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STATISTICAL LIMIT POINTS

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ABSTRACT. Following the concept of a statistically convergent sequence x, we define a statistical limit point of x as a number λ that is the limit of a subsequence $\{x_{k(j)}\}$ of x such that the set $\{k(j): j \in \mathbb{N}\}$ does not have density zero. Similarly, a statistical cluster point of x is a number γ such that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N}: |x_k - \gamma| < \varepsilon\}$ does not have density zero. These concepts, which are not equivalent, are compared to the usual concept of limit point of a sequence. Statistical analogues of limit point results are obtained. For example, if x is a bounded sequence then x has a statistical cluster point but not necessarily a statistical limit point. Also, if the set $M := \{k \in \mathbb{N}: x_k > x_{k+1}\}$ has density one and x is bounded on M, then x is statistically convergent.

1. INTRODUCTION AND BACKGROUND

In [4] Fast introduced the concept of statistical convergence for real number sequences; in [10] Zygmund called it "almost convergence" and established a relation between it and strong summability. In [2, 3, 5, 6, 9] this concept was studied as a nonmatrix summability method. In the present paper we return to the view of statistical convergence as a sequential limit concept, and we extend this concept in a natural way to define a statistical analogue of the set of limit points or cluster points of a number sequence. In §2 we give the basic properties of statistical limit points and cluster points. This section develops the similarities and differences between these points and ordinary limit points. Section 3 presents statistical analogues of some of the well-known completeness properties of the real numbers.

If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The "natural density" of K (see [8, Chapter 11]) is given by $\delta(K) = \lim_n n^{-1}|K_n|$. A (real) number sequence x is *statistically convergent* to L provided that for every $\varepsilon > 0$ the set $K(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero; in this case we write st-lim $x = L$.

In [5, Theorem 1] it is proved that st-lim $x = L$ if and only if there is a (convergent) sequence y such that $\lim y = L$ and $\delta\{k \in \mathbb{N} : x_k \neq y_k\} = 0$.

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The zero density property is described succinctly as "$x_k = y_k$ for almost all k". Sets of density zero play an important role, so we introduce some convenient terminology and notation for working with them. If x is a sequence we write $\{x_k : k \in \mathbb{N}\}$ to denote the range of x. If $\{x_{k(j)}\}$ is a subsequence of x and $K = \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$. In case $\delta(K) = 0$, $\{x\}_K$ is called a *subsequence of density zero*, or a *thin subsequence*. On the other hand, $\{x\}_K$ is a *nonthin* subsequence of x if K does not have density zero. It should be noted that $\{x\}_K$ is a nonthin subsequence of x if either $\delta(K)$ is a positive number or K fails to have natural density.

2. Definitions and basic properties

The number L is an ordinary limit point of a sequence x if there is a subsequence of x that converges to L; therefore we define a statistical limit point by considering the density of such a subsequence.

Definition 1. The number λ is a *statistical limit point* of the number sequence x provided that there is a nonthin subsequence of x that converges to λ.

Notation. For any number sequence x, let Λ_x denote the set of statistical limit points of x, and let L_x denote the set of ordinary limit points of x.

Example 1. Let $x_k = 1$ if k is a square and $x_k = 0$ otherwise; then $L_x = \{0, 1\}$ and $\Lambda_x = \{0\}$.

It is clear that $\Lambda_x \subseteq L_x$ for any sequence x. To show that Λ_x and L_x can be very different, we give a sequence x for which $\Lambda_x = \varnothing$ while $L_x = \mathbb{R}$, the set of real numbers.

Example 2. Let $\{r_k\}_{k=1}^{\infty}$ be a sequence whose range is the set of all rational numbers and define

$$x_k = \begin{cases} r_n, & \text{if } k = n^2 \text{ for } n = 1, 2, 3, \dots, \\ k, & \text{otherwise}. \end{cases}$$

Since the set of squares has density zero, it follows that $\Lambda_x = \varnothing$, while the fact that $\{r_k : k \in \mathbb{N}\}$ is dense in $\mathbb{R}$ implies that $L_x = \mathbb{R}$.

A limit point L of a sequence x can be characterized by the statement "every open interval centered at L contains infinitely many terms of x". To form a statistical analogue of this criterion we require the open interval to contain a nonthin subsequence, but we must avoid calling the center of the interval a statistical limit point for reasons that will be apparent shortly.

Definition 2. The number γ is a *statistical cluster point* of the number sequence x provided that for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \gamma| < \varepsilon\}$ does not have density zero.

For a given sequence x, we let Γ_x denote the set of all statistical cluster points of x. It is clear that $\Gamma_x \subseteq L_x$ for every sequence x. The inclusion relationship between Γ_x and Λ_x is a bit more subtle.

Proposition 1. *For any number sequence* x, $\Lambda_x \subseteq \Gamma_x$.

Proof. Suppose $\lambda \in \Lambda_x$, say $\lim_j x_{k(j)} = \lambda$, and

$$\limsup_n \frac{1}{n}|\{k(j) \leq n\}| = d > 0.$$

For each $\varepsilon > 0$, $\{j : |x_{k(j)} - \lambda| \geq \varepsilon\}$ is a finite set, so

$$\{k \in \mathbb{N} : |x_k - \lambda| < \varepsilon\}| \supseteq \{k(j) : j \in \mathbb{N}\} \sim \{\text{finite set}\}.$$

Therefore,

$$\frac{1}{n}|\{k \leq n : |x_k - \lambda| < \varepsilon\} \geq \frac{1}{n}|\{k(j) \leq n\}| - \frac{1}{n}O(1) \geq \frac{d}{2}$$

for infinitely many n. Hence, $\delta\{k \in \mathbb{N} : |x_k - \lambda| < \varepsilon\} \neq 0$, which means that $\lambda \in \Gamma_x$.

Although our experience with ordinary limit points may lead us to expect that Λ_x and Γ_x are equivalent, the next example shows that this is not always the case.

Example 3. Define the sequence x by

$$x_k = 1/p, \qquad \text{where } k = 2^{p-1}(2q+1);$$

i.e., $p-1$ is the number of factors of 2 in the prime factorization of k. It is easy to see that for each p, $\delta\{k : x_k = 1/p\} = 2^{-p} > 0$, whence $1/p \in \Lambda_x$. Also, $\delta\{k : 0 < x_k < 1/p\} = 2^{-p}$, so $0 \in \Gamma_x$, and we have $\Gamma_x = \{0\} \cup \{1/p\}_{p=1}^{\infty}$. Now we assert that $0 \notin \Lambda_x$; for, if $\{x\}_K$ is a subsequence that has limit zero, then we can show that $\delta(K) = 0$. This is done by observing that for each p,

$$\begin{aligned} |K_n| &= |\{k \in K_n : x_k \geq 1/p\}| + |\{k \in K_n : x_k < 1/p\}| \\ &\leq O(1) + |\{k \in \mathbb{N} : x_k < 1/p\}| \leq O(1) + n/2^p. \end{aligned}$$

Thus $\delta(K) \leq 2^{-p}$, and since p is arbitrary this implies that $\delta\{K\} = 0$.

It is easy to prove that if x is a statistically convergent sequence, say st-lim $x = \lambda$, then Λ_x and Γ_x are both equal to the singleton set $\{\lambda\}$. The converse is not true, as one can see by taking $x_k = [1 + (-1)^k]k$. The following example presents a sequence x for which Γ_x is an interval while $\Lambda_x = \varnothing$.

Example 4. Let x be the sequence $\{0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots\}$. This sequence is uniformly distributed in $[0, 1]$ (see [7]), so we have not only that $L_x = [0, 1]$ but also the density of the x_k's in any subinterval of length d is d itself. Therefore for any γ in $[0, 1]$,

$$\delta\{k \in \mathbb{N} : x_k \in (\gamma - \varepsilon, \gamma + \varepsilon)\} \geq \varepsilon > 0.$$

Hence, $\Gamma_x = [0, 1]$. On the other hand, if $\lambda \in [0, 1]$ and $\{x\}_K$ is a subsequence that converges to λ, then we claim that $\delta\{K\} = 0$. To prove this assertion, let $\varepsilon > 0$ be given and note that for each n,

$$\begin{aligned} |K_n| &\leq |\{k \in K_n : |x_k - \lambda| < \varepsilon\}| + |\{k \in K_n : |x_k - \lambda| \geq \varepsilon\}| \\ &\leq 2\varepsilon n + O(1). \end{aligned}$$

Consequently, $\delta\{k(j)\} \leq 2\varepsilon$, and since ε is arbitrary, we conclude that $\delta\{k(j)\} = 0$. Hence, $\Lambda_x = \varnothing$.

From Example 3 we see that Λ_x need not be a closed point set. The next result states that Γ_x, like L_x, is always a closed set.

Proposition 2. *For any number sequence x, the set Γ_x of statistical cluster points of x is a closed point set.*

Proof. Let p be an accumulation point of Γ_x: if $\varepsilon > 0$ then Γ_x contains some point γ in $(p-\varepsilon, p+\varepsilon)$. Choose ε' so that $(\gamma-\varepsilon', \gamma+\varepsilon') \subseteq (p-\varepsilon, p+\varepsilon)$. Since $\gamma \in \Gamma_x$, $\delta\{k : x_k \in (\gamma-\varepsilon', \gamma+\varepsilon')\} \neq 0$, which implies that $\delta\{k : x_k \in (p-\varepsilon, p+\varepsilon)\} \neq 0$. Hence, $p \in \Gamma_x$.

For a given sequence x its statistical convergence or nonconvergence is not altered by changing the values of a thin subsequence. (See [5, Theorem 1].) We now show that the same is true for statistical limit points and cluster points.

Theorem 1. *If x and y are sequences such that $x_k = y_k$ for almost all k, then $\Lambda_x = \Lambda_y$ and $\Gamma_x = \Gamma_y$.*

Proof. Assume $\delta\{k : x_k \neq y_k\} = 0$ and let $\lambda \in \Lambda_x$, say $\{x\}_K$ is a nonthin subsequence of x that converges to λ. Since $\delta\{k : k \in K \text{ and } x_k \neq y_k\} = 0$, it follows that $\{k : k \in K \text{ and } x_k = y_k\}$ does not have density zero. Therefore the latter set yields a nonthin subsequence $\{y\}_{K'}$ of $\{y\}_K$ that converges to λ. Hence, $\lambda \in \Lambda_y$ and $\Lambda_x \subseteq \Lambda_y$. By symmetry we see that $\Lambda_y \subseteq \Lambda_x$, whence $\Lambda_x = \Lambda_y$. The assertion that $\Gamma_x = \Gamma_y$ is proved by a similar argument.

In the next theorem we establish a strong connection between statistical cluster points and ordinary limit points.

Theorem 2. *If x is a number sequence then there exists a sequence y such that $L_y = \Gamma_x$ and $y_k = x_k$ for almost all k; moreover, the range of y is a subset of the range of x.*

Proof. If Γ_x is a proper subset of L_x, then for each ξ in $L_x \sim \Gamma_x$ choose an open interval I_ξ with center ξ such that $\delta\{k : x_k \in I_\xi\} = 0$. The collection of all such I_ξ's is an open cover of $L_x \sim \Gamma_x$, and by the Lindelöf Covering Property there exists a countable subcover, say $\{I_j\}_{j=1}^{\infty}$. Thus each I_j contains a thin subsequence of x. By a result of Connor [3, Corollary 9], this countable collection of sets, each having density zero, yields a single set Ω such that $\delta(\Omega) = 0$ and for each j, $\{k : x_k \in I_j\} \sim \Omega$ is a finite set. Let $\mathbb{N} \sim \Omega := \{j(k) : k \in \mathbb{N}\}$, and define the sequence y by

$$y_k := \begin{cases} x_{j(k)}, & \text{if } k \in \Omega, \\ x_k, & \text{if } k \in \mathbb{N} \sim \Omega. \end{cases}$$

Obviously $\delta\{k : y_k \neq x_k\} = 0$, and Theorem 1 ensures that $\Gamma_y = \Gamma_x$. Since the subsequence $\{y\}_\Omega$ has no limit point that is not also a statistical limit point of y, it follows that $L_y = \Gamma_y$; hence, $L_y = \Gamma_x$.

Remark. The conclusion of Theorem 2 is not valid if Γ_x is replaced by Λ_x, because L_y is always a closed set while Λ_x need not be closed (as in Example 3).

3. Statistical analogues of completeness theorems

There are several well-known theorems that are equivalent to the completeness of the real number system. When such a theorem concerns sequences we can attempt to formulate and prove a statistical analogue of that theorem by replacing ordinary limits with statistical limits. For example, in [5, Theorem

1] it is proved that a number sequence is statistically convergent if and only if it is a statistically Cauchy sequence. A sequential version of the Least Upper Bound Axiom (in $\mathbb{R}$) is the Monotone Sequence Theorem: if the (real) number sequence x is nondecreasing and bounded above, then x is convergent. The following result, which is an easy consequence of [5, Theorem 1], is a statistical analogue of that theorem.

Proposition 3. *Suppose x is a number sequence and $M := \{k \in \mathbb{N} : x_k \leq x_{k+1}\}$; if $\delta\{M\} = 1$ and x is bounded on M, then x is statistically convergent.*

Another completeness result for $\mathbb{R}$ is the Bolzano-Weierstrass Theorem which asserts that $L_x \neq \varnothing$ for a bounded sequence x. Example 4 shows that a bounded sequence might have $\Lambda_x = \varnothing$, but there is an analogue of the Bolzano-Weierstrass Theorem that uses statistical *cluster* points.

Theorem 3. *If x is a number sequence that has a bounded nonthin subsequence, then x has a statistical cluster point.*

Proof. Given such an x, Theorem 2 ensures that there exists a sequence y such that $L_y = \Gamma_x$ and $\delta\{k \in \mathbb{N} : y_k \neq x_k\} = 0$. Then y must have a bounded nonthin subsequence, so by the Bolzano-Weierstrass Theorem $L_y \neq \varnothing$, whence $\Gamma_x \neq \varnothing$.

Corollary. *If x is a bounded number sequence, then x has a statistical cluster point.*

The next result is a statistical analogue of the Heine-Borel Covering Theorem. If x is a bounded number sequence, let $\overline{X}$ denote the compact set $\{x_k : k \in \mathbb{N}\} \cup L_x$. A sequential version of the Heine-Borel Theorem tells us that if $\{J_n\}$ is a collection of open sets that covers $\overline{X}$, then there is a finite subcollection of $\{J_n\}$ that covers $\overline{X}$. To form a statistical analogue of this result we replace L_x with Γ_x and define the set

$$X := \{x_k : k \in \mathbb{N}\} \cup \Gamma_x,$$

which we might call the *statistical closure* of x. It is easy to see that X need not be a closed set; indeed, X is a closed set if and only if X equals $\{x_k : k \in \mathbb{N}\} \cup L_x$, the ordinary closure of x.

Theorem 4. *If x is a bounded number sequence, then it has a thin subsequence $\{x\}_K$ such that $\{x_k : k \in \mathbb{N} \sim K\} \cup \Gamma_x$ is a compact set.*

Proof. Using Theorem 2 we can choose a bounded sequence y such that $L_y = \Gamma_x$, $\{y_k : k \in \mathbb{N}\} \subseteq \{x_k : k \in \mathbb{N}\}$, and $\delta(K) = 0$, where $K = \{k \in \mathbb{N} : x_k \neq y_k\}$. This yields

$$\{x_k : k \in \mathbb{N} \sim K\} \cup \Gamma_x = \{y_k : k \in \mathbb{N}\} \cup L_y,$$

and the right-hand member is a compact set.

It is easy to see that the proof of Theorem 4 remains valid even for unbounded x provided that x is bounded for almost all k; i.e., there is a thin sequence $\{x\}_M$ such that $\{x_k : k \in \mathbb{N} \sim M\}$ is a bounded set.

Finally, we note that for the compact set in Theorem 4 we cannot use Λ_x in place of Γ_x. In Example 3, $\Lambda_x = \{1/p : p \in \mathbb{N}\}$ and for each p in $\mathbb{N}$, $\delta\{k \in \mathbb{N} : x_k = 1/p\} = 2^{-p}$. If $\{x\}_K$ is any thin subsequence then for each p, $\delta\{k \in \mathbb{N} \sim K : x_k = 1/p\} = 2^{-p}$, and therefore $\{x_k : k \in \mathbb{N} \sim K\}$ still has zero as a limit point. Consequently, $\{x_k : k \in \mathbb{N} \sim K\} \cup \Lambda_x$ is not compact.

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