# Statistical manifolds with almots contact structures and its statistical submersions

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Abstract: In this paper, we discuss statistical manifolds with almost contact sturctures. We define a Sasaki-like statistical manifold. Moreover, we consider Sasaki-like statistical submersions, and we study Sasaki-like statistical submersion with the property that the curvature tensor with respect to the affine connection of the total space satisfies the condition (2.12).

Key words: affine connection, conjugate connection, statistical manifold, statistical submersion, semi-Riemannian manifold, semi-Riemannian submersion.

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## 1 Introduction

Let $M$ and $B$ be two Riemannian manifolds of class $C^\infty$. A Riemannian submersion $\pi : M \to B$ is a mapping of $M$ onto $B$ such that $\pi$ has maximal rank and $\pi_*$ preserves lengths of horizontal vectors ([5], [6], [11], [17]). If $\pi : M \to B$ is a Riemannian submersion such that $M$ is a Sasakian manifold with almost contact structure $(\phi, \xi, \eta)$, each fiber is a $\phi$-invariant submanifold of $M$ and tangent to the vector $\xi$, then $\pi$ is said to be a Sasakian submersion ([7], [8], [13], [16]). If $\pi$ is a Sasakian submersion, then $B$ is Kählerian and each fiber is Sasakian. B. H. Kim ([8]) and the author ([13]) investigated a Sasakian submersion with vanishing contact Bochner curvature tensor. It is known that ([7], [13])

Theorem A. *Let $\pi : M \to B$ be a Sasakian submersion. If $M$ is a space of constant $\phi$-holomorphic sectional curvature $c$, then $B$ is of constant holomorphic sectional curvature $c + 3\,(\leq 0)$.*

Next, let $M$ and $B$ be two semi-Riemannian manifolds. A semi-Riemannian submersion $\pi : M \to B$ is a submersion such that all fibers are semi-Riemannian submanifolds of $M$, and $\pi_*$ preserves lengths of horizontal vectors ([12]). Recently, N. Abe and K. Hasegawa ([1]) studied an affine submersion with horizontal distribution. They investigated when the total space is the statistical manifold. Also, the author ([14]) studied statistical manifolds with almost complex structure and its statistical submersions.

Let $M$ be a manifold with a non-degenerate metric $g$ and a torsion-free affine connection $\nabla$. If $\nabla g$ is symmetric, then $(M, \nabla, g)$ is called a statistical manifold. In [9], M. Noguchi studied statistical manifolds. On the statistical manifold, we define another connection, called the conjugate (or dual) connection ([3], [10]). This concept was widely studied in information geometry ([2], [3]). The statistical models in information geometry have a Fisher metric as Riemannian metric, and admit an affine connection which is constructed from the mean of the probability distribution. This affine connection is called $\alpha$-connection, and conjugate relative to the Fisher metric is the so called $(-\alpha)$-connection, where $\alpha$ is a real number. The 0-connection is the Levi-Civita connection with respect to the Fisher metric. Also, O. E. Barndorff-Nielsen and P. E. Jupp ([4]) studied a Riemannian submersion from the viewpoint of statistics. In [15], we studied the statistical submersion of the space of the multivariate normal distribution.

In this paper, we study a statistical submersion. In §2, we introduce statistical manifolds with almost complex structure (resp. almost contact metric manifold), and define a Kähler-like (resp. Sasaki-like) statistical manifold. In §3, we describe a semi-Riemannian submersion with affine connection and define a statistical submersion. We consider a Sasaki-like statistical submersion in §4. In §5, we discuss Sasaki-like statistical submersions such that the curvature tensor of the total space satisfies the type (2.12) with $c$ and show results similar to Theorem A.

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## 2 Statistical manifolds with certain structures

An $n$-dimensional semi-Riemannian manifold is a smooth manifold $M^n$ equipped with a metric tensor $g$, where $g$ is a symmetric nondegenerate tensor field on $M$ of constant index. The common value $\nu$ of index $g$ on $M$ is called the index of $M$ $(0 \leq \nu \leq n)$ and we denote a semi-Riemannian manifold by $M^n_\nu$. If $\nu = 0$, then $M$ is a Riemannian manifold. For each $p \in M$, a tangent vector $E$ in $M$ is spacelike (resp. null, timelike) if $g(E, E) > 0$ or $E = 0$, (resp. $g(E, E) = 0$ and $E \neq 0$, $g(E, E) < 0$). Let $\mathbf{R}^n_\nu$ be an $n$-dimensional real vector space with an inner product of signature $(\nu, n - \nu)$ given by

$$\langle x, x \rangle = -\sum_{i=1}^{\nu} x_i^2 + \sum_{i=\nu+1}^{n} x_i^2,$$

where $x = (x_1, \ldots, x_n)$ is the natural coordinate of $\mathbf{R}^n_\nu$. $\mathbf{R}^n_\nu$ is called an $n$-dimensional semi-Euclidean space.

Let $M$ be a semi-Riemannian manifold. Denote a torsion-free affine connection by $\nabla$. The triple $(M, \nabla, g)$ is called a statistical manifold if $\nabla g$ is symmetric. For the statistical manifold $(M, \nabla, g)$, we define another affine connection $\nabla^*$ by

$$Eg(F, G) = g(\nabla_E F, G) + g(F, \nabla^*_E G) \tag{2.1}$$

for vector fields $E, F$ and $G$ on $M$. The affine connection $\nabla^*$ is called conjugate (or dual) to $\nabla$ with respect to $g$. The affine connection $\nabla^*$ is torsion-free, $\nabla^* g$ is symmetric and satisfies $(\nabla^*)^* = \nabla$. Clearly, the triple $(M, \nabla^*, g)$ is statistical. We denote by $R$ and $R^*$ the curvature tensors on $M$ with respect to the affine connection $\nabla$ and its conjugate $\nabla^*$, respectively. Then we find

$$g(R(E, F)G, H) = -g(G, R^*(E, F)H)$$

for vector fields $E, F, G$ and $H$ on $M$, where $R(E, F)G = [\nabla_E, \nabla_F]G - \nabla_{[E,F]}G$.

An almost complex structure on a manifold $M$ is a tensor field $\phi$ of type (1,1) such that $\phi^2 = -I$, where $I$ stands for the identity transformation. An almost complex manifold is such a manifold with a fixed almost complex structure. An almost complex manifold is necessarily orientable and must have even dimension. We consider the semi-Riemannian manifold on the almost complex manifold $M$. If $\phi$ preserves the metric $g$, that is,

$$g(\phi E, \phi F) = g(E, F) \tag{2.2}$$

for vector fields $E$ and $F$ on $M$, then $(M, g, \phi)$ is an almost Hermitian manifold. Now, we consider the semi-Riemannian manifold $(M, g)$ with the almost complex structure $\phi$ which has another tensor field $\phi^*$ of type (1,1) satisfying

$$g(\phi E, F) + g(E, \phi^* F) = 0 \tag{2.3}$$

for vector fields $E$ and $F$. Then $(M, g, \phi)$ is called an almost Hermite-like manifold. We see that $(\phi^*)^* = \phi$, $(\phi^*)^2 = -I$ and $g(\phi E, \phi^* F) = g(E, F)$. According to $\phi^2 = -I$, the tensor field $\phi$ is not symmetric relative to $g$. Thus $\phi + \phi^*$ does not vanish everywhere. The tensor field $\phi - \phi^*$ is symmetric and $\phi + \phi^*$ is skew symmetric with respect to $g$. We consider the statistical manifold on the almost Hermite-like manifold. If $\phi$ is parallel with respect to $\nabla$, then $(M, \nabla, g, \phi)$ is called a Kähler-like statistical manifold. Also, we find $R(E, F)\phi = \phi R(E, F)$. By virtue of (2.3), we get

$$g((\nabla_G \phi)E, F) + g(E, (\nabla^*_G \phi^*)F) = 0 \tag{2.4}$$

for vector fields $E, F$ and $G$ on $M$. Hence $(M, \nabla, g, \phi)$ is a Kähler-like statistical manifold if and only if so is $(M, \nabla^*, g, \phi^*)$.

For vector fields $E, F$ and $G$ on the Kähler-like statistical manifold, we consider the curvature tensor $R$ with respect to $\nabla$ such that

$$
\begin{aligned}
(2.5)\qquad R(E,F)G = \frac{c}{4}[&g(F,G)E - g(E,G)F - g(F,\phi G)\phi E + g(E,\phi G)\phi F \\
&+\{g(E,\phi F) - g(\phi E,F)\}\phi G],
\end{aligned}
$$

where $c$ is a constant. Changing $\phi$ for $\phi^*$ in (2.5), we get the curvature tensor $R^*$.

REMARK 2.1. If $M$ is a Kählerian manifold, then $M$, satisfying (2.5), is a space of constant holomorphic sectional curvature $c$.

EXAMPLE 2.1. Let $\mathbf{R}_n^{2n}$ be a $2n$-dimensional semi-Euclidean space with a local coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ which admits the following almost complex structure $\phi$, the metric $g$

$$
\phi = \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} 2\delta_{ij} & 0 \\ 0 & -\delta_{ij} \end{pmatrix}
$$

and the flat affine connection $\nabla$. It is easy to see that $(\mathbf{R}_n^{2n}, \nabla, g, \phi)$ is a Kähler-like statistical manifold. The conjugate is flat and

$$
\phi^* = \frac{1}{2}\begin{pmatrix} 0 & -\delta_{ij} \\ 4\delta_{ij} & 0 \end{pmatrix}.
$$

Next, let $M$ be an odd dimensional manifold and $\phi, \xi, \eta$ be a tensor field of type (1,1), a vector field, a 1-form on $M$ respectively. If $\phi, \xi$ and $\eta$ satisfy the following conditions

$$
(2.6)\qquad \eta(\xi) = 1, \qquad \phi^2 E = -E + \eta(E)\xi
$$

for arbitrary vector field $E$ on $M$, then $M$ is said to have an almost contact structure $(\phi, \xi, \eta)$ and is called an almost contact manifold.

The semi-Riemannian manifold $(M, g)$ is called an almost contact metric manifold if

$$
(2.7)\qquad g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F)
$$

for vector fields $E$ and $F$ on $M$. We consider the semi-Riemannian manifold $(M, g)$ with the almost contact structure $(\phi, \xi, \eta)$ which has an another tensor field $\phi^*$ of type (1,1) satisfying

$$
(2.8)\qquad g(\phi E, F) + g(E, \phi^* F) = 0
$$

for vector fields $E$ and $F$. Then $(M, g, \phi, \xi, \eta)$ is called an almost contact metric manifold of certain kind. Obviously, we find $(\phi^*)^2 E = -E + \eta(E)\xi$ and

$$
(2.9)\qquad g(\phi E, \phi^* F) = g(E, F) - \eta(E)\eta(F).
$$

Because of (2.6), the tensor field $\phi$ is not symmetric with respect to $g$. This means that $\phi + \phi^*$ does not vanish everywhere. Equations $\phi\xi = 0$ and $\eta(\phi E) = 0$ hold on the almost contact manifold. We obtain $\phi^*\xi = 0$ and $\eta(\phi^* E) = 0$ on the almost contact metric manifold of certain kind.

Now, we consider the statistical manifold on the almost contact metric manifold of certain kind. If

$$\nabla_E \xi = -\phi E, \qquad (\nabla_E \phi)F = g(E, F)\xi - \eta(F)E, \tag{2.10}$$

then $(M, \nabla, g, \phi, \xi, \eta)$ is called a Sasaki-like statistical manifold. From $\eta(\xi) = 1$, we find $\eta(\nabla^*_E \xi) = 0$. Operating $\nabla_E$ to $\eta(\phi F) = 0$, we get $g(E, F) - \eta(E)\eta(F) + g(\phi F, \nabla^*_E \xi) = 0$. Moreover, changing $F$ to $\phi F$, we see $\nabla^*_E \xi = -\phi^* E$. Hence we have

LEMMA 2.1. *The pair $(M, g, \phi, \xi, \eta)$ is an almost contact metric manifold of certain kind if and only if so is $(M, g, \phi^*, \xi, \eta)$. Moreover, $(M, \nabla, g, \phi, \xi, \eta)$ is a Sasaki-like statistical manifold if and only if so is $(M, \nabla^*, g, \phi^*, \xi, \eta)$.*

On the Sasaki-like statistical manifold, we get

$$\begin{aligned} &R(E, F)\phi G - \phi R(E, F)G \\ &= -g(F, G)\phi E + g(E, G)\phi F + g(F, \phi G)E - g(E, \phi G)F \end{aligned} \tag{2.11}$$

for vector fields $E, F, G$. We consider the curvature tensor $R$ with respect to $\nabla$ such that

$$\begin{aligned} R(E, F)G = &\frac{1}{4}(c + 3)\{g(F, G)E - g(E, G)F\} \\ &+ \frac{1}{4}(c - 1)[\eta(E)\eta(G)F - \eta(F)\eta(G)E + g(E, G)\eta(F)\xi \\ &\qquad -g(F, G)\eta(E)\xi - g(F, \phi G)\phi E + g(E, \phi G)\phi F \\ &\qquad +\{g(E, \phi F) - g(\phi E, F)\}\phi G], \end{aligned} \tag{2.12}$$

where $c$ is a constant. Changing $\phi$ for $\phi^*$ in (2.12), we get the curvature tensor $R^*$.

REMARK 2.2. If $M$ is a Sasakian manifold, then $M$ satisfying (2.12) is a space of constant $\phi$-holomorphic sectional curvature $c$.

A Killing vector field on a statistical manifold is a vector field $E$ for which the Lie derivative of the metric tensor vanishes, that is, $\mathcal{L}_E g = 0$, where $\mathcal{L}$ is the Lie derivative. Then we have

PROPOSITION 2.1. *Let $(M, \nabla, g)$ be a statistical manifold. Then the following conditions on a vector field $E$ are equivalent*:

(1) *$E$ is Killing, that is, $\mathcal{L}_E g = 0$,*

(2) $Eg(F,G) = g([E,F],G) + g(F,[E,G])$ *for vector fields* $F, G$ *on* $M$,

(3) $g(\nabla_F E, G) + g(F, \nabla^*_G E) = 0$ *for vector fields* $F$ *and* $G$ *on* $M$.

Hence, we have

Lemma 2.2. *The structure vector field* $\xi$ *is Killing on the Sasaki-like statistical manifold.*

Next, we give an example of a Sasaki-like statistical manifold such that the curvature tensor with respect to the affine connection satisfies the equation (2.12).

Example 2.2. Let $\mathrm{R}^{2m+1}_m$ be a $(2m+1)$-dimensional affine space with the standard coordinate $(x_1, \ldots, x_m, y_1, \ldots, y_m, z)$. We define a semi-Riemannian metric $g$ on $\mathrm{R}^{2m+1}_m$ by

$$g = \begin{pmatrix} 2\delta_{ij} + y_i y_j & 0 & -y_i \\ 0 & -\delta_{ij} & 0 \\ -y_j & 0 & 1 \end{pmatrix}.$$

We define the affine connection $\nabla$ by

$$\begin{aligned}
&\nabla_{\partial_{x_i}} \partial_{x_j} = -y_j\, \partial_{y_i} - y_i\, \partial_{y_j},\\
&\nabla_{\partial_{x_i}} \partial_{y_j} = \nabla_{\partial_{y_j}} \partial_{x_i} = y_i\, \partial_{x_j} + (y_i y_j - 2\delta_{ij})\partial_z,\\
&\nabla_{\partial_{x_i}} \partial_z = \nabla_{\partial_z} \partial_{x_i} = \partial_{y_i},\\
&\nabla_{\partial_{y_i}} \partial_z = \nabla_{\partial_z} \partial_{y_i} = -\partial_{x_i} - y_i\, \partial_z,\\
&\nabla_{\partial_{y_i}} \partial_{y_j} = \nabla_{\partial_z} \partial_z = 0,
\end{aligned}$$

where $\partial_{x_i} = \partial/\partial x_i,\ \partial_{y_i} = \partial/\partial y_i$ and $\partial_z = \partial/\partial z$. Then its conjugate $\nabla^*$ is given as follows:

$$\begin{aligned}
&\nabla^*_{\partial_{x_i}} \partial_{x_j} = 2y_j\, \partial_{y_i} + 2y_i\, \partial_{y_j},\\
&\nabla^*_{\partial_{x_i}} \partial_{y_j} = \nabla^*_{\partial_{y_j}} \partial_{x_i} = -\frac{y_i}{2}\, \partial_{x_j} - \frac{1}{2}(y_i y_j - 2\delta_{ij})\partial_z,\\
&\nabla^*_{\partial_{x_i}} \partial_z = \nabla^*_{\partial_z} \partial_{x_i} = -2\, \partial_{y_i},\\
&\nabla^*_{\partial_{y_i}} \partial_z = \nabla^*_{\partial_z} \partial_{y_i} = \frac{1}{2}\, \partial_{x_i} + \frac{y_i}{2}\, \partial_z,\\
&\nabla^*_{\partial_{y_i}} \partial_{y_j} = \nabla^*_{\partial_z} \partial_z = 0.
\end{aligned}$$

Now we define $\phi,\ \xi$ and $\eta$ by

$$\phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix}, \qquad \xi = \partial_z = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and $\eta = (-y_1, 0, -y_2, 0, \ldots, -y_m, 0, 1)$. Then we can verify that $(\mathbf{R}_m^{2m+1}, \nabla, g, \phi, \xi, \eta)$ is a Sasaki-like statistical manifold such that the curvature tensor of $\mathbf{R}_m^{2m+1}$ satisfies the type (2.12) with $c = -3$. Also we find

$$\phi^* = \frac{1}{2}\begin{pmatrix} 0 & -\delta_{ij} & 0 \\ 4\delta_{ij} & 0 & 0 \\ 0 & -y_j & 0 \end{pmatrix}.$$

This manifold is not Sasakian with respect to the Levi-Civita connection.

# 3 Statistical submersions

Let $\pi : M \to B$ be a semi-Riemannian submersion. We put $\dim M = m$ and $\dim B = n$. For each point $x \in B$, the semi-Riemannian submanifold $\pi^{-1}(x)$ with the induced metric $\overline{g}$ is called a fiber and denoted by $\overline{M}_x$ or $\overline{M}$ simply. We notice that the dimension of each fiber is always $m-n(=s)$. A vector field on $M$ is vertical if it is always tangent to fibers, horizontal if always orthogonal to fibers. We denote the vertical and horizontal subspace in the tangent space $T_pM$ of the total space $M$ by $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$ for each point $p \in M$, and the vertical and horizontal distributions in the tangent bundle $TM$ of $M$ by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. Then $TM$ is the direct sum of $\mathcal{V}(M)$ and $\mathcal{H}(M)$. The projection mappings are denoted $\mathcal{V} : TM \to \mathcal{V}(M)$ and $\mathcal{H} : TM \to \mathcal{H}(M)$, respectively. We call a vector field $X$ on $M$ projectable if there exists a vector field $X_*$ on $B$ such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$, and say that $X$ and $X_*$ are $\pi$-related. Also, a vector field $X$ on $M$ is called basic if it is projectable and horizontal. Then we have ([11], [12])

Lemma B. *If $X$ and $Y$ are basic vector fields on $M$ which are $\pi$-related to $X_*$ and $Y_*$ on $B$, then*

(1) $g(X, Y) = g_B(X_*, Y_*) \circ \pi$, *where $g$ is the metric on $M$ and $g_B$ the metric on $B$,*

(2) $\mathcal{H}[X, Y]$ *is basic and is $\pi$-related to $[X_*, Y_*]$.*

Let $(M, \nabla, g)$ be a statistical manifold and $\pi : M \to B$ be a semi-Riemannian submersion. We denote the affine connections of $\overline{M}$ by $\overline{\nabla}$ and $\overline{\nabla}^*$. Notice that $\overline{\nabla}_U V$ and $\overline{\nabla}^*_U V$ are well-defined vertical vector fields on $M$ for vertical vector fields $U$ and $V$ on $M$, more precisely $\overline{\nabla}_U V = \mathcal{V}\nabla_U V$ and $\overline{\nabla}^*_U V = \mathcal{V}\nabla^*_U V$. Moreover, $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion-free and conjugate to each other with respect to $\overline{g}$. We put $S = \nabla - \nabla^*$. Then $S$ is symmetric, that is, $S_E F = S_F E$ for vector fields $E$ and $F$ on $M$. Let $\widehat{\nabla}$ be an affine connection on $B$. We call $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ is a statistical submersion if $\pi : M \to B$ satisfies $\pi_*(\nabla_X Y)_p = (\widehat{\nabla}_{X_*} Y_*)_{\pi(p)}$ for basic vector fileds $X, Y$ and $p \in M$. The letters $U, V, W$ will always denote vertical vector fields, and $X, Y, Z$ horizontal vector fields. The tensor fields $T$ and $A$ of type (1,2) defined by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \qquad A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

for vector fields $E$ and $F$ on $M$. Changing $\nabla$ for $\nabla^*$ in the above equations, we define $T^*$ and $A^*$, respectively. Then we find $T^{**} = T$ and $A^{**} = A$. For vertical vector fields, $T$ and $T^*$ have the symmetry property. For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$, we obtain

$$g(T_U V, X) = -g(V, T^*_U X), \qquad g(A_X Y, U) = -g(Y, A^*_X U).$$

Thus, $T$ (resp. $A$) vanishes identically if and only if $T^*$ (resp. $A^*$) vanishes identically. Since $A$ is related to the integrability of $\mathcal{H}(M)$, $A$ is symmetric for horizontal vectors if and only if $\mathcal{H}(M)$ is integrable with respect to $\nabla$. Moreover, if $A$ and $T$ vanish identically, then the total space is a product space of the base space and the fiber. It is known that ([1])

Theorem C. *Let $\pi : M \to B$ be a semi-Riemannian submersion. Then $(M, \nabla, g)$ is a statistical manifold if and only if the following conditions hold*:

1. (1) $\mathcal{H}S_V X = A_X V - A^*_X V$ *for* $X \in \mathcal{H}(M)$ *and* $V \in \mathcal{V}(M)$,
2. (2) $\mathcal{V}S_X V = T_V X - T^*_V X$ *for* $X \in \mathcal{H}(M)$ *and* $V \in \mathcal{V}(M)$,
3. (3) $(\overline{M}, \overline{\nabla}, \overline{g})$ *is a statistical manifold for each* $x \in B$,
4. (4) $(B, \widehat{\nabla}, g_B)$ *is a statistical manifold.*

For the statistical submersion $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$, we have the following Lemmas ([14]).

Lemma D. *If $X$ and $Y$ are horizontal vector fields, then $A_X Y = -A^*_Y X$.*

Lemma E. *For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$ we have*

$$\begin{aligned}
\nabla_U V &= T_U V + \overline{\nabla}_U V, & \nabla^*_U V &= T^*_U V + \overline{\nabla}^*_U V,\\
\nabla_U X &= \mathcal{H}\nabla_U X + T_U X, & \nabla^*_U X &= \mathcal{H}\nabla^*_U X + T^*_U X,\\
\nabla_X U &= A_X U + \mathcal{V}\nabla_X U, & \nabla^*_X U &= A^*_X U + \mathcal{V}\nabla^*_X U,\\
\nabla_X Y &= \mathcal{H}\nabla_X Y + A_X Y, & \nabla^*_X Y &= \mathcal{H}\nabla^*_X Y + A^*_X Y.
\end{aligned}$$

*Furthermore, if $X$ is basic, then $\mathcal{H}\nabla_U X = A_X U$ and $\mathcal{H}\nabla^*_U X = A^*_X U$.*

We define the covariant derivatives $\nabla T$ and $\nabla A$ by

$$\begin{aligned}
(\nabla_E T)_F V &= \nabla_E (T_F V) - T_{\nabla_E F} V - T_F(\nabla_E V),\\
(\nabla_E A)_F Y &= \nabla_E (A_F Y) - A_{\nabla_E F} Y - A_F(\nabla_E Y)
\end{aligned}$$

for $E, F \in TM$, $Y \in \mathcal{H}(M)$ and $V \in \mathcal{V}(M)$. We change $\nabla$ to $\nabla^*$, then the covariant derivatives $\nabla^* T, \nabla^* A$ are defined similarly. We consider the curvature tensor on the statistical submersion. Let $\overline{R}$ (resp. $\overline{R}^*$) be the curvature tensor with respect to the induced affine connection $\overline{\nabla}$ (resp. $\overline{\nabla}^*$ ) of each fiber. Also, let $\widehat{R}(X, Y)Z$ (resp. $\widehat{R}^*(X, Y)Z$) be horizontal vector field such that $\pi_*(\widehat{R}(X, Y)Z) = \widehat{R}(\pi_* X, \pi_* Y)\pi_* Z$ (resp. $\pi_*(\widehat{R}^*(X, Y)Z) = \widehat{R}^*(\pi_* X, \pi_* Y)\pi_* Z$) at each $p \in M$, where $\widehat{R}$ (resp. $\widehat{R}^*$) is the curvature tensor on $B$ of the affine connection $\widehat{\nabla}$ (resp. $\widehat{\nabla}^*$). Then we have ([14])

Theorem F. *If $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ is a statistical submersion, then we obtain for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$*

$$\begin{aligned}
g(R(U,V)W,W') &= g(\overline{R}(U,V)W,W') + g(T_U W, T_V^* W') - g(T_V W, T_U^* W'),\\
g(R(U,V)W,X) &= g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X),\\
g(R(U,V)X,W) &= g((\nabla_U T)_V X, W) - g((\nabla_V T)_U X, W),\\
g(R(U,V)X,Y) &= g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) + g(T_U X, T_V^* Y)\\
&\quad - g(T_V X, T_U^* Y) - g(A_X U, A_Y^* V) + g(A_X V, A_Y^* U),\\
g(R(X,U)V,W) &= g([\mathcal{V}\nabla_X, \overline{\nabla}_U]V, W) - g(\nabla_{[X,U]}V, W) - g(T_U V, A_X^* W)\\
&\quad + g(T_U^* W, A_X V),\\
g(R(X,U)V,Y) &= g((\nabla_X T)_U V, Y) - g((\nabla_U A)_X V, Y) + g(A_X U, A_Y^* V)\\
&\quad - g(T_U X, T_V^* Y),\\
g(R(X,U)Y,V) &= g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y, V) + g(T_U X, T_V Y)\\
&\quad - g(A_X U, A_Y V),\\
g(R(X,U)Y,Z) &= g((\nabla_X A)_Y U, Z) - g(T_U X, A_Y^* Z) - g(T_U Y, A_X^* Z)\\
&\quad + g(A_X Y, T_U^* Z),\\
g(R(X,Y)U,V) &= g((\nabla_X T)_U Y, V) - g((\nabla_Y T)_U X, V) - g((\nabla_U \theta)_X Y, V)\\
&\quad + g(T_U X, T_V Y) - g(T_V X, T_U Y) - g(A_X U, A_Y V)\\
&\quad + g(A_X V, A_Y U),\\
g(R(X,Y)U,Z) &= g((\nabla_X A)_Y U, Z) - g((\nabla_Y A)_X U, Z) + g(T_U^* Z, \theta_X Y),\\
g(R(X,Y)Z,U) &= g((\nabla_X A)_Y Z, U) - g((\nabla_Y A)_X Z, U) - g(T_U Z, \theta_X Y),\\
g(R(X,Y)Z,Z') &= g(\widehat{R}(X,Y)Z,Z') - g(A_Y Z, A_X^* Z') + g(A_X Z, A_Y^* Z')\\
&\quad + g(\theta_X Y, A_Z^* Z'),
\end{aligned}$$

*where we put $\theta_X = A_X + A_X^*$.*

For each $p \in M$, we denote by $\{E_1, \ldots, E_m\}$, $\{X_1, \ldots, X_n\}$ and $\{U_1, \ldots, U_s\}$ local orthonormal basis of $T_pM$, $\mathcal{H}_p(M)$ and $\mathcal{V}_p(M)$, respectively such that $E_i = X_i$ $(i = 1, \ldots, n)$ and $E_{n+\alpha} = U_\alpha$ $(\alpha = 1, \ldots, s)$. Denote respectively by $\omega_a^b$ and $\omega_a^{*b}$ the connection forms in terms of local coordinates with respect to $\{E_1, \ldots, E_m\}$ of the affine connection $\nabla$ and its conjugate $\nabla^*$, where $a, b$ run over the range $\{1, \ldots, m\}$. Set $\varepsilon_a = g(E_a, E_a) = +1$ or $-1$ according as $E_a$ is spacelike or timelike, respectively. Owing to equation (2.1), we have

$$\omega_b^{*a} = -\varepsilon_a \varepsilon_b \omega_a^b. \tag{3.1}$$

We put

$$g(TX, TY) = \sum_{\alpha=1}^{s} \varepsilon_\alpha g(T_{U_\alpha}X, T_{U_\alpha}Y),$$

$$g(TX, SE) = \sum_{\alpha=1}^{s} \varepsilon_\alpha g(T_{U_\alpha}X, S_{U_\alpha}E)$$

for $X, Y \in \mathcal{H}(M)$ and $E \in TM$. The mean curvature vector of the fiber with respect to the affine connection $\nabla$ is given by the horizontal vector field

$$N = \sum_{\alpha=1}^{s} \varepsilon_\alpha T_{U_\alpha} U_\alpha.$$

Lemma 3.1. *We have*

$$\sum_{\alpha=1}^{s} \varepsilon_\alpha g((\nabla_E T)_{U_\alpha} U_\alpha, X) = g(\nabla_E N, X) + g(T^*X, SE)$$

*for $X \in \mathcal{H}(M)$ and $E \in TM$.*

Proof. From (3.1), we get

$$\begin{aligned} \sum \varepsilon_\alpha g(\nabla_E U_\alpha, T^*_{U_\alpha}X) &= \sum \varepsilon_\alpha \, \omega_\alpha{}^\beta(E) \, g(U_\beta, T^*_{U_\alpha}X) \\ &= -\sum \varepsilon_\beta \, \omega_\beta^{*\alpha}(E) \, g(U_\alpha, T^*_{U_\beta}X) \\ &= -\sum \varepsilon_\beta g(\nabla^*_E U_\beta, T^*_{U_\beta}X), \end{aligned}$$

that is,

$$\sum_{\alpha=1}^{s} \varepsilon_\alpha g(\nabla_E U_\alpha, T^*_{U_\alpha}X) = -\sum_{\alpha=1}^{s} \varepsilon_\alpha g(\nabla^*_E U_\alpha, T^*_{U_\alpha}X). \tag{3.2}$$

For $U, V \in \mathcal{V}(M)$, we find

$$(\nabla_E T)_U V = \nabla_E (T_U V) - T_V(\mathcal{V}\nabla_E U) - T_U(\mathcal{V}\nabla_E V) - T_U(\mathcal{H}\nabla_E V).$$

Then we have from (3.2)

$$\begin{aligned}
&\sum \varepsilon_\alpha g((\nabla_E T)_{U_\alpha} U_\alpha, X) \\
&= \sum \varepsilon_\alpha g(\nabla_E (T_{U_\alpha} U_\alpha), X) - 2\sum \varepsilon_\alpha g(T_{U_\alpha}(\mathcal{V}\nabla_E U_\alpha), X) \\
&= g(\nabla_E N, X) + 2\sum \varepsilon_\alpha g(\nabla_E U_\alpha, T^*_{U_\alpha} X) \\
&= g(\nabla_E N, X) + \sum \varepsilon_\alpha g(\nabla_E U_\alpha, T^*_{U_\alpha} X) - \sum \varepsilon_\alpha g(\nabla^*_E U_\alpha, T^*_{U_\alpha} X) \\
&= g(\nabla_E N, X) + \sum \varepsilon_\alpha g(T^*_{U_\alpha} X, S_E U_\alpha) \\
&= g(\nabla_E N, X) + g(T^* X, SE).
\end{aligned}$$

## 4 Sasaki-like statistical submersions

If $\pi : M \to B$ is a semi-Riemannian submersion such that $(M, g, \phi, \xi, \eta)$ is an almost contact metric manifold of certain kind, each fiber is a $\phi$-invariant semi-Riemannian submanifold of $M$ and tangent to the vector $\xi$, then $\pi$ is said to be an almost contact metric submersion of certain kind. The horizontal and vertical distributions are $\phi$-invariant if and only if are $\phi^*$-invariant. If $X$ is basic on $M$ which is $\pi$-related to $X_*$ on $B$, then $\phi X$ (resp. $\phi^* X$) is basic and $\pi$-related to $\widehat{\phi} X_*$ (resp. $\widehat{\phi}^* X_*$), where $\widehat{\phi}$ and $\widehat{\phi}^*$ are tensor fields of type (1,1) such that $g_B(\widehat{\phi} X_*, Y_*) + g_B(X_*, \widehat{\phi}^* Y_*) = 0$ with respect to the metric $g_B$ on $B$. We say that a statistical submersion $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ is a Sasaki-like statistical submersion if $(M, \nabla, g, \phi, \xi, \eta)$ is a Sasaki-like statistical manifold, each fiber is a $\phi$-invariant semi-Riemannian submanifold of $M$ and tangent to the vector $\xi$. Then we have

Theorem 4.1. *Let $\pi : M \to B$ be an almost contact metric submersion of certain kind. Then the base space is an almost Hermite-like manifold and each fiber is an almost contact metric manifold of certain kind.*

Also, it is clear from (2.10) that the following Lemmas hold.

Lemma 4.1. *Let $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ be a Sasaki-like statistical submersion. Then we have for $X \in \mathcal{H}(M)$ and $U \in \mathcal{V}(M)$*

$$\begin{aligned}
&A_X \xi = -\phi X, \\
&\mathcal{V}\nabla_X \xi = 0, \\
&T_U \xi = 0, \\
&\overline{\nabla}_U \xi = -\overline{\phi} U.
\end{aligned}$$

Lemma 4.2. *If* $\pi : (M,\nabla,g) \to (B,\widehat{\nabla},g_B)$ *is a Sasaki-like statistical submersion, then we have for* $X, Y \in \mathcal{H}(M)$ *and* $U, V \in \mathcal{V}(M)$

$$\begin{aligned}
&(\mathcal{H}\nabla_X\phi)Y = 0,\\
&A_X(\phi Y) - \overline{\phi}(A_X Y) = g(X,Y)\xi,\\
&A_X(\overline{\phi}U) - \phi(A_X U) = -\eta(U)X,\\
&(\mathcal{V}\nabla_X\overline{\phi})U = 0,\\
&A_{\phi X}U = \phi(A_X U),\\
&T_U(\phi X) = \overline{\phi}(T_U X),\\
&T_U(\overline{\phi}V) = \phi(T_U V),\\
&(\overline{\nabla}_U\overline{\phi})V = g(U,V)\xi - \eta(V)U.
\end{aligned}$$

Lemma 4.3. *Let* $\pi : (M,\nabla,g) \to (B,\widehat{\nabla},g_B)$ *be a Sasaki-like statistical submersion. If* $\dim \overline{M} = 1$*, then we have* $A_X Y = -g(X,\phi Y)\xi$*.*

Moreover, we have

Theorem 4.2. *If* $\pi : (M,\nabla,g) \to (B,\widehat{\nabla},g_B)$ *is a Sasaki-like statistical submersion, then the base space* $(B,\widehat{\nabla},g_B,\widehat{\phi})$ *is a Kähler-like statistical manifold and each fiber* $(\overline{M},\overline{\nabla},\overline{g},\overline{\phi},\xi,\eta)$ *is a Sasaki-like statistical manifold.*

By virtue of Lemmas E and 4.2, we get

$$(\overline{\phi} + \overline{\phi}^*)A_X Y = 0$$

for $X, Y \in \mathcal{H}(M)$. Thus we have

Theorem 4.3. *Let* $\pi : (M,\nabla,g) \to (B,\widehat{\nabla},g_B)$ *is a Sasaki-like statistical submersion. If* $\operatorname{rank}(\overline{\phi} + \overline{\phi}^*) = \dim \overline{M} - 1$*, then we have* $A_X Y = -g(X,\phi Y)\xi$ *for* $X, Y \in \mathcal{H}(M)$*.*

Corollary 4.1. *Let* $\pi : (M,\nabla,g) \to (B,\widehat{\nabla},g_B)$ *is a Sasaki-like statistical submersion. If* $\overline{\phi} = \overline{\phi}^*$*, then we have* $A_X Y = -g(X,\phi Y)\xi$ *for* $X, Y \in \mathcal{H}(M)$*.*

Remark. If $\pi : M \to B$ is a Sasakian submersion, then $A_X Y = -g(X,\phi Y)\xi$ holds ([7], [8]).

# 5 Sasaki-like statistical submersions satisfying the certain condition

Let $\pi : (M,\nabla,g) \to (B,\widehat{\nabla},g_B)$ be a Sasaki-like statistical submersion. We assume that the curvature tensor of $(M,\nabla,g,\phi,\xi,\eta)$ satisfies the type (2.12) with $c$, that is, for $E, F, G, G' \in TM$

$$\begin{aligned}
&g(R(E,F)G,G') \\
&= \frac{1}{4}(c+3)\{g(F,G)g(E,G') - g(E,G)g(F,G')\} \\
&+\frac{1}{4}(c-1)[\eta(E)\eta(G)g(F,G') - \eta(F)\eta(G)g(E,G') + g(E,G)\eta(F)\eta(G') \\
&\quad -g(F,G)\eta(E)\eta(G') - g(F,\phi G)g(\phi E,G') + g(E,\phi G)g(\phi F,G') \\
&\quad +\{g(E,\phi F) - g(\phi E,F)\}g(\phi G,G')],
\end{aligned}$$

where $c$ is a constant. Then we see from Theorem F

$$\begin{aligned}
(5.1)\qquad &g(\overline{R}(U,V)W,W') + g(T_U W, T_V^* W') - g(T_V W, T_U^* W') \\
&= \frac{1}{4}(c+3)\{g(V,W)g(U,W') - g(U,W)g(V,W')\} \\
&+\frac{1}{4}(c-1)[\eta(U)\eta(W)g(V,W') - \eta(V)\eta(W)g(U,W') \\
&\qquad +g(U,W)\eta(V)\eta(W') - g(V,W)\eta(U)\eta(W') \\
&\qquad -g(V,\overline{\phi}W)g(\overline{\phi}U,W') + g(U,\overline{\phi}W)g(\overline{\phi}V,W') \\
&\qquad +\{g(U,\overline{\phi}V) - g(\overline{\phi}U,V)\}g(\overline{\phi}W,W')],
\end{aligned}$$

$$(5.2)\qquad g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) = 0,$$

$$(5.3)\qquad g((\nabla_U T)_V X, W) - g((\nabla_V T)_U X, W) = 0,$$

$$\begin{aligned}
(5.4)\qquad &g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) + g(T_U X, T_V^* Y) \\
&-g(T_V X, T_U^* Y) - g(A_X U, A_Y^* V) + g(A_X V, A_Y^* U) \\
&= \frac{1}{4}(c-1)\{g(U,\overline{\phi}V) - g(\overline{\phi}U,V)\}g(\phi X,Y),
\end{aligned}$$

$$\begin{aligned}
(5.5)\qquad &g([\mathcal{V}\nabla_X, \overline{\nabla}_U]V, W) - g(\nabla_{[X,U]}V, W) - g(T_U V, A_X^* W) \\
&+g(T_U^* W, A_X V) = 0,
\end{aligned}$$

$$\begin{aligned}
(5.6)\qquad &g((\nabla_X T)_U V, Y) - g((\nabla_U A)_X V, Y) + g(A_X U, A_Y^* V) - g(T_U X, T_V^* Y) \\
&= \frac{1}{4}(c+3)g(U,V)g(X,Y) \\
&-\frac{1}{4}(c-1)\{\eta(U)\eta(V)g(X,Y) + g(U,\overline{\phi}V)g(\phi X,Y)\},
\end{aligned}$$

$$\begin{aligned}
(5.7)\qquad &g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y, V) + g(T_U X, T_V Y) - g(A_X U, A_Y V) \\
&= -\frac{1}{4}(c+3)g(X,Y)g(U,V) \\
&+\frac{1}{4}(c-1)\{g(X,Y)\eta(U)\eta(V) + g(X,\phi Y)g(\overline{\phi}U,V)\},
\end{aligned}$$

(5.8) $g((\nabla_X A)_Y U, Z) - g(T_U X, A^*_Y Z) - g(T_U Y, A^*_X Z) + g(A_X Y, T^*_U Z) = 0,$

(5.9)
$$g((\nabla_X T)_U Y, V) - g((\nabla_Y T)_U X, V) - g((\nabla_U \theta)_X Y, V) + g(T_U X, T_V Y)$$
$$-g(T_V X, T_U Y) - g(A_X U, A_Y V) + g(A_Y U, A_X V)$$
$$= \frac{1}{4}(c-1)\{g(X, \phi Y) - g(\phi X, Y)\} g(\overline{\phi} U, V),$$

(5.10) $g((\nabla_X A)_Y U, Z) - g((\nabla_Y A)_X U, Z) + g(T^*_U Z, \theta_X Y) = 0,$

(5.11) $g((\nabla_X A)_Y Z, U) - g((\nabla_Y A)_X Z, U) - g(T_U Z, \theta_X Y) = 0,$

(5.12)
$$g(\widehat{R}(X,Y)Z, Z') - g(A_Y Z, A^*_X Z') + g(A_X Z, A^*_Y Z') + g(\theta_X Y, A^*_Z Z')$$
$$= \frac{1}{4}(c+3)\{g(Y,Z)g(X,Z') - g(X,Z)g(Y,Z')\}$$
$$+\frac{1}{4}(c-1)[-g(Y,\phi Z)g(\phi X, Z') + g(X, \phi Z)g(\phi Y, Z')$$
$$+\{g(X,\phi Y) - g(\phi X, Y)\} g(\phi Z, Z')]$$

for $U, V, W, W' \in \mathcal{V}(M)$ and $X, Y, Z, Z' \in \mathcal{H}(M)$. We have from Lemma 4.3, Theorem 4.3 and (5.12)

Theorem 5.1. *Let* $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ *be a Sasaki-like statistical submersion. If* $\operatorname{rank}(\overline{\phi} + \overline{\phi}^*) = \dim \overline{M} - 1$ *and the curvature tensor of the total space satisfies the type* (2.12) *with c, then the curvature tensor of the base space satisfies the type* (2.5) *with* $c + 3$.

Corollary 5.1. *Let* $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ *be a Sasaki-like statistical submersion. If* $\dim \overline{M} = 1$ *and the curvature tensor of the total space satisfies the type* (2.12) *with c, then the curvature tensor of the base space satisfies the type* (2.5) *with* $c + 3$.

By virtue of and Lemma 4.1 and Theorem 4.3, equation (5.6) can be rewritten as follows:

$$g((\nabla_X T)_U V, Y) - g(T_U X, T^*_V Y)$$
$$= \frac{1}{4}(c+3)[g(X,Y)\{g(U,V) - \eta(U)\eta(V)\} - g(\phi X, Y) g(U, \overline{\phi} V)]$$

which implies from Lemma 3.1 that

$$g(\nabla_X N, Y) - g(T^* X, T^* Y) = \frac{1}{4}(c+3)\{(s-1)g(X,Y) - (\operatorname{tr}\overline{\phi})g(\phi X, Y)\}.$$

If $\mathcal{H}\nabla_X N = 0$, then we obtain $c + 3 = 0$ or $\operatorname{tr}\overline{\phi} = 0$. Therefore we have

Theorem 5.2. *Let* $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ *be a Sasaki-like statistical submersion such that the curvature tensor of the total space satisfies the type* (2.12) *with c. We assume that* $\operatorname{rank}(\overline{\phi} + \overline{\phi}^*) = \dim \overline{M} - 1$ *and* $\mathcal{H}\nabla_X N = 0$ *for* $X \in \mathcal{H}(M)$. *Then*

(1) *if* $c+3=0$*, then the base space is flat and each fiber is a totally geodesic submanifold of* $M$ *such that the curvature tensor satisfies the type* (2.12) *with* $-3$,

(2) *in the case of* $\operatorname{tr}\overline{\phi}=0$ *and* $s>1$,

  (i) *if* $g$ *is positive definite, then* $c+3\leq 0$,

  (ii) $c+3<0$ *and* $X$ *is spacelike* (*resp. timelike*) *or* $c+3>0$ *and* $X$ *is timelike* (*resp. spacelike*) *if and only if* $T^*X$ *is spacelike* (*resp. timelike*),

  (iii) *the horizontal vector* $X$ *is null if and only if* $T^*X$ *is null.*

Corollary 5.2. *Let* $\pi:(M,\nabla,g)\to(B,\widehat{\nabla},g_B)$ *be a Sasaki-like statistical submersion such that the curvature tensor of the total space satisfies the type* (2.12) *with* $c$*. If* $\operatorname{rank}(\overline{\phi}+\overline{\phi}^*)=\dim\overline{M}-1$ *and* $N$ *is constant, then results similar to Theorem* 5.2 *hold.*

Also, it is easy to see from (5.7) that

$$\begin{aligned}&g((\nabla^*_XT^*)_UV,Y)-g(T^*_UX,T_VY)\\&=\frac{1}{4}(c+3)[g(X,Y)\{g(U,V)-\eta(U)\eta(V)\}-g(X,\phi Y)g(\overline{\phi}U,V)].\end{aligned}$$

Thus by virtue of Lemma 3.1, we get

$$g(\nabla^*_XN^*,Y)-g(TX,TY)=\frac{1}{4}(c+3)\{(s-1)g(X,Y)-(\operatorname{tr}\overline{\phi})g(X,\phi Y)\}.$$

If $\mathcal{H}\nabla^*_XN^*=0$, then we find $c+3=0$ or $\operatorname{tr}\overline{\phi}=0$. Hence we have

Theorem 5.3. *Let* $\pi:(M,\nabla,g)\to(B,\widehat{\nabla},g_B)$ *be a Sasaki-like statistical submersion such that the curvature tensor of the total space satisfies the type* (2.12) *with* $c$*. We assume that* $\operatorname{rank}(\overline{\phi}+\overline{\phi}^*)=\dim\overline{M}-1$ *and* $\mathcal{H}\nabla^*_XN^*=0$ *for* $X\in\mathcal{H}(M)$*. Then*

(1) *if* $c+3=0$*, then the base space is flat and each fiber is a totally geodesic submanifold of* $M$ *such that the curvature tensor satisfies the type* (2.12) *with* $-3$,

(2) *in the case of* $\operatorname{tr}\overline{\phi}=0$ *and* $s>1$,

  (i) *if* $g$ *is positive definite, then* $c+3\leq 0$,

  (ii) $c+3<0$ *and* $X$ *is spacelike* (*resp. timelike*) *or* $c+3>0$ *and* $X$ *is timelike* (*resp. spacelike*) *if and only if* $TX$ *is spacelike* (*resp. timelike*),

  (iii) *the horizontal vector* $X$ *is null if and only if* $TX$ *is null.*

Corollary 5.3. *Let $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ be a Sasaki-like statistical submersion that the curvature tensor of the total space satisfies the type* (2.12) *with c. If* $\operatorname{rank}(\overline{\phi} + \overline{\phi}^*) = \dim \overline{M} - 1$ *and $N^*$ is constant, then results similar to Theorem* 5.3 *hold.*

Next, we consider $\pi$ as a statistical submersion with conformal fibers. For $U$ and $V \in \mathcal{V}(M)$ if $T_U V = 0$ (resp. $T_U V = \frac{1}{s} g(U, V) N$) holds, then $\pi$ is called a statistical submersion with isometric fibers (resp. conformal fibers). Then we can get from $T_U \xi = 0$ of Lemma 4.1

Lemma 5.1. *If $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ is a Sasaki-like statistical submersion with conformal fibers, then $\pi$ has isometric fibers.*

Theorem 5.4. *Let $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ be a Sasaki-like statistical submersion with conformal fibers such that the curvature tensor of the total space satisfies the type* (2.12) *with c. Then each fiber is a totally geodesic submanifold of $M$ such that the curvature tensor satisfies the type* (2.12) *with c.*

Furthermore, we find from (5.6)

Theorem 5.5. *Let $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ be a Sasaki-like statistical submersion with conformal fibers such that the curvature tensor of the total space satisfies the type* (2.12) *with c. If* $\operatorname{rank}(\overline{\phi} + \overline{\phi}^*) = \dim \overline{M} - 1$*, then*

(1) *the total space satisfies the type* (2.12) *with $c = -3$,*

(2) *the base space is flat,*

(3) *each fiber satisfies the type* (2.12) *with* $-3$.

Finally, we give an example of a Sasaki-like statistical submersion such that the curvature tensor satisfies the type (2.12).

Example. Let $(\mathbf{R}^{2n}_n, \widehat{\nabla}, \widehat{g}, \widehat{\phi})$ and $(\mathbf{R}^{2m+1}_m, \nabla, g, \phi, \xi, \eta)$ be a Kähler-like statistical manifold in Example 2.1 and Sasaki-like statistical manifold in Example 2.2, respectively. We define the statistical submersion $\pi : (\mathbf{R}^{2m+1}_m, \nabla, g) \to (\mathbf{R}^{2n}_n, \widehat{\nabla}, \widehat{g})$ by

$$\pi(x_1, \ldots, x_m, y_1, \ldots, y_m, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \qquad (n \leq m).$$

Then $\pi$ is a Sasaki-like statistical submersion such that the curvature tensor of $\mathbf{R}^{2m+1}_m$ satisfies the type (2.12) with $c = -3$. Each fiber is a totally geodesic submanifold of $\mathbf{R}_m^{2m+1}$. Because of $\partial_{x_i} + y_i\partial_z,\ \partial_{y_i} \in \mathcal{H}(\mathbf{R}_m^{2m+1})$, we find

$$\begin{aligned}
&A_{\partial_{x_i}+y_i\partial_z}(\partial_{x_j} + y_j\partial_z) = -A^*_{\partial_{x_j}+y_j\partial_z}(\partial_{x_i} + y_i\partial_z) = 0,\\
&A_{\partial_{x_i}+y_i\partial_z}\partial_{y_j} = -A^*_{\partial_{y_j}}(\partial_{x_i} + y_i\partial_z) = -2\delta_{ij}\,\partial_z,\\
&A_{\partial_{y_j}}(\partial_{x_i} + y_i\partial_z) = -A^*_{\partial_{x_i}+y_i\partial_z}\partial_{y_j} = -\delta_{ij}\,\partial_z,\\
&A_{\partial_{y_i}}\partial_{y_j} = -A^*_{\partial_{y_j}}\partial_{y_i} = 0
\end{aligned}$$

for $i, j \in \{1, \ldots, n\}$. Hence we find $A_XY = -g(X, \phi Y)\xi$ for $X, Y \in \mathcal{H}(\mathbf{R}_m^{2m+1})$.

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