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Statistical inferences on location parameters of bivariate exponential distributions

Hiroto HYAKUTAKE

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Summary

In this paper we consider some statistical inferences on location parameters of Marshall and Olkin's bivariate exponential distribution. Two-stage sampling procedures are given for constructing a fixed-size confidence region and selecting the best population. Some test procedures for a structure of location parameters are proposed.

0. Introduction

Exponential distributions have been introduced in a rich literature as a simple model for statistical analysis of lifetimes. In survival analysis, the hazard function is a constant if and only if the failure time has an exponential distribution, see e.g., Cox and Oakes [11] or Kalbfleisch and Prentice [30]. Then the hazard rate is equal to the scale parameter of the exponential distribution. In reliability analysis, the location parameter is viewed as a guaranteed lifetime of a system. Bain [3] and Barlow and Proschan [5] dealt with statistical procedures in reliability theory by using exponential distributions.

There is an extensive literature on the construction of bivariate exponential models, for example, Gumbel [21], Freund [18], Downton [13], Block and Basu [8] and so on. Marshall and Olkin [36] proposed a multivariate extension of exponential distributions which is much of interest in both theoretical developments and applications. Marshall and Olkin [36] derived a bivariate exponential (BVE) distribution by supposing that failure is caused by three types of Poisson shocks on a system containing two components. The BVE distribution has three scale parameters. We note that the parameters of the BVE distribution are not scale parameters in the usual definition of scale parameters; however we call them scale parameters. Statistical inferences for scale parameters have been considered by many authors. For example, Arnold [1] and Bemis, Bain and Higgins [7] derived estimators for the scale parameters. Awad, Azzam and Hamdan [2] considered the problem of estimating $P(X > Y)$ in the BVE and Ebrahimi [17] applied the BVE to an accelerated life test. Lu and Battacharyya [35] proposed a bivariate extension of the Weibull model along the line of Marshall and Olkin [36]. However, there is no literature which deals with the locations of BVE distribution. In this paper, we propose a BVE distribution having location parameters and consider statistical inferences on the location parameters.

In Section 1, we define a BVE distribution with location and scale parameters, which is an extension of Marshall and Olkin [36], and state some properties of the BVE. The problem of constructing a fixed-size confidence region for location parameters has been studied by many authors for the normal distribution. Stein [43] derived a two-stage sampling procedure of obtaining a confidence interval for the normal mean. Healy [23] extended this two-stage procedure to the construction of simultaneous confidence intervals for multivariate normal means. Chatterjee [9], Dudewicz and Bishop [14], Mukhopadhyay and Al-Mousawi [39], Hyakutake and Siotani [27] developed Stein's [43] two-stage procedures for multivariate normal populations. Chatterjee's [9] method could be implemented in practice by applying Hyakutake [24] and Hyakutake, et. al. [28]. A two-stage sampling procedure for the univariate exponential population has been considered by Ghurye [19]. For the BVE distribution, we derive a two-stage procedure of constructing a fixed-size confidence region in Section 2. The procedure is applied to the ranking and selection problem in Section 3, where we treat two BVE populations with the same scale parameters. Desu, Narula and Villarreal [12], Lee and Kim [32] and Mukhopadhyay [38] applied Ghurye's [19] procedure to the problem of selecting the best of several univariate exponential populations. Dudewicz and Dalal [15] gave a two-stage selection procedure for univariate normal populations. Dudewicz and Taneja [16] and Hyakutake [25] derived a general treatment for selecting the best multivariate normal population based on Stein's [43] two-stage procedure. In Section 4, we propose two-step procedures of testing a hypothesis on a structure of location parameters. The null hypothesis is the equality of two location parameters of the BVE distribution. The two-step procedure can be viewed as a preliminary test, see e.g., Bancroft [4]. Takeuchi [44] and Lehmann [34] considered a test for the location parameter of univariate exponential distribution.

1. BVE distribution and its properties

The location parameters of exponential distributions can be viewed as guaranteed lifetimes of systems or their components. We define a BVE distribution of Marshall and Olkin [36] with location parameters as the distribution whose joint survival function is

$$
\begin{aligned}
\bar{F}(x, y) &= P(X > x, Y > y) \\
&= \exp\{-\lambda_1(x - \tau_1) - \lambda_2(y - \tau_2) - \lambda_0 \max(x - \tau_1, y - \tau_2)\} ,
\end{aligned} \tag{1.1}
$$

where $x > \tau_1$, $y > \tau_2$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_0 \geq 0$ (see Hyakutake [26]). For convenience, we denote $(X, Y) \sim BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$, where $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_0)$, if a random vector (X, Y) is distributed as a BVE with the survival function (1.1). It is easily seen that the BVE distribution has exponential marginals, say the distribution of X (or Y) is exponential with the location parameter τ_1 (or τ_2) and the scale parameter $\lambda_1 + \lambda_0$ (or $\lambda_2 + \lambda_0$). Marshall and Olkin [36] has given some basic properties for $BVE(\mathbf{0}, \boldsymbol{\lambda})$. Modifying their derivations, the moment generating function of $BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$ is given by

$$\psi(s, t) = \frac{(\lambda_1 + \lambda_2 + \lambda_0 + s + t)(\lambda_1 + \lambda_0)(\lambda_2 + \lambda_0) + st\lambda_0}{(\lambda_1 + \lambda_2 + \lambda_0 + s + t)(\lambda_1 + \lambda_0 + s)(\lambda_2 + \lambda_0 + t)} e^{s\tau_1 + t\tau_2} .$$

The mean vector and the covariance matrix of (X, Y) are

$$E[(X, Y)] = \left(\tau_1 + \frac{1}{\lambda_1 + \lambda_0}, \tau_2 + \frac{1}{\lambda_2 + \lambda_0}\right)$$

and

$$V[(X, Y)] = \begin{bmatrix} \dfrac{1}{(\lambda_1 + \lambda_0)^2} & \dfrac{\lambda_0}{(\lambda_1 + \lambda_2 + \lambda_0)(\lambda_1 + \lambda_0)(\lambda_2 + \lambda_0)} \\ \dfrac{\lambda_0}{(\lambda_1 + \lambda_2 + \lambda_0)(\lambda_1 + \lambda_0)(\lambda_2 + \lambda_0)} & \dfrac{1}{(\lambda_2 + \lambda_0)^2} \end{bmatrix},$$

respectively. The correlation coefficient of X and Y is $\lambda_0/(\lambda_1 + \lambda_2 + \lambda_0)$. Hence, X and Y are independent if and only if $\lambda_0 = 0$, that is, the correlation (or the covariance) is zero.

The BVE distribution is not absolutely continuous with respect to the usual Lebesgue measure; however, let

$$f(x, y) = \begin{cases} \lambda_1(\lambda_2 + \lambda_0)\bar{F}(x, y), & y - \tau_2 > x - \tau_1 > 0, \\ \lambda_2(\lambda_1 + \lambda_0)\bar{F}(x, y), & x - \tau_1 > y - \tau_2 > 0, \\ \lambda_0 \bar{F}(x, x), & x - \tau_1 = y - \tau_2 > 0. \end{cases} \tag{1.2}$$

The function (1.2) may be considered to be a density for the BVE distribution, if it could be seen that the first two terms are densities with respect to two-dimensional Lebesgue measure and the third term is a density with respect to one-dimensional Lebesgue measure (see Bemis, Bain and Higgins [7]). From Arnold [1] or Awad, Azzam and Hamdan [2], we obtain the following lemma.

LEMMA 1.1. *If* $(X, Y) \sim BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$, *then*

$$P(X - Y < \delta) = \lambda_1/(\lambda_1 + \lambda_2 + \lambda_0),$$
$$P(X - Y > \delta) = \lambda_2/(\lambda_1 + \lambda_2 + \lambda_0),$$
$$P(X - Y = \delta) = \lambda_0/(\lambda_1 + \lambda_2 + \lambda_0),$$

where $\delta = \tau_1 - \tau_2$.

PROOF. It is easily seen from (1.2) that

$$P(X - Y < \delta) = \iint_{y-\tau_2 > x-\tau_1 > 0} f(x, y)\, dx\, dy = \lambda_1/(\lambda_1 + \lambda_2 + \lambda_0),$$

$$P(X - Y > \delta) = \iint_{x-\tau_1 > y-\tau_2 > 0} f(x, y)\, dx\, dy = \lambda_2/(\lambda_1 + \lambda_2 + \lambda_0),$$

and $P(X - Y = \delta) = 1 - P(X - Y < \delta) - P(X - Y > \delta)$.

The p.d.f. of $Z = X - Y$ is given by

$$(1.3) \qquad h(z) = \begin{cases} \dfrac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_0}(\lambda_1 + \lambda_0) \exp[-(\lambda_1 + \lambda_0)(z - \delta)], & z > \delta, \\[2ex] \dfrac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_0}(\lambda_2 + \lambda_0) \exp[(\lambda_2 + \lambda_0)(z - \delta)], & z < \delta, \\[2ex] \dfrac{\lambda_0}{\lambda_1 + \lambda_2 + \lambda_0}, & z = \delta, \end{cases}$$

which is obtained by (1.2) and a change of variables. Using (1.3), we have the following lemma.

LEMMA 1.2. *If* $(X, Y) \sim BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$,

$$(1.4) \qquad P(X > Y) = \begin{cases} 1 - \dfrac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_0} \exp[-(\lambda_2 + \lambda_0)\delta], & \delta > 0, \\[2ex] \dfrac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_0} \exp[(\lambda_1 + \lambda_0)\delta], & \delta \leq 0. \end{cases}$$

Let $\{(X_r, Y_r), r = 1, 2, \dots\}$ be a sequence of independent and identically random vectors having the joint survival function (1.1). By using (1.2), the likelihood function of the random sample of size n may be expressed as

$$\prod_{r=1}^{n} f(x_r, y_r) = \exp\{-\lambda_1 \sum_r (x_r - \tau_1) - \lambda_2 \sum_r (y_r - \tau_2) \tag{1.5}$$
$$- \lambda_0 \sum_r \max(x_r - \tau_1, y_r - \tau_2) + n_+ \log[\lambda_1(\lambda_2 + \lambda_0)]$$
$$+ n_- \log[\lambda_2(\lambda_1 + \lambda_0)] + m \log \lambda_0\},$$

where $n_+ = \Sigma_r I(X_r - Y_r < \delta)$, $n_- = \Sigma_r I(X_r - Y_r > \delta)$, $m = \Sigma_r I(X_r - Y_r = \delta)$, $I(X - Y < \delta) = 1$ if $X - Y < \delta$, $= 0$ otherwise, $n = n_+ + n_- + m$ and Σ_r means $\Sigma_{r=1}^{n}$. The following lemma and theorem are obtained.

LEMMA 1.3 (Arnold [1]). *(n_+, n_-, m) is distributed as the trinomial distribution with parameter $(n, \lambda_1/(\lambda_1 + \lambda_2 + \lambda_0), \lambda_2/(\lambda_1 + \lambda_2 + \lambda_0), \lambda_0/(\lambda_1 + \lambda_2 + \lambda_0))$.*

THEOREM 1.1. *Let (X_r, Y_r), $r = 1, \ldots, n$, be a random sample of size n from $BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$, and $X_{n,1} = \min(X_1, \ldots, X_n)$ and $Y_{n,1} = \min(Y_1, \ldots, Y_n)$. Then*
(i) *the maximum likelihood estimate (MLE) of $\boldsymbol{\tau}$ is $(X_{n,1}, Y_{n,1})$.*
(ii) $(X_{n,1}, Y_{n,1}) \sim BVE(\boldsymbol{\tau}, n\boldsymbol{\lambda})$.

PROOF. (i) is easily seen from (1.5). By the definition of the BVE distribution and $(X_{n,1}, Y_{n,1})$, we find that

$$\begin{aligned} &P(X_{n,1} > x, Y_{n,1} > y) \\ &= P(X_r > x, Y_r > y, r = 1, \ldots, n) \\ &= \prod_{r=1}^{n} P(X_r > x, Y_r > y) \\ &= \exp\{-n\lambda_1(x - \tau_1) - n\lambda_2(y - \tau_2) - n\lambda_0 \max(x - \tau_1, y - \tau_2)\}, \end{aligned}$$

which proves (ii).

When $\boldsymbol{\tau} = \mathbf{0}$, Proschan and Sullo [40] gave the MLE of $\boldsymbol{\lambda}$ if n_+, n_- or m is zero. Klein and Basu [31] obtained the MLE in the case of Block and Basu's [8] absolutely continuous BVE distribution with $\lambda_1 = \lambda_2$. If all of n_+, n_- and m are not equal to zero, the MLE can not be written in a closed form, but it can be obtained by an iteration method. When $\boldsymbol{\tau}$ is unknown, it is quite difficult to derive the MLE since (n_+, n_-, m) depends on $\boldsymbol{\tau}$ (or δ).

REMARK. If X is distributed as an exponential, then aX is distributed as exponential for all $a > 0$. However, if $(X, Y) \sim BVE$, then $(a_1 X, a_2 Y) \sim BVE$ only if $a_1 = a_2 > 0$ (Marshall and Olkin [36]).

2. Estimation of location parameters

Here we consider two estimation problems, which are constructing a fixed-size confidence region and estimators with a bounded risk. To provide such a confidence region is basic in multiple comparisons (or simultaneous statistical inferences), see Miller [37]. The problems are solved by using the MLE $(X_{n,1}, Y_{n,1})$ of (τ_1, τ_2).

2.1. Fixed-size confidence region when λ is known

Since the MLE of τ is $(X_{n,1}, Y_{n,1})$, $X_{n,1} > \tau_1$ and $Y_{n,1} > \tau_2$, it is natural to consider a confidence region for τ such that

$$(2.1)\qquad P(X_{n,1} - \ell \leq \tau_1 \leq X_{n,1}, Y_{n,1} - \ell \leq \tau_2 \leq Y_{n,1}) \geq 1 - \alpha ,$$

where $\ell > 0$ and α $(0 < \alpha < 1)$ are predetermined constants. We consider the problem of determining the sample size n satisfying (2.1). Jones [29] constructed a fixed-size rectangular confidence region like (2.1) for a multivariate mean by a sequential procedure. For simplicity, we write the left-side of (2.1) as $P(A_{n,1})$. By using Bonferroni's inequality, we have

$$(2.2)\qquad P(A_{n,1}) \geq 1 - P(X_{n,1} > \tau_1 + \ell) - P(Y_{n,1} > \tau_2 + \ell) .$$

By Theorem 1.1 and the fact that the BVE has exponential marginals,

$$P(X_{n,1} > \tau_1 + \ell) = \int_{\tau_1+\ell}^{\infty} n(\lambda_1 + \lambda_0)e^{-n(\lambda_1+\lambda_0)(x-\tau_1)}\, dx = \int_{2n(\lambda_1+\lambda_0)\ell}^{\infty} g_2(t)\, dt$$

and similarly

$$P(Y_{n,1} > \tau_2 + \ell) = \int_{2n(\lambda_2+\lambda_0)\ell}^{\infty} g_2(t)\, dt ,$$

where $g_q(\cdot)$ is the probability density function (p.d.f.) of a chi-square distribution with q degrees of freedom (d.f.). From the above argument, we obtain

$$P(A_{n,1}) \geq 1 - 2\int_{2n(\lambda_{\min}+\lambda_0)\ell}^{\infty} g_2(t)\, dt ,$$

where $\lambda_{\min} = \min(\lambda_1, \lambda_2)$. Hence we have the following theorem.

THEOREM 2.1. *Suppose that λ is known. Then the required condition* (2.1) *is satisfied if*

$$(2.3)\qquad n \geq \frac{1}{(\lambda_{\min} + \lambda_0)\ell}\frac{1}{2}\chi_2^2(\alpha/2) ,$$

where $\chi_q^2(a)$ is the upper $100a$ % point of a chi-square distribution with q d.f.

We choose the sample size n equal to the smallest integer satisfying (2.3). Tables give the sample size n and the exact confidence coefficient $P(A_{n,1})$ for some given values of ℓ and λ when $\alpha = 0.05$. It is seen that $P(A_{n,1})$ is fairly closed to the required confidence coefficient when the difference of λ_1 and λ_2 is small.

$\lambda = (.1, .1, .1)$

ℓ	n	$P(A_{n,1})$
.1	185	.9544
.2	93	.9553
.3	62	.9553
.4	47	.9569
.5	37	.9544

$\lambda = (.1, .2, .1)$

ℓ	n	$P(A_{n,1})$
.1	185	.9720
.2	93	.9725
.3	62	.9725
.4	47	.9737
.5	37	.9720

$\lambda = (.1, .5, .1)$

ℓ	n	$P(A_{n,1})$
.1	185	.9752
.2	93	.9757
.3	62	.9757
.4	47	.9767
.5	37	.9752

$\lambda = (.1, .1, .2)$

ℓ	n	$P(A_{n,1})$
.1	123	.9573
.2	62	.9585
.3	41	.9573
.4	31	.9585
.5	25	.9597

2.2. Fixed-size confidence region when λ is unknown; two-stage estimation

It is well known that there exists no fixed-size confidence region for the location parameter of location-scale family with the fixed sample size if the scale parameter is unknown (see Lehmann [33] or Singh [41]). Hence, in order to construct a fixed-size confidence region, it is necessary to use at least two-stage sampling schemes.

We first take a sample of size n_0 (≥ 2) from the BVE population having the survival function (1.1) and compute

$$S_1 = \frac{1}{n_0 - 1} \sum_{r=1}^{n_0} (X_r - X_{n_0,1}) \qquad \text{and} \qquad S_2 = \frac{1}{n_0 - 1} \sum_{r=1}^{n_0} (Y_r - Y_{n_0,1}) .$$

Define N by

$$N = \max \{n_0, [c_1 S_1] + 1, [c_1 S_2] + 1\} , \tag{2.4}$$

where $[b]$ denotes the greatest integer not greater than b and c_1 is a positive constant. Next we take $N - n_0$ additional observations and compute $(X_{N,1}, Y_{N,1}) = (\min\{X_1, \ldots, X_N\}, \min\{Y_1, \ldots, Y_N\})$. From (2.2) and (2.4), it can be shown that

$$\begin{aligned} P(A_{N,1}) &\geq 1 - P(X_{N,1} > \tau_1 + \ell) - P(Y_{N,1} > \tau_2 + \ell) \\ &= 1 - P\{N(X_{N,1} - \tau_1) > N\ell\} - P\{N(Y_{N,1} - \tau_2) > N\ell\} \\ &\geq 1 - P\{N(X_{N,1} - \tau_1)/S_1 > c_1\ell\} - P\{N(Y_{N,1} - \tau_2)/S_2 > c_1\ell\}, \end{aligned}$$

where both of $N(X_{N,1} - \tau_1)/S_1$ and $N(Y_{N,1} - \tau_2)/S_2$ have an F-distribution with $(2, 2(n_0 - 1))$ d.f. (see Desu, Nalura and Villareal [12]). Hence we have

$$P(A_{N,1}) \geq 1 - 2\int_{c_1\ell}^{\infty} h_{2,2(n_0-1)}(t)\, dt,$$

where $h_{q_1,q_2}(\cdot)$ is the p.d.f. of the F-distribution with (q_1, q_2) d.f. Let $F_{q_1,q_2}(a)$ be the upper $100a$ % point of the F-distribution with (q_1, q_2) d.f. We can summarize the above argument in the following

THEOREM 2.2. *If the constant* c_1 *in* (2.4) *is chosen so that*

(2.5) $$c_1 \geq \frac{1}{\ell} F_{2,2(n_0-1)}(\alpha/2),$$

then the fixed-size confidence region as (2.1) *can be constructed.*

In the above two-stage sampling procedure, we choose the first stage sample size n_0 satisfying

(2.6) $$\lim_{\ell\to 0} n_0 = \infty \quad \text{and} \quad \lim_{\ell\to 0} (\ell n_0) = 0.$$

Then

$$\lim_{\ell\to 0} S_i = \frac{1}{\lambda_i + \lambda_0} \quad \text{(almost surely)}, \qquad i = 1, 2.$$

Hence we have

$$\lim_{\ell\to 0} \max(S_1, S_2) = \frac{1}{\lambda_{\min} + \lambda_0} \quad \text{(almost surely)}.$$

Furthermore, it holds that

$$\max(c_1 S_1, c_1 S_2) \leq N \leq \max(c_1 S_1, c_1 S_2) + n_0$$

from (2.4). These imply the following theorem.

THEOREM 2.3. *In the two-stage sampling procedure defined by* (2.4), *suppose that* $c_1 = F_{2,2(n_0-1)}(\alpha/2)/\ell$ *and* n_0 *satisfies* (2.6). *Then*

$$\text{(2.7)} \qquad \lim_{\ell \to 0} \frac{N}{n} = 1 \ (\textit{almost surely}) \qquad \textit{and} \qquad \lim_{\ell \to 0} \frac{E(N)}{n} = 1\,,$$

if $n = \chi_2^2(\alpha/2)/\{2\ell(\lambda_{\min} + \lambda_0)\}$ *in Theorem* 2.1.

In the sense of Chow and Robbins [10], that is, satisfying the second property of (2.7), the two-stage sampling procedure (2.4) is called asymptotically efficient for the single-stage procedure in the previous subsection.

2.3. Bounded risk

Let $\boldsymbol{d}_n$ be an estimator of $\boldsymbol{\tau}$, based on a random sample of size n from $BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$. Let $L(\boldsymbol{d}_n, \boldsymbol{\tau})$ be the loss function defined by the sum of absolute errors, that is, $L(\boldsymbol{d}_n, \boldsymbol{\tau}) = \Sigma\,|d_{n,i} - \tau_i|$. Then the risk function (or expected loss) is

$$R_n(\boldsymbol{d}_n, \boldsymbol{\tau}) = E\{L(\boldsymbol{d}_n, \boldsymbol{\tau})\}\,.$$

For a given $\varepsilon > 0$, we wish to decide the sample size n such that

$$\text{(2.8)} \qquad R_n(\boldsymbol{d}_n, \boldsymbol{\tau}) \le \varepsilon \qquad \text{for all } (\boldsymbol{\tau}, \boldsymbol{\lambda})\,.$$

Now we consider the problem of determining such a sample size when $\boldsymbol{d}_n = (X_{n,1}, Y_{n,1})$ or $\boldsymbol{d}_N = (X_{N,1}, Y_{N,1})$ according to the case when $\boldsymbol{\lambda}$ is known or unknown. By Theorem 1.1 (ii) we have

$$R_n(\boldsymbol{d}_n, \boldsymbol{\tau}) = \frac{1}{n}\left(\frac{1}{\lambda_1 + \lambda_0} + \frac{1}{\lambda_2 + \lambda_0}\right).$$

Hence, when λ is known, the requirement (2.8) is satisfied if n is chosen so that

$$n \ge \frac{1}{\varepsilon}\left(\frac{1}{\lambda_1 + \lambda_0} + \frac{1}{\lambda_2 + \lambda_0}\right).$$

If $\boldsymbol{\lambda}$ is unknown, it can be solved by using the estimator $\boldsymbol{d}_N = (X_{N,1}, Y_{N,1})$ based on the two-stage sampling scheme described in Subsection 2.2. By using the two-stage procedure, we get

$$\begin{aligned} R_N(\boldsymbol{d}_N, \boldsymbol{\tau}) &= E_N[E\{|X_{N,1} - \tau_1| + |Y_{N,1} - \tau_2||N\}] \\ &= \left(\frac{1}{\lambda_1 + \lambda_0} + \frac{1}{\lambda_2 + \lambda_0}\right) E\left\{\frac{1}{N}\right\} \\ &\le \frac{1}{\lambda_1 + \lambda_0} E\left\{\frac{1}{c_1 S_1}\right\} + \frac{1}{\lambda_2 + \lambda_0} E\left\{\frac{1}{c_1 S_2}\right\}. \end{aligned}$$

Since $2(n_0 - 1)(\lambda_i + \lambda_0)S_i$ $(i = 1, 2)$ is distributed as a chi-square with $2(n_0 - 1)$

d.f., $E[\{2(n_0-1)(\lambda_i+\lambda_0)S_i\}^{-1}] = 1/[2(n_0-2)]$ for $n_0 > 2$. Hence, if the constant $c_1 > 0$ in (2.4) is chosen so that $c_1 \geq 2(n_0-1)/\{\varepsilon(n_0-2)\}$, the requirement (2.8) is satisfied.

3. Ranking and selection problem

Suppose that there are two BVE populations Π_1 and Π_2 with different location parameters and common scale parameters, that is, $\Pi_1 : BVE(\boldsymbol{\tau}_1, \boldsymbol{\lambda})$ and $\Pi_2 : BVE(\boldsymbol{\tau}_2, \boldsymbol{\lambda})$, where $\boldsymbol{\tau}_j = (\tau_{j1}, \tau_{j2})$ $(j = 1, 2)$. The problem is to select one population as "better" in an experimenter-specified sense. In the multivariate case, almost all ranking and selection procedures are based on some device by which the original multivariate parameter is univariatized (see Chapter 15 of Gibbons, Olkin and Sobel [20] or Chapter 7 of Gupta and Panchapakesan [22]). Here we consider a simple scalar function $\theta_j = \tau_{j1} + \tau_{j2}$ as a univariatized parametric function. If the ordered values are denoted by $\theta_{[1]} \leq \theta_{[2]}$, and corresponding populations by $\Pi_{[1]}$ and $\Pi_{[2]}$, then we select $\Pi_{[2]}$ as the better population. The better population can also be viewed as the one associated with the larger mean of the sum of the components.

3.1. The case when $\boldsymbol{\lambda}$ is known

A natural selection procedure is to estimate θ_j (or $\boldsymbol{\tau}_j$) by $T_{j,n} = X_{n,1}^{(j)} + Y_{n,1}^{(j)}$ $(j = 1, 2)$ and select the population having the larger $T_{j,n}$. However, since the order of $T_{1,n}$ and $T_{2,n}$ does not always equal to the order of θ_1 and θ_2 due to sampling variation, we need to have a probability statement with a specified confidence that the procedure leads to a correct selection (CS). We use the indifference zone approach introduced by Bochhofer [6] in which he considered the problem of ranking k normal means. For a fixed P^* $(1/2 < P^* < 1)$, we wish to attain the probability of correct selection P^* in the indifference zone formulation, that is

$$\inf_{\Omega_P} P(CS) = P^* , \tag{3.1}$$

where Ω_P is a subset of the parameter space Ω and is called the preference zone.

Let $\Omega_\tau = \{(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) : -\infty < \tau_{ji} < +\infty, i, j = 1, 2\}$ and $\Omega_\lambda = \{\boldsymbol{\lambda} : \lambda_i > 0, i = 1, 2, 0\}$. We define $\Omega_{P(\tau)} = \{(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2) : \theta_{[2]} \geq \theta_{[1]} + \delta^*, \delta^* > 0\}$ and use $T_{[1],n}$ and $T_{[2],n}$ as estimates for $\theta_{[1]}$ and $\theta_{[2]}$, respectively. Then the probability requirement (3.1) is written by $\inf_{\Omega_{P(\tau)}} P(CS) = P^*$, that is

$$P(T_{[1],n} < T_{[2],n}) \geq P^* \quad \text{whenever} \quad \theta_{[2]} - \theta_{[1]} \geq \delta^* \tag{3.2}$$

for given P^* $(1/2 < P^* < 1)$ and $\delta^* > 0$. When $\boldsymbol{\lambda}$ is known, we obtain the following theorem.

THEOREM 3.1. *The probability requirement* (3.2) *is satisfied, if the sample size n from each population satisfies*

$$n \geq \frac{2}{(\lambda_{\min} + \lambda_0)\delta^*} \log (1 - P^*)^{-1} . \tag{3.3}$$

PROOF. Using $\theta_{[2]} - \theta_{[1]} \geq \delta^*$ and Bonferroni's inequality, we have

$$\begin{aligned} (3.4) \quad & P(T_{[1],n} < T_{[2],n}) \\ & \geq P(T_{[1],n} - \theta_{[1]} < T_{[2],n} - \theta_{[2]} + \delta^*) \\ & \geq P\left(X_{n,1}^{[1]} - \tau_{[11]} < X_{n,1}^{[2]} - \tau_{[21]} + \frac{\delta^*}{2}, Y_{n,1}^{[1]} - \tau_{[12]} < Y_{n,1}^{[2]} - \tau_{[22]} + \frac{\delta^*}{2} \right) \\ & \geq 1 - P\left(X_{n,1}^{[1]} - \tau_{[11]} \geq X_{n,1}^{[2]} - \tau_{[21]} + \frac{\delta^*}{2} \right) \\ & \quad - P\left(Y_{n,1}^{[1]} - \tau_{[12]} \geq Y_{n,1}^{[2]} - \tau_{[22]} + \frac{\delta^*}{2} \right) , \end{aligned}$$

where $T_{[j],n} = X_{n,1}^{[j]} + Y_{n,1}^{[j]}$ $(j = 1, 2)$. The second and third terms of the last inequality in (3.4) are expressed as

$$\int_0^\infty \int_{v_2 + n(\lambda_1 + \lambda_0)\delta^*}^\infty g_2(v_1) g_2(v_2) \, dv_1 \, dv_2$$

and

$$\int_0^\infty \int_{v_2 + n(\lambda_2 + \lambda_0)\delta^*}^\infty g_2(v_1) g_2(v_2) \, dv_1 \, dv_2 ,$$

respectively. Hence we have

$$\begin{aligned} P(T_{[1],n} < T_{[2],n}) &\geq 1 - 2 \int_0^\infty \int_{v_2 + n(\lambda_{\min} + \lambda_0)\delta^*}^\infty g_2(v_1) g_2(v_2) \, dv_1 \, dv_2 \\ &= 1 - \exp \{ -n(\lambda_{\min} + \lambda_0)\delta^*/2 \} . \end{aligned}$$

If we choose n such that $1 - \exp(-n(\lambda_{\min} + \lambda_0)\delta^*/2) \geq P^*$, which implies (3.3), then (3.2) is satisfied.

In the above procedure, the upper bound for $P(CS)$ is given in the following theorem.

THEOREM 3.2. *In the least favorable configuration,* $\theta_{[2]} = \theta_{[1]} + \delta^*$, $P(CS)$ *is bounded by*

$$1 - e^{-2} e^{-n(\lambda_{\min} + \lambda_0)\delta^*} .$$

PROOF. It is easily seen that

$$P(T_{[j],n} < t) \le 1 - \exp\{-n(\lambda_{\min} + \lambda_0)(t - \theta_{[j]})\}$$

for $t > \theta_{[j]}$. Hence we have

$$\begin{aligned} &P(T_{[1],n} - \theta_{[1]} < T_{[2],n} - \theta_{[2]} + \delta^*) \\ &\quad = E_{T_{[2],n}}\{P(T_{[1],n} - \theta_{[1]} < T_{[2],n} - \theta_{[2]} + \delta^* \mid T_{[2],n})\} \\ &\quad \le E_{T_{[2],n}}(1 - \exp\{-n(\lambda_{\min} + \lambda_0)(T_{[2],n} - \theta_{[2]} + \delta^*)\}) . \end{aligned}$$

On the other hand, by using Jensen's inequality, we find that

$$\begin{aligned} &E(\exp\{-n(\lambda_{\min} + \lambda_0)(T_{[2],n} - \theta_{[2]})\}) \\ &\quad \ge \exp\{-n(\lambda_{\min} + \lambda_0)E(T_{[2],n} - \theta_{[2]})\} \\ &\quad = \exp\left\{-n(\lambda_{\min} + \lambda_0)\left(\frac{1}{n(\lambda_1 + \lambda_0)} + \frac{1}{n(\lambda_2 + \lambda_0)}\right)\right\} \\ &\quad \ge e^{-2} . \end{aligned}$$

Thus we obtain

$$P(T_{[1],n} - \theta_{[1]} < T_{[2],n} - \theta_{[2]} + \delta^*) \le 1 - e^{-2}e^{-n(\lambda_{\min} + \lambda_0)\delta^*} .$$

3.2. The case when $\boldsymbol{\lambda}$ is unknown; Two-stage procedure

When $\boldsymbol{\lambda}$ is unknown, the required sample size n as in (3.3) cannot be determined. Furthermore, the probability requirement (3.1) should be replaced by $\inf_{\Omega_{P(\tau)}, \Omega_{\lambda}} P(CS) = P^*$. If one knows lower bounds on all elements of $\boldsymbol{\lambda}$, say $\boldsymbol{\lambda}_L$, one can obtain n replacing $\boldsymbol{\lambda}$ by $\boldsymbol{\lambda}_L$ in (3.3). However, this may lead to over sampling. Thus we propose a two-stage sampling procedure, which is an extension of the one considered by Desu, Narula and Villarreal [12].

We first take samples of size n_0 (≥ 2) from each of two populations and compute

$$S_x = \frac{1}{2(n_0 - 1)} \sum_{j=1}^{2} \sum_{r=1}^{n_0} (X_r^{[j]} - X_{n_0,1}^{[j]})$$

and

$$S_y = \frac{1}{2(n_0 - 1)} \sum_{j=1}^{2} \sum_{r=1}^{n_0} (Y_r^{[j]} - Y_{n_0,1}^{[j]}) .$$

Define N by

$$N = \max\{n_0, [c_2 S_x] + 1, [c_2 S_y] + 1\} , \tag{3.5}$$

where $[b]$ denotes the greatest integer not greater than b and c_2 is a positive constant. Next we take $N - n_0$ additional observations and compute $(X_{N,1}^{[j]}, Y_{N,1}^{[j]})$ based on N observations ($j = 1, 2$). By (3.4) and (3.5), it can be shown that

$$
\begin{aligned}
&P(T_{[1],N} < T_{[2],N}) \\
&\geq 1 - P\left(X_{N,1}^{[1]} - \tau_{[11]} \geq X_{N,1}^{[2]} - \tau_{[21]} + \frac{\delta^*}{2} \right) \\
&\quad - P\left(Y_{N,1}^{[1]} - \tau_{[12]} \geq Y_{N,1}^{[2]} - \tau_{[22]} + \frac{\delta^*}{2} \right) \\
&\geq 1 - P\{2N(\lambda_1 + \lambda_0)(X_{N,1}^{[1]} - \tau_{[11]}) \geq 2N(\lambda_1 + \lambda_0)(X_{N,1}^{[1]} - \tau_{[21]}) \\
&\quad + c_2\delta^*(\lambda_1 + \lambda_0)S_x\} \\
&\quad - P\{2N(\lambda_2 + \lambda_0)(Y_{N,1}^{[1]} - \tau_{[12]}) \geq 2N(\lambda_2 + \lambda_0)(Y_{N,1}^{[1]} - \tau_{[22]}) \\
&\quad + c_2\delta^*(\lambda_2 + \lambda_0)S_y\} \\
&= 1 - 2\int_0^\infty \int_0^\infty \int_{v_2 + (c_2\delta^* u/4(n_0-1))}^\infty g_2(v_1)g_2(v_2)g_{4(n_0-1)}(u)\, dv_1\, dv_2\, du \\
&= 1 - \left\{1 + \frac{c_2\delta^*}{4(n_0-1)}\right\}^{-4(n_0-1)/2} .
\end{aligned}
$$

Hence we obtain the following theorem.

THEOREM 3.3. *The probability requirement* (3.2) *is satisfied, if the constant* c_2 *in* (3.5) *is chosen so that*

$$c_2 \geq \frac{2}{\delta^*} 2(n_0 - 1)\{(1 - P^*)^{-1/(2(n_0-1))} - 1\} .$$

The procedure proposed in this section is based on the marginal distributions. Hence, it is possible to use the procedure for the selection with respect to linear combinations of the locations, that is, $\theta_j = a_1\tau_{j1} + a_2\tau_{j2}$ ($j = 1, 2$) with $a_1 > 0$ and $a_2 > 0$, since $a_1 X_r$ and $a_2 Y_r$ are distributed as exponentials.

4. Tests for a structure of location parameters

We consider the null hypothesis $H_0 : \delta = 0$ (say, $\tau_1 = \tau_2$) against the alternative hypothesis $H_1 : \delta \neq 0$ in the $BVE(\boldsymbol{\tau}, \boldsymbol{\lambda})$ population. Here we assume that $\lambda_1 = \lambda_2$ $(= \bar{\lambda})$. Without loss of generality, we may assume that there is no tied observation except for the point δ with probability one, which means that

$$P(X_r - Y_r = X_{r'} - Y_{r'} \neq \delta) = 0 \qquad \text{for all } r \neq r'$$

and

$$P(X_r - Y_r = X_{r'} - Y_{r'} = \delta | \lambda_0 > 0) > 0 \qquad \text{for some } r \neq r' .$$

Since $\lambda_1 = \lambda_2$, the hypothesis can be viewed as the one for the equality of the means of two components in the BVE distribution. Thus the hypothesis is one of the contrasts of means (e.g., see Chapter 5 of Siotani, Hayakawa and Fujikoshi [42]). We propose two-step procedures for testing the hypothesis H_0, based on a sample of size n whose size is fixed.

4.1. Conditional binomial test

From Lemma 1.3, we may employ (n_+, n_-, m) to derive a simple test for the hypothesis H_0. A two-step procedure is described in the following.

STEP I.

We first examine the difference of the components, say $X_r - Y_r$, of each of n observations. By the assumption of no tied observation, if we could observe that

$$X_{r_1} - Y_{r_1} = X_{r_2} - Y_{r_2} = \cdots = X_{r_m} - Y_{r_m} \qquad (2 \leq m \leq n) ,$$

then we decide $X_{r_1} - Y_{r_1} = \delta$ with probability one. Hence, for $m \geq 2$, the most obvious test is to accept H_0 when $X_{r_1} - Y_{r_1} = X_{r_2} - Y_{r_2} = 0$ for some $r_1 \neq r_2$ and to reject H_0 when $X_{r_1} - Y_{r_1} = X_{r_2} - Y_{r_2} \neq 0$ for some $r_1 \neq r_2$. It is easily seen that both of the probabilities of type I error and type II error are zero for this test.

We cannot always make a decision at Step I whether to accept or reject the hypothesis, since $P(m \geq 2) < 1$. When $m < 2$, δ is not determined exactly. In general, n_+ and n_- depend on δ. However, under the null hypothesis, it is possible to compute n'_+ (or n'_-), that is

$$\text{(4.1)} \qquad n'_+ = \textstyle\sum_{r=1}^{n} I(X_r - Y_r > 0) , \qquad n'_- = \textstyle\sum_{r=1}^{n} I(X_r - Y_r < 0) .$$

Now we may use the statistics n'_+ (or n'_-) for test of the hypothesis in the next step.

STEP II.

The test procedure in Step I is available only for the case of $m \geq 2$. We employ n'_+ in (4.1) as a test statistics and reject H_0 if n'_+ or $n - n'_+$ is small. In order to obtain the critical region, the conditional distribution of n'_+ given $m < 2$ is needed. The following theorem is useful in determining the critical region.

THEOREM 4.1. *Under H_0, the conditional probability function of n'_+ given $m < 2$ is*

$$Q_0(n'_+|m < 2) = \frac{n!}{n'_+!(n - n'_+)!}\left(\frac{1}{2}\right)^n\left(\frac{p^+(n - n'_+)(1 - 2p)}{p + n(1 - 2p)/2}\right), \tag{4.2}$$

where $p = \bar{\lambda}/(2\bar{\lambda} + \lambda_0)$.

PROOF. By Lemma 1.3, we find

$$Q_0(n'_+|m < 2) = Q_0(n'_+|m = 0, 1)$$

$$= \frac{\dfrac{n!}{n'_+!(n - n'_+)!}p^n + \dfrac{n!}{n'_+!(n - n'_+ - 1)!1!}p^{n-1}(1 - 2p)}{\dfrac{n!}{n!}(2p)^n + \dfrac{n!}{(n - 1)!1!}(2p)^{n-1}(1 - 2p)}$$

$$= \frac{n!}{n'_+!(n - n'_+)!}\left(\frac{1}{2}\right)^n\left(\frac{p^+(n - n'_+)(1 - 2p)}{p + n(1 - 2p)/2}\right).$$

The result (4.2) implies the following corollary.

COROLLARY 4.1. *Under H_0,*

$$Q_0(n'_+|m < 2) + Q_0(n - n'_+|m < 2) = 2\frac{n!}{n'_+!(n - n'_+)!}\left(\frac{1}{2}\right)^n. \tag{4.3}$$

By Corollary 4.1, we can determine the critical region with significance level α as

$$C_{\zeta_\alpha} = \{n'_+ : n'_+ \leq \zeta_\alpha, n - n'_+ \leq \zeta_\alpha\}, \tag{4.4}$$

where ζ_α is the greatest integer such that

$$2\sum_{v=0}^{\zeta_\alpha} \frac{v!}{v!(n - v)!}\left(\frac{1}{2}\right)^n \leq \alpha.$$

The above test is available for both cases when $\boldsymbol{\lambda}$ is known or $\boldsymbol{\lambda}$ is unknown, since the test statistics and the critical region do not depend on $\boldsymbol{\lambda}$.

In order to calculate the power function, it is needed to find the probability function of n'_+ (or n'_-) under the alternatives. From (1.4) and the proof of Theorem 4.1, we obtain

$$Q_1(n'_+|m < 2, \delta > 0) = \frac{n!}{n'_+!(n - n'_+)!}\left\{1 - \frac{1}{2}e^{-(\bar{\lambda}+\lambda_0)\delta}\right\}^{n'_+}\left\{\frac{1}{2}e^{-(\bar{\lambda}+\lambda_0)\delta}\right\}^{n-n'_+}$$

$$\cdot\frac{p + (n - n'_+)(1 - 2p)e^{(\bar{\lambda}+\lambda_0)\delta}}{p + n(1 - 2p)/2},$$

$$Q_1(n'_+|m<2,\delta<0)=\frac{n!}{n'_+!(n-n'_+)!}\left\{\frac{1}{2}e^{(\bar{\lambda}+\lambda_0)\delta}\right\}^{n'_+}\left\{1-\frac{1}{2}e^{(\bar{\lambda}+\lambda_0)\delta}\right\}^{n-n'_+}\cdot\frac{p+(n-n'_+)(1-2p)e^{-(\bar{\lambda}+\lambda_0)\delta}}{p+n(1-2p)/2}\,.$$

The power of the test is given by

$$\textstyle\sum_{n'_+\in C_0} Q_1(n'_+|m<2,\delta>0 \text{ (and/or } \delta<0)),$$

where $C_0 = C_{\xi_\alpha}$.

4.2. Conditional exponential test

It is interesting to derive an alternative test procedure based on $(X_{n,1}, Y_{n,1})$ instead of n_+ in Step II of the previous subsection. We shall derive such a test under the assumption that λ is known. Letting $Z_n = X_{n,1} - Y_{n,1}$, it is natural to reject the null hypothesis when and only when

$$|Z_n| > \eta\,, \tag{4.5}$$

where η is a positive constant determined by the significance level. The p.d.f. of Z_n under H_0 is given by

$$h(z)=\begin{cases}\dfrac{2\bar{\lambda}}{2\bar{\lambda}+\lambda_0}\dfrac{1}{2}n(\bar{\lambda}+\lambda_0)e^{-n(\bar{\lambda}+\lambda_0)|z|}, & z\neq 0\,,\\[2ex] \dfrac{\lambda_0}{2\bar{\lambda}+\lambda_0}, & z=0\,,\end{cases} \tag{4.6}$$

which follows from (1.3) with $\lambda_1 = \lambda_2 = \bar{\lambda}$. In order to determine η, we need to evaluate

$$P\{|Z_n| > \eta | m < 2\}\,, \tag{4.7}$$

under H_0. For this, let

$$A = \{-\eta \le Z_n \le \eta\}, \qquad A^c = \{|Z_n| > \eta\}\,,$$

$$B_0 = \{X_r - Y_r \neq 0, r = 1, \dots, n\}\,,$$

$$B_{1s} = \{X_s - Y_s = 0 \text{ and } X_r - Y_r \neq 0 \ (r = 1, \dots, s-1, s+1, \dots, n)\}\,.$$

Then (4.7) can be rewritten as

$$1 - P\{A|B_0 \cup (\textstyle\bigcup_{s=1}^n B_{1s})\} = 1 - \dfrac{P(A\cap B_0) + \sum_{s=1}^n P(A\cap B_{1s})}{P(B_0) + \sum_{s=1}^n P(B_{1s})}\,. \tag{4.8}$$

LEMMA 4.1. *Under* H_0,

(i) $P(B_0) + \sum P(B_{1s}) = \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^n + n\left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^{n-1}\left(\frac{\lambda_0}{2\bar{\lambda} + \lambda_0}\right)$,

(ii) $$P(A \cap B_0) = \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^n \{1 - e^{-n(\bar{\lambda}+\lambda_0)\eta}\},$$

(iii) $$\sum P(A \cap B_{1s}) = n\left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^{n-1}\left(\frac{\lambda_0}{2\bar{\lambda} + \lambda_0}\right)$$
$$\cdot\left\{\frac{1}{n} + \left(1 - \frac{1}{n}\right)[1 - e^{-((2\bar{\lambda}+\lambda_0)+(n-1)(\bar{\lambda}+\lambda_0))\eta}]\right\}.$$

PROOF. (i) follows from Lemma 1.3. By using (4.6), we have

$$\begin{aligned} P(A \cap B_0) &= P(B_0)P(A \mid B_0) \\ &= \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^n \int_{-\eta}^{\eta} \frac{n}{2}(\bar{\lambda} + \lambda_0)e^{-n(\bar{\lambda}+\lambda_0)|z|}\,dz \\ &= \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^n \{1 - e^{-n(\bar{\lambda}+\lambda_0)\eta}\}, \end{aligned}$$

which proves (ii). In order to evaluate $P(A \cap B_{1s})$, we put

$$B_{1s}^1 = \{X_s < X_{(s)} < Y_{(s)}\}, \qquad B_{1s}^2 = \{X_s < Y_{(s)} < X_{(s)}\},$$
$$B_{1s}^3 = \{X_{(s)} < X_s < Y_{(s)}\}, \qquad B_{1s}^4 = \{Y_{(s)} < X_s < X_{(s)}\},$$
$$B_{1s}^5 = \{X_{(s)} < Y_{(s)} < X_s\}, \qquad B_{1s}^6 = \{Y_{(s)} < X_{(s)} < X_s\},$$

where $X_{(s)} = \min(X_1, \ldots, X_{s-1}, X_{s+1}, \ldots, X_n)$, $Y_{(s)} = \min(Y_1, \ldots, Y_{s-1}, Y_{s+1}, \ldots, Y_n)$ and $X_s = Y_s$. Since $B_{1s} = \bigcup_{\kappa=1}^6 B_{1s}^\kappa$ and $B_{1s}^1, \ldots, B_{1s}^6$ are mutually disjoint,

$$P(A \cap B_{1s}) = \textstyle\sum_{\kappa=1}^6 P(A \cap B_{1s}^\kappa) = \sum_{\kappa=1}^6 P(A \cap B_{1s}^\kappa \mid B_{1s})P(B_{1s}). \tag{4.9}$$

From (1.2) and Lemma 1.1,

$$f_0(x_{(s)}, y_{(s)} \mid x_{(s)} \neq y_{(s)})$$
$$= \begin{cases} \frac{2\bar{\lambda} + \lambda_0}{2\bar{\lambda}}(n-1)\bar{\lambda}e^{-(n-1)\bar{\lambda}(x_{(s)}-\tau)}(n-1)(\bar{\lambda} + \lambda_0)e^{-(n-1)(\bar{\lambda}+\lambda_0)(y_{(s)}-\tau)}, & y_{(s)} > x_{(s)}, \\ \frac{2\bar{\lambda} + \lambda_0}{2\bar{\lambda}}(n-1)(\bar{\lambda} + \lambda_0)e^{-(n-1)(\bar{\lambda}+\lambda_0)(x_{(s)}-\tau)}(n-1)\bar{\lambda}e^{-(n-1)\bar{\lambda}(y_{(s)}-\tau)}, & x_{(s)} > y_{(s)}, \end{cases}$$

and

$$f_1(x_s, y_s \mid x_s = y_s) = (2\bar{\lambda} + \lambda_0)e^{-(2\bar{\lambda}+\lambda_0)(x_s-\tau)},$$

where $\tau = \tau_1 = \tau_2$. Hence we obtain

$$P(A \cap B_{1s}^1 | B_{1s})$$

$$= P(-\eta \le X_s - Y_s \le \eta, X_s < X_{(s)} < Y_{(s)})$$

$$= P(X_s < X_{(s)} < Y_{(s)})$$

$$= \int_\tau^\infty \int_{x_s}^\infty \int_{x_{(s)}}^\infty (2\bar{\lambda} + \lambda_0) e^{-(2\bar{\lambda}+\lambda_0)(x_s - \tau)} \frac{2\bar{\lambda} + \lambda_0}{2\bar{\lambda}} (n-1)\bar{\lambda} e^{-(n-1)\bar{\lambda}(x_{(s)} - \tau)}$$

$$\cdot (n-1)(\bar{\lambda} + \lambda_0) e^{-(n-1)(\bar{\lambda}+\lambda_0)(y_{(s)} - \tau)} \, dy_{(s)} \, dx_{(s)} \, dx_s$$

$$= 1/2n \,,$$

$$P(A \cap B_{1s}^3 | B_{1s})$$

$$= P(-\eta \le X_{(s)} - Y_s \le \eta, X_{(s)} < X_s < Y_{(s)})$$

$$= P(X_{(s)} - X_s \le X_{(s)} + \eta, X_{(s)} < X_s < Y_{(s)})$$

$$= \int_\tau^\infty \int_{x_{(s)}}^{x_{(s)}+\eta} \int_{x_s}^\infty (2\bar{\lambda} + \lambda_0) e^{-(2\bar{\lambda}+\lambda_0)(x_s - \tau)} \frac{2\bar{\lambda} + \lambda_0}{2\bar{\lambda}} (n-1)\bar{\lambda} e^{-(n-1)\bar{\lambda}(x_{(s)} - \tau)}$$

$$\cdot (n-1)(\bar{\lambda} + \lambda_0) e^{-(n-1)(\bar{\lambda}+\lambda_0)(y_{(s)} - \tau)} \, dy_{(s)} \, dx_s \, dx_{(s)}$$

$$= \frac{1}{2} \frac{(n-1)(2\bar{\lambda} + \lambda_0)}{n(2\bar{\lambda} + \lambda_0)} \frac{2\bar{\lambda} + \lambda_0}{(2\bar{\lambda} + \lambda_0) + (n-1)(\bar{\lambda} + \lambda_0)} [1 - e^{-\{(2\bar{\lambda}+\lambda_0)+(n-1)(\bar{\lambda}+\lambda_0)\}\eta}] \,,$$

$$P(A \cap B_{1s}^5 | B_{1s})$$

$$= P(-\eta \le X_{(s)} - Y_{(s)} \le \eta, X_{(s)} < Y_{(s)} < X_s)$$

$$= P(X_{(s)} < Y_{(s)} < X_{(s)} + \eta, X_{(s)} < Y_{(s)} < X_s)$$

$$= \int_\tau^\infty \int_{x_{(s)}}^{x_{(s)}+\eta} \int_{y_{(s)}}^\infty (2\bar{\lambda} + \lambda_0) e^{-(2\bar{\lambda}+\lambda_0)(x_s - \tau)} \frac{2\bar{\lambda} + \lambda_0}{2\bar{\lambda}} (n-1)\bar{\lambda} e^{-(n-1)\bar{\lambda}(x_{(s)} - \tau)}$$

$$\cdot (n-1)(\bar{\lambda} + \lambda_0) e^{-(n-1)(\bar{\lambda}+\lambda_0)(y_{(s)} - \tau)} \, dx_s \, dy_{(s)} \, dx_{(s)} \,.$$

$$= \frac{1}{2} \frac{(n-1)(2\bar{\lambda} + \lambda_0)}{n(2\bar{\lambda} + \lambda_0)} \frac{(n-1)(\bar{\lambda} + \lambda_0)}{(2\bar{\lambda} + \lambda_0) + (n-1)(\bar{\lambda} + \lambda_0)} [1 - e^{-\{(2\bar{\lambda}+\lambda_0)+(n-1)(\bar{\lambda}+\lambda_0)\}\eta}] \,.$$

It is easily seen that

$$P(A \cap B_{1s}^1 | B_{1s}) = P(A \cap B_{1s}^2 | B_{1s}) \,,$$

$$P(A \cap B_{1s}^3 | B_{1s}) = P(A \cap B_{1s}^4 | B_{1s}) \,,$$

$$P(A \cap B_{1s}^5 | B_{1s}) = P(A \cap B_{1s}^6 | B_{1s}) \,.$$

Hence, we get (4.9) by substituting $P(B_{1s})$ and $P(A \cap B_{1s}^{\kappa} | B_{1s})$ $(\kappa = 1, \ldots, 6)$. This completes the proof of (iii).

THEOREM 4.2. *If we choose the critical point*

$$\eta = \frac{1}{n(\bar{\lambda} + \lambda_0)} \log \left[\frac{2\bar{\lambda} + n\lambda_0}{2\bar{\lambda} + (n-1)\lambda_0} \alpha \right]^{-1},$$

then $P(|Z_n| > \eta | m < 2) \leq \alpha$ *when* H_0 *is true.*

PROOF. Substituting (i), (ii) and (iii) of Lemma 4.1 into (4.8), we obtain

$$P[A | B_0 \cup (\textstyle\bigcup_{s=1}^{n} B_{1s})]$$

$$= \left[\left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0} \right)^n [1 - e^{-n(\bar{\lambda}+\lambda_0)\eta}] \right.$$

$$\left. + n \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0} \right)^{n-1} \left(\frac{\lambda_0}{2\bar{\lambda} + \lambda_0} \right) \left\{ \frac{1}{n} + \left(1 - \frac{1}{n}\right) [1 - e^{-((2\bar{\lambda}+\lambda_0)+(n-1)(\bar{\lambda}+\lambda_0))\eta}] \right\} \right]$$

$$\cdot \left[\left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0} \right)^n + n \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0} \right)^{n-1} \left(\frac{\lambda_0}{2\bar{\lambda} + \lambda_0} \right) \right]^{-1}$$

$$= 1 - \frac{1}{2\bar{\lambda} + n\lambda_0} \{ 2\bar{\lambda} e^{-n(\bar{\lambda}+\lambda_0)\eta} + (n-1)\lambda_0 e^{-((2\bar{\lambda}+\lambda_0)+(n-1)(\bar{\lambda}+\lambda_0))\eta} \},$$

which is bounded by

$$1 - \frac{2\bar{\lambda} + (n-1)\lambda_0}{2\bar{\lambda} + n\lambda_0} e^{-n(\bar{\lambda}+\lambda_0)\eta}. \tag{4.10}$$

Hence the proof is completed by letting the second term of (4.10) is equal to α.

This theorem shows that the probability of Type I error of the proposed two-step test procedure is not greater than α, since the error probability in Step I is zero.

When λ is unknown, it is suggested to use

$$W_n = Z_n / S_0$$

instead of Z_n, where

$$S_0 = \frac{1}{2(n-1)} \textstyle\sum_{r=1}^{n} \{ (X_r - X_{n,1}) + (Y_r - Y_{n,1}) \} \tag{4.11}$$

is an unbiased estimate of $1/(\bar{\lambda} + \lambda_0)$. However, the distribution of W_n depends on λ. It is complicated to derive the conditional distribution of W_n as in (4.2) or (4.7).

4.3. Asymptotic test

In this subsection, we construct an asymptotic test procedure based on the mean difference, say $\bar{X} - \bar{Y}$, in Step II, when λ is unknown. Under H_0, since the mean and the variance of $X_r - Y_r$ are zero and $\left(1 - \frac{\lambda_0}{2\bar{\lambda} + \lambda_0}\right)\frac{2}{(\bar{\lambda} + \lambda_0)^2}$, respectively, the mean and the variance of $\bar{X} - \bar{Y}$ are zero and $\frac{1}{n}\left(1 - \frac{\lambda_0}{2\bar{\lambda} + \lambda_0}\right) \cdot \frac{2}{(\bar{\lambda} + \lambda_0)^2}$. Hence the asymptotic distribution of

$$\sqrt{n}(\bar{X} - \bar{Y})\left\{\left(1 - \frac{\lambda_0}{2\bar{\lambda} + \lambda_0}\right)\frac{2}{(\bar{\lambda} + \lambda_0)^2}\right\}^{-1/2} \tag{4.12}$$

is the standard normal distribution $N(0, 1)$ by the central limit theorem. Since the statistics (4.12) includes an unknown parameter λ, it is suggested to use the MLE m/n and the unbiased estimate S_0 in (4.11) for $\lambda_0/(2\bar{\lambda} + \lambda_0)$ and $1/(\bar{\lambda} + \lambda_0)$, respectively, into (4.12). From the condition $\{m < 2\}$, m/n tends to zero as $n \to \infty$. Thus the suggested test statistics becomes to $\sqrt{n/2}(\bar{X} - \bar{Y})/S_0$. The critical region is given by

$$\sqrt{n/2}\frac{|\bar{X} - \bar{Y}|}{S_0} > \phi(\alpha/2), \tag{4.13}$$

where $\phi(\alpha/2)$ is the upper $100(\alpha/2)\,\%$ point of $N(0, 1)$.

The test procedure mentioned above is based on the unconditional distribution. We should derive the conditional distribution as in the previous subsections, if the procedure is used in Step II. However, one can assume that the two components X and Y are independent in the asymptotic case, because the MLE of the correlation coefficient, say m/n, tends to zero as $n \to \infty$. Hence the testing hypothesis can be treated as a two sample problem, which shows that the test procedure based on (4.13) is reasonable. We note that it is possible to make a decision at Step I for large sample sizes with $\lambda_0 > 0$, since

$$P(m < 2) = \left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^n + n\left(\frac{2\bar{\lambda}}{2\bar{\lambda} + \lambda_0}\right)^{n-1}\left(\frac{\lambda_0}{2\bar{\lambda} + \lambda_0}\right)$$

tends to zero as $n \to \infty$. In other words, if we use only Step I, the null hypothesis is accepted when

$$X_{r_1} - Y_{r_1} = \cdots = X_{r_m} - Y_{r_m} = 0 \qquad (2 \leq m \leq n)$$

and is rejected otherwise. Then the probability of Type I error is fairly closed to zero. Clearly, the probability of Type II error is zero.

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References

[1] B. C. Arnold, Parameter estimation for a multivariate exponential distribution, J. Amer. Statist. Assoc., **63**(1968), 848–852.

[2] A. M. Awad, M. M. Azzam and M. A. Hamdan, Some inference result on $Pr(X > Y)$ in the bivariate exponential model, Commun. Statist., **A10**(1981), 2515–2525.

[3] L. J. Bain, Statistical Analysis of Reliability and Life-Testing Models, Marcel Dekker, New York, 1978.

[4] T. A. Bancroft, Some recent advances in inference procedures using preliminary tests of significance, Statistical Papers in Honor of G. W. Snedecor, 1972, 19–30.

[5] R. E. Barlow and F. Proschan, Statistical Theory of Reliability and Life Testing, Holt, Reinhart and Winston, New York, 1975.

[6] R. E. Bechhofer, A single-sample multiple decision procedure for ranking means of normal populations with known variances, Ann. Math. Statist., **25**(1954), 16–39.

[7] B. M. Bemis, L. J. Bain and J. J. Higgins, Estimation and hypothesis testing for the parameters of a bivariate exponential distribution, J. Amer. Statist. Assoc., **67**(1972), 927–929.

[8] H. W. Block and A. P. Basu, A continuous bivariate exponential extension, J. Amer. Statist. Assoc., **69**(1974), 1031–1037.

[9] S. K. Chatterjee, On an extension of Stein's two-sample procedure to the multi-normal problem, Calcutta Statist. Assoc. Bull., **8**(1959), 121–148.

[10] Y. S. Chow and H. Robbins, On the asymptotic theory of fixed width confidence intervals for the mean, Ann. Math. Statist., **36**(1965), 457–462.

[11] D. R. Cox and D. Oakes, Analysis of Survival Data, Chapman and Hall, London, 1984.

[12] M. M. Desu, S. C. Narula and B. Villareal, A two-stage procedure for selecting the best of k exponential distributions, Commun. Statist., **A6**(1977), 1223–1230.

[13] F. Downton, Bivariate exponential distributions in reliability theory, J. Royal Statist. Soc., **33B**(1970), 408–417.

[14] E. J. Dudewicz and T. A. Bishop, The heteroscedastic method, Optimizing Methods in Statistics (ed. J. S. Rustagi), Academic Press, New York, (1979), 183–203.

[15] E. J. Dudewicz and S. R. Dalal, Allocation of observations in ranking and selection with unknown variances, Sankhyā, **37B**(1975), 28–78.

[16] E. J. Dudewicz and V. S. Taneja, A multivariate solution of the ranking and selection problem, Commun. Statist., **A10**(1981), 1849–1868.

[17] N. Ebrahimi, Analysis of bivariate accelerated life test data for the bivariate exponential distribution, Amer. J. Math. Manage. Sci., **7**(1987), 175–190.

[18] J. E. Freund, A bivariate extension of exponential distribution, J. Amer. Statist. Assoc., **56**(1961), 971–977.

[19] S. G. Ghurye, Note on sufficient statistics and two-stage procedures, Ann. Math. Statist., **29**(1958), 155–166.

[20] J. D. Gibbons, I. Olkin and M. Sobel, Selecting and Ordering Populations: A New Statistical Methodology, Wiley, New York, 1977.

[21] E. J. Gumbel, Bivariate exponential distribution, J. Amer. Statist. Assoc., **55**(1960), 698–707.

[22] S. S. Gupta and S. Panchapakesan, Multiple Decision Procedures: Theory and Methodology of Selecting and Ranking Populations, Wiley, New York, 1979.

[23] W. C. Healy, Jr., Two-sample procedure in simultaneous estimation, Ann. Math. Statist., **27**(1956), 687–702.

[24] H. Hyakutake, A construction method of certain matrices required in the multivariate heteroscedastic method, Ann. Inst. Statist. Math., **38**(1986), 523–528.

[25] H. Hyakutake, A general treatment for selecting the best of several multivariate normal populations, Sequential Analysis, **7**(1988), 239–251.

[26] H. Hyakutake, A fixed-size confidence region for locations of a bivariate exponential distribution, Tech. Rep. No. 231, Statist. Research Group, Hiroshima University, 1988.

[27] H. Hyakutake and M. Siotani, The multivariate heteroscedastic method: Distributions of statistics and an application, Amer. J. Math. Manage. Sci., **7**(1987), 89–111.

[28] H. Hyakutake, M. Siotani, C. Y. Li and Mustafid, Distributions of some statistics in heteroscedastic inference method—Power functions and percentage points—, J. Japan Statist. Soc., **16**(1986), 7–20.

[29] E. R. Jones, Fixed-size rectangular confidence regions, Commun. Statist., **A6**(1977), 251–264.

[30] J. D. Kalbfleisch and R. L. Prentice, The Statistical Analysis of Failure Time Data, Wiley, New York, 1980.

[31] J. P. Klein and A. P. Basu, Estimating reliability for bivariate exponential distributions, Sankhyā, **47B**(1985), 346–353.

[32] S. H. Lee and W. C. Kim, An elimination type two-stage selection procedure for exponential distributions, Commun. Statist., **14**(1985), 2563–2571.

[33] E. L. Lehmann, Mimeographed note on the theory of estimation, University of California Press, 1951.

[34] E. L. Lehmann, Testing Statistical Hypothesis (2nd ed.), Wiley, New York, 1986.

[35] J. C. Lu and G. K. Battacharyya, Some bivariate extensions of the Weibull distribution, Tech. Rep., No. 821, Dept. Statist., University of Wisconsin, 1988.

[36] A. W. Marshall and I. Olkin, A multivariate exponential distribution, J. Amer. Statist. Assoc., **62**(1967), 30–44.

[37] R. G. Miller, Jr., Simultaneous Statistical Inference (2nd ed.), Springer-Verlag, New York, 1981.

[38] N. Mukhopadhyay, Sequential and two-stage procedures for selecting the better exponential population covering the case of unknown and unequal scale parameters, J. Statist. Planning and Inference, **9**(1984), 33–43.

[39] N. Mukhopadhyay and J. S. Al-Mousawi, Fixed-size confidence regions for the mean vector of a multinormal distribution, Sequential Analysis, **5**(1986), 139–168.

[40] F. Proschan and P. Sullo, Estimating the parameters of a multivariate exponential distribution, J. Amer. Statist. Assoc., **71**(1976), 465–472.

[41] R. Singh, Existence of bounded length confidence intervals, Ann. Math. Statist., **34**(1963), 1474–1485.

[42] M. Siotani, T. Hayakawa and Y. Fujikoshi, Modern Multivariate Statistical Analysis: A Graduate Course and Handbook, American Sciences Press, Columbus, Ohio, 1985.

[43] C. Stein, A two-sample test for a linear hypothesis whose power is independent of the variance, Ann. Math. Statist., **16**(1945), 243–258.

[44] K. Takeuchi, A Note on the test for the location parameter of an exponential distribution, Ann. Math. Statist., **40**(1969), 1838–1839.

Department of Mathematics,
Faculty of Engineering
Kumamoto University