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Statistical Mechanical Methods in Particle Structure Analysis of Lattice Field Theories

II. Scalar and Surface Models

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Abstract. We illustrate on simple examples a new method to analyze the particle structure of lattice field theories. We prove that the two-point function in Ising and rotator models has an Ornstein-Zernike correction at high temperature. We extend this to Ising models at low temperatures if the lattice dimension $d \geqq 3$. We prove that the energy-energy correlation function at high temperatures (for Ising or $N=2$ rotators) decays according to mean field theory (i.e. with the square of the Ornstein-Zernike correction) if $d \geqq 4$. We also study some surface models mimicking the strong-coupling expansion of the glueball correlation function. In the latter model, besides Ornstein-Zernike decay, we establish the presence of two nearly degenerate bound states.

I. Introduction

There has recently been some renewed interest in the analysis of the particle structure of (lattice) field theories, in particular of gauge theories [1–11]. In [11] we have outlined a new method leading to various results on the particle structure of scalar and gauge lattice field theories. In this paper, we explain our method in mathematical detail on the simplest examples and we prove some of the results claimed in [11].

We ask the following questions: what is the precise long-distance behaviour of the two-point function, or of higher-order correlation functions in lattice field theories? What information on the spectrum of the theory can one obtain from this behaviour? The connection between both questions is provided by the Källen-Lehman representation, which, for a continuum Euclidean theory, is:

$$\langle \phi_0 \phi_x \rangle = \int (-\Delta + a^2)^{-1}(0,x)\, d\varrho(a)\,, \tag{1.1}$$

where $d\varrho$ is a positive measure. An analogous, slightly weaker, formula holds for a reflection positive correlation function in a lattice theory. From (1.1) it is clear that, if we can prove that

$$\langle \phi_0 \phi_x \rangle \underset{|x| \to \infty}{\sim} \frac{e^{-m|x|}}{|x|^{(d-1)/2}} \sim (-\Delta + m^2)^{-1}(0,x) \tag{1.2}$$

for some $m>0$, then $d\varrho(a)$ contains a point measure $\delta(a-m)$ at the bottom of its support. Moreover, subleading corrections to (1.2) will give information on bound states, upper mass gap, etc. ....

Thus we want a detailed analysis of the correlation functions. This will be possible only when we have a convergent expansion (high-temperature, low-temperature, etc. ...). In this situation we can use this expansion in order to convert the expectation value in the spin system into a sum over random geometrical objects. For example, a two point function $\langle\phi_0\phi_x\rangle$ can be rewritten as a sum over lines (with appropriate weights) joining 0 to x. In lattice gauge theories, sums over random surfaces enter. Our next step is to decompose, say a random line, into a gas of excitations as follows: first of all, we observe that the simplest line corresponds to a straight line joining 0 and x. Then we also see that, in general, a line contains some straight parts and other parts. It is the latter that we call "excitations." We set up a one-to-one correspondence between random lines and sets of excitations. Our sum over lines becomes a partition function for a "gas" of these excitations. A similar analysis was used by Gallavotti [12] when he studied the phase separation line in the two dimensional Ising model. Now, it turns out that for the values of the parameters (such as temperature) that we consider, the gas of excitations is very dilute and its Mayer expansion converges.

The main results of our analysis going from the correlation function to the gas of excitations are: the pressure of the gas is related to the "mass gap" [m in (1.2)] of the original model and the (Gaussian) fluctuations of the gas are closely related to the power-law correction in (1.2). Moreover, the bound states and the upper gap in $d\varrho(a)$ can be analyzed in terms of the rate at which the pressure of the gas in a finite box approaches its thermodynamic limit. None of the above statements are obvious but see [11] for a more detailed heuristic discussion.

The results that are proven in this paper with our method are fairly simple and most of them have been derived elsewhere (with different methods). However we want to explain our method with all details in the simplest cases. More elaborate results will be published elsewhere [13]. Our first result concerns the two-point function for Ising models or $O(N)$ rotators in the high-temperature region. We prove Ornstein-Zernike decay of the two point function and the presence of an upper gap roughly equal to 3 m in the distribution $\varrho(a)$ entering the lattice analogue of (1.1). This kind of result was previously obtained in [14–16, 3].

We consider also low temperatures: for the Ising model and for lattice dimensions $d\geqq 3$, we prove the same results as in the high-temperature situation: Ornstein-Zernike decay and the existence of an upper gap. This is essentially a result of Schor [17, see also 4, 7]. Moreover, we can understand from our point of view the anomalous decay in $d=2$, see Sect. 7.

We consider also a model of self-avoiding random surfaces [18] in three dimensions (this restriction is for simplicity but is not essential). The latter can be viewed as a simplification of the random surface models occurring in the strong-coupling expansion of lattice gauge theories [19]: The analogue of the plaquette-plaquette correlation has an Ornstein-Zernike decay and there is an upper gap. Moreover we can establish the presence of two nearly degenerate bound states below the continuum threshold. With little extra work, one could extend these results to the real gauge models in the strong-coupling regime and also, by duality, to the 3 dimensional Ising model at low temperatures.

Coming back to the spin models, we also discuss the decay of the energy-energy correlation function, which is a truncated four-point function. There has been some debate in the literature [20, 21] as to what is the exact decay of these correlation functions at high temperatures. The exponential decay is $\exp(-2m|x|)$, where m comes from (1.2), but what is the power-law correction? Polyakov arrives at the following conclusion [20]: $|x|^{-2}$ for $d=2$; $|x|^{-2}(\ln|x|)^{-2}$ for $d=3$ and $|x|^{-d+1}$ for $d \geqq 4$. However, a different calculation, in [21], leads to the prediction of an $|x|^{-d}$ correction for all d. The $d=2$ result is known exactly for the Ising model [22]. Here we prove that Polyakov's result is correct for $d \geqq 4$ (for the Ising model or the $N=2$ rotator model); the $d=3$ case remains open, but we have some heuristic arguments supporting Polyakov's result (see Sect. 7). This part of our paper is somewhat special, because the proof is based on correlation inequalities rather than on our general method. The inequalities that we use are similar to those of [23, 24], leading to mean field behaviour at the critical point in $d \geqq 5$. Indeed, as we explain in more details in Sect. 7, we regard these power law corrections as quite similar, but simpler, to the power law decay at the critical point. There is a critical dimension, equal to 3 here, above which mean field theory is exact: The truncated four-point function decays like the square of the two-point function. In $d=3$ there are logarithmic corrections to this behaviour and, below 3 dimensions, power law corrections. Moreover, these phenomena are related to intersection properties of random walks, just as the critical phenomena are.

Outline. We state our results in Sect. 2. Each of the following four sections is devoted the proof of one of our results: The decay of the two-point function in $O(N)$ models at high temperatures (3). The one of the energy-energy correlation for $N=1$ or 2 (4). The low temperature Ising model (5). The surface models (6). In Sect. 7, we discuss some extension of our ideas.

II. The Main Results

In part A below we consider the spin models [Ising model and $O(N)$-invariant rotators] in the high-temperature region. In part B, we analyse the low-temperature phase of the Ising model. Finally, in Part C, we extend our analysis to surface models and discuss the applications to lattice gauge theories in the strong-coupling regime.

A. Spin Models at High Temperatures

At each site $x \in \mathbb{Z}^d$ we have an N-component spin variable $\mathbf{s}_x \in S^{N-1}$ of unit length. The Hamiltonian in a finite box $\Lambda \subset \mathbb{Z}^d$ is:

$$-H_\Lambda = \sum_{\langle xy \rangle \subset \Lambda} \mathbf{s}_x \cdot \mathbf{s}_y ,$$

where the sum runs over pairs of nearest-neighbour sites. The Gibbs measure in Λ, at inverse temperature β, is

$$d\mu_\Lambda(\mathbf{s}) = Z_\Lambda^{-1} \exp(-\beta H_\Lambda) \prod_{x \in \Lambda} \delta(|\mathbf{s}_x|^2 - 1) d\mathbf{s}_x ,$$

where Z_Λ is the partition function. $\lim_{\Lambda \uparrow \mathbb{Z}^d} d\mu_\Lambda = d\mu$ exists and, for β small, μ is the unique Gibbs state of this model. We are interested in the asymptotic behaviour, when $|x| \to \infty$, of the two-point function:

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_x \rangle = \int \mathbf{s}_0 \cdot \mathbf{s}_x d\mu(\mathbf{s}) \quad \text{for } \beta \text{ small}.$$

Since we have ferromagnetic nearest-neighbour interactions, this model has a positive transfer matrix, T, $0 < T \leqq 1$ [17]. Using this and the spectral theorem, one arrives at a spectral representation for the two-point function, which is nothing but the lattice version of the (Euclidean) Källen-Lehman representation [17, 11]:

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_{(t,\mathbf{x})} \rangle = \int_0^1 \int_{T^{d-1}} \lambda^t e^{i\mathbf{p}\cdot\mathbf{x}} d\varrho(\lambda, \mathbf{p}) \quad \text{for} \quad t \geqq 0, \tag{2.1}$$

where we have written $x = (t, \mathbf{x})$ choosing the first axis as the "time" direction. T^{d-1} is the $(d-1)$-dimensional torus, and $d\varrho(\lambda, \mathbf{p})$ is a positive measure on $[0, 1] \times T^{d-1}$.

Summing (2.1) over $\mathbf{x} \in \mathbb{Z}^{d-1}$ we get

$$\sum_{\mathbf{x} \in \mathbb{Z}^{d-1}} \langle \mathbf{s}_0 \cdot \mathbf{s}_{(t,\mathbf{x})} \rangle = \int_0^1 \lambda^t d\varrho(\lambda), \tag{2.2}$$

where we write $d\varrho(\lambda, \mathbf{0}) = d\varrho(\lambda)$. The right-hand side of (2.2) can be written in a more suggestive form as

$$\int_0^\infty e^{-mt} d\mu(m). \tag{2.3}$$

Notation. Let A be a subset of $\mathbb{Z}^d$. We write $f(x) \sim g(x)$ for $x \in A$ to mean that there exist two constants $c_1 \neq 0$, $c_2 < \infty$, such that $c_1 g(x) \leqq f(x) \leqq c_2 g(x)$ $\forall x \in A$.

Theorem 1. *For β small enough, we have*

a)
$$\langle \mathbf{s}_0 \cdot \mathbf{s}_{(t,0)} \rangle \sim \frac{\exp(-m_0 t)}{(\sqrt{t})^{d-1}} \quad \textit{for} \quad t > 0, \tag{2.4}$$

where

$$m_0(\beta) = -\ln \beta + \ln N - P(\beta) \tag{2.5}$$

with $P(\beta)$ analytic in β around 0.

b)
$$\mu(m) = c\delta(m - m_0) + \mu'(m),$$

where $c > 0$, $\operatorname{supp} \mu' \subset [m_1, \infty[$ *and* $\lim_{\beta \to 0} \frac{m_1(\beta)}{3m_0(\beta)} = 1$.

Remarks. 1) This result has already been derived using other methods in [3, 14–16]. We shall call m_0 the mass gap and m_1 the upper gap.

2) Statement a) can be extended to $\langle \mathbf{s}_0 \cdot \mathbf{s}_{(t,\mathbf{x})} \rangle$ provided $\frac{|\mathbf{x}|^2}{t}$ remains bounded.

3) Our results can be extended to more general single-spin distributions than $\delta(|\mathbf{s}_x|^2 - 1) d\mathbf{s}_x$. Moreover, the Ornstein-Zernike decay (2.4) holds for general two-spin interactions of exponential decay.

However, for such general interactions, the transfer matrix need not be symmetric, or positive, and, therefore, the spectral representation of the two-point function need not always be valid.

We analyse now the energy-energy correlation function. Since our proof uses correlation inequalities, we restrict ourselves to $N=1$ or 2.

Theorem 2. *Let $N=1$ or 2 and $d \geqq 4$. Then, for β small,*

$$\sum_{\substack{|x|=1 \\ |y-(t,0)|=1}} \langle \mathbf{s}_0 \cdot \mathbf{s}_x ; \mathbf{s}_{(t,0)} \cdot \mathbf{s}_y \rangle \sim \frac{e^{-2m_0 t}}{t^{d-1}} \quad \text{for} \quad t>0\,, \tag{2.6}$$

where $\langle f; g\rangle \equiv \langle fg\rangle - \langle f\rangle \langle g\rangle$ and m_0 is equal to (2.5)

Remarks. 1) This results extends to other even correlation functions.

2) For $d=2$ and $N=1$ the power-law correction is t^{-2} [22], and for $d=3$ Polyakov conjectures a correction $(t \ln t)^{-2}$ [20]. We cannot prove this at present but we discuss this problem in Sect. 7.

B. Ising Model at Low Temperatures

In this section, we analyse the low temperature phase of the Ising model ($N=1$) in $d \geqq 3$ dimensions. For $N \geqq 2$ and $d \geqq 3$, there is no exponential decay, for large β, due to the appearance of "Goldstone bosons" [25, 26]. For $d=2$ and $N=1$, there is an anomalous power-law correction t^{-2} to the exponential decay (see Sect. 7); For $N=2$ we have a temperature dependent power-law decay at low temperatures [27] and, for $N \geqq 3$, the question remains open.

For $N=1$ and β large, $d\varrho(\lambda, \mathbf{p})$ in (2.1) contains a delta function at $\lambda=1, \mathbf{p}=0$. Its weight is $(m^*)^2$ where m^* is the spontaneous magnetization. We are therefore interested in the decay of $\langle s_0; s_x \rangle = \langle s_0 s_x \rangle - (m^*)^2$, where $\langle\ \rangle$ is the infinite volume Gibbs state obtained with "+" boundary conditions.

Theorem 3. *Let $d=3$ and $N=1$. For β large enough,*

a)
$$\langle s_0; s_{(t,0)} \rangle \sim \frac{\exp(-m_0 t)}{(\sqrt{t})^{d-1}} \quad \text{for} \quad t>0\,, \tag{2.7}$$

with $m_0(\beta) = 4\beta - P(\beta)$ where $P(\beta)$ is analytic in $z=e^{-2\beta}$ around $z=0$.

b) *$\mu(m) = (m^*)^2 \delta(m) + c\delta(m-m_0) + \mu'(m)$ where $c>0$, $\operatorname{supp}\mu' \subset [m_1, \infty[$ and*

$$\lim_{\beta \to \infty} \frac{2m_1(\beta)}{3m_0(\beta)} = 1\,. \tag{2.8}$$

Remarks. 1) Closely related results have been derived by other methods in [17, 4, 7].

2) The results extend to all dimensions $d \geqq 3$, with appropriate changes of the numerical constants. For the anomalous law in $d=2$, see Sect. 7.

C. Surface Models

In his extension of the Krammers-Wannier duality, Wegner [28] realized that the $3d$ Ising model is dual to the $\mathbb{Z}_2$ lattice gauge theory. Thus the low-temperature

Ising model is "equivalent" to the strongly coupled Ising gauge model. From this point of view, it is natural to ask whether our methods give some information on the spectrum of lattice gauge theories. In the strong coupling regime all the relevant correlation functions have a convergent "high-temperature" expansion.

This expansion can be expressed as a sum over surfaces ("flux sheets") with appropriate weights. This holds not only for $\mathbb{Z}_2$ but for general gauge groups [19]. Since all expressions in these theories can be reduced to summing over surfaces, it is sufficient to consider, for simplicity, a model of surfaces defined directly as such. We claim that one can extend, with little extra work, the results obtained for surface models to general (compact) lattice gauge theories.

Such results have already been obtained by different methods (see [1, 2, 6] for gauge theories and [7] for some surface models). Our only purpose is to illustrate our new technique in simple cases. The new results that it allows to prove, in particular for the gauge-matter systems, are published elsewhere [13].

In the following we focus our attention to a model of self-avoiding random surfaces [18]. By a self-avoiding surface in $\mathbb{Z}^d$ we mean a collection of (non-oriented) plaquettes in $\mathbb{Z}^d$ constituting a connected set in $\mathbb{R}^d$ in such a way that each link in S is contained in, at most, two plaquettes.

The boundary ∂S of S is the set of links that are contained in exactly one plaquette in S. If $\gamma=\gamma_1\bigcup\ldots\bigcup\gamma_n$, where $\gamma_1,\ldots,\gamma_n$ are n closed non self-intersecting loops in $\mathbb{Z}^d$, we let

$$G_\tau(\gamma)=\sum_{\substack{S:\\ \partial S=\gamma}} e^{-\tau|S|},$$

where the sums runs over all self-avoiding surfaces with boundary γ and $|S|$ is the number of plaquettes in S.

In particular, we are interested in several two-point functions $G_\tau(\gamma)$ with $\gamma=\gamma_1\bigcup\gamma_2$.

The relation with gauge theories is as follows: if $\gamma=\gamma_1\bigcup\ldots\bigcup\gamma_n$, $G_\tau(\gamma)$ is related to the strong coupling expansion of the connected expectation $\langle\mathrm{Tr}(U_{\gamma_1});\ldots;\mathrm{Tr}(U_{\gamma_n})\rangle$, where the U_γ's are Wilson loops. However, the weights tend to be more complicated and the ensemble of surfaces to be summed over is larger; the coefficient τ equals $|\ln\beta_p|=\ln g^2$.

Spectral representations such as (2.2) are not known for the self-avoiding surface model. However, they hold for various correlation functions, in particular, the plaquette-plaquette correlation in lattice gauge theories [19]. Therefore our results concerning the decay of correlations in surface models have a direct interpretation in terms of the structure of the excitations in gauge theories.

Now we fix the dimension $d=3$ and we define

$p(t,\mathbf{x})$: the boundary of a plaquette perpendicular to the time axis whose first point in lexicographic order is $(t,\mathbf{x})$.

A *rectangle* is the boundary of the union of two adjacent plaquettes. If a rectangle is perpendicular to the t axis, we call it horizontal if it is oriented along the first spatial axis x_1, and vertical if it is oriented along x_2.

$r_v(t,\mathbf{x})$: the vertical rectangle perpendicular to the time axis whose first point is $(t,\mathbf{x})$.

$r_h(t, \mathbf{x})$: the horizontal rectangle perpendicular to the time axis whose first point is $(t, \mathbf{x})$.

We look at the following correlation functions:

$$P(t) = \sum_{\mathbf{x} \in \mathbb{Z}^2} G_\tau(p(0), p(t, \mathbf{x})),$$

$$R_{v+h}(t) = \sum_{\mathbf{x} \in \mathbb{Z}^2} [G_\tau(r_v(0), r_v(t, \mathbf{x})) + G_\tau(r_v(0), r_h(t, \mathbf{x}))],$$

$$R_{v-h}(t) = \sum_{\mathbf{x} \in \mathbb{Z}^2} [G_\tau(r_v(0), r_v(t, \mathbf{x})) - G_\tau(r_v(0), r_h(t, \mathbf{x}))].$$

Theorem 4. *Let $d=3$; For τ large enough,*

a) $P(t) \sim C_0^p e^{-m_0 t} + C_1^p e^{-m_1 t} + O(e^{-m_2 t})$ *with* $m_0/4\tau \to 1$ *as* $\tau \to \infty$, $3m_0/2m_1 \to 1$ *as* $\tau \to \infty$, *and* $2m_0/m_2 \to 1$ *as* $\tau \to \infty$; C_0^p, $C_1^p > 0$.

b) $R_{v+h}(t) \sim C_0^{v+h} e^{-m_0 t} + C_1^{v+h} e^{-m_1 t} + O(e^{-m_2 t})$ *with* m_0, m_1, m_2 *as in* a).

c) $R_{v-h}(t) \sim C_1^{v-h} e^{-m_1' t} + O(e^{-m_2 t})$, *where* $3m_0/2m_1' \to 1$ *as* $\tau \to \infty$ *and* $m_1 - m_1' \sim 8e^{-2\tau}$ *as* $\tau \to \infty$.

Remarks. 1) The interpretation in terms of gauge theories is as follows: there is a stable glueball (one particle state) of mass $m_0 \sim 4\tau$. Furthermore there are two nearly degenerate bound states of mass m_1 and m_1'. One bound state appears in the spectral decomposition of the plaquette-plaquette correlation or of the symmetric combination of r_v and r_h. The other one appears only in the antisymmetric R_{v-h}.

2) By a duality transformation [28], the three dimensional $\mathbb{Z}_2$ gauge theory at strong coupling (i.e. $\beta = 1/g^2$ small) is mapped onto the Ising model at low temperatures. For example, the truncated plaquette-plaquette correlation (summed over all orientations of the plaquettes) is equal to the energy-energy correlation function in the corresponding Ising model. R_{v+h} and R_{v-h} are also equal to sums over (more complicated) correlation functions in the Ising model. Therefore, the bound state analysis, outlined in Remark 1 above for the gauge theory, applies to the Ising model at low temperatures.

3) Our methods extend to $d=4$ but the results are more complicated because the excitation spectrum becomes richer. See [6] for an analysis of the $d=4$ pure gauge theories.

III. Proof of Theorem 1

1. The first step consists in reviewing some properties of the small β expansion of the $O(N)$ models. We follow the presentation in [29, 30] where the proofs of Lemma 1 and 2 below can be found.

Lemma 1 [29, Theorem 2.1].

a) $Z_\Lambda = c(N/2)^{|\Lambda|} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \sum_{\omega_1, \ldots, \omega_n}^{\sim} \prod_{i=1}^{n} \beta^{|\omega_i|} \prod_{x \in \Lambda} \frac{c(n_x(\omega_1, \ldots, \omega_n))}{c(N/2)}$,

where

– $c(n) = \left(2^{n-1} \int_0^\infty t^{n-1} e^{-t} dt\right)^{-1}$, $n \geq \frac{1}{2}$,

– $n_x(\omega_1, \ldots, \omega_n) = n(x, \omega_1) + \ldots + n(x, \omega_n) + \frac{N}{2}$, *$n(x,\omega)$ = number of visits of x by ω.*

– *$|\omega|$ = number of steps in ω.*

– *The sum over $\omega_1, \ldots, \omega_n$ is over all sequences of (not necessarily distinct) loops in Λ.*

– *The symbol $\tilde{\sum_{\omega}}$ means*

$$\sum_{x \in \Lambda} \sum_{\substack{\omega \subset \Lambda \\ n(x,\omega) \neq 0}} \frac{1}{|\omega|} = \sum_{\omega \subset \Lambda} \frac{\#\{x \mid n(x,\omega) \neq 0\}}{|\omega|}.$$

b) $\langle \mathbf{s}_0 \cdot \mathbf{s}_x \rangle = Z_\Lambda^{-1} N \sum_{\omega : 0 \to x} Z_\Lambda(\omega)$, *where*

$$Z_\Lambda(\omega) = c\left(\frac{N}{2}\right)^{|\Lambda|} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{N}{2}\right)^n \tilde{\sum_{\omega_1, \ldots, \omega_n}} \beta^{|\omega|} \prod_{i=1}^{n} \beta^{|\omega_i|} \prod_{x \in \Lambda} \frac{c(n_x(\omega, \omega_1, \ldots, \omega_n))}{c(N/2)}.$$

We can transform these formulas in two ways: first of all, one observes that the value of a term does not depend on the ordering in the sequence $(\omega_1, \ldots, \omega_n)$.

Define Ω (a "generalized set" or "g set" as in [30]) to be a map from the set of loops into $\mathbb{N}$. We write:

– $\Omega! = \prod_{\omega} \Omega(\omega)!$

– $W(\Omega) = \prod_{\omega} \beta^{|\omega| \Omega(\omega)} \prod_{x} \frac{c(n_x(\Omega))}{c(N/2)}$, with $n_x(\Omega) = \sum_{\omega} \Omega(\omega) n_x(\omega)$,

– $z_0(\Omega) = \left(\frac{N}{2}\right)^{|\Omega|}$, with $|\Omega| = \sum_{\omega} \Omega(\omega)$.

Then, if we define $\tilde{Z}_\Lambda = Z_\Lambda / c(N/2)^{|\Lambda|}$, we can write

$$\tilde{Z}_\Lambda = \sum_{\Omega}^{\Lambda} \frac{1}{\Omega!} W(\Omega) z_0(\Omega),$$

where the sum $\sum^{\Lambda}$ is restricted to those Ω's where $\Omega(\omega) = 0$ unless $\omega \subset \Lambda$.

Next, we observe that $W(\Omega) = W(\Omega_1) \cdot W(\Omega_2)$ if $\Omega_1 \cap \Omega_2 = \emptyset$ in the sense that the support of Ω_1 is disconnected from the support of Ω_2: if $\Omega_1(\omega) \neq 0$ and $\Omega_2(\omega') \neq 0 \Rightarrow \omega \cap \omega' = \emptyset$.

Decompose Ω into maximally connected g sets called *polymers*: $\Omega = \Omega_1^c + \ldots + \Omega_n^c$, where $\Omega_i^c \cap \Omega_j^c = \emptyset$ and Ω_i^c is connected in the sense that it cannot be decomposed further. Letting

$$z(\Omega^c) = \frac{z_0(\Omega^c) W(\Omega^c)}{\Omega^c!}, \tag{3.1}$$

we have

$$\frac{z_0(\Omega) W(\Omega)}{\Omega!} = \prod_{i=1}^{n} z(\Omega_i^c).$$

Define

$$g(\Omega_i^c, \Omega_j^c) = \begin{cases} 0 & \text{if } \Omega_i^c \cap \Omega_j^c = \emptyset \\ -1 & \text{otherwise} \end{cases}.$$

Then

$$\tilde{Z}_\Lambda = \sum_{r=0}^{\infty} \sum_{\{\Omega_1^c, \ldots, \Omega_r^c\}} \prod_{k=1}^{r} z(\Omega_k^c) \prod_{1 \le k < k' \le r} (1 + g(\Omega_k^c, \Omega_{k'}^c)), \tag{3.2}$$

where $\{\Omega_1^c, \ldots, \Omega_r^c\}$ are (unordered) sets of polymers.

We say that Ω_1^c is a $0 \to x$ *polymer* if: $\Omega_1^c(\omega) = 1$ for one $\omega : 0 \to x$, the support of Ω_1^c is otherwise made of loops and Ω_1^c is connected.

If Ω_1^c is a $0 \to x$ polymer we let

$$z(\Omega_1^c) = \frac{W(\Omega_1^c)}{\Omega_1^c!} \left(\frac{N}{2}\right)^{|\Omega_1^c| - 1}. \tag{3.3}$$

$Z_\Lambda(\omega)$ has a representation similar to (3.2) but with the constraint that $\Omega_1^c(\omega) = 1$.

Now we use the polymer formalism (see e.g. [19, 30]) to obtain

Lemma 2 [30, Lemmata 3.1, 3.2, 3.3]. *For β small enough,*

a)
$$\langle \mathbf{s}_0 \cdot \mathbf{s}_x \rangle = N \sum_{X : 0 \to x} \frac{1}{X!} \phi^T(X) z(X),$$

where the sum runs over all g-sets X of polymers, i.e. over all maps X from the set of polymers into $\mathbb{N}$ with the restriction that $X(\Omega^c) = 1$ for exactly one $0 \to x$ polymer,

$$X! = \prod_{\Omega^c} X(\Omega^c)!,$$

$$z(X) = \prod_{\Omega^c} z(\Omega^c)^{X(\Omega^c)},$$

$$\phi^T(X) = \sum_{G \in g(X)} (-1)^{L(G)}, \tag{3.4}$$

where the sum runs over all connected subgraphs G of $g(X)$ containing all vertices of $g(X)$. $g(X)$ is the graph whose vertices are the polymers Ω^c such that $X(\Omega^c) \neq 0$ and whose lines joint Ω_i^c to Ω_j^c whenever $\Omega_i^c \cap \Omega_j^c \neq \emptyset$.

b) *There exists some $c < \infty$ such that, for all X,*

$$\left| \frac{\phi^T(X) z(X)}{X!} \right| \le (c\beta)^{|X|},$$

with

$$|X| = \sum_{\Omega, \omega} X(\Omega) \Omega(\omega) |\omega|. \tag{3.5}$$

2. Geometrical Analysis of X

Given a nearest-neighbour bond b and a loop ω, one defines

$$\omega(b) = \begin{cases} 1 & \text{if } b \text{ belongs to } \omega \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define $\Omega(b) = \sum_\omega \Omega(\omega)\omega(b)$, and $X(b) = \sum_{\Omega, \omega} X(\Omega)\Omega(\omega)\omega(b)$.

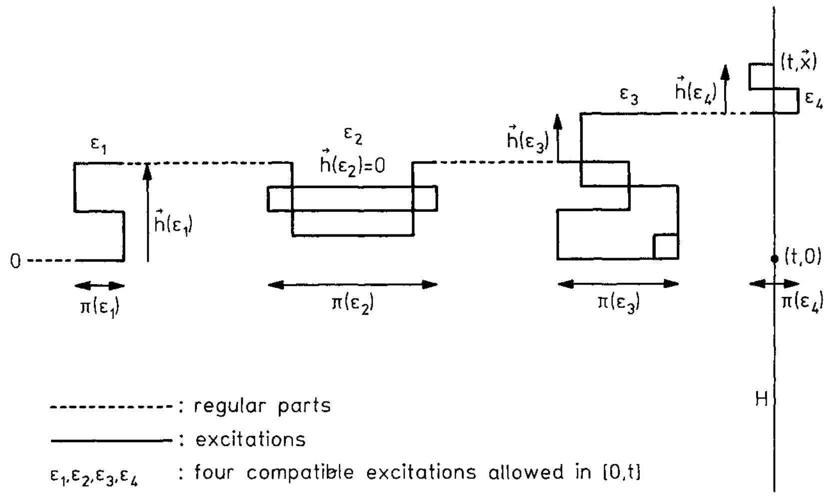

Fig. 1

We shall decompose $X: 0 \to (t, \mathbf{x})$ into a set of "regular parts" (or "ground-state" parts) and of "excitations." The regular parts will correspond to the simplest possible subset of X, namely a straight line parallel to the t axis (see Fig. 1 for the remainder of this subsection).

Given $X: 0 \to (t, \mathbf{x})$, a bond b is *regular* if

- $X(b) = 1$,
- b is parallel to the t axis,
- there is no other bond b' satisfying $X(b') \neq 0$ and having the same projection as b on the t axis.

Now one can decompose $\underline{X} \equiv \{b | X(b) \neq 0\}$ into maximally connected sets of regular bonds (called *straight lines*) and maximally connected irregular ones.

Let ε be the restriction of X to an irregular part of $\underline{X}$. An *excitation* is an equivalence class (also denoted ε) of such objects, modulo translations by vectors of $\mathbb{Z}^{d-1}$ in the directions perpendicular to the time axis (these excitations are called "jumps" in Gallavotti's paper [12]).

This definition is motivated by the fact that, as we shall now see, there is a one-to-one correspondence between the X's entering the sum in Lemma 2a) and suitable families of excitations. It is already clear that to each X corresponds a set of excitations $\{\varepsilon_1, \ldots, \varepsilon_n\}$ such that

$$\pi(\varepsilon_i) \cap \pi(\varepsilon_j) = \emptyset, \quad \text{if} \quad i \neq j, \tag{3.6}$$

where $\pi(\varepsilon)$ = projection of ε on the time axis. Indeed, (3.6) holds because different excitations are separated by regular parts of X. $\pi(\varepsilon)$ is a segment of the time axis also denoted $[t^-, t^+]$.

If ε is an excitation of $X: 0 \to (t, \mathbf{x})$, then: $[t^-, t^+] \cap [0, t] \neq \emptyset$.

Now we define the *height* of an excitation ε: Let $H \cong \mathbb{Z}^{d-1}$ be an hyperplane perpendicular to the time axis. An excitation ε is, by definition, adjacent to two straight lines (one to the left, one to the right). The projection of each of those on H

is a point (noted $\mathbf{x}_l$ and $\mathbf{x}_r$). The vector joining them: $\mathbf{h}(\varepsilon)=\mathbf{x}_r-\mathbf{x}_l \in H \cong \mathbb{Z}^{d-1}$ is the *height* of ε. Note that this definition does not depend on which representative we take in the equivalence class ε. Notice also that a height can be zero.

Let the *excess length* of ε, $l(\varepsilon)$, be defined as $|\varepsilon|-|\pi(\varepsilon)|$, where $|\pi(\varepsilon)|=t^+-t^-$ and $|\varepsilon|$ is defined the same way as $|X|$ in (3.5).

We say that an excitation ε is *allowed in* $[0,t]$ if there exists an $\mathbf{x}\in\mathbb{Z}^{d-1}$ and an $X:0\to(t,\mathbf{x})$ such that ε is an excitation of X. By extension, an excitation ε is allowed in $[0,\infty[$ if there exists a t_0 such that ε is allowed in $[0,t]$ for all $t>t_0$, similarly for intervals $]-\infty,t]$, $]-\infty,+\infty[$.

Finally a set $E=(\varepsilon_1,\ldots,\varepsilon_n)$ of excitations is *compatible* if $\pi(\varepsilon_i)\cap\pi(\varepsilon_j)=\emptyset\;\forall i\neq j$.

To each compatible set of excitations we associate a g-set of polymers X by adding straight lines between the excitations and by choosing the appropriate representative in each equivalence class, so that the line to the left of an excitation has the same projection on H as the one to the right of the preceding excitation.

If each $\varepsilon\in E$ is allowed in $[0,t]$, the resulting X will be going from 0 to $x=(t,\mathbf{x})$, where $\mathbf{x}=\mathbf{h}(E)\equiv\sum_{i=1}^{n}\mathbf{h}(\varepsilon_i)$. So, in this sense, the X's are equivalent to compatible sets of (allowed) excitations.

We write $X(E)$ for the reconstructed X and $\pi(E)=\bigcup_{i=1}^{n}\pi(\varepsilon_i)$, $l(E)=\sum_{i=1}^{n}l(\varepsilon_i)$. We observe that

$$|X(E)|=l(E)+t\,. \tag{3.7}$$

We have, from Lemma 2 and the preceding construction,

$$\langle\mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle=N\tilde{\sum_{E}}^{0,t}\frac{1}{X(E)!}\phi^T(X(E))z(X(E))\delta(\mathbf{h}(E)-\mathbf{x})\,, \tag{3.8}$$

and the sum runs over all compatible sets of excitations allowed in $[0,t]$.

Now the geometrical analysis is finished: the two-point function $\langle\mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle$ is written in (3.8) as a sum over sets of excitations. In the next paragraph, we shall study the gas of excitations as such and show that it enjoys the properties of a dilute gas. Then the conclusions on $\langle\mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle$ will be easy to derive.

3. The Statistical Mechanics of Excitations

Let ε be an excitation of X. This means that the support of ε can be decomposed into a subset of the polymer Ω_1^c, in supp$X$, going from 0 to x and possibly some other closed polymers. In other words ε is nothing but a special type of X, going from some y to some y'; therefore we can define $g(\varepsilon)$ and $\phi^T(\varepsilon)$ just as we did with X in Lemma 2. We have an important factorization property:

Lemma 3. *Let $E=(\varepsilon_1,\ldots,\varepsilon_n)$ be a compatible set of polymers allowed in $[0,t]$. Then*

$$\phi^T(X(E))=\prod_{i=1}^{n}\phi^T(\varepsilon_i)\,, \tag{3.9}$$

$$z(X(E))=(\beta/N)^t\prod_{i=1}^{n}\tilde{z}(\varepsilon_i)\,,$$

where

$$\tilde{z}(\varepsilon)=(N/\beta)^{(t^+-t^-)}N\prod_{\Omega^c} z(\Omega^c)^{\varepsilon(\Omega^c)},$$

and $z(\Omega^c)$ *is given by* (3.1), *or by* (3.3) *for the "open" polymer going from y to y′*,

$$X(E)!=\prod_{i=1}^{n}\varepsilon_i!.$$

Proof. Consider the graph $g(X(E))$. Since different excitations are separated by straight lines, all the polymers contributing to one excitation do not overlap with those contributing to another excitation. Therefore, there is no line in $g(X(E))$ joining a polymer Ω_i^c with $\varepsilon_i(\Omega_i^c)\neq 0$ to a polymer Ω_j^c with $\varepsilon_j(\Omega_j^c)\neq 0$ for $i\neq j$. This observation and the formula defining $\phi^T(X)$ (3.4) proves the factorization (3.9). The other two formulas of the lemma follow from the definitions and the computation

$$\frac{c\left(\frac{N}{2}+1\right)}{c\left(\frac{N}{2}\right)}=N^{-1}.$$

If we define

$$\zeta(\varepsilon)=\frac{\tilde{z}(\varepsilon)\phi^T(\varepsilon)}{\varepsilon!}, \tag{3.10}$$

and

$$g(\varepsilon,\varepsilon')=\begin{cases}0 & \text{if } \pi(\varepsilon)\cap\pi(\varepsilon')=\emptyset\\ -1 & \text{otherwise},\end{cases}$$

we may rewrite (3.8) as

$$\langle \mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle=(\beta/N)^t\left(\sum_E^{0,t}\prod_{i=1}^{n}\zeta(\varepsilon_i)\prod_{i,j=1}^{n}(1+g(\varepsilon_i,\varepsilon_j))\right)\delta(\mathbf{h}(E)-\mathbf{x}), \tag{3.11}$$

where the sum runs over all sets of excitations allowed in $(0,t)$ [compatible or not, since the compatibility condition is automatically implied by the product over $(1+g(\varepsilon_i,\varepsilon_j))$].

If we omit the prefactor $(\beta/N)^t$ in (3.11), which plays the role of a "ground state energy," the formula for $\langle \mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle$ looks like the partition function of a gas of excitations with activities $\zeta(\varepsilon)$ and hard-core interactions $(1+g(\varepsilon,\varepsilon'))$. The δ-function is a global constraint which is similar to working in a canonical ensemble. This constraint is removed by summing over $\mathbf{x}\in H_t$. For β small, $\tilde{z}(\varepsilon)$ and thus the gas activities $\zeta(\varepsilon)$ are small, and we can use a convergent Mayer series for this gas:

Lemma 4. *For* β *small enough*,

a)

$$\langle \mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle=(\beta/N)^t\left(\frac{1}{2\pi}\right)^{d-1}\int_{-\pi}^{\pi}d\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}}\exp\left(\sum_E^{(0,t)}\frac{\phi^T(E)}{E!}\zeta(E)e^{-i\mathbf{k}\cdot\mathbf{h}(E)}\right),$$

where the integral is over the $(d-1)$ dimensional torus:

– the sum over E's runs now over g-sets of excitations, each of which is allowed in $(0,t)$.

– $\phi^T(E)$ *and* $E!$ *are as usual, and* $\zeta(E)=\prod_{\varepsilon}\zeta(\varepsilon)^{E(\varepsilon)}$.

b)
$$\sum_{\mathbf{x}\in\mathbb{Z}^{d-1}}\langle \mathbf{s}_0\cdot\mathbf{s}_{(t,\mathbf{x})}\rangle=(\beta/N)^t\exp\left(\sum_E^{(0,t)}\frac{\phi^T(E)\zeta(E)}{E!}\right).$$

Proof. Given formula (3.11), part b is the standard formula for Mayer series. The convergence follows from the estimates in Lemma 5 below. For part a, we write

$$\delta(\mathbf{h}(E)-\mathbf{x})=\frac{1}{(2\pi)^{d-1}}\int_{-\pi}^{\pi}d\mathbf{k}\exp(i\mathbf{k}\cdot(\mathbf{h}(E)-\mathbf{x})),$$

and we define a complex activity $\tilde{\zeta}(\varepsilon)=\zeta(\varepsilon)e^{i\mathbf{k}\cdot\mathbf{h}(\varepsilon)}$, and then use the Mayer expansion again.

Lemma 5. *There exists a* $\beta_0>0$ *and a* $c<\infty$ *such that, for all* $\beta<\beta_0$,

a)
$$\sum_{\substack{E:\\ t^-(E)=0}}\left|\frac{\phi^T(E)\zeta(E)}{E!}\right|\left(\frac{\beta_0}{\beta}\right)^{l(E)}\leqq c\beta_0,$$

where $t^-(E)=\inf\{t^-(\varepsilon)\,|\,E(\varepsilon)\neq 0\}$

b)
$$\sum_{\substack{E:\\ \pi(E)\supset[a,b]}}\left|\frac{\phi^T(E)\zeta(E)}{E!}\right|\leqq c\beta^{2(b-a)}.$$

The sums over E runs over all g-sets of excitations allowed in $(-\infty,+\infty)$ *or in* $[0,\infty[$ *or in* $[0,t]$.

Proof. We do not give a detailed proof of this lemma, since it follows from the polymer formalism [19], once we have the following bounds on the activities $\zeta(\varepsilon)$ of the excitations: By definition,

$$\zeta(\varepsilon)=\frac{\tilde{z}(\varepsilon)\phi^T(\varepsilon)}{\varepsilon!}=\frac{z(\varepsilon)\phi^T(\varepsilon)N(N/\beta)^{\pi(\varepsilon)}}{\varepsilon!}.$$

By Lemma 2b),

$$\left|\frac{z(\varepsilon)\phi^T(\varepsilon)}{\varepsilon!}\right|\leqq(c\beta)^{|\varepsilon|},$$

so $|\zeta(\varepsilon)|\leqq c^{|\varepsilon|}\beta^{l(\varepsilon)}$ or, $|\zeta(\varepsilon)|\left(\frac{\beta_0}{\beta}\right)^{l(\varepsilon)}\leqq c^{|\varepsilon|}\beta_0^{l(\varepsilon)}$. Now,

$$\pi(\varepsilon)\leqq\tfrac{1}{3}|\varepsilon| \tag{3.12}$$

because, by connectedness of ε, the number of bonds contained in ε (counting multiplicities) that have a given bond b as projection on the time axis is odd. So,

$$l(\varepsilon)\geqq\tfrac{2}{3}|\varepsilon|, \tag{3.13}$$

and

$$|\zeta(\varepsilon)|\left(\frac{\beta_0}{\beta}\right)^{l(\varepsilon)} \leqq c^{|\varepsilon|}\beta_0^{2|\varepsilon|/3}.$$

The number of excitations ε with $|\varepsilon|=n$ and $t^-(\varepsilon)$ fixed is bounded by $(c')^n$; therefore, the proof of a) follows from standard estimates in the polymer formalism, e.g.

$$\left|\frac{\phi^T(E)}{E!}\right| \begin{array}{ll} \leqq (c'')^{|E|} & \text{if } E \text{ is connected}, \\ =0 & \text{otherwise}. \end{array}$$

b) follows from a) and the fact that $l(E)\geqq\frac{2}{3}|E|\geqq 2|\pi(E)|$ by (3.12) and (3.13), so that

$$\left(\frac{1}{\beta}\right)^{2(b-a)} \leqq \left(\frac{1}{\beta}\right)^{l(E)}$$

if $\pi(E)\supset[a,b]$.

Since $\sum_E^{0,t}\frac{\phi^T(E)\zeta(E)}{E!}$ is nothing but the pressure of the gas of excitations, expanded in a Mayer series, we expect that it is the sum of

– a bulk term, proportional to t, plus

– a boundary term, equal to a constant, since the gas considered here is one-dimensional, plus

– an exponentially small (in t) correction.

This is precisely the content of the next lemma:

Lemma 6. *There exists a (t-independent) constant c such that*

$$\sum_E^{0,t}\frac{\phi^T(E)\zeta(E)}{E!}=tP+c+O(\beta^{2t}), \tag{3.14}$$

where

$$P=\sum_{\substack{E \\ t^-(E)=0}}\frac{\phi^T(E)\zeta(E)}{E!}, \tag{3.15}$$

with the sum over E's allowed in $(-\infty,\infty)$.

Proof. We write the left-hand side of (3.14) as

$$\sum_{s=0}^{t-1}\sum_{\substack{E \\ t^-(E)=s}}\frac{\phi^T(E)\zeta(E)}{E!}+c_0+c_t+f(0,t), \tag{3.16}$$

where

$$c_0=\sum_{\substack{E \\ t^-(E)<0}}^{0}\frac{\phi^T(E)\zeta(E)}{E!},$$

and $\sum^0$ = sum over E's allowed in $[0, \infty[$,

$$c_t = \sum_{\substack{E \\ t \in \pi(E)}}^{t} \frac{\phi^T(E)\zeta(E)}{E!} - \sum_{\substack{E \\ t \in \pi(E)}} \frac{\phi^T(E)\zeta(E)}{E!},$$

and $\sum^t$: E's allowed in $]-\infty, t]$, $\sum$: E's allowed in $]-\infty, +\infty[$,

$$f(0,t) = \Big(\sum_E^{0,t} - \sum_{id}^{0} - \sum_{id}^{t} + \sum_{id}\Big) \frac{\phi^T(E)\zeta(E)}{E!}$$

in each sum over $E: \pi(E) \supset [0,t]$.

By translation invariance, the sum over s in (3.16) equals tP, and c_0 and c_t are t-independent constants ($c = c_0 + c_t$). Finally, by Lemma 5b), $|f(0,t)| \leqq O(\beta^{2t})$.

Proof of Theorem 1. a) We write

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_{(t,0)} \rangle = (\beta/N)^t \exp\left(\sum_E^{0,t} \frac{\phi^T(E)\zeta(E)}{E!}\right) \cdot \frac{1}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} \exp\left(\sum_E^{0,t} \frac{\phi^T(E)\zeta(E)}{E!} (exp(i\mathbf{k} \cdot \mathbf{h}(E)) - 1)\right) d\mathbf{k}.$$

By Lemma 6 we know that

$$(\beta/N)^t \exp\left(\sum_E^{0,t} \frac{\phi^T(E)\zeta(E)}{E!}\right) \sim \exp(-m_0 t), \quad \text{with} \quad m_0 = -\ln\beta + \ln N - P, \tag{3.17}$$

where P is defined by (3.15).

This proves the claim about the mass $m_0(\beta)$, since $P(\beta)$ is analytic in $\beta(\zeta(E)$ is a power of β and the series over E is convergent by Lemma 5).

All we have so show is that

$$\int_{-\pi}^{\pi} \exp\left(\sum_E^{0,t} \frac{\phi^T(E)\zeta(E)}{E!} (\exp(i\mathbf{k} \cdot \mathbf{h}(E)) - 1)\right) d\mathbf{k} \cong \frac{1}{(\sqrt{t})^{d-1}}.$$

This is an argument typical of a central limit theorem (see [12]). We split the sum over E into two parts: the sum over the "unit excitations" $E(\varepsilon') = \delta_{\varepsilon,\varepsilon'}$ for ε such that $|\mathbf{h}(\varepsilon)| = 1$, $l(\varepsilon) = 1$, and the rest.

By explicit computation the first sum gives (there are $d-1$ unit excitations):

$$2tO(\beta) \sum_{\alpha=1}^{d-1} (\cos k_\alpha - 1).$$

For the rest, we write

$$e^{i\mathbf{k} \cdot \mathbf{h}} - 1 = i\mathbf{k} \cdot \mathbf{h} - \frac{(\mathbf{k} \cdot \mathbf{h})^2}{2} e^{i\theta \mathbf{k} \cdot \mathbf{h}}$$

for some θ, $0 < \theta < 1$. The linear term, $i\mathbf{k} \cdot \mathbf{h}(E)$, vanishes when we sum over E, by symmetry. The quadratic term is of order $O(\beta^2)t|\mathbf{k}|^2$. The β^2 comes from the fact that the smallest excitations in this sum have at least $l(\varepsilon) \geqq 2$. The sum is of order t by an estimate similar to Lemma 5a): we have to include $|\mathbf{h}(E)|^2$ in the sum, but

$|\mathbf{h}(E)| \leqq l(E) \leqq \exp l(E)$, and since we have an exponentially decreasing factor, $\beta^{l(E)}$ in $\zeta(E)$, the extension is straightforward.

The final result is that the first term dominates for small β, since $\cos k_\alpha - 1 \leqq -\frac{2}{\pi^2} k_\alpha^2$, for $|k_\alpha| \leqq \pi$, and, if we integrate,

$$\int_{-\pi}^{\pi} \exp\left(O(\beta)t \sum_{\alpha=1}^{d-1} (\cos k_\alpha - 1)\right) d\mathbf{k} \sim \frac{1}{(\sqrt{\beta t})^{d-1}}, \tag{3.18}$$

since the dominant contribution comes from small $|\mathbf{k}|$.

b) Using (2.3) and Lemma 4b, we have

$$\sum_{\mathbf{x} \in \mathbf{Z}^{d-1}} \langle \mathbf{s}_0 \cdot \mathbf{s}_{(t,\mathbf{x})} \rangle = \int_0^\infty e^{-mt} d\mu(m) = (\beta/N)^t \exp\left(\sum_E{}^{0,t} \frac{\phi^T(E)\zeta(E)}{E!} \right).$$

By Lemma 6, we can write this last expression,

$$= \exp(-m_0 t + c + O(\beta^{2t})) \sim e^c e^{-m_0 t}(1 + O(\beta^{2t})), \tag{3.19}$$

with m_0 given in (3.17). Therefore we have

$$\lim_{t \to \infty} \int_0^\infty e^{-(m-m_0)t} d\mu(m) = e^c,$$

which proves the presence of a point measure in μ at $m = m_0$ with coefficient e^c. If we write

$$\mu = e^c \delta(m - m_0) + \mu',$$

we have by (3.19)

$$\int_0^\infty e^{-mt} d\mu'(t) \sim \beta^{2t} e^{-m_0 t},$$

which proves that $\operatorname{supp} \mu' \subset [m_1, \infty[$ with $m_1(\beta) \sim 3|\ln \beta|$, i.e.

$$m_0(\beta)/m_1(\beta) \to 1/3 \quad \text{as} \quad \beta \to 0.$$

IV. Proof of Theorem 2

We write the proof for $N=1$; when $N=2$ it follows from similar correlation inequalities. Our proof is based on Lebowitz' [31] and Aizenman's inequalities [23, 24] that yield upper and lower bounds on the truncated energy-energy correlation function in terms of sums of products of two-point functions. Then we use Theorem 1 to prove that, for $d \geqq 4$, these upper and lower bounds have indeed the same decay, namely

$$c \frac{\exp(-2m_0 t)}{(\beta t)^{d-1}}, \tag{4.1}$$

with c independent of β.

Let $|x| = 1$, $y = t + x$, where we write t for $(t, 0)$, and consider $\langle s_0 s_x; s_t s_y \rangle$. This is bounded from above, using Lebowitz' inequality [31] by

$\langle s_0 s_t \rangle \langle s_x s_y \rangle + \langle s_0 s_y \rangle \langle s_x s_t \rangle$, which indeed behaves like (4.1), by Theorem 1. The β dependence of the proportionality constant in Theorem 1 is made explicit here. It can be found by looking at (3.18).

For the lower bound, we use Fröhlich's version [24] of Aizenman's inequality:

$$\langle s_0 s_x ; s_t s_y \rangle \geqq \langle s_0 s_t \rangle \langle s_x s_y \rangle + \langle s_0 s_y \rangle \langle s_x s_t \rangle - \beta^2 R_1 - \beta R_2 , \tag{4.2}$$

where R_1 is a finite sum of terms of the form

$$\sum_{\substack{u,v,w \\ |v-w|=1 \\ |v-u|=1}} \langle s_0 s_v \rangle \langle s_w s_x \rangle \langle s_y s_u \rangle \langle s_v s_t \rangle ,$$

and R_2 is a sum of terms of the form:

$$\sum_{\substack{u: \\ |u-t|=1}} \langle s_0 s_u \rangle \langle s_x s_t \rangle \langle s_t s_y \rangle .$$

"Of the form" means that we have to sum over some permutations of indices. Now we show that there exists a β independent constant c such that

$$R_1 \leqq \frac{c}{\beta} \frac{\exp(-2m_0 t)}{(\beta t)^{d-1}} , \tag{4.3}$$

and

$$R_2 \leqq c \frac{\exp(-2m_0 t)}{(\beta t)^{d-1}} ,$$

since, in (4.2) β^2 multiplies R_1 and β multiplies R_2, we have a lower bound of the same form, (4.1), as the upper bound, and having a *positive coefficient* for β small.

For simplicity we consider, of all the terms contributing to R_1, the one where $w = u = v + x$, and we take x perpendicular to the t axis; thus we have, by translation invariance,

$$\sum_{v \in \mathbb{Z}^d} \langle s_0 s_v \rangle \langle s_x s_{v+x} \rangle \langle s_{v+x} s_y \rangle \langle s_v s_t \rangle = \sum_{v \in \mathbb{Z}^d} \langle s_0 s_v \rangle^2 \langle s_v s_t \rangle^2 .$$

All the other terms are similar; we write

$$\sum_{v \in \mathbb{Z}^d} \ldots = \sum_{s=-\infty}^{+\infty} \sum_{\substack{v: \\ v_1 = s}} \ldots = 2 \sum_{s=-\infty}^{t/2} \sum_{\substack{v: \\ v_1 = s}} \ldots , \tag{4.4}$$

where $v_1 = (v_1, 0)$ is the projection of v on the t axis and the last equality follows from symmetry.

The following inequalities follow from reflection positivity (see e.g. [32] and references therein): Let $v_1 < v_1' < t$,

$$\langle s_v s_t \rangle \leqq \langle s_{v_1} s_t \rangle \leqq \langle s_{v_1'} s_t \rangle . \tag{4.5}$$

Moreover, if $v_1 \leqq t/2$, we have, by Theorem 1,

$$\langle s_{v_1} s_t \rangle \leqq c \frac{\exp(-m_0(t-v_1))}{(\sqrt{\beta(t-v_1)})^{d-1}} \leqq c' \frac{\exp(-m_0(t-v_1))}{(\sqrt{\beta t})^{d-1}} . \tag{4.6}$$

In (4.6), $c'=c(\sqrt{2})^{d-1}$, and we made explicit the β dependence, using (3.18), i.e., c is independent of β.

Now the sum over $s\in]-\infty,0]$ can be handled easily: by (4.5), (4.6), if $v_1\leqq 0$,

$$\langle s_v s_t\rangle^2 \leqq \langle s_0 s_t\rangle^2 \leqq c'\frac{\exp(-2m_0 t)}{(\beta t)^{d-1}},$$

and

$$\sum_{\substack{v:\\ v_1\leqq 0}} \langle s_0 s_v\rangle^2 \leqq O(1).$$

Therefore, we have (4.3) for that part of the sum (even without the β^{-1} factor).

Now we consider the sum $\sum_{s=1}^{t/2}$ in (4.4): we bound $\langle s_v s_t\rangle^2$ by $\langle s_{v_1} s_t\rangle^2$, and we use (4.6). So,

$$\sum_{s=1}^{t/2} \sum_{\substack{v:\\ v_1=s}} \langle s_0 s_v\rangle^2 \langle s_v s_t\rangle^2 \leqq \frac{c'}{(\beta t)^{d-1}} \sum_{s=1}^{t/2} \exp(-2m_0(t-s)) \sum_{\substack{v:\\ v_1=s}} \langle s_0 s_v\rangle^2. \tag{4.7}$$

We use the method of proof of Theorem 1 to bound $\sum_{\substack{v:\\ v_1=s}} \langle s_0 s_v\rangle^2$. Write $v=(v_1,\mathbf{v})$ and consider:

$$\frac{\sum_{\substack{u,v\\ u_1=v_1=s}} \langle s_0 s_u\rangle \langle s_0 s_v\rangle \delta(\mathbf{u}-\mathbf{v})}{\sum_{\substack{u,v\\ u_1=v_1=s}} \langle s_0 s_u\rangle \langle s_0 s_v\rangle} \left(\sum_{\substack{u:\\ u_1=s}} \langle s_0 s_u\rangle\right)^2. \tag{4.8}$$

The second factor is bounded, using Theorem 1, by

$$c\exp(-2m_0 s), \tag{4.9}$$

where c is independent of β. The first factor is $\langle\delta(\mathbf{u}-\mathbf{v})\rangle$, where the expectation value is taken in an ensemble of pairs of g-sets of polymers going from 0 to the hyperplane H_s passing through s. It is easy to see, by going through the proof of Theorem 1, that this constraint plays exactly the same role as the delta function $\delta(\mathbf{h}-\mathbf{x})$ in an ensemble made of single g-sets of polymers. Therefore, we obtain as in (3.18),

$$\langle\delta(\mathbf{u}-\mathbf{v})\rangle \leqq \frac{1}{(2\pi)^{d-1}} \int_{-\pi}^{\pi} \exp(-c\beta s|\mathbf{k}|^2)d\mathbf{k},$$

where c is β-independent. Inserting this bound and (4.8), (4.9) into the sum (4.7) we get

$$(4.7) \leqq \frac{c'c}{(\beta t)^{d-1}} \exp(-2m_0 t) \int_{-\pi}^{\pi} \sum_{s=1}^{t/2} \exp(-c\beta s|\mathbf{k}|^2))d\mathbf{k}.$$

The last integral is bounded by

$$\int_{-\pi}^{\pi} \frac{1}{1-\exp(-c\beta|\mathbf{k}|^2)} d\mathbf{k} \leqq O\left(\frac{1}{\beta}\right),$$

if $d-1 \geqq 3$.

This finishes the proof of the bound (4.3) on R_1. The bound on R_2 is easier: we sum a finite number of terms, each of which is bounded by

$$\frac{\exp(-2m_0 t)}{(\beta t)^{d-1}}.$$

V. Proof of Theorem 3

The proof of Theorem 3, as well as the one of Theorem 4, will be a variation on the theme of the proof of Theorem 1. Therefore, we shall not repeat all the details.

The main new step of the proof of Theorem 3 consists in rewriting $\langle s_0 s_x \rangle - (m^*)^2$ in a suitable form; for this we use a representation due to Kunz and Souillard [33], and based on the low temperature expansion of Minlos and Sinai [34]:

Lemma 7. *For β large enough,*

$$\langle s_0 s_x \rangle - (m^*)^2 = (m^*)^2 \left(\exp\left[4 \sum_{\substack{\Gamma: \\ 0, x \in \operatorname{Int}\Gamma}} \frac{\phi^T(\Gamma)}{\Gamma!} z(\Gamma) \right] - 1 \right), \tag{5.1}$$

where the sum is over all functions $\Gamma: \{\text{closed contours } \gamma\} \to \mathbb{N}$, $\phi^T(\Gamma)$, $\Gamma!$ are defined as in Lemma 2. $z(\Gamma) = \prod_\gamma z(\gamma)^{\Gamma(\gamma)}$ and $z(\gamma) = \exp(-2\beta J|\gamma|)$, $|\gamma| =$ number of broken bonds in γ. Since γ is closed, it has an interior, denoted Intγ. *We define* IntΓ *as follows:*

Let

$$s_x(\gamma) = \begin{pmatrix} -1 & \text{if} & x \in \operatorname{Int}\gamma \\ +1 & \text{if} & x \notin \operatorname{Int}\gamma, \end{pmatrix}$$

and

$$s_x(\Gamma) = \prod_\gamma s_x(\gamma)^{\Gamma(\gamma)}.$$

Then $x \in \operatorname{Int}\Gamma$, by definition, if $s_x(\Gamma) = -1$.

Proof. First of all, if β is large, we can work with the $\langle\,\rangle^+$ state obtained with $+$ boundary conditions instead of the $\langle\,\rangle$ state (they coincide in the thermodynamic limit on even correlation functions). Then,

$$Z_\Lambda^+ = c(\Lambda) \sum_{\Gamma \subset \Lambda} z(\Gamma) \prod_{\gamma, \gamma' \in \Gamma} (1 + g(\gamma, \gamma')),$$

where $c(\Lambda)$ is a Λ-dependent constant. The sum runs over maps satisfying $\Gamma! = 1$;

$$\langle s_A \rangle_\Lambda^+ = \frac{\sum_{\Gamma \subset \Lambda} \phi(\Gamma) z(\Gamma) s_A(\Gamma)}{\sum_{\Gamma \subset \Lambda} \phi(\Gamma) z(\Gamma)},$$

with

$$\begin{aligned} \phi(\Gamma) &= \prod_{\gamma,\gamma'\in\Gamma} (1+g(\gamma,\gamma')), \\ g(\gamma,\gamma') &= \begin{cases} -1 & \text{if } \gamma\cap\gamma' \neq \emptyset \\ 0 & \text{otherwise}, \end{cases} \\ s_A(\Gamma) &= \prod_\gamma s_A(\gamma)^{\Gamma(\gamma)}, \\ s_A(\gamma) &= \prod_{x\in A} s_x(\gamma). \end{aligned} \tag{5.2}$$

If one uses the multiplicative form of $s_A(\gamma)$, (5.2), one gets:

$$\frac{\langle s_0 s_x\rangle^+}{\langle s_0\rangle^+ \langle s_x\rangle^+} = \exp\left(\sum_\Gamma \frac{\phi^T(\Gamma)}{\Gamma!} z(\Gamma)\,(s_0s_x(\Gamma) - s_0(\Gamma) - s_x(\Gamma) + 1)\right),$$

and since

$$s_0s_x(\Gamma) - s_0(\Gamma) - s_x(\Gamma) + 1 = (s_0(\Gamma)-1)\,(s_x(\Gamma)-1)$$

is non-zero only if $0, x\in \operatorname{Int}\Gamma$, we have proven (5.1).

Remark. This lemma holds also for $d=2$.

Proof of Theorem 3. Since $0<m^*<1$ for β large, and since $\exp y - 1 \sim y$ if $y\to 0$, it is enough to show that

$$\sum_{\substack{\Gamma:\\ 0,(t,0)\in\operatorname{Int}\Gamma}} \frac{\phi^T(\Gamma)}{\Gamma!} z(\Gamma) \sim \frac{\exp(-mt)}{t^{(d-1)/2}}.$$

Since this formula is similar to the one of Lemma 2, the proof will follow the same pattern. However, the contours are not made of lines and we have to perform a slightly different geometrical analysis. A contour in $d=3$ is a connected set of plaquettes (elementary squares) crossing the broken bonds of the configuration. If p is such a plaquette we write

$$\gamma(p) = \begin{cases} 1 & \text{if } p\in\gamma \\ 0 & \text{if } p\notin\gamma, \end{cases}$$

and

$$\Gamma(p) = \sum_\gamma \Gamma(\gamma)\gamma(p).$$

A plaquette is *regular* if
- $\Gamma(p)=1$
- p is parallel to the t axis
- there are only 4 plaquettes (including p) having the same projection on the t axis.

A connected set of regular plaquettes is a *regular tube*. In order to motivate such definitions, consider the smallest contour γ such that $\operatorname{Int}\gamma \ni 0$, $(t,0)$.

Let us decompose $\Gamma = \{p \mid \Gamma(p) \geqq 1\}$ into maximally connected regular tubes and maximally connected sets of irregular plaquettes. The latter are called *excitations*. From now on, the proof simply repeats the one of Theorem 1.

In order to estimate the size of the upper gap, we must find the excitations ε such that $t^-(\varepsilon)<0$ and $t^+(\varepsilon)\geqq t$ with the lowest possible energy; these are given by long tubes whose intersection with each plane $\mathbb{Z}^2$ perpendicular to the time axis, between 0 and t, is a rectangle made of two adjacent plaquettes. The weight of such an excitation is: $\zeta(\varepsilon)\sim O(e^{-(6\beta-4\beta)t})$. Therefore the sum over these excitations [corresponding to $f(0,t)$ in Lemma 6] is $O(e^{-2\beta t})$. So $m_0\sim 4\beta$ and $m_1\sim(4+2)\beta=6\beta$, which justifies (2.8).

VI. Proof of Theorem 4

a) The analysis of $P(t)$ is quite similar to the one of the sum over contours in the preceding section:

$$P(t)=\sum_{\mathbf{x}\in\mathbb{Z}^{d-1}} G_\tau(p(0),p(t,\mathbf{x}))=\sum_{\mathbf{x}\in\mathbb{Z}^{d-1}}\ \sum_{\substack{S:\\ \partial S=p(0)\cup p(t,\mathbf{x})}} e^{-\tau|S|}.$$

We decompose each surface into regular tubes and excitations as in Sect. 5. Since we sum over $\mathbf{x}$, the sum over excitations is unconstrained and we have

$$P(t)=e^{-4\tau t}\exp\left(\sum_E{}^{0,t}\frac{\phi^T(E)\zeta(E)}{E!}\right).$$

By an argument similar to the proof of Lemma 6,

$$\sum{}^{0,t}\frac{\phi^T(E)\zeta(E)}{E!}=tP+c+f(0,t)\,, \tag{6.1}$$

where P is the pressure of the gas of excitations, c is the boundary term and $f(0,t)$ is a sum over excitations such that $\pi(E)\supset[0,t]$ [see (3.14)]. Of course,

$$e^{-m_0t}=e^{-4\tau t+tP}\,, \tag{6.2}$$

and $|f(0,t)|\leqq O(e^{-2t\tau})$, as in the Ising case. However, we can do better than just bound $f(0,t)$. We shall decompose the E's entering the sum in $f(0,t)$ into new "regular parts" and new "excitations" and then use our previous expansion another time to show that

$$f(0,t)\cong c_1e^{-m_1t}+O(e^{-m_2t})\,, \tag{6.3}$$

which, inserted in (6.1) together with suitable estimates on m_1 and m_2, proves a).

As we mentioned in the proof of Theorem 3, the excitations with lowest energy having the property that $\pi(E)\supset[0,t]$ consist of rectangular tubes; we define a plaquette to be *regular*, in the new sense of the word, if it is parallel to the time axis and if there are only six plaquettes (including itself) having the same projection on the time axis. Then one has new excitations defined as connected sets of irregular plaquettes.

Given these definitions, we can apply our expansion to $f(0,t)$; it is completely similar to our previous proofs and it leads to (6.3).

b) We can just repeat for $R_{v+h}(t)$ the analysis made for $P(t)$. The regular pieces are tubes of 4 plaquettes parallel to the time axis. Physically, one could say that the state corresponding to r_v or r_h jumps into a lower energy state represented by p. However, this is not possible for $R_{v-h}(t)$:

c) In $R_{v-h}(t)$ we have a "superselection rule" that eliminates all regular tubes made of four plaquettes parallel to the time axis. Indeed, consider a surface S containing such a tube. Construct the surface S' which is identical to S to the left of that tube and including it, but where everything to the right of that tube is rotated by $\frac{\pi}{2}$. Then, if S contributes to the sum defining $G_\tau(r_v(0), r_v(t, \mathbf{x}))$, S' contributes to $G_\tau(r_v(0), r_h(t, \mathbf{x}))$ and vice-versa, so they cancel each other in R_{v-h}. Thus we are left with surfaces with no regular piece in the sense of a 4 plaquettes tube.

Now we define a new regular part as a tube of 6 plaquettes (just like in the expansion giving the bound state in $P(t)$ above). However, we still have to keep track of the minus sign in the definition of R_{v-h}. In order to achieve this we assign a $+$ or $-$ sign to each excitation as follows:

By definition, there is a regular tube to the left and one to the right of each excitation. If they have the same orientation (both horizontal or both vertical) we assign a $+$ sign to the excitation. If they have different orientations, we put a minus sign. Making the product over the excitations gives the correct sign since there must be an even number of changes of orientation in $G_\tau(r_v(0), r_v(t, \mathbf{x}))$ and an odd number in $G_\tau(r_v(0), r_h(t, \mathbf{x}))$. From then on we can simply use our method to finish the proof of c).

The estimate on the difference $m_1 - m_1'$ between the two bound state masses follows from a simple calculation to first order in perturbation theory:

The mass m_1 and m_1' are both given, to leading order, by 6τ. The corresponding pressures P, P' have a convergent Mayer expansion in $e^{-\tau}$. Let us compute the first term in both expansions. They come from the smallest excitations: Remembering that the regular tubes here consist of 6 plaquettes, we see that the smallest excitations consist of 2 plaquettes connecting two tubes; one being rotated by $\frac{\pi}{2}$ or translated upward or downward by one unit with respect to the other. However, the difference between P and P' comes from the – sign that is given to a rotation of $\frac{\pi}{2}$.

Doing the arithmetic (there are 6 such excitations, two translation and 4 rotations) gives

$$P \cong 6e^{-2\tau},$$
$$P' \cong (2-4)e^{-2\tau} = -2e^{-2\tau}.$$

And, therefore, to leading order in $e^{-\tau}$, $|m_1 - m_1'| \sim 8e^{-2\tau}$.

VII. Concluding Remarks

1) In the two dimensional Ising model at low temperatures, it is well known that the Ornstein-Zernike decay fails. The power-law correction is t^{-2} and not $t^{-1/2}$. This is fairly easy to understand from the point of view of our method. In the contour expansion (5.1), all contributions come from contours containing 0 and x. In two dimensions, such a contour is approximately given by two lines joining 0 and x. The two lines fluctuate independently except for the fact that they cannot cross each other. Imposing the constraint that one line (or rather its projection on the vertical line perpendicular to the t axis) comes back to the origin after a "time" t induces a decay like $t^{-1/2}$, just like in the high-temperature situation. Now, if we

look at the random process given by the difference between both lines, it feels the constraint of coming back to the origin *for the first time* after time t (because the two lines cannot cross each other): This produces a decay factor $t^{-3/2}$. The product of both factors (the two conditions are essentially independent) gives t^{-2}. It is easy to understand that the addition of a next nearest neighbour coupling or of an external field restores the Ornstein-Zernike decay: in this situation the two lines become bound together and fluctuate as one single random line, thus producing a $t^{-1/2}$ correction to the exponential decay (see also [35, 11]).

2) The first remark can be extended to the analysis of the energy-energy correlation function for β small in all dimensions. Using the high-temperature expansion, one sees that the dominant contributions come from two non-intersecting random lines joining 0 and t. In two dimensions, this should, for the reasons explained in Remark 1, produce a t^{-2} decay. This is the exact behaviour [22] (the relation between this decay and the low temperature one for the two-point function was analyzed in [36] using duality).

However, the only interaction between the lines is a repulsive one and moreover, it is a kind of point-like interaction. Therefore we expect that this interaction plays no role in dimensions d greater than 3 and the two lines fluctuate as if they were independent: we have to consider $d-1$ dimensional random walks obtained by projecting the random lines on the "space" hyperplane perpendicular to the "time" axis. In quantum mechanics a repulsive point interaction plays no role in dimensions greater than or equal to two [37] (i.e. here $d-1 \geqq 2$). Therefore we expect a decay of the energy-energy correlation function just like the square of the two-point function when $d \geqq 4$ (this is the content of Theorem 2) and possibly logarithmic corrections in the "critical" dimension 3 (since we are on a lattice).

This is in agreement with Polyakov's calculation predicting a t^{-d+1} power law for $d \geqq 4$ and a $t^{-2}(\ln t)^{-2}$ decay for $d=3$. Another way to understand this result is simply to count a $t^{-(d-1)/2}$ decay because one walk has to come back to the origin at time t, and to multiply it by the appropriate decay factor due to the fact that the difference of the two walks comes back to the origin *for the first time* at time t: this event has a probability of order $t^{-(d-1)/2}$ for $d-1 \geqq 3$ because the walk is transient and the constraint that a return be the first return plays no role asymptotically. However, for $d=3$ we get a decay $t^{-1}(\ln t)^{-2}$ for this first return. This, multiplied by t^{-1} again leads to Polyakov's result. However, in this case, we have no proof.

There is a striking similarity between these power-law corrections and the power-law decay at the critical point in at least two respects: there is a critical dimension, equal to three here, above which mean field theory is correct and which is characterized by logarithmic corrections to mean field theory. Moreover, the above discussion shows the relevance of intersection properties of random walks, a typical feature of critical point theory. However, here we deal with intersections of two walks at the same time, while it appears that the intersection of the paths is the relevant quantity for critical phenomena [23, 24].

Although one is dealing with a high-temperature situation where the theory is massive and non-critical, the similarity with critical phenomena is as follows: the high-temperature lines joining 0 and x have a weight that is exponentially decreasing with their length, and this produces the mass. However, the power-law corrections, as our analyses have demonstrated, have their origin in the transverse fluctuations of the random line. These are not massive in any sense, and the question of their Gaussian or non-Gaussian nature is equivalent to the Ornstein-Zernike (or mean field) decay or to violations of this decay.

3) In this remark, of a more technical character, we relate our method to the one of Gallavotti [12]. Instead of using a complete expansion, as we did in Lemma 2, we could resum part of the expansion and write:

$$\langle \mathbf{s}_0 \cdot \mathbf{s}_x \rangle = \sum_{\omega : 0 \to x} \mathfrak{z}(\omega)$$

(see [11]) where

$$\mathfrak{z}(\omega) \geqq 0, \quad \mathfrak{z}(\omega) = \exp(-c(\beta)|\omega| + U(\omega)),$$

with $c(\beta) = |\ln \beta|$, and $U(\omega)$ can be written, using the high-temperature expansion and the Möbius inversion formula, as a sum of many-body potentials representing the interactions in the gas of excitations. These potentials are nicely decaying and small for β small. The factor $\exp(-c(\beta)|\omega|)$ gives a small activity to the excitations, and thus we obtain a convergent Mayer expansion for this gas. This is closer technically to Gallavotti's paper [12].

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