \big(\tfrac{i}{k_n}\big)^{-1-\varepsilon}\wedge 1.$$

Then

(A.6)
$$\mathbb{E}\Bigg[\Bigg|\frac{1}{k_n}\int_{t-k_n^p\delta_n}^{([t/\delta_n]-\ell k_n)\delta_n} \sum_{i=[t/\delta_n]-[r/\delta_n]-(\ell+2)k_n+1}^{\infty} \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})$$
$$\times \Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n})|\boldsymbol{\eta}_r|^2 dr\Bigg|\Bigg] \lesssim \frac{1}{k_n}k_n^{1+\varepsilon}\int_{t-k_n^p\delta_n}^{([t/\delta_n]-\ell k_n)\delta_n}\mathbb{E}[|\boldsymbol{\eta}_r|^2]dr \lesssim k_n^{\varepsilon+p}\delta_n.$$

The reader can verify that for any $H\in(0,\frac{1}{2})$, if p is close enough to $1+(1-\kappa)/(4\kappa(1-H))$ and ε is small enough, then $k_n^{\varepsilon+p}\delta_n = o((k_n\delta_n)^{1/2})$.

If $H=\frac{1}{2}$, we choose $p\in(1,\frac{3}{2})$. By (A.4), the left-hand side of (A.5) is simply zero because $p>1$. Similarly, the summation in (A.6) only involves $O(k_n)$ many terms, so the left-hand side of (A.6) is $O(k_n^p\delta_n)$, which is $(k_n\delta_n)^{1/2}$ since $p<\frac{3}{2}$. □

PROOF OF LEMMA 3.6. If $H=\frac{1}{2}$, we have $\Delta_1^2 G_{1/2}(v)=0$ for $v\notin(-2,0)$. Thus, $\Phi_\ell^{1/2}=0$ for all $\ell\geq 2$. For $H\in(0,\frac{1}{2})$, it is possible to compute Φ_ℓ^H using properties of fractional Brownian motion and integration by parts. But in order to prepare for upcoming (and more involved) calculations, we show how to obtain (3.25) using Fourier methods. An advantage of this approach is that it yields a formula for arbitrary $\ell\in\mathbb{R}$ (not just $\ell\in\{2,3,\dots\}$), without the need to differentiate between multiple cases. First notice that there is no harm to extend the integral in (3.25) to $-\infty$, because $\Delta_1^2 G_H(v)=0$ for all $v<-2$. Therefore, by Parseval's formula,

$$\int_{-2}^{\infty}\Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\ell)dv = \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}[\Delta_1^2 G_H](\xi)\overline{\mathcal{F}[\Delta_1^2 G_H](\xi)}e^{-i\ell\xi}d\xi,$$

where $\mathcal{F}[\varphi](\xi)=\int_{\mathbb{R}}\varphi(x)e^{-ix\xi}d\xi$ denotes the Fourier transform of an L^2-function (which can be extended to the space of tempered distributions) and $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We need a few definitions and facts regarding Fourier transforms, which can be found in [25, Section 3.2 and Example 7.1.17]: for $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$,

$$\begin{aligned} x_\pm^\alpha &= (\pm x)^\alpha \mathbf{1}_{\{\pm x > 0\}}, \quad \mathcal{F}[x_\pm^\alpha](\xi) = \Gamma(\alpha+1) e^{\mp i\pi(\alpha+1)/2} (\xi \mp i0)^{-\alpha-1}, \\ (x \pm i0)^\alpha &= x_+^\alpha + e^{\pm i\pi\alpha} x_-^\alpha, \quad \mathcal{F}[(x \pm i0)^\alpha](\xi) = 2\pi e^{\pm i\pi\alpha/2} \Gamma(-\alpha)^{-1} \xi_\pm^{-\alpha-1}. \end{aligned} \tag{A.7}$$

In particular, still for $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$,

$$\begin{aligned} \mathcal{F}[|x|^\alpha](\xi) &= \Gamma(\alpha+1)(e^{-i\pi(\alpha+1)/2}(\xi - i0)^{-\alpha-1} + e^{i\pi(\alpha+1)/2}(\xi + i0)^{-\alpha-1}) \\ &= 2\Gamma(\alpha+1)\cos(\tfrac{\pi(\alpha+1)}{2})|\xi|^{-\alpha-1}. \end{aligned} \tag{A.8}$$

Moreover, by the fact that $\mathcal{F}[\varphi(\cdot + h)](\xi) = e^{ih\xi}\mathcal{F}[\varphi](\xi)$, the operator Δ_1^2 in the time domain corresponds to multiplication with $e^{2i\xi} - 2e^{i\xi} + 1$ in the Fourier domain. Therefore, recalling (3.16), we have the right-hand side of (3.25) equals

$$\begin{aligned} &\frac{K_H^{-2}\Gamma(H+\frac{3}{2})^2}{2\pi(H+\frac{1}{2})^2} \int_{\mathbb{R}} e^{-i\ell\xi} e^{-\frac{1}{2}i\pi(H+\frac{3}{2})}(\xi - i0)^{-H-3/2} e^{\frac{1}{2}i\pi(H+\frac{3}{2})}(\xi + i0)^{-H-3/2} \\ &\quad \times (e^{2i\xi} - 2e^{i\xi} + 1)(e^{-2i\xi} - 2e^{-i\xi} + 1) d\xi. \end{aligned}$$

Observe that

$$(e^{2i\xi} - 2e^{i\xi} + 1)(e^{-2i\xi} - 2e^{-i\xi} + 1) = e^{2i\xi} - 4e^{i\xi} + 6 - 4e^{-i\xi} + e^{-2i\xi} = (e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix})^4,$$

which corresponds to δ_1^4 in the time domain. Moreover, by (A.7),

$$\begin{aligned} (\xi - i0)^{-\alpha}(\xi + i0)^{-\alpha} &= \xi_+^{-2\alpha} + e^{i\pi\alpha}\xi_-^{-\alpha}\xi_+^{-\alpha} + \xi_+^{-\alpha}e^{-i\pi\alpha}\xi_-^{-\alpha} + \xi_-^{-2\alpha} \\ &= \xi_+^{-2\alpha} + \xi_-^{-2\alpha} = |\xi|^{-2\alpha}. \end{aligned} \tag{A.9}$$

Therefore, using the last formula in (A.7) and with the convention that δ_1^4 acts on the variable ℓ, we obtain

$$\begin{aligned} \Phi_\ell^H &= \frac{K_H^{-2}\Gamma(H+\frac{3}{2})^2}{2\pi(H+\frac{1}{2})^2} \int_{\mathbb{R}} e^{-i\ell\xi}(\xi - i0)^{-H-3/2}(\xi + i0)^{-H-3/2}(e^{\frac{1}{2}ix} - e^{-\frac{1}{2}ix})^4 d\xi \\ &= \frac{\Gamma(H+\frac{1}{2})^2\Gamma(-2H-2)}{2\pi K_H^2}\delta_1^4(e^{i\pi(H+1)}(\ell - i0)^{2H+2} + e^{-i\pi(H+1)}(\ell + i0)^{2H+2}) \\ &= \frac{\Gamma(H+\frac{1}{2})^2\Gamma(-2H-2)}{2\pi K_H^2}(e^{i\pi(H+1)} + e^{-i\pi(H+1)})\delta_1^4(\ell_+^{2H+2} + \ell_-^{2H+2}) \\ &= \frac{2\cos(\pi(H+1))\Gamma(H+\frac{1}{2})^2\Gamma(-2H-2)}{2\pi K_H^2}\delta_1^4|\ell|^{2H+2}. \end{aligned}$$

Using (2.7) and properties of the Gamma function, one can show that the factor in front of $\delta_1^4|\ell|^{2H+2}$ is equal to $1/(2(2H+1)(2H+2))$, proving (3.25). □

PROOF OF LEMMA 3.7. Recall (3.1). In a first step, we show that the contributions of A^η and $A^{\hat\eta}$ to (3.27) are negligible at a rate of $(k_n\delta_n)^{1/2}$. We only consider A^η, as our arguments apply to $A^{\hat\eta}$ analogously. If $j \leq (k_n/\delta_n)^{1/2}$, then $\sum_{j=1}^{[(k_n/\delta_n)^{1/2}]} \frac{1}{k_n\delta_n} \int_{(j-1)\delta_n}^{j\delta_n} \int_{(j-1)\delta_n}^{r}$ is of size $(\delta_n/k_n)^{1/2}$, the sum over i of the terms in $\{\cdots\}$ can be bounded by a multiple of $k_n^{1+\varepsilon}$, where $\varepsilon > 0$ can be as small as we want (cf. (A.6)), and $A_r^\eta - A_u^\eta$ is of size δ_n^H by Lemma A.1. So in total, the contribution of terms with $j \leq (k_n/\delta_n)^{1/2}$ is of size $(k_n\delta_n)^{1/2}k_n^\varepsilon\delta_n^H$, which is $o((k_n\delta_n)^{1/2})$ if ε is sufficiently small. If $j > (k_n/\delta_n)^{1/2}$, then $u \geq (k_n\delta_n)^{1/2}$ and therefore, by similar arguments, the contribution of the terms with $j > (k_n/\delta_n)^{1/2}$ is of size $k_n^{-1}k_n^{1+\varepsilon}(k_n\delta_n)^{-H/2}\delta_n^{(2H+1/2)\wedge 1}$, which is $o((k_n\delta_n)^{1/2})$ if ε is small.

It remains to analyze the expressions $\sum_{j=1}^{[t/\delta_n]-\ell k_n} \xi_j^n$ and $\sum_{j=1}^{[t/\delta_n]-\ell k_n} \hat{\xi}_j^n$, where

$$
\text{(A.10)}\quad \begin{aligned}
\xi_j^n = \frac{1}{k_n\delta_n} \sum_{i=1-2k_n}^{\infty} \int_{(j-1)\delta_n}^{j\delta_n} \int_{(j-1)\delta_n}^{r} \Big\{ &\Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{r/\delta_n\}}{k_n}) \\
&- \Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n})\Delta_1^2 G_H(\tfrac{i+\ell k_n-\{u/\delta_n\}}{k_n})\Big\} \int_0^r \Delta_{r-u} g_{H_\eta}(u-s)\boldsymbol{\theta}_s d\bar{\boldsymbol{W}}_s du dr
\end{aligned}
$$

and $\hat{\xi}_j^n$ is defined in the same way but with ϑ instead of θ. Clearly, it suffices to consider ξ_j^n. To this end, if $H_\eta \in (0, \frac{1}{2})$, we consider a sequence of numbers $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_Q < \lambda_{Q+1} = \infty$, whose values shall be determined at a later stage, and define $\lambda_n^{(q)} = [\delta_n^{-\lambda_q}]$ for all $q = 0, \ldots, Q+1$. In particular, $1 = \lambda_n^{(0)} \ll \lambda_n^{(1)} \ll \cdots \ll \lambda_n^{(Q)} < \lambda_n^{(Q+1)} = \infty$. Accordingly, we can define $\xi_j^{n,q}$ by the same formula as in (A.10), except we replace $\int_0^r \cdots d\bar{\boldsymbol{W}}_s$ by $\int_{(j+1-\lambda_n^{(q+1)})\delta_n}^{(j+1-\lambda_n^{(q)})\delta_n\wedge r} \cdots d\bar{\boldsymbol{W}}_s$. Then clearly

$$
\sum_{j=1}^{[t/\delta_n]-\ell k_n} \xi_j^n = \sum_{q=0}^{Q} \sum_{j=1}^{[t/\delta_n]-\ell k_n} \xi_j^{n,q}.
$$

Since $u, r \in ((j-1)\delta_n, j\delta_n)$, we have by the mean-value theorem (for $q = 1, \ldots, Q$) and a change of variables (for $q = 0$) that

$$
\text{(A.11)}\quad \begin{aligned}
\int_{(j+1-\lambda_n^{(q+1)})\delta_n}^{(j+1-\lambda_n^{(q)})\delta_n\wedge r} (\Delta_{r-u} g_{H_\eta}(u-s))^2 ds &\lesssim \begin{cases} \delta_n^2 \displaystyle\int_{(j+1-\lambda_n^{(q+1)})\delta_n}^{(j+1-\lambda_n^{(q)})\delta_n} (u-s)^{2H_\eta-3} ds & \text{if } q \geq 1, \\ \delta_n^{2H_\eta} \displaystyle\int_{j+1-\lambda_n^{(1)}}^{r/\delta_n} (\Delta_{\frac{r-u}{k_n\delta_n}} g_{H_\eta}(\tfrac{u}{\delta_n} - s))^2 ds & \text{if } q = 0. \end{cases} \\
&\lesssim \delta_n^{2H_\eta}(\lambda_n^{(q)})^{2H_\eta-2},
\end{aligned}
$$

which, in combination with previous arguments for the contribution of A^η, shows that

$$
\text{(A.12)}\quad \mathbb{E}[(\xi_j^{n,q})^2]^{1/2} \lesssim k_n^\varepsilon \delta_n^{1+H_\eta}(\lambda_n^{(q)})^{H_\eta-1}
$$

uniformly in n and j, with arbitrarily small $\varepsilon > 0$. Next, observe that ξ_j^n is $\mathcal{F}_{j\delta_n}$-measurable with $\mathbb{E}[\xi_j^n \mid \mathcal{F}_{(j+1-\lambda_n^{(q+1)})\delta_n}] = 0$. Therefore, using a martingale size estimate for $q = 0, \ldots, Q-1$ and a standard size estimate for $q = Q$ (see [11, Appendix A]), we obtain

$$
\text{(A.13)}\quad \begin{aligned}
&\mathbb{E}\Bigg[\sup_{t\in[0,T]} \Bigg(\sum_{j=1}^{[t/\delta_n]-\ell k_n} \xi_j^{n,q}\Bigg)^2\Bigg]^{1/2} \\
&\qquad\lesssim \begin{cases} (\lambda_n^{(q+1)}/\delta_n)^{1/2} k_n^\varepsilon \delta_n^{1+H_\eta}(\lambda_n^{(q)})^{H_\eta-1} & \text{if } q \leq Q-1, \\ k_n^\varepsilon \delta_n^{H_\eta}(\lambda_n^{(Q)})^{H_\eta-1} & \text{if } q = Q. \end{cases}
\end{aligned}
$$

We want this to go to zero faster than $(k_n\delta_n)^{1/2}$ for all $q = 0, \ldots, Q$. Because we can replace ε by $\frac{1}{2}\varepsilon$ in the last display, it suffices to start with $\lambda_0 = 0$ and then define $\lambda_1, \lambda_2, \ldots$ iteratively using the relation

$$
\text{(A.14)}\quad \begin{aligned}
&-\tfrac{1}{2}\lambda_{q+1} + (\tfrac{1}{2} - \varepsilon)\kappa + H_\eta + (1-H_\eta)\lambda_q = 0 \\
&\iff \lambda_{q+1} = (1-2\varepsilon)\kappa + 2H_\eta + 2(1-H_\eta)\lambda_q.
\end{aligned}
$$

The solution to this recurrence equation is

$$\lambda_q = \frac{((1-2\varepsilon)\kappa + 2H_\eta)((2-2H_\eta)^q - 1)}{1-2H_\eta}, \tag{A.15}$$

from which we see that $\lambda_q \to \infty$ if we keep iterating. Let Q be the smallest Q such that λ_Q, computed from the formula (A.15), is bigger than $(\frac{1-\kappa}{2} + \kappa\varepsilon - H_\eta)/(1-H_\eta)$, which is smaller than 1 if ε is small. Replacing λ_Q by a number between this threshold and 1 (if λ_Q from (A.15) exceeds 1), we obtain $\mathbb{E}[\sup_{t\in[0,T]}(\sum_{j=1}^{[t/\delta_n]-\ell k_n} \xi_j^{n,q})^2]^{1/2} = o((k_n\delta_n)^{1/2})$ for all $q = 0, \dots, Q$, proving the lemma for $H_\eta \in (0, \frac{1}{2})$.

If $H_\eta = \frac{1}{2}$, things are much simpler. Indeed, in this case, $\int_0^r \Delta_{r-u} g_{H_\eta}(u-s)\boldsymbol{\theta}_s d\bar{\boldsymbol{W}}_s = \int_u^r \boldsymbol{\theta}_s d\bar{\boldsymbol{W}}_s$, so $(k_n\delta_n)^{-1/2}\sum_{j=1}^{[t/\delta_n]-\ell k_n} \xi_j^n = O_{\mathbb{P}}(\delta_n^{-1/4}\delta_n^{-1}(k_n\delta_n)^{-1}k_n\delta_n^2\delta_n^{1/2}) = O_{\mathbb{P}}(\delta_n^{1/4})$. □

APPENDIX B: PROOF OF PROPOSITIONS 3.1 AND 3.3

PROOF OF PROPOSITION 3.3. The proposition follows from Lemmas B.1–B.3. □

LEMMA B.1. *Recall* (3.19) *and that* $y_t = \int_0^t \sigma_s dW_s$. *Under Assumption CLT', we have* $(k_n\delta_n)^{-1/2}(Z_3^{n,\ell}(t) - \tilde{Z}_3^{n,\ell}(t)) \stackrel{L^1}{\Longrightarrow} 0$ *and* $(k_n\delta_n)^{-1/2}(Z_3'^{n,\ell}(t) - M_3'^{n,\ell}(t)) \stackrel{L^1}{\Longrightarrow} 0$ *as* $n\to\infty$, *where*

$$\begin{aligned}\tilde{Z}_3^{n,\ell}(t) = {}& \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{1,i+\ell k_n}^n \\ & \times \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \chi\Big(\tfrac{[s/\delta_n]-i+1}{2k_n-1}\Big)(y_s - y_{[s/\delta_n]\delta_n})dy_s.\end{aligned} \tag{B.1}$$

PROOF. We only consider the approximation of $Z_3^{n,\ell}(t)$; the arguments for $Z_3'^{n,\ell}(t)$ are analogous. Using the equality $xy - x_0y_0 = (x-x_0)y_0 + x(y-y_0)$, we can decompose the difference $Z_3^{n,\ell}(t) - \tilde{Z}_3^{n,\ell}(t) = E_1^n(t) + E_2^n(t) + E_3^n(t)$, where

$$\begin{aligned}E_1^n(t) = {}& \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{1,i+\ell k_n}^n \\ & \times \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \chi\Big(\tfrac{[s/\delta_n]-i+1}{2k_n-1}\Big) \int_{[s/\delta_n]\delta_n}^s b_r dr dy_s,\\
E_2^n(t) = {}& \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{1,i+\ell k_n}^n \\ & \times \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \chi\Big(\tfrac{[s/\delta_n]-i+1}{2k_n-1}\Big)(x_s - x_{[s/\delta_n]\delta_n})b_s ds,\\
E_3^n(t) = {}& \frac{2(k_n\delta_n)^{-1-2H}}{k_n} \sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} D_{2,i+\ell k_n}^n \\ & \times \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \chi\Big(\tfrac{[s/\delta_n]-i+1}{2k_n-1}\Big)(x_s - x_{[s/\delta_n]\delta_n})dx_s.\end{aligned}$$

The first term is the easiest to deal with. The dy_s-integral is of order $\delta_n(k_n\delta_n)^{1/2}$, while $D^n_{1,i+\ell k_n}$ is of order $(k_n\delta_n)^{1+H}$ by (A.2). Hence,

$$E^n_1(t) = O_{\mathbb{P}}(k_n\delta_n)^{-2-2H}\delta_n(k_n\delta_n)^{1/2}(k_n\delta_n)^{1+H} = o((k_n\delta_n)^{1/2})$$

for any $\kappa \geq \frac{2H}{2H+1}$.

Next, consider $E^n_3(t)$ and denote the ds-integral by Y^n_i. Clearly, we have $\mathbb{E}[(Y^n_i)^2]^{1/2} \lesssim \delta_n^{1/2}(k_n\delta_n)^{1/2}$, uniformly in n and i. Interchanging summation over i with the integral defining $D^n_{2,i+\ell k_n}$ in (3.19), we have that

$$E^n_3(t) = \frac{2(k_n\delta_n)^{-1-2H}}{k_n}\int_{\ell k_n\delta_n}^{([t/\delta_n]-k_n)\delta_n} \sum_{i=[s/\delta_n]-(\ell+1)k_n+2}^{[s/\delta_n]-\ell k_n+1} Y^n_i(A_{s+k_n\delta_n} - A_s)ds.$$

The sum ranges over $O(k_n)$ many terms only. Thus, by Lemma A.1,

$$\mathbb{E}\left[\sup_{t\in[0,T]}|E^n_3(t)|\right] \lesssim (k_n\delta_n)^{-1-2H}\delta_n^{1/2}(k_n\delta_n)^{1/2}(k_n\delta_n)^{(2H+1/2)\wedge 1}\int_0^T(1+s^{-H})ds.$$

Distinguishing the two cases $H \leq \frac{1}{4}$ and $H \in (\frac{1}{4},\frac{1}{2})$, one can verify that the last line is $o((k_n\delta_n)^{1/2})$ for all $\kappa \in [\frac{2H}{2H+1},\frac{1}{2}]$ and $H \in (0,\frac{1}{2})$. We postpone the analysis of $E^n_3(t)$ if $H = \frac{1}{2}$ to the end of this proof.

The term $E^n_2(t)$ is more complicated. Let us first try a power-counting argument as before: the dy_s-integral is of order $\delta_n^{1/2}k_n\delta_n$, while $D^n_{1,i+\ell k_n} = O_{\mathbb{P}}((k_n\delta_n)^{1+H})$, so $E^n_2(t) = O_{\mathbb{P}}((k_n\delta_n)^{-2-2H}\delta_n^{1/2}k_n\delta_n(k_n\delta_n)^{1+H})$, which as the reader can check, is $o_{\mathbb{P}}((k_n\delta_n)^{1/2}$ if $\kappa > \frac{2H}{2H+1}$ but unfortunately only $O_{\mathbb{P}}((k_n\delta_n)^{1/2})$ if $\kappa = \frac{2H}{2H+1}$. While this simple approach fails for the boundary case $\kappa = \frac{2H}{2H+1}$, it shows one important point: when trying to improve the bound, we are allowed to make any modifications that generate an asymptotically vanishing error (the speed can be arbitrarily slow). For instance, we may replace x by y and, thanks to (3.5), b_s by $b_{(i-1)\delta_n}$ in the definition of $E^n_2(t)$, so that we only have to analyze

$$\begin{aligned}\text{(B.2)}\quad &\frac{2(k_n\delta_n)^{-1-2H}}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} b_{(i-1)\delta_n}\int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n}\chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})ds\\ &\times\int_0^{(i-1+(\ell+2)k_n)\delta_n}\Delta^2_{k_n\delta_n}G_H((i+\ell k_n-1)\delta_n - r)(\eta_r dW_r + \hat{\eta}_r d\hat{W}_r).\end{aligned}$$

Since $y_s - y_{[s/\delta_n]\delta_n} = \int_{[s/\delta_n]\delta_n}^s \sigma_r dW_r$, we can use the stochastic Fubini theorem to rewrite the ds-integral above as

$$\int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n}\int_r^{(r+\delta_n)\wedge(i-1+2k_n)\delta_n}\chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})ds\sigma_r dW_r.$$

We do not really need the explicit form of the new ds-integral, so let us denote it by ψ^n_r and only remark that $\mathbb{E}[\sup_{r\in[0,T]}|\psi^n_r|^p]^{1/p} \lesssim \delta_n^{3/2}$ for all $p > 0$. Using integration by parts, we can now write (B.2) as $E^n_{21}(t) + E^n_{22}(t) + E^n_{23}(t)$, where

$$\begin{aligned}E^n_{21}(t) = (k_n\delta_n)^{-1-2H}\frac{2}{k_n}&\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} b_{(i-1)\delta_n}\int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n}\psi^n_r\sigma_r\\ &\times\int_0^r \Delta^2_{k_n\delta_n}G_H((i+\ell k_n-1)\delta_n - u)(\eta_u dW_u + \hat{\eta}_u d\hat{W}_u)dW_r,\end{aligned}$$

$$E_{22}^n(t) = (k_n\delta_n)^{-1-2H}\frac{2}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} b_{(i-1)\delta_n} \times \int_{(i-1)\delta_n}^{(i-1+(\ell+2)k_n)\delta_n} \Delta_{k_n\delta_n}^2 G_H((i+\ell k_n-1)\delta_n - r) \times \int_{(i-1)\delta_n}^{r} \psi_u^n \sigma_u dW_u (\eta_r dW_r + \hat{\eta}_r d\hat{W}_r),$$

$$E_{23}^n(t) = (k_n\delta_n)^{-1-2H}\frac{2}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1} b_{(i-1)\delta_n} \times \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \Delta_{k_n\delta_n}^2 G_H((i+\ell k_n-1)\delta_n - r)\psi_r^n\sigma_r\eta_r dr.$$

Since

$$\text{(B.3)}\quad \int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} |\Delta_{k_n\delta_n}^2 G_H((i+\ell k_n-1)\delta_n - r)| dr = (k_n\delta_n)^{H+3/2}\int_{\ell-2}^{\ell} |\Delta_1^2 G_H(r)| dr,$$

we have that $E_{23}^n(t) = O_{\mathbb{P}}((k_n\delta_n)^{-2-2H}(k_n\delta_n)^{H+3/2}\delta_n^{3/2}) = o_{\mathbb{P}}((k_n\delta_n)^{1/2})$ for all $\kappa \geq \frac{2H}{2H+1}$. For both $E_{21}^n(t)$ and $E_{22}^n(t)$, note that the ith term is $\mathcal{F}_{(i-1+(\ell+2)k_n)\delta_n}$-measurable with zero $\mathcal{F}_{(i-1)\delta_n}$-conditional expectation. Moreover, using (A.2), we have that each summand is of order $(k_n\delta_n)^{1+H}(k_n\delta_n)^{1/2}\delta_n^{3/2}$. We can therefore apply a martingale size estimate (see [11, Appendix A]) to both terms and obtain

$$\text{(B.4)}\quad \mathbb{E}\left[\sup_{t\in[0,T]} |E_{21}^n(t) + E_{22}^n(t)|\right] \lesssim (k_n\delta_n)^{-1-2H}k_n^{-1}(k_n/\delta_n)^{1/2}(k_n\delta_n)^{1+H}(k_n\delta_n)^{1/2}\delta_n^{3/2},$$

which is $o((k_n\delta_n)^{1/2})$.

Lastly, let us come back to $E_3^n(t)$ if $H = \frac{1}{2}$. As in the case of $E_2^n(t)$, bounding term by term leads to an $O_{\mathbb{P}}((k_n\delta_n)^{1/2})$ estimate, which is just not enough at the considered rate. But we are allowed to modify $E_3^n(t)$ in the following way at no cost: we replace σ (which appears in y) by $\sigma_{(i-1)\delta_n}$ and a_r (which appears in $A_{s+k_n\delta_n} - A_s$, which in turn appears in $D_{2,i+\ell k_n}^n$) by $a_{(i-1)\delta_n}$. Once these changes are made, the ith term in $E_3^n(t)$ will be $\mathcal{F}_{(i-1+2k_n)\delta_n}$-measurable with zero $\mathcal{F}_{(i-1)\delta_n}$-conditional expectation, so we can conclude by a martingale size estimate. □

Next, using integration by parts, we have that

$$\tilde{Z}_3^{n,\ell}(t) = M_{31}^{n,\ell}(t) + M_{32}^{n,\ell}(t) + Q_3^{n,\ell}(t),$$

where

$$Q_3^{n,\ell}(t) = (k_n\delta_n)^{-1-2H}\frac{2}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\int_{(i-1)\delta_n}^{(i-1+2k_n)\delta_n} \Delta_{k_n\delta_n}^2 G_H((i-1+\ell k_n)\delta_n - u) \times \chi\left(\tfrac{[u/\delta_n]-i+1}{2k_n-1}\right)(y_u - y_{[u/\delta_n]\delta_n})\sigma_u\eta_u du.$$

LEMMA B.2. *Under Assumption CLT', if* $\kappa > \frac{2H}{2H+1}$, *then* $M_{31}^{n,\ell} + M_{32}^{n,\ell} + M_3'^{n,\ell} \stackrel{L^1}{\Longrightarrow} 0$.

PROOF. By (A.2), the ith term in the summation in $M_{31}^{n,\ell}(t)$, $M_{32}^{n,\ell}(t)$ and $M_3'^{n,\ell}(t)$ is of order $(k_n\delta_n)^{1+H}(k_n\delta_n)^{1/2}\delta_n^{1/2}$. Therefore, by a martingale size argument, very similarly to how we obtained (B.4), it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|M_{31}^{n,\ell}(t)+M_{32}^{n,\ell}(t)+M_3'^{n,\ell}(t)|\right] \lesssim (k_n\delta_n)^{-1-2H}k_n^{-1}(k_n/\delta_n)^{1/2}(k_n\delta_n)^{1+H}(k_n\delta_n)^{1/2}\delta_n^{1/2},$$

which is $o((k_n\delta_n)^{1/2})$ if (and only if) $\kappa > \frac{2H}{2H+1}$. □

LEMMA B.3. *Under Assumption CLT', we have* $(k_n\delta_n)^{-1/2}(Q_3^{n,\ell}(t) - \mathcal{A}_t^{n,\ell,k_n}) \overset{L^1}{\Longrightarrow} 0$ *for any* $\ell\geq 2$, *where* $\mathcal{A}_t^{n,\ell,k_n}$ *is defined in* (2.13). *In addition, we have that* $\mathcal{A}_t^{n,\ell,k_n} = O_{\mathbb{P}}(k_n^{-1/2-H})$. *In particular, if* $\kappa > \frac{1}{2+2H}$, *then* $(k_n\delta_n)^{-1/2}\mathcal{A}^{n,\ell,k_n} \overset{L^1}{\Longrightarrow} 0$. *The last condition is satisfied with* $\kappa = \frac{2H}{2H+1}$ *if and only if* $H > \frac{1}{4}(\sqrt{5}-1)\approx 0.3090$.

PROOF. In a first step, we decompose $Q_3^{n,\ell}(t) = Q_{31}^{n,\ell}(t) + Q_{32}^{n,\ell}(t)$, where

$$\begin{aligned}
Q_{31}^{n,\ell}(t) &= 2(k_n\delta_n)^{-1-2H}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\sum_{j=0}^{2k_n-1}\chi(\tfrac{j}{2k_n-1}) \\
&\qquad\times\int_{(i-1+j)\delta_n}^{(i+j)\delta_n}\Delta_{k_n\delta_n}^2 G_H((i-1+\ell k_n)\delta_n-u) \\
&\qquad\times(y_u - y_{(i+j-1)\delta_n})(\sigma_u\eta_u - \sigma_{(i+j-1)\delta_n}\eta_{(i+j-1)\delta_n})\,du, \\
Q_{32}^{n,\ell}(t) &= 2(k_n\delta_n)^{-1-2H}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\sum_{j=0}^{2k_n-1}\chi(\tfrac{j}{2k_n-1}) \\
&\qquad\times\int_{(i-1+j)\delta_n}^{(i+j)\delta_n}\Delta_{k_n\delta_n}^2 G_H((i-1+\ell k_n)\delta_n-u) \\
&\qquad\times(y_u - y_{(i+j-1)\delta_n})\sigma_{(i-1+j)\delta_n}\eta_{(i-1+j)\delta_n}\,du.
\end{aligned} \tag{B.5}$$

Let us consider $Q_{32}^{n,\ell}(t)$ first and interchange the sums over i and j. For every fixed j, we observe that the ith term is $\mathcal{F}_{(i+j)\delta_n}$-measurable with vanishing $\mathcal{F}_{(i+j-1)\delta_n}$-conditional expectation. Moreover, similarly to (B.3),

$$\int_{(i-1+j)\delta_n}^{(i+j)\delta_n}|\Delta_{k_n\delta_n}^2 G_H((i+\ell k_n-1)\delta_n-u)|du = (k_n\delta_n)^{H+3/2}\int_{\ell-\frac{j+1}{k_n}}^{\ell-\frac{j}{k_n}}|\Delta_1^2 G_H(u)|du \lesssim \frac{(k_n\delta_n)^{H+3/2}}{k_n}.$$

Therefore, for every j, the sum over i is a martingale sum, which yields

$$\mathbb{E}\left[\sup_{t\in[0,T]}|Q_{32}^{n,\ell}(t)|\right] \lesssim (k_n\delta_n)^{-1-2H}\frac{(k_n\delta_n)^{H+3/2}}{k_n}\delta_n^{-1/2}\delta_n^{1/2} = o((k_n\delta_n)^{1/2})$$

for all $\kappa\geq\frac{2H}{2H+1}$.

Consequently, we only have to consider $Q_{31}^{n,\ell}(t)$ further, which can be rewritten as

$$
\begin{aligned}
Q_{31}^{n,\ell}(t) &= 2(k_n\delta_n)^{-1/2-H}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\int_{(i-1)\delta_n}^{(i+2k_n-1)\delta_n}\chi(\tfrac{[u/\delta_n]-i+1}{2k_n-1}) \\
&\quad\times \Delta_1^2 G_H(\tfrac{i-1-u/\delta_n}{k_n}+\ell)(y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du \\
&= 2(k_n\delta_n)^{-1/2-H}\frac{1}{k_n}\int_0^{([t/\delta_n]-\ell k_n)\delta_n}\sum_{i=([u/\delta_n]-2k_n+2)\vee 1}^{([u/\delta_n]+1)\wedge([t/\delta_n]-(\ell+2)k_n+1)}\chi(\tfrac{[u/\delta_n]-i+1}{2k_n-1}) \\
&\quad\times \Delta_1^2 G_H(\tfrac{i-1-u/\delta_n}{k_n}+\ell)(y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du \\
&= 2(k_n\delta_n)^{-1/2-H}\frac{1}{k_n}\int_0^{([t/\delta_n]-\ell k_n)\delta_n}\sum_{i=(1-2k_n)\vee(-[u/\delta_n])}^{0\wedge([t/\delta_n]-[u/\delta_n]-(\ell+2)k_n)}\chi(\tfrac{-i}{2k_n-1}) \\
&\quad\times \Delta_1^2 G_H(\tfrac{i-\{u/\delta_n\}}{k_n}+\ell)(y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du.
\end{aligned}
$$

Similarly to how we proved Lemma 3.5, one can use (A.4) to show that $(k_n\delta_n)^{-1/2}(Q_{31}^{n,\ell}(t)-\tilde{Q}_{31}^{n,\ell}(t))\stackrel{L^1}{\Longrightarrow}0$, where

$$
\begin{aligned}
\tilde{Q}_{31}^{n,\ell}(t) &= 2(k_n\delta_n)^{-1/2-H}\frac{1}{k_n}\int_0^{([t/\delta_n]-\ell k_n)\delta_n}\sum_{i=0}^{2k_n-1}\chi(\tfrac{i}{2k_n-1})\Delta_1^2 G_H(\tfrac{-i-\{u/\delta_n\}}{k_n}+\ell) \\
&\quad\times (y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du.
\end{aligned}
$$

In fact, we can further change the upper bound of the integral and replace $\tilde{Q}_{31}^{n,\ell}(t)$ by

$$
\begin{aligned}
\hat{Q}_{31}^{n,\ell}(t) &= 2(k_n\delta_n)^{-1/2-H}\frac{1}{k_n}\int_0^t\sum_{i=0}^{2k_n-1}\chi(\tfrac{i}{2k_n-1})\Delta_1^2 G_H(\tfrac{-i-\{u/\delta_n\}}{k_n}+\ell) \\
&\quad\times (y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du.
\end{aligned}
\tag{B.6}
$$

Indeed, by (A.4), $\mathbb{E}[\sup_{t\in[0,T]}|\tilde{Q}_{31}^{n,\ell}(t)-\hat{Q}_{31}^{n,\ell}(t)|]\lesssim(k_n\delta_n)^{1/2-H}\delta_n^{1/2+H}=o((k_n\delta_n)^{1/2})$.

Now recall the definition of $\chi(t)$, which is -1 for $t<\frac{1}{2}$ and 1 for $t\geq\frac{1}{2}$. Therefore,

$$
\begin{aligned}
\hat{Q}_{31}^{n,\ell}(t) &= \frac{2(k_n\delta_n)^{-1/2-H}}{k_n}\int_0^t\sum_{i=0}^{k_n-1}\Big\{\Delta_1^2 G_H(\tfrac{-i-\{u/\delta_n\}}{k_n}+\ell-1) \\
&\qquad -\Delta_1^2 G_H(\tfrac{-i-\{u/\delta_n\}}{k_n}+\ell)\Big\}(y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du \\
&= -\frac{2(k_n\delta_n)^{-1/2-H}}{k_n}\int_0^t\sum_{i=0}^{k_n-1}\Delta_1^3 G_H(\ell-1-\tfrac{i+\{u/\delta_n\}}{k_n}) \\
&\qquad\qquad \times (y_u-y_{[u/\delta_n]\delta_n})(\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n})du,
\end{aligned}
$$

which shows that $\hat{Q}_{31}^{n,\ell}(t)$ is nothing else but the bias term $\mathcal{A}_t^{n,\ell,k_n}$. This establishes the first claim of the lemma. The second follows from (2.13) by observing that $\Delta_1^3 G_H$ is a bounded function (and, of course, that $y_u-y_{[u/\delta_n]\delta_n}$ and $\sigma_u\eta_u-\sigma_{[u/\delta_n]\delta_n}\eta_{[u/\delta_n]\delta_n}$ are of order $\delta_n^{1/2}$ and δ_n^H, respectively). The last two assertions are obvious. □

PROOF OF PROPOSITION 3.1. By (3.11), we have that

$$J^n_{1,i} = \frac{2}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i+k_n-1)\delta_n} (y_s - y_{[s/\delta_n]\delta_n})dy_s + \frac{2}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i+k_n-1)\delta_n} (y_s - y_{[s/\delta_n]\delta_n})b_s ds$$
$$+ \frac{2}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i+k_n-1)\delta_n}\int_{[s/\delta_n]\delta_n}^{s} b_r dr dy_s,$$

which implies

$$J^n_{1,i+k_n} - J^n_{1,i} = \frac{2}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i+2k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})dy_s$$
$$+ \frac{2}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i+2k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})b_s ds$$
$$+ \frac{2}{k_n\delta_n}\int_{(i-1)\delta_n}^{(i+2k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})\int_{[s/\delta_n]\delta_n}^{s} b_r dr dy_s.$$

Clearly, the last term is $O_{\mathbb{P}}(\sqrt{\delta_n/k_n})$, while $J^n_{1,i+k_n} - J^n_{1,i} = O_{\mathbb{P}}(\sqrt{1/k_n})$. Therefore, the contribution of the former to $Z^{n,\ell}_1(t)$ is $O_{\mathbb{P}}((k_n\delta_n)^{1-2H}(k_n\delta_n)^{-1}k_n^{-1/2}(\delta_n/k_n)^{1/2}) = O_{\mathbb{P}}(k_n^{-1-2H}\delta_n^{1/2-2H})$, which, as the reader may verify, is $o_{\mathbb{P}}((k_n\delta_n)^{1/2})$ for all $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$. Therefore,

$$(k_n\delta_n)^{-1/2}(Z^{n,\ell}_1(t) - M^{n,\ell}_1(t)) \approx (k_n\delta_n)^{-1/2}(F^n_1(t) + F^n_2(t)),$$

where

$$F^n_1(t) = 4(k_n\delta_n)^{-1-2H}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\int_{(i-1)\delta_n}^{(i+2k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})dy_s$$
$$\times \int_{(i+\ell k_n-1)\delta_n}^{(i+(\ell+2)k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i-\ell k_n+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})b_s ds,$$

$$F^n_2(t) = 4(k_n\delta_n)^{-1-2H}\frac{1}{k_n}\sum_{i=1}^{[t/\delta_n]-(\ell+2)k_n+1}\int_{(i+\ell k_n-1)\delta_n}^{(i+(\ell+2)k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i-\ell k_n+1}{2k_n-1})$$
$$\times (y_s - y_{[s/\delta_n]\delta_n})dy_s\int_{(i-1)\delta_n}^{(i+2k_n-1)\delta_n} \chi(\tfrac{[s/\delta_n]-i+1}{2k_n-1})(y_s - y_{[s/\delta_n]\delta_n})b_s ds.$$

Because $\ell \geq 2$, the ith term in $F^n_2(t)$ is $\mathcal{F}_{(i+(\ell+2)k_n-1)\delta_n}$-measurable with a vanishing $\mathcal{F}_{(i+\ell k_n-1)\delta_n}$-conditional expectation. Thus, by a martingale size estimate (see [11, Appendix A]) and the bounds found in the previous paragraph, we obtain that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|F^n_2(t)|\Big] \lesssim (k_n\delta_n)^{-1-2H}k_n^{-1}\sqrt{k_n/\delta_n}(k_n\delta_n^2)^{1/2}k_n\delta_n^{3/2} = (k_n\delta_n)^{-2H}\delta_n$$
$$= o((k_n\delta_n)^{1/2})$$

for all $\kappa \in [\frac{2H}{2H+1}, \frac{1}{2}]$. Regarding $F^n_1(t)$, observe that if we just applied a term-by-term size estimate, we would obtain $(k_n\delta_n)^{-1/2}F^n_1(t) = o_{\mathbb{P}}(1)$ if $\kappa > \frac{2H}{2H+1}$ but only $O_{\mathbb{P}}(1)$ if $\kappa = \frac{2H}{2H+1}$. To handle the latter case, note that we can replace b_s in $F^n_1(t)$ by $b_{(i-1)\delta_n}$ (by the preceding arguments and (3.5), the error is $o_{\mathbb{P}}((k_n\delta_n)^{1/2})$). After doing so, the ith in $F^n_1(t)$

will have a zero $\mathcal{F}_{(i+\ell k_n-1)\delta_n}$-conditional expectation, so applying another martingale size estimate yields $F_1^n(t) = o_{\mathbb{P}}((k_n\delta_n)^{1/2})$.

It remains to prove the last statement of the proposition. Because $\ell \geq 2$, it is easy to see that the ith term in (3.13) is $\mathcal{F}_{(i-1+(\ell+2)k_n)\delta_n}$-measurable while having a zero $\mathcal{F}_{(i-1+\ell k_n)\delta_n}$-conditional mean. By yet another martingale size estimate, it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M_1^{n,\ell}(t)|\right] \lesssim (k_n\delta_n)^{-1-2H} k_n^{-1}\sqrt{k_n/\delta_n} k_n \delta_n^2 = o((k_n\delta_n)^{1/2})$$

for $\kappa > \frac{2H}{2H+1}$. □

APPENDIX C: DETAILS FOR SECTION 3.2

PROOF OF PROPOSITION 3.8. Let us start with $M_2^{n,\ell,k_n}(t)$. Interchanging summation over i with the $d\boldsymbol{W}_r$-integral in (3.15) and breaking the latter into small pieces of length δ_n, we can rewrite $\delta_n^{-(1-\kappa)/2} M_2^{n,\ell,k_n}(t) = \sum_{j=1}^{[t/\delta_n]} \tilde{\zeta}_2^{n,j,\ell,k_n}$, where

$$\begin{aligned}
\tilde{\zeta}_2^{n,j,\ell,k_n} &= \frac{\delta_n^{-(1-\kappa)/2}(k_n\delta_n)^{-1-2H}}{k_n} \int_{(j-1)\delta_n}^{j\delta_n} \int_0^r \sum_{i=([r/\delta_n]-(\ell+2)k_n+2)\vee 1}^{[t/\delta_n]-(\ell+2)k_n+1} \\
&\quad \times \Big\{ \Delta_{k_n\delta_n}^2 G_H((i-1)\delta_n - r)\Delta_{k_n\delta_n}^2 G_H((i+\ell k_n-1)\delta_n - u) \\
&\qquad + \Delta_{k_n\delta_n}^2 G_H((i+\ell k_n-1)\delta_n - r)\Delta_{k_n\delta_n}^2 G_H((i-1)\delta_n - u)\Big\} \boldsymbol{\eta}_u d\boldsymbol{W}_u \boldsymbol{\eta}_r d\boldsymbol{W}_r
\end{aligned}$$

$$\begin{aligned}
\text{(C.1)} \qquad &= \delta_n^{-(1-\kappa)/2} \int_{(j-1)\delta_n}^{j\delta_n} \int_0^r \frac{1}{k_n}\Bigg(\sum_{i=(1-(\ell+2)k_n)\vee(-[r/\delta_n])}^{[t/\delta_n]-[r/\delta_n]-(\ell+2)k_n} \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n} + \tfrac{r-u}{k_n\delta_n} + \ell) + \sum_{i=(1-2k_n)\vee(\ell k_n-[r/\delta_n])}^{[t/\delta_n]-[r/\delta_n]-2k_n} \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n} + \tfrac{r-u}{k_n\delta_n} - \ell)\Bigg) \boldsymbol{\eta}_u d\boldsymbol{W}_u \boldsymbol{\eta}_r d\boldsymbol{W}_r.
\end{aligned}$$

Let us bound the pth moment of $\tilde{\zeta}_2^{n,j,\ell,k_n}$ for $p \geq 2$ and draw some conclusions. By the Burkholder–Davis–Gundy inequality and similar steps to (3.22) and (3.23), we have that

$$\begin{aligned}
\mathbb{E}[|\tilde{\zeta}_2^{n,j,\ell,k_n}|^p] &\lesssim \delta_n^{-(1-\kappa)p/2} \Bigg(\int_{(j-1)\delta_n}^{j\delta_n} \int_0^r \Bigg(\frac{1}{k_n} \sum_{i=1-2k_n}^{\infty} \Big\{ \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n} + \tfrac{r-u}{k_n\delta_n} + \ell) + \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{r/\delta_n\}}{k_n} + \tfrac{r-u}{k_n\delta_n} - \ell)\Big\}\Bigg)^2 du dr \Bigg)^{p/2}.
\end{aligned}$$

Changing $(r-u)/(k_n\delta_n)$ to u and noticing that $\frac{1}{k_n}\sum_{i=1-2k_n}^{\infty}$ is a Riemann sum, we obtain from (A.1) that

(C.2)
$$\begin{aligned}\mathbb{E}[|\tilde{\zeta}_2^{n,j,\ell,k_n}|^p] \lesssim \Bigg(\int_{(j-1)\delta_n}^{j\delta_n}\int_0^\infty\Bigg(\int_{-2}^\infty\Big\{\Delta_1^2 G_H(v)\Delta_1^2 G_H(v+u+\ell)\\ + \Delta_1^2 G_H(v)\Delta_1^2 G_H(v+u-\ell)\Big\}dv\Bigg)^2 dudr\Bigg)^{p/2}\\ \lesssim \delta_n^{p/2}.\end{aligned}$$

Consequently, $\tilde{\zeta}_2^{n,j,\ell,k_n}$ is of size $\delta_n^{1/2}$, uniformly in i. Because (3.15) is a sum of martingale increments (note that $\mathbb{E}[\tilde{\zeta}_2^{n,j,\ell,k_n} \mid \mathcal{F}_{(j-1)\delta_n}] = 0$), it follows that (3.15) is $O_{\mathbb{P}}(1)$. This is, of course, expected because (3.15) is supposed to contribute to the CLT. But what this calculation also shows is that before we try to find the limit of (3.15), we can make any modifications that lead to an $o_{\mathbb{P}}(1)$ error. For example, we can replace $\boldsymbol{\eta}_r$ by $\boldsymbol{\eta}_{(j-1)\delta_n}$ (this incurs an $O_{\mathbb{P}}(\delta_n^H)$ error) and replace $\frac{1}{k_n}$ times the two sums after second equality in (C.1) by

$$\int_{-2}^\infty\Big\{\Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\tfrac{r-u}{k_n\delta_n}+\ell) + \Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\tfrac{r-u}{k_n\delta_n}-\ell)\Big\}dv$$

(for modifying the upper and lower bounds of the summation, see the discussion after (3.23); for the integral approximation, the error is at most $k_n^{-1/2-H}$ because $\Delta_1^2 G_H$ is $(\frac{1}{2}+H)$-Hölder continuous). We will make two more changes, after which we will arrive at ζ_2^{n,j,ℓ,k_n}, hence proving the second relation in (3.28): first, we replace change the boundaries of the $d\boldsymbol{W}_u$-integral from $\int_0^r$ to $\int_{r-k_n\delta_n^{1-\varepsilon}}^r$, where $\varepsilon > 0$ is a small but fixed number. Similarly to (C.2), one can show that the resulting error is $\delta_n^{\varepsilon(1-H)}$. And second, we replace $\boldsymbol{\eta}_u$ first by $\boldsymbol{\eta}_{r-k_n\delta_n^{1-\varepsilon}}$ and then by $\boldsymbol{\eta}_{(j-1)\delta_n}$, which leads to an $O_{\mathbb{P}}((k_n\delta_n^{1-\varepsilon})^H)$ error.

Similar arguments can be employed to show the other two approximations in (3.28). Note that thanks to Proposition 3.1 and Lemma B.2, we only have to consider the case where $\kappa = \frac{2H}{2H+1}$. In order to show the first approximation in (3.28), we interchange summation and double integration in (3.13) and obtain

$$\begin{aligned}M_1^{n,\ell,k_n}(t) = 4(k_n\delta_n)^{-1-2H}\frac{1}{k_n}\int_{\ell k_n}^{[t/\delta_n]\delta_n}\int_{([s/\delta_n]-(\ell+2)k_n+1)\delta_n\vee 0}^{([s/\delta_n]-(\ell-2)k_n)\delta_n\wedge([t/\delta_n]-\ell k_n)\delta_n}\\ \sum_{i=(2+[s/\delta_n]-(\ell+2)k_n)\vee(2+[r/\delta_n]-2k_n)\vee 1}^{([s/\delta_n]-\ell k_n+1)\wedge([r/\delta_n]+1)\wedge([t/\delta_n]-(\ell+2)k_n+1)}\chi(\tfrac{[s/\delta_n]-i-\ell k_n+1}{2k_n-1})\chi(\tfrac{[r/\delta_n]-i+1}{2k_n-1})\\ \times(y_r - y_{[r/\delta_n]\delta_n})\sigma_r dW_r(y_s - y_{[s/\delta_n]\delta_n})\sigma_s dW_s.\end{aligned}$$

We change $i+\ell k_n - 1 - [s/\delta_n]$ to i and, with similar arguments to those after (C.2), omit the last $\wedge(\cdots)$ and $\vee(\cdots)$ in the boundaries of both the dW_r-integral and the sum over i. As a result,

$$\begin{aligned}\delta_n^{-(1-\kappa)/2}M_1^{n,\ell,k_n}(t) \approx 4\delta_n^{-(1-\kappa)/2}(k_n\delta_n)^{-1-2H}\frac{1}{k_n}\int_{\ell k_n}^{[t/\delta_n]\delta_n}\int_{([s/\delta_n]-(\ell+2)k_n+1)\delta_n}^{([s/\delta_n]-(\ell-2)k_n)\delta_n}\\ \sum_{i=(1-2k_n)\vee([r/\delta_n]-[s/\delta_n]+(\ell-2)k_n+1)}^{0\wedge([r/\delta_n]-[s/\delta_n]+\ell k_n}\chi(\tfrac{-i}{2k_n-1})\chi(\tfrac{[r/\delta_n]-[s/\delta_n]-i+\ell k_n}{2k_n-1})\\ \times(y_r - y_{[r/\delta_n]\delta_n})\sigma_r dW_r(y_s - y_{[s/\delta_n]\delta_n})\sigma_s dW_s.\end{aligned}$$

Because

$$\sum_{i=(1-2k_n)\vee(1-2k_n+m)}^{0\wedge m} \chi(\tfrac{-i}{2k_n-1})\chi(\tfrac{-i+m}{2k_n-1}) = 2k_n\xi(\tfrac{m}{2k_n}),$$

the first approximation in (3.28) follows by replacing all σ's by $\sigma_{[s/\delta_n]\delta_n}$.

Regarding the last approximation in (3.28), we have analogously to (C.1) that

$$\begin{aligned}
&\delta_n^{-(1-\kappa)/2} M_{31}^{n,\ell,k_n}(t) = \sum_{j=1}^{[t/\delta_n]-\ell k_n} \tilde{\zeta}_{31}^{n,j,\ell,k_n}, \quad \delta_n^{-(1-\kappa)/2} M_{32}^{n,\ell,k_n}(t) = \sum_{j=1}^{[t/\delta_n]} \tilde{\zeta}_{32}^{n,j,\ell,k_n}, \\
&\delta_n^{-(1-\kappa)/2} M_3^{\prime n,\ell,k_n}(t) = \sum_{j=1+\ell k_n}^{[t/\delta_n]} \tilde{\zeta}_3^{\prime n,j,\ell,k_n},
\end{aligned} \tag{C.3}$$

where

$$\begin{aligned}
\tilde{\zeta}_{31}^{n,j,\ell,k_n} &= \frac{2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2}}{k_n} \int_{(j-1)\delta_n}^{j\delta_n} \int_0^s \sum_{i=(1-2k_n)\vee(-[s/\delta_n])}^{0\wedge([t/\delta_n]-[s/\delta_n]-(\ell+2)k_n)} \chi(\tfrac{-i}{2k_n-1}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{s/\delta_n\}}{k_n} + \tfrac{s-r}{k_n} + \ell)\boldsymbol{\eta}_r d\boldsymbol{W}_r (y_s - y_{[s/\delta_n]\delta_n})\sigma_s dW_s, \\
\tilde{\zeta}_{32}^{n,j,\ell,k_n} &= \frac{2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2}}{k_n} \int_{(j-1)\delta_n}^{j\delta_n} \int_{([r/\delta_n]-(\ell+2)k_n+1)\delta_n\vee 0}^{r\wedge([t/\delta_n]-\ell k_n)\delta_n} (y_s - y_{[s/\delta_n]\delta_n}) \\
&\quad \times \sum_{i=(1-2k_n)\vee([r/\delta_n]-[s/\delta_n]-(\ell+2)k_n+1)\vee(-[s/\delta_n])}^{0\wedge([t/\delta_n]-[s/\delta_n]-(\ell+2)k_n)} \chi(\tfrac{-i}{2k_n-1}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{s/\delta_n\}}{k_n} - \tfrac{r-s}{k_n} + \ell)\sigma_s dW_s \boldsymbol{\eta}_r d\boldsymbol{W}_r, \\
\tilde{\zeta}_3^{\prime n,j,\ell,k_n} &= \frac{2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2}}{k_n} \int_{(j-1)\delta_n}^{j\delta_n} \int_0^{([s/\delta_n]-(\ell-2)k_n)\delta_n\wedge([t/\delta_n]-\ell k_n)\delta_n} \sigma_s \\
&\quad \times (y_s - y_{[s/\delta_n]\delta_n}) \sum_{i=(1-2k_n)\vee([r/\delta_n]-[s/\delta_n]+(\ell-2)k_n+1)\vee(\ell k_n-[s/\delta_n])}^{0\wedge([t/\delta_n]-[s/\delta_n]-2k_n)} \chi(\tfrac{-i}{2k_n-1}) \\
&\quad \times \Delta_1^2 G_H(\tfrac{i-\{s/\delta_n\}}{k_n} + \tfrac{s-r}{k_n} - \ell)\boldsymbol{\eta}_r d\boldsymbol{W}_r dW_s.
\end{aligned}$$

Since $\kappa = \frac{2H}{2H+1}$, it can be shown similarly to (C.2) and the subsequent paragraph that each of the three terms defined in (C.3) is of order $O_{\mathbb{P}}(1)$. Therefore, by the same type of modifications (i.e., discretization of $\boldsymbol{\eta}$ and σ, dropping $\wedge(\cdots)$ and $\vee(\cdots)$ in the summation over i, approximating sums by integrals, and restricting the $d\boldsymbol{W}_r$-integral in $\tilde{\zeta}_{31}^{n,j,\ell,k_n}$ and $\tilde{\zeta}_{31}^{\prime n,j,\ell,k_n}$ to $r \geq s - k_n\delta_n^{1-\varepsilon}$), we obtain $\delta_n^{-(1-\kappa)/2}(M_{31|32}^{n,\ell,k_n}(t) - \bar{M}_{31|32}^{n,\ell,k_n}(t)) \stackrel{L^1}{\Longrightarrow} 0$ and $\delta_n^{-(1-\kappa)/2}(M_3^{\prime n,\ell,k_n}(t) - \bar{M}_3^{\prime n,\ell,k_n}(t)) \stackrel{L^1}{\Longrightarrow} 0$, where

$$\bar{M}_{31|32}^{n,\ell,k_n}(t) = \sum_{j=1}^{[t/\delta_n]-\ell k_n} \bar{\zeta}_{31|32}^{n,j,\ell,k_n}, \qquad \bar{M}_3^{\prime n,\ell,k_n}(t) = \sum_{j=1+\ell k_n}^{[t/\delta_n]} \bar{\zeta}_3^{\prime n,j,\ell,k_n}$$

and

$$\bar{\zeta}_{31}^{n,j,\ell,k_n} = 2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^2_{(j-1)\delta_n}\int_{s-k_n\delta_n^{1-\varepsilon}}^{s}\int_0^2\chi(\tfrac{u}{2})$$
$$\times\,\Delta_1^2 G_H(\tfrac{s-r}{k_n\delta_n}+\ell-u)du\,\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_r(W_s-W_{(j-1)\delta_n})dW_s,$$
$$\bar{\zeta}_{32}^{n,j,\ell,k_n} = 2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^2_{(j-1)\delta_n}\int_{([r/\delta_n]-(\ell+2)k_n+1)\delta_n}^{r}(W_s-W_{[s/\delta_n]\delta_n})$$
$$\times\int_0^{2-((\frac{r-s}{k_n\delta_n}-\ell)\vee 0)}\chi(\tfrac{u}{2})\Delta_1^2 G_H(\ell-\tfrac{r-s}{k_n\delta_n}-u)du dW_s\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_r, \tag{C.4}$$
$$\bar{\zeta}_{3}^{\prime n,j,\ell,k_n} = 2(k_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)/2}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^2_{(j-1)\delta_n}\int_{s-k_n\delta_n^{1-\varepsilon}}^{([s/\delta_n]-(\ell-2)k_n)\delta_n}(W_s-W_{(j-1)\delta_n})$$
$$\times\int_0^{2-((\ell-\frac{s-r}{k_n\delta_n})\vee 0)}\chi(\tfrac{u}{2})\Delta_1^2 G_H(\tfrac{s-r}{k_n\delta_n}-\ell-u)du\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_r dW_s.$$

Let us make three observations: First, for any of the three terms in (C.4), by a straightforward power-counting argument, if we restrict the inner integral to $((j-1)\delta_n, s)$ or $((j-1)\delta_n, r)$, respectively, the second moment of the resulting term will be of order $\delta_n^{1+2H/(2H+1)} = o(\delta_n)$, showing that the latter is asymptotically negligible (cf. (C.2) and the subsequent arguments). Second, by the definition of $\chi(t)$,

$$\int_0^2 \chi(\tfrac{u}{2})\Delta_1^2 G_H(\tfrac{s-r}{k_n\delta_n}+\ell-u)du = -\int_0^1 \Delta_1^3 G_H(\tfrac{s-r}{k_n\delta_n}+\ell-u-1)du.$$

And finally, because $\Delta_1^2 G_H(v)=0$ for $v\leq -2$, there is no harm in extending the du-integral in $\bar{\zeta}_{32}^{n,j,\ell,k_n}$ and $\bar{\zeta}_{3}^{\prime n,j,\ell,k_n}$ up to the upper bound 2. The aforementioned modifications turn $\bar{\zeta}_{32}^{n,j,\ell,k_n}$ into $\zeta_{32}^{n,j,\ell,k_n}$ and the sum $\bar{\zeta}_{31}^{n,j,\ell,k_n}+\bar{\zeta}_{3}^{\prime n,j,\ell,k_n}$ into $\zeta_{31}^{n,j,\ell,k_n}$, which establishes the last relation in (3.28). □

PROOF OF EQUATION (3.35). For any $\nu\in\{1,2,3\}$, we have seen in the proof of Proposition 3.8 that $\mathbb{E}[|\zeta_\nu^{n,j,\ell,k_n}|^p]\lesssim \delta_n^{p/2}$, uniformly in j. Setting $p=4$, we easily obtain that the left-hand side of (3.35) is $O_{\mathbb{P}}(\delta_n)$. □

PROOF OF EQUATION (3.36). We only show (3.36) for $\nu=2$ as the arguments for $\nu=1$ and $\nu=3$ are similar. Note that ζ_2^{n,j,ℓ,k_n} can be decomposed into two parts, $\zeta_{21}^{n,j,\ell,k_n}$ and $\zeta_{22}^{n,j,\ell,k_n}$, which are defined in the same way as ζ_2^{n,j,ℓ,k_n} in (3.30), except that the $d\boldsymbol{W}_u$-integral is restricted to $(r-k_n\delta_n^{1-\varepsilon},(j-1)\delta_n)$ and $((j-1)\delta_n, r)$, respectively. By definition, $\zeta_{22}^{n,j,\ell,k_n}$ belongs to the second Wiener chaos with respect to $\boldsymbol{W}$, conditionally on $\mathcal{F}_{(j-1)\delta_n}$. Thus, $\mathbb{E}[\zeta_{22}^{n,j,\ell,k_n}(N_{j\delta_n}-N_{(j-1)\delta_n})\mid\mathcal{F}_{(j-1)\delta_n}]=0$ by the orthogonality of Wiener chaoses of different orders if $N\in\{W,\hat{W}\}$ and by the orthogonality of N and W otherwise. If N is orthogonal to W, we also have $\mathbb{E}[\zeta_{21}^{n,j,\ell,k_n}(N_{j\delta_n}-N_{(j-1)\delta_n})\mid\mathcal{F}_{(j-1)\delta_n}]=0$, so let us assume that $N=W$ or $N=\hat{W}$. Since the two cases are completely analogous, we take $N=W$. Then

$$\mathbb{E}[\zeta_{21}^{n,j,\ell,k_n}(N_{j\delta_n}-N_{(j-1)\delta_n})\mid\mathcal{F}_{(j-1)\delta_n}]$$
$$=\delta_n^{-(1-\kappa)/2}\int_{(j-1)\delta_n}^{j\delta_n}\int_{r-k_n\delta_n^{1-\varepsilon}}^{(j-1)\delta_n}\int_{-2}^{\infty}\Big\{\Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\tfrac{r-u}{k_n\delta_n}+\ell)$$

$$+ \Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\tfrac{r-u}{k_n\delta_n}-\ell)\Big\}dv\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_u\eta_{(j-1)\delta_n}dr.$$

Since taking conditional expectation is a contraction on L^2, this term is still of size $O_{\mathbb{P}}(\delta_n)$. Consequently, for the purpose of showing (3.36), we may replace $\eta_{(j-1)\delta_n}$ and $\boldsymbol{\eta}_{(j-1)\delta_n}$ in the previous display by $\eta_{(j-1-k_n\delta_n^{-\varepsilon})\delta_n}$ and $\boldsymbol{\eta}_{(j-1-k_n\delta_n^{-\varepsilon})\delta_n}$, respectively. Once we have done so, the resulting expression will be $\mathcal{F}_{(j-1)\delta_n}$-measurable with vanishing $\mathcal{F}_{(j-1-k_n\delta_n^{-\varepsilon})\delta_n}$-conditional expectation. Therefore, by a martingale size estimate (see [11, Appendix A]), it follows that

$$\text{(C.5)}\qquad \mathbb{E}\Bigg[\Bigg|\sum_{j=1}^{[t/\delta_n]}\mathbb{E}[\zeta_{21}^{n,j,\ell,k_n}(N_{j\delta_n}-N_{(j-1)\delta_n})\mid\mathcal{F}_{(j-1)\delta_n}]\Bigg|\Bigg]\lesssim(k_n\delta_n^{-\varepsilon-1})^{1/2}\delta_n\to 0,$$

proving (3.36) for $\nu=2$. □

PROOF OF EQUATION (3.33). Again let us start with $\nu=2$. There is no loss of generality to restrict ourselves to $m=1$ and $m'=2$, in which case we simply write $k_n=k_n^{(1)}$ and $k_n'=k_n^{(2)}$. We want to find the limit of

$$\text{(C.6)}\qquad R_{22}^{n,\ell_1,k_n,\ell_2,k_n'}(t)=\sum_{j=1}^{[t/\delta_n]}\mathbb{E}[\zeta_2^{n,j,\ell_1,k_n}\zeta_2^{n,j,\ell_2,k_n'}\mid\mathcal{F}_{(j-1)\delta_n}],$$

where $\ell_1,\ell_2\geq 2$, $k_n\sim\theta_1\delta_n^{-\kappa}$ and $k_n'\sim\theta_2\delta_n^{-\kappa}$. Moreover, by the flexibility we have in the truncation of the $d\boldsymbol{W}_u$-integral in (3.30), we may and will assume that it runs from $r-k_n\delta_n^{1-\varepsilon}$ to r for both ζ_1^{n,j,ℓ,k_n} and ζ_2^{n,j,ℓ,k_n}. By Itô's isometry, we then have $R_{22}^{n,\ell_1,k_n,\ell_2,k_n'}(t)=\sum_{\iota=1}^3 R_{22,\iota}^{n,\ell_1,k_n,\ell_2,k_n'}(t)$ where

$$\begin{aligned}
R_{22,1}^{n,\ell_1,k_n,\ell_2,k_n'}(t)={}&\sum_{j=1}^{[t/\delta_n]}\delta_n^{-(1-\kappa)}|\boldsymbol{\eta}_{(j-1)\delta_n}|^4\int_{(j-1)\delta_n}^{j\delta_n}\int_{r-k_n\delta_n^{1-\varepsilon}}^{r}\int_{-2}^{\infty}\Big\{\Delta_1^2G_H(v)\\
&\times\Delta_1^2G_H(v+\tfrac{r-u}{k_n\delta_n}+\ell_1)+\Delta_1^2G_H(v)\Delta_1^2G_H(v+\tfrac{r-u}{k_n\delta_n}-\ell_1)\Big\}dv\\
&\times\int_{-2}^{\infty}\Big\{\Delta_1^2G_H(w)\Delta_1^2G_H(w+\tfrac{r-u}{k_n'\delta_n}+\ell_2)\\
&+\Delta_1^2G_H(w)\Delta_1^2G_H(w+\tfrac{r-u}{k_n'\delta_n}-\ell_2)\Big\}dw\,du\,dr,\\
R_{22,2}^{n,\ell_1,k_n,\ell_2,k_n'}(t)={}&\sum_{j=1}^{[t/\delta_n]}\delta_n^{-(1-\kappa)}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\int_{(j-1)\delta_n}^{j\delta_n}\int_{r-k_n\delta_n^{1-\varepsilon}}^{(j-1)\delta_n}\int_{-2}^{\infty}\Big\{\Delta_1^2G_H(v)\\
&\times\Delta_1^2G_H(v+\tfrac{r-u}{k_n\delta_n}+\ell_1)+\Delta_1^2G_H(v)\Delta_1^2G_H(v+\tfrac{r-u}{k_n\delta_n}-\ell_1)\Big\}dv\\
&\times\int_{r-k_n\delta_n^{1-\varepsilon}}^{u}\int_{-2}^{\infty}\Big\{\Delta_1^2G_H(w)\Delta_1^2G_H(w+\tfrac{r-u'}{k_n'\delta_n}+\ell_2)+\Delta_1^2G_H(w)\\
&\times\Delta_1^2G_H(w+\tfrac{r-u'}{k_n'\delta_n}-\ell_2)\Big\}dw\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_{u'}\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_u dr,\\
R_{22,3}^{n,\ell_1,k_n,\ell_2,k_n'}(t)={}&\sum_{j=1}^{[t/\delta_n]}\delta_n^{-(1-\kappa)}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\int_{(j-1)\delta_n}^{j\delta_n}\int_{r-k_n\delta_n^{1-\varepsilon}}^{(j-1)\delta_n}\int_{-2}^{\infty}\Big\{\Delta_1^2G_H(v)\\
&\times\Delta_1^2G_H(v+\tfrac{r-u}{k_n'\delta_n}+\ell_2)+\Delta_1^2G_H(v)\Delta_1^2G_H(v+\tfrac{r-u}{k_n'\delta_n}-\ell_2)\Big\}dv
\end{aligned}$$

$$\times \int_{r-k_n\delta_n^{1-\varepsilon}}^{u} \int_{-2}^{\infty} \Big\{ \Delta_1^2 G_H(w) \Delta_1^2 G_H(w + \tfrac{r-u'}{k_n\delta_n} + \ell_1) + \Delta_1^2 G_H(w) \times \Delta_1^2 G_H(w + \tfrac{r-u'}{k_n\delta_n} - \ell_1) \Big\} dw \boldsymbol{\eta}_{(j-1)\delta_n} d\boldsymbol{W}_{u'} \boldsymbol{\eta}_{(j-1)\delta_n} d\boldsymbol{W}_u dr.$$

Repeating the argument leading to (C.5) shows that $R_{22,2}^{n,\ell_1,k_n,\ell_2,k_n'}(t)$ and $R_{22,3}^{n,\ell_1,k_n,\ell_2,k_n'}(t)$ are $O_{\mathbb{P}}((k_n\delta_n^{-\varepsilon}/\delta_n)^{1/2}\delta_n) = O_{\mathbb{P}}((k_n\delta_n^{1-\varepsilon})^{1/2}) = o_{\mathbb{P}}(1)$, and hence they do not contribute to the limit of (C.6). So only $R_{22,1}^{n,\ell_1,k_n,\ell_2,k_n'}(t)$ is asymptotically relevant.

By a change of variables ($(r-u)/\delta_n^{1-\kappa}$ to u),

$$R_{22,1}^{n,\ell_1,k_n,\ell_2,k_n'}(t) = \sum_{j=1}^{[t/\delta_n]} |\boldsymbol{\eta}_{(j-1)\delta_n}|^4 \int_{(j-1)\delta_n}^{j\delta_n} \int_0^{k_n\delta_n^{\kappa-\varepsilon}} \int_{-2}^{\infty} \Big\{ \Delta_1^2 G_H(v) \times \Delta_1^2 G_H(v + u\tfrac{\delta_n^{-\kappa}}{k_n} + \ell_1) + \Delta_1^2 G_H(v) \Delta_1^2 G_H(v + u\tfrac{\delta_n^{-\kappa}}{k_n} - \ell_1) \Big\} dv \times \int_{-2}^{\infty} \Big\{ \Delta_1^2 G_H(w) \Delta_1^2 G_H(w + u\tfrac{\delta_n^{-\kappa}}{k_n'} + \ell_2) + \Delta_1^2 G_H(w) \Delta_1^2 G_H(w + u\tfrac{\delta_n^{-\kappa}}{k_n'} - \ell_2) \Big\} dw du dr,$$

so we obtain $R_{22,1}^{n,\ell_1,k_n,\ell_2,k_n'}(t) \sim \gamma_2^{\ell_1,\theta_1,\ell_2,\theta_2}(H)\Gamma_2(t)$ once we establish

$$\gamma_2^{\ell_1,\theta_1,\ell_2,\theta_2}(H) = \int_0^{\infty} \int_{-2}^{\infty} \Big\{ \Delta_1^2 G_H(v) \Delta_1^2 G_H(v + u/\theta_1 + \ell_1) + \Delta_1^2 G_H(v) \Delta_1^2 G_H(v + u/\theta_1 - \ell_1) \Big\} dv \times \int_{-2}^{\infty} \Big\{ \Delta_1^2 G_H(w) \Delta_1^2 G_H(w + u/\theta_2 + \ell_2) + \Delta_1^2 G_H(w) \Delta_1^2 G_H(w + u/\theta_2 - \ell_2) \Big\} dw du. \tag{C.7}$$

By (2.12) (and its extension to $\ell \in \mathbb{R}$ as shown in the proof), the right-hand side equals

$$\begin{aligned} &\frac{1}{4(2H+1)^2(2H+2)^2} \int_0^{\infty} (\delta_1^4|u/\theta_1 + \ell_1|^{2H+2} + \delta_1^4|u/\theta_1 - \ell_1|^{2H+2}) \times (\delta_1^4|u/\theta_2 + \ell_2|^{2H+2} + \delta_1^4|u/\theta_2 - \ell_2|^{2H+2}) du \\ &= \frac{1}{4(2H+1)^2(2H+2)^2} \Big(\int_{\mathbb{R}} \delta_1^4|u/\theta_1 + \ell_1|^{2H+2} \delta_1^4|u/\theta_2 + \ell_2|^{2H+2} du + \int_{\mathbb{R}} \delta_1^4|u/\theta_1 + \ell_1|^{2H+2} \delta_1^4|u/\theta_2 - \ell_2|^{2H+2} du \Big), \end{aligned} \tag{C.8}$$

where the second step follows by symmetry. By Parseval's identity,

$$\begin{aligned} &\int_{\mathbb{R}} \delta_1^4|u/\theta_1 + \ell_1|^{2H+2} \delta_1^4|u/\theta_2 + \ell_2|^{2H+2} du \\ &= \frac{1}{(\theta_1\theta_2)^{2H+2}} \int_{\mathbb{R}} \delta_{\theta_1}^4|u + \ell_1\theta_1|^{2H+2} \delta_{\theta_2}^4|u + \ell_2\theta_2|^{2H+2} du \\ &= \frac{1}{2\pi(\theta_1\theta_2)^{2H+2}} \int_{\mathbb{R}} e^{i(\ell_1\theta_1 - \ell_2\theta_2)\xi} \mathcal{F}[|x|^{2H+2}](\xi)^2 \times (e^{\frac{1}{2}i\theta_1\xi} - e^{-\frac{1}{2}i\theta_1\xi})^4 (e^{\frac{1}{2}i\theta_2\xi} - e^{-\frac{1}{2}i\theta_2\xi})^4 d\xi. \end{aligned} \tag{C.9}$$

The product $(e^{\frac{1}{2}i\theta_1\xi} - e^{-\frac{1}{2}i\theta_1\xi})^4(e^{\frac{1}{2}i\theta_2\xi} - e^{-\frac{1}{2}i\theta_2\xi})^4$ translates into $\delta_{\theta_1}^4\delta_{\theta_2}^4$ in the time domain. Together with (A.8), this yields

$$\begin{aligned}&\int_{\mathbb{R}} \delta_1^4|u/\theta_1 + \ell_1|^{2H+2}\delta_1^4|u/\theta_2 + \ell_2|^{2H+2}du \\ &\quad = \frac{2\cos^2(\pi(H+\frac{3}{2}))\Gamma(2H+3)^2}{\pi(\theta_1\theta_2)^{2H+2}} \int_{\mathbb{R}} e^{i(\ell_1\theta_1-\ell_2\theta_2)\xi}|\xi|^{-4H-6} \\ &\qquad\qquad \times (e^{\frac{1}{2}i\theta_1\xi} - e^{-\frac{1}{2}i\theta_1\xi})^4(e^{\frac{1}{2}i\theta_2\xi} - e^{-\frac{1}{2}i\theta_2\xi})^4 d\xi \\ &\quad = \frac{4\cos^2(\pi(H+\frac{3}{2}))\cos(\pi(2H+\frac{5}{2}))\Gamma(2H+3)^2\Gamma(-5-4H)}{\pi(\theta_1\theta_2)^{2H+2}}\delta_{\theta_1}^4\delta_{\theta_2}^4|\ell_2\theta_2 - \ell_1\theta_1|^{4H+5},\end{aligned}$$

where the last step is valid for all $H \in (0, \frac{1}{2}) \setminus \{\frac{1}{4}\}$. Inserting this into (C.8) and simplifying the resulting expression, we finally obtain (C.7) if $H \notin \{\frac{1}{4}, \frac{1}{2}\}$. To obtain the results for $H \in \{\frac{1}{4}, \frac{1}{2}\}$, it suffices by the dominated convergence theorem to let $H \to \frac{1}{4}$ and $H \to \frac{1}{2}$ in the formula established for $H \in (0, \frac{1}{2}) \setminus \{\frac{1}{4}\}$. As there is no singularity at $H = \frac{1}{2}$, this formula continues to hold for $H = \frac{1}{2}$. For $H = \frac{1}{4}$, it suffices to note that $(1 - 1/\cos(2\pi H))(H - \frac{1}{4}) \to \frac{1}{2\pi}$ as $H \to \frac{1}{4}$ and that

$$\lim_{H\to\frac{1}{4}} \frac{\delta_{\theta_1}^4\delta_{\theta_2}^4|\ell_2\theta_2 - \ell_1\theta_1|^{4H+5}}{H - \frac{1}{4}} = 4\delta_{\theta_1}^4\delta_{\theta_2}^4[|\ell_2\theta_2 - \ell_1\theta_1|^6 \log|\ell_2\theta_2 - \ell_1\theta_1|]$$

by L'Hôpital's rule.

Next, we consider $\nu = 1$. As in (C.6) we want to find the limit of

$$R_{11}^{n,\ell_1,k_n,\ell_2,k_n'}(t) = \sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_1^{n,j,\ell_1,k_n}\zeta_1^{n,j,\ell_2,k_n'} \mid \mathcal{F}_{(j-1)\delta_n}],$$

where $\ell_1, \ell_2 \geq 2$, $k_n \sim \theta_1\delta_n^{-\kappa}$ and $k_n' \sim \theta_2\delta_n^{-\kappa}$ with $\kappa = \frac{2H}{2H+1}$.

By Itô's isometry,

$$\begin{aligned}R_{11}^{n,\ell_1,k_n,\ell_2,k_n'}(t) &= 64\delta_n^{\kappa-1}(\theta_1\theta_2)^{-1-2H}\delta_n^{-2}\sum_{j=1}^{[t/\delta_n]}\int_{(j-1)\delta_n}^{j\delta_n} \sigma_{(j-1)\delta_n}^8 \\ &\quad \times \mathbb{E}\Bigg[\int_{([s/\delta_n]-(\ell_1+2)k_n+1)\delta_n}^{([s/\delta_n]-(\ell_1-2)k_n)\delta_n} \xi(\tfrac{[r/\delta_n]-[s/\delta_n]+\ell_1k_n}{2k_n})(W_r - W_{[r/\delta_n]\delta_n})dW_r \\ &\quad \times \int_{([s/\delta_n]-(\ell_2+2)k_n'+1)\delta_n}^{([s/\delta_n]-(\ell_2-2)k_n')\delta_n} \xi(\tfrac{[r/\delta_n]-[s/\delta_n]+\ell_2k_n'}{2k_n'})(W_r - W_{[r/\delta_n]\delta_n})dW_r \\ &\quad \times (W_s - W_{[s/\delta_n]\delta_n})^2 \,\Big|\, \mathcal{F}_{(j-1)\delta_n}\Bigg]ds.\end{aligned}$$

Further conditioning on $\mathcal{F}_{[s/\delta_n]\delta_n} = \mathcal{F}_{(j-1)\delta_n}$, we can replace $(W_s - W_{[s/\delta_n]\delta_n})^2$ simply by $s - (j-1)\delta_n$. Hereafter, we can further remove the boundaries of the two dW_r-integrals because $\xi(t) = 0$ for $|t| > 1$. Consequently,

$$\begin{aligned}R_{11}^{n,\ell_1,k_n,\ell_2,k_n'}(t) &= 64\delta_n^{\kappa-1}(\theta_1\theta_2)^{-1-2H}\delta_n^{-2}\sum_{j=1}^{[t/\delta_n]} \sigma_{(j-1)\delta_n}^8 \int_{(j-1)\delta_n}^{j\delta_n} (s - (j-1)\delta_n) \\ &\quad \times \int_{\mathbb{R}} \xi(\tfrac{[r/\delta_n]-[s/\delta_n]}{2k_n} + \tfrac{\ell_1}{2})\xi(\tfrac{[r/\delta_n]-[s/\delta_n]}{2k_n'} + \tfrac{\ell_2}{2})(r - [r/\delta_n]\delta_n)drds.\end{aligned}$$

Changing $r-[s/\delta_n]\delta_n$ to u, we can write

$$R_{11}^{n,\ell_1,k_n,\ell_2,k_n'}(t) = 64\delta_n^{\kappa-1}(\theta_1\theta_2)^{-1-2H}\delta_n^{-2}\sum_{j=1}^{[t/\delta_n]}\sigma_{(j-1)\delta_n}^8\int_{(j-1)\delta_n}^{j\delta_n}(s-(j-1)\delta_n)$$
$$\times\int_{\mathbb{R}}\xi(\tfrac{[u/\delta_n]}{2k_n}+\tfrac{\ell_1}{2})\xi(\tfrac{[u/\delta_n]}{2k_n'}+\tfrac{\ell_2}{2})(u-[u/\delta_n]\delta_n)duds$$
$$= 64\delta_n^{\kappa-1}(\theta_1\theta_2)^{-1-2H}\delta_n^{-2}\sum_{j=1}^{[t/\delta_n]}\sigma_{(j-1)\delta_n}^8\int_{(j-1)\delta_n}^{j\delta_n}(s-(j-1)\delta_n)$$
$$\times\sum_{i=-\infty}^{\infty}\int_{(i-1)\delta_n}^{i\delta_n}\xi(\tfrac{i-1}{2k_n}+\tfrac{\ell_1}{2})\xi(\tfrac{i-1}{2k_n'}+\tfrac{\ell_2}{2})(u-(i-1)\delta_n)duds.$$

Computing the $duds$-integrals and observing that $\delta_n^\kappa\sum_{i=-\infty}^{\infty}$ and $\delta_n\sum_{j=1}^{[t/\delta_n}$ are Riemann sums, we have that

$$R_{11}^{n,\ell_1,k_n,\ell_2,k_n'}(t)\sim 16(\theta_1\theta_2)^{-1-2H}\int_0^t\sigma_s^8ds\int_{\mathbb{R}}\xi(\tfrac{v}{2\theta_1}+\tfrac{\ell_1}{2})\xi(\tfrac{v}{2\theta_2}+\tfrac{\ell_2}{2})dv.$$

Next, we realize that $\xi(t)$ is equal to $-\frac{1}{4}\delta_1^4|x|$ evaluated at $x=2t$. Thus,

$$R_{11}^{n,\ell_1,k_n,\ell_2,k_n'}(t)\sim(\theta_1\theta_2)^{-2-2H}\Gamma_1(t)\int_{\mathbb{R}}\delta_{\theta_1}^4|v+\ell_1\theta_1|\delta_{\theta_2}^4|v+\ell_2\theta_2|dv.$$

It remains to derive a closed-form expression for the integral. By Parseval's identity and (A.8) (and a limit argument noting that $2\Gamma(\alpha+1)\cos(\frac{\pi(\alpha+1)}{2})\to\frac{\pi}{6}$ as $\alpha\to-4$), it is given by

$$\frac{2}{\pi}\int_{\mathbb{R}}e^{i(\ell_1\theta_1-\ell_2\theta_2)\xi}|\xi|^{-4}(e^{\frac{1}{2}i\theta_1\xi}-e^{-\frac{1}{2}i\theta_1\xi})^4(e^{\frac{1}{2}i\theta_2\xi}-e^{-\frac{1}{2}i\theta_2\xi})^4d\xi=\frac{1}{3}\delta_{\theta_1}^4\delta_{\theta_2}^4|\ell_1\theta_1-\ell_2\theta_2|^3,$$

which completes the proof of (3.33) for $\nu=1$.

Finally, let us consider $\nu=3$ and, as a first step, note that

$$R_{31,32}^{n,\ell_1,k_n,\ell_2,k_n'}(t)=\sum_{j=1}^{[t/\delta_n]}\mathbb{E}[\zeta_{31}^{n,j,\ell_1,k_n}\zeta_{32}^{n,j,\ell_2,k_n'}\mid\mathcal{F}_{(j-1)\delta_n}]=0 \tag{C.10}$$

because $\mathbb{E}[W_s-W_{(j-1)\delta_n}\mid\mathcal{F}_{(j-1)\delta_n}]=0$. Thus, it remains to find the limits of

$$R_{31|32,31|32}^{n,\ell_1,k_n,\ell_2,k_n'}(t)=\sum_{j=1}^{[t/\delta_n]}\mathbb{E}[\zeta_{31|32}^{n,j,\ell_1,k_n}\zeta_{31|32}^{n,j,\ell_2,k_n'}\mid\mathcal{F}_{(j-1)\delta_n}]. \tag{C.11}$$

To this end, we define

$$\mathcal{G}_H(t)=\frac{K_H^{-1}}{(H+\frac{1}{2})(H+\frac{3}{2})}t_+^{H+3/2}, \tag{C.12}$$

such that $\int_0^1 G_H(t-u)^{H+1/2}du=\mathcal{G}_H(t)-\mathcal{G}_H(t-1)$ for all $t\in\mathbb{R}$ and therefore,

$$\int_0^1\Delta_1^3G_H(\tfrac{s-r}{k_n\delta_n}+\ell-u-1)du=\delta_1^4\mathcal{G}_H(\tfrac{s-r}{k_n\delta_n}+\ell) \tag{C.13}$$

for all $\ell \in \mathbb{R}$. Analogously to the arguments between (C.6) and (C.7), it suffices to consider, instead of $R^{n,\ell_1,k_n,\ell_2,k'_n}_{31|32,31|32}(t)$, the simpler terms

$$\tilde{R}^{n,\ell_1,k_n,\ell_2,k'_n}_{31,31}(t) = 4(k_n\delta_n)^{-1/2-H}(k'_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)}\sum_{j=1}^{[t/\delta_n]}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^4_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2$$

$$\text{(C.14)}\qquad \times\int_{s-k_n\delta_n^{1-\varepsilon}}^{(j-1)\delta_n}(\delta_1^4\mathcal{G}_H(\tfrac{s-r}{k_n\delta_n}+\ell_1)+\delta_1^4\mathcal{G}_H(\tfrac{s-r}{k_n\delta_n}-\ell_1))$$

$$\times(\delta_1^4\mathcal{G}_H(\tfrac{s-r}{k'_n\delta_n}+\ell_2)+\delta_1^4\mathcal{G}_H(\tfrac{s-r}{k'_n\delta_n}-\ell_2))dr(s-(j-1)\delta_n)ds$$

and

$$\tilde{R}^{n,\ell_1,k_n,\ell_2,k'_n}_{32,32}(t) = 4(k_n\delta_n)^{-1/2-H}(k'_n\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)}\sum_{j=1}^{[t/\delta_n]}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^4_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2$$

$$\text{(C.15)}\qquad \times\int_{r-((\ell_1+2)k_n\wedge(\ell_2+2)k'_n)\delta_n}^{(j-1)\delta_n}\delta_1^4\mathcal{G}_H(\ell_1-\tfrac{r-s}{k_n\delta_n})\delta_1^4\mathcal{G}_H(\ell_2-\tfrac{r-s}{k'_n\delta_n})(s-[s/\delta_n]\delta_n)dsdr.$$

In (C.14), changing $(s-r)/\delta_n^{1-\kappa}$ to u and $s-(j-1)\delta_n$ to v, we obtain

$$\begin{aligned}\tilde{R}^{n,\ell_1,k_n,\ell_2,k'_n}_{31,31}(t) &= 4(k_n\delta_n)^{-1/2-H}(k'_n\delta_n)^{-1/2-H}\sum_{j=1}^{[t/\delta_n]}\int_0^{\delta_n}\sigma^4_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\\ &\quad\times\int_{\frac{v}{\delta_n^{1-\kappa}}}^{k_n\delta_n^{\kappa-\varepsilon}}(\delta_1^4\mathcal{G}_H(u\tfrac{\delta_n^{-\kappa}}{k_n}+\ell_1)+\delta_1^4\mathcal{G}_H(u\tfrac{\delta_n^{-\kappa}}{k_n}-\ell_1))\\ &\quad\times(\delta_1^4\mathcal{G}_H(u\tfrac{\delta_n^{-\kappa}}{k'_n}+\ell_2)+\delta_1^4\mathcal{G}_H(u\tfrac{\delta_n^{-\kappa}}{k'_n}-\ell_2))du\cdot vdv\\ &\sim\rho^{\ell_1,\theta_1,\ell_2,\theta_2}_{31,31}\Gamma_3(t),\end{aligned}$$

where

$$\text{(C.16)}\qquad \rho^{\ell_1,\theta_1,\ell_2,\theta_2}_{31,31} = \frac{2}{(\theta_1\theta_2)^{1/2+H}}\int_0^\infty(\delta_1^4\mathcal{G}_H(u/\theta_1+\ell_1)+\delta_1^4\mathcal{G}_H(u/\theta_1-\ell_1))\times(\delta_1^4\mathcal{G}_H(u/\theta_2+\ell_2)+\delta_1^4\mathcal{G}_H(u/\theta_2-\ell_2))du.$$

Similarly, changing $(r-s)/\delta_n^{1-\kappa}$ to u, we derive

$$\begin{aligned}\tilde{R}^{n,\ell_1,k_n,\ell_2,k'_n}_{32,32}(t) &= 4(k_n\delta_n)^{-1/2-H}(k'_n\delta_n)^{-1/2-H}\sum_{j=1}^{[t/\delta_n]}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^4_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\\ &\quad\times\int_{\frac{r-(j-1)\delta_n}{\delta_n^{1-\kappa}}}^{\frac{(\ell_1+2)k_n\wedge(\ell_2+2)k'_n}{\delta_n^{-\kappa}}}(\delta_1^4\mathcal{G}_H(\ell_1-u\tfrac{\delta_n^{-\kappa}}{k_n})+\delta_1^4\mathcal{G}_H(\ell_2-u\tfrac{\delta_n^{-\kappa}}{k'_n}))\\ &\quad\times(r-u\delta_n^{1-\kappa}-[r/\delta_n-u\delta_n^{-\kappa}]\delta_n)dudr\\ &\sim 4(k_n\delta_n)^{-1/2-H}(k'_n\delta_n)^{-1/2-H}\sum_{j=1}^{[t/\delta_n]}\sigma^4_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\end{aligned}$$

$$\times \int_0^{\frac{(\ell_1+2)k_n\wedge(\ell_2+2)k_n'}{\delta_n^{-\kappa}}} (\delta_1^4\mathcal{G}_H(\ell_1 - u\tfrac{\delta_n^{-\kappa}}{k_n}) + \delta_1^4\mathcal{G}_H(\ell_2 - u\tfrac{\delta_n^{-\kappa}}{k_n'}))$$

$$\times \int_{(j-1)\delta_n}^{j\delta_n} (r - u\delta_n^{1-\kappa} - [r/\delta_n - u\delta_n^{-\kappa}]\delta_n)drdu.$$

Note that the last dr-integral equals $\int_0^{\delta_n} rdr = \frac{1}{2}\delta_n^2$, so that

$$\tilde{R}_{32,32}^{n,\ell_1,k_n,\ell_2,k_n'}(t) \sim \rho_{32,32}^{\ell_1,\theta_1,\ell_2,\theta_2} \int_0^t \sigma_s^4 |\boldsymbol{\eta}_s|^2 ds,$$

where

$$\text{(C.17)}\quad \rho_{32,32}^{\ell_1,\theta_1,\ell_2,\theta_2} = \frac{2}{(\theta_1\theta_2)^{1/2+H}} \int_0^{(\ell_1+2)\theta_1\wedge(\ell_2+2)\theta_2} \delta_1^4\mathcal{G}_H(\ell_1 - u/\theta_1)\delta_1^4\mathcal{G}_H(\ell_2 - u/\theta_2)du.$$

Using the fact that $\delta_1^4\mathcal{G}_H(t) = 0$ for $t \leq -2$, we can extend the previous integral up to $+\infty$, which shows that

$$\rho_{31,31}^{\ell_1,\theta_1,\ell_2,\theta_2} + \rho_{32,32}^{\ell_1,\theta_1,\ell_2,\theta_2} = \frac{2}{(\theta_1\theta_2)^{1/2+H}} \int_{\mathbb{R}} (\delta_1^4\mathcal{G}_H(u/\theta_1 + \ell_1) + \delta_1^4\mathcal{G}_H(u/\theta_1 - \ell_1))$$

$$\times (\delta_1^4\mathcal{G}_H(u/\theta_2 + \ell_2) + \delta_1^4\mathcal{G}_H(u/\theta_2 - \ell_2))du.$$

We want to show that this is exactly $\gamma_3^{\ell_1,\theta_1,\ell_2,\theta_2}(H)$, which would then finish the proof of (3.33). Switching to the Fourier domain, we use (A.7), (A.8), (A.9) and (2.7) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \delta_1^4\mathcal{G}_H(u/\theta_1 + \ell_1)\delta_1^4\mathcal{G}_H(u/\theta_2 + \ell_2)du \\
&= \frac{K_H^{-2}}{(\theta_1\theta_2)^{H+3/2}(H+\frac{1}{2})^2(H+\frac{3}{2})^2} \int_{\mathbb{R}} \delta_{\theta_1}^4 (u+\ell_1\theta_1)_+^{H+3/2} \delta_{\theta_2}^4 (u+\ell_2\theta_2)_+^{H+3/2} du \\
&= \frac{K_H^{-2}\Gamma(H+\frac{1}{2})^2}{2\pi(\theta_1\theta_2)^{H+3/2}} \int_{\mathbb{R}} e^{i\xi(\ell_1\theta_1-\ell_2\theta_2)} e^{-i\pi(H+5/2)/2}(\xi - i0)^{-H-5/2} \\
&\qquad \times e^{i\pi(H+5/2)/2}(\xi+i0)^{-H-5/2}(e^{\frac{1}{2}i\theta_1\xi} - e^{-\frac{1}{2}i\theta_1\xi})^4(e^{\frac{1}{2}i\theta_2\xi} - e^{-\frac{1}{2}i\theta_2\xi})^4 d\xi \\
&= \frac{\sin(\pi H)\Gamma(2H+1)}{2\pi(\theta_1\theta_2)^{H+3/2}} \int_{\mathbb{R}} e^{i\xi(\ell_1\theta_1-\ell_2\theta_2)}|\xi|^{-2H-5}(e^{\frac{1}{2}i\theta_1\xi} - e^{-\frac{1}{2}i\theta_1\xi})^4(e^{\frac{1}{2}i\theta_2\xi} - e^{-\frac{1}{2}i\theta_2\xi})^4 d\xi \\
&= \frac{\sin(\pi H)\Gamma(2H+1)\Gamma(-2H-4)\cos(\pi(H+2))}{\pi}(\theta_1\theta_2)^{-H-3/2}\delta_{\theta_1}^4\delta_{\theta_2}^4|\ell_2\theta_2 - \ell_1\theta_1|^{2H+4}
\end{aligned}$$

for $H \in (0, \frac{1}{2})$. The last fraction is equal to $-1/(32(H+\frac{1}{2})(H+1)(H+\frac{3}{2})(H+2))$, which shows that $\rho_{31,31}^{\ell_1,\theta_1,\ell_2,\theta_2} + \rho_{32,32}^{\ell_1,\theta_1,\ell_2,\theta_2} = \gamma_3^{\ell_1,\theta_1,\ell_2,\theta_2}(H)$ for $H \in (0, \frac{1}{2})$. As before, the expression for $H = \frac{1}{2}$ can be obtained by letting $H \to \frac{1}{2}$, and since there is no singularity at $H = \frac{1}{2}$ in the formula defining $\gamma_3^{\ell_1,\theta_1,\ell_2,\theta_2}(H)$, it remains valid for $H = \frac{1}{2}$. □

PROOF OF EQUATION (3.34). Let us start by showing that

$$R_{2,31|32}^{n,\ell_1,k_n,\ell_2,k_n'}(t) = \sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_2^{n,j,\ell_1,k_n}\zeta_{31|32}^{n,j,\ell_2,k_n'} \mid \mathcal{F}_{(j-1)\delta_n}] \xrightarrow{\mathbb{P}} 0.$$

By (3.30) and (3.31) and Itô's isometry, we have

$$R^{n,\ell_1,k_n,\ell_2,k_n'}_{2,31}(t) = -2\sum_{j=1}^{[t/\delta_n]}(k_n'\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^2_{(j-1)\delta_n}\eta^2_{(j-1)\delta_n}$$
$$\times\int_{s-k_n'\delta_n^{1-\varepsilon}}^{(j-1)\delta_n}\int_0^1\Big\{\Delta_1^3 G_H(\tfrac{s-r}{k_n'\delta_n}+\ell_2-u-1)$$
$$+\Delta_1^3 G_H(\tfrac{s-r}{k_n'\delta_n}-\ell_2-u-1)\Big\}du\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_r$$
$$\times\int_{(j-1)\delta_n}^{s}\int_{-2}^{\infty}\Big\{\Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\tfrac{s-w}{k_n\delta_n}+\ell_1)$$
$$+\Delta_1^2 G_H(v)\Delta_1^2 G_H(v+\tfrac{s-w}{k_n\delta_n}-\ell_1)\Big\}dvdwds.$$

For each j, we know from the analysis of $R^{n,\ell_1,k_n,\ell_2,k_n'}_{22}$ and $R^{n,\ell_1,k_n,\ell_2,k_n'}_{31,31}$ that ζ_2^{n,j,ℓ_1,k_n} and $\zeta_{31}^{n,j,\ell_2,k_n'}$ are of order $O_{\mathbb{P}}(\delta_n^{1/2})$, uniformly in i. Therefore, we are free to modify terms in the previous display as long as it leads to an asymptotically vanishing error. For example, for any fixed j, we may replace $\sigma^2_{(j-1)\delta_n}\eta^2_{(j-1)\delta_n}$ and $\boldsymbol{\eta}_{(j-1)\delta_n}$ by $\sigma^2_{(j-1-k_n'\delta_n^{-\varepsilon})\delta_n}\eta^2_{(j-1-k_n'\delta_n^{-\varepsilon})\delta_n}$ and $\boldsymbol{\eta}_{(j-1-k_n'\delta_n^{-\varepsilon})\delta_n}$, respectively. Once we have done so, the resulting term, for fixed j, will be $\mathcal{F}_{(j-1)\delta_n}$-measurable with vanishing $\mathcal{F}_{(j-1-k_n'\delta_n^{-\varepsilon})\delta_n}$-conditional expectation. Thus, by a martingale size estimate (see [11, Appendix A]), the sum over j will be of magnitude $O_{\mathbb{P}}(\delta_n^{-(\kappa+\varepsilon+1)/2}\delta_n^{1/2}\delta_n^{1/2}) = o_{\mathbb{P}}(1)$, proving $R^{n,\ell_1,k_n,\ell_2,k_n'}_{2,31}\approx 0$.

The reasoning for $R^{n,\ell_1,k_n,\ell_2,k_n'}_{2,32}$ is similar. Again by Itô's isometry,

$$R^{n,\ell_1,k_n,\ell_2,k_n'}_{2,32}(t) = -2\sum_{j=1}^{[t/\delta_n]}(k_n'\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^2_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2$$
$$\times\int_{r-(\ell_2+1)k_n'\delta_n}^{(j-1)\delta_n}\int_0^1\Delta_1^3 G_H(\ell_2-\tfrac{r-s}{k_n'\delta_n}-u-1)du(W_s-W_{[s/\delta_n]\delta_n})dW_s$$
$$\times\int_{r-k_n\delta_n^{1-\varepsilon}}^{(j-1)\delta_n}\int_{-2}^{\infty}\Delta_1^2 G_H(v)\Big\{\Delta_1^2 G_H(v+\tfrac{r-w}{k_n\delta_n}+\ell_1)$$
$$+\Delta_1^2 G_H(v+\tfrac{r-w}{k_n\delta_n}-\ell_1)\Big\}dv\boldsymbol{\eta}_{(j-1)\delta_n}d\boldsymbol{W}_w dr.$$

We can now use integration by parts to expand the product of the dW_s-integral and the $d\boldsymbol{W}_w$-integral. As in the analysis of $R^{n,\ell_1,k_n,\ell_2,k_n'}_{2,31}$ above, the martingale terms can be shown to be negligible. So only the quadratic variation part remains and

$$R^{n,\ell_1,k_n,\ell_2,k_n'}_{2,32}(t)\approx -2\sum_{j=1}^{[t/\delta_n]}(k_n'\delta_n)^{-1/2-H}\delta_n^{-(1-\kappa)}\int_{(j-1)\delta_n}^{j\delta_n}\sigma^2_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\eta_{(j-1)\delta_n}$$
$$\times\int_{r-(\ell_2+1)k_n'\delta_n}^{(j-1)\delta_n}\int_0^1\Delta_1^3 G_H(\ell_2-\tfrac{r-s}{k_n'\delta_n}-u-1)du(W_s-W_{[s/\delta_n]\delta_n})$$
$$\times\int_{-2}^{\infty}\Delta_1^2 G_H(v)\Big\{\Delta_1^2 G_H(v+\tfrac{r-s}{k_n\delta_n}+\ell_1)+\Delta_1^2 G_H(v+\tfrac{r-s}{k_n\delta_n}-\ell_1)\Big\}dvdsdr.$$

Now we apply the same trick as before: we first shift the index of $\sigma^2_{(j-1)\delta_n}|\boldsymbol{\eta}_{(j-1)\delta_n}|^2\eta_{(j-1)\delta_n}$ to $(j-1-(\ell_2+1)k_n')\delta_n$ and then realize that the conditional expectation of the resulting expression given $\mathcal{F}_{(j-1-(\ell_2+1)k'_n)\delta_n}$ is zero. Thus, by another martingale size estimate, we obtain $R^{n,\ell_1,k_n,\ell_2,k'_n}_{2,32} \approx 0$. Since the proof of

$$R^{n,\ell_1,k_n,\ell_2,k'_n}_{12|1,31}(t) = \sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_1^{n,j,\ell_1,k_n} \zeta_{2|31}^{n,j,\ell_2,k'_n} \mid \mathcal{F}_{(j-1)\delta_n}] \xrightarrow{\mathbb{P}} 0$$

is very similar, we omit the details and leave it to the reader. Lastly, by Itô's isometry, we have

$$R^{n,\ell_1,k_n,\ell_2,k'_n}_{1,32}(t) = \sum_{j=1}^{[t/\delta_n]} \mathbb{E}[\zeta_1^{n,j,\ell_1,k_n} \zeta_{32}^{n,j,\ell_2,k'_n} \mid \mathcal{F}_{(j-1)\delta_n}] \equiv 0. \qquad \square$$

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