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# Statistical Mechanics of Lattice Systems

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**Abstract.** We study the thermodynamic limit for a classical system of particles on a lattice and prove the existence of infinite volume correlation functions for a “large” set of potentials and temperatures.

## § 1. Introduction and Notations

In this article we shall study the statistical mechanics of a classical system on a  $\nu$ -dimensional lattice  $Z^\nu$ . We assume that at each lattice point there can be either 0 or 1 particle. We suppose that the particles interact through symmetric translationally invariant many body potentials  $\Phi^{(k)}(x_1 \dots x_k)$ . Let  $X = \{x_1, \dots, x_N\}$  be a finite subset of  $Z^\nu$ , then the potential energy  $U$  of  $N$  particles located at  $x_1, x_2, \dots, x_N$  is:

$$U(X) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{\# \\ (i_1, \dots, i_k) \in \{1, \dots, N\}}} \Phi^{(k)}(x_{i_1}, \dots, x_{i_k}) \quad (1)$$

where  $\sum^{\#}$  extends over all  $k$ -ples  $i_1, \dots, i_k$  of distinct indices (between 1 and  $N$ ); in particular  $U(\emptyset) = 0$ . We shall consider only interactions  $\Phi = (\Phi^{(k)})_{k \geq 1}$  such that

$$\|\Phi\| = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{0 \neq x_1, \dots, x_{k-1} \in Z^\nu} |\Phi^{(k)}(0, x_1, \dots, x_{k-1})| < +\infty \quad (2)$$

where the second sum extends over all  $(k-1)$ -ples  $x_1, \dots, x_{k-1}$  of distinct lattice points different from the origin 0 of  $Z^\nu$ . With respect to the norm (2) the set  $\mathcal{B}$  of interactions  $\Phi$  such that  $\|\Phi\| < +\infty$  is a (real) Banach space.

## § 2. Definitions and Inequalities

From (1) and (2) we deduce the following stability property:

$$|U(\{x_1, \dots, x_N\})| \leq N \|\Phi\|. \quad (3)$$

We define a subspace  $\mathcal{B}'$  of  $\mathcal{B}$  by

$$\mathcal{B}' = \{\Phi \in \mathcal{B} : \Phi^{(1)} = 0\}.$$

We may write  $\Phi = (-\mu, \Phi')$  for every  $\Phi \in \mathcal{B}$  with  $\mu = -\Phi^{(1)}$  and  $\Phi' \in \mathcal{B}'$ . We interpret  $\mu$  as chemical potential and denote by  $U'$  the potential energy corresponding to  $\Phi'$ .

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If  $\beta$  is the inverse temperature, the grand partition function corresponding to the region (finite set)  $\mathcal{A}$  is then, if  $N(X)$  is the number of points of  $X$ :

$$\Xi(\beta, \mu, \Phi') = \sum_{X \subseteq \mathcal{A}} e^{-\beta U(X)} = \sum_{X \subseteq \mathcal{A}} e^{\beta \mu N(X)} e^{-\beta U'(X)}. \quad (4)$$

It is notationally convenient to define

$$Z_{\mathcal{A}}(\Phi) = \sum_{X \subseteq \mathcal{A}} e^{-U(X)} \quad (5)$$

and, if  $V(\mathcal{A})$  is the number of points of  $\mathcal{A}$ :

$$P_{\mathcal{A}}(\Phi) = V(\mathcal{A})^{-1} \log Z_{\mathcal{A}}(\Phi).$$

Then

$$\Xi(\beta, \mu, \Phi') = Z_{\mathcal{A}}(\beta(-\mu, \Phi')); \quad Z_{\mathcal{A}}(\Phi^{(1)}, \Phi') = \Xi(1, -\Phi^{(1)}, \Phi'). \quad (6)$$

**Proposition 1.** *If  $\Phi', \Psi' \in \mathcal{B}'$  then*

$$Z_{\mathcal{A}}(\Phi^{(1)} + \|\Psi'\|, \Phi') \leq Z_{\mathcal{A}}(\Phi^{(1)}, \Phi' + \Psi') \leq Z_{\mathcal{A}}(\Phi^{(1)} - \|\Psi'\|, \Phi'). \quad (7)$$

Let indeed  $V'$  be the potential energy corresponding to  $\Psi'$  then

$$\begin{aligned} (\Phi^{(1)} - \|\Phi'\|) N(X) + U'(X) &\leq \Phi^{(1)} N(X) + U'(X) + V'(X) \leq \\ &\leq (\Phi^{(1)} + \|\Psi'\|) N(X) + U'(X). \end{aligned}$$

Where  $N(X)$  is the number of points of  $X$ . The result then follows taking the exponentials and summing over  $X$ .

**Proposition 2.** *If  $\Phi \in \mathcal{B}$ ,*

$$\log(1 + e^{-\Phi^{(1)} - \|\Phi'\|}) \leq P_{\mathcal{A}}(\Phi) \leq \log(1 + e^{-\Phi^{(1)} + \|\Phi'\|}). \quad (8)$$

This is obtained from proposition 1 by taking  $\Phi' = 0$ .

**Proposition 3.** *The function  $\Phi \rightarrow P_{\mathcal{A}}(\Phi)$  is convex on  $\mathcal{B}$ .*

The proof is standard<sup>1</sup> and will be omitted.

**Proposition 4.** *If  $\Phi' \in \mathcal{B}'$ , the following inequality holds:*

$$|P_{\mathcal{A}}(\Phi^{(1)} - \lambda, \Phi') - P_{\mathcal{A}}(\Phi^{(1)} + \lambda, \Phi')| \leq 2\lambda \quad \forall \Phi^{(1)} \in R, \quad \forall \lambda \geq 0. \quad (9)$$

This follows from the fact that the derivative of  $P_{\mathcal{A}}(\Phi^{(1)}, \Phi')$  with respect to  $\Phi^{(1)}$  is the expectation value of  $[-N(X)/V(\mathcal{A})]$ :

$$\frac{dP_{\mathcal{A}}(\Phi^{(1)}, \Phi')}{d\Phi^{(1)}} = - \frac{\sum_{X \subseteq \mathcal{A}} e^{-U(X)} N(X)}{Z_{\mathcal{A}}(\Phi) V(\mathcal{A})}$$

and is therefore contained in the interval  $[-1, 0]$ .

### § 3. Existence and Properties of the Thermodynamic Limit

Let  $\mathcal{B}_0 \subset \mathcal{B}$  consist of those  $\Phi$  which have finite range i. e.  $\Phi^{(k)} \equiv 0$  for  $k$  sufficiently large and  $\Phi^{(k)}(0, x_1, \dots, x_{k-1})$  vanishes except for a finite number of values of  $x_1, \dots, x_{k-1}$ .

<sup>1</sup> It follows from the convexity criteria for many variables functions and the Schwartz inequality.

**Proposition 5.** *If  $\Phi \in \mathcal{B}_0$ , the following limit exists*

$$P(\Phi) = \lim_{\Lambda \rightarrow \infty} P_\Lambda(\Phi). \tag{10}$$

This result is well known (see [1], [2], [3], [4], [5]).

In this proposition and in the following  $\Lambda$  may be taken a parallelo-piped and  $\Lambda \rightarrow \infty$  means that each side of  $\Lambda$  tends to  $\infty$ . It is also possible to let  $\Lambda$  to go to  $\infty$  in a more general manner (see [6], [7] for a definition of Van-Hove convergence to  $\infty$ ).

**Theorem 1.** *If  $\Phi \in \mathcal{B}$ , the following limit exists*

$$P(\Phi) = \lim_{\Lambda \rightarrow \infty} P_\Lambda(\Phi) \tag{11}$$

and satisfies the following properties

i) if  $\Phi', \Psi' \in \mathcal{B}'$  then

$$P(\Phi^{(1)} + \|\Psi'\|, \Phi') \leq P(\Phi^{(1)}, \Phi' + \Psi') \leq P(\Phi^{(1)} - \|\Psi'\|, \Phi'); \tag{12}$$

$$\text{ii) } \log(1 + e^{-\Phi^{(1)} - \|\Phi'\|}) \leq P(\Phi) \leq \log(1 + e^{-\Phi^{(1)} + \|\Phi'\|}); \tag{13}$$

iii) the functional  $P(\cdot)$  is convex and continuous on the Banach space  $\mathcal{B}$ .

Let  $\Phi'_n \in \mathcal{B}_0$  be such that  $\lim_{n \rightarrow \infty} \|\Phi'_n - \Phi'\| = 0$ . From proposition 1 we obtain:

$$P_\Lambda(\Phi^{(1)} + \|\Phi' - \Phi'_n\|, \Phi'_n) \leq P_\Lambda(\Phi) \leq P_\Lambda(\Phi^{(1)} - \|\Phi' - \Phi'_n\|, \Phi'_n). \tag{14}$$

On the other hand  $Z_\Lambda(\Phi^{(1)}, \Phi')$  is a decreasing function of  $\Phi^{(1)}$  so that:

$$P_\Lambda(\Phi^{(1)} + \|\Phi' - \Phi'_n\|, \Phi'_n) \leq P_\Lambda(\Phi^{(1)}, \Phi'_n) \leq P_\Lambda(\Phi^{(1)} - \|\Phi' - \Phi'_n\|, \Phi'_n) \tag{15}$$

from proposition 4, the difference between extreme terms in (14) and (15) is bounded by  $2\|\Phi' - \Phi'_n\|$ , hence

$$|P_\Lambda(\Phi) - P_\Lambda(\Phi^{(1)}, \Phi'_n)| \leq 2\|\Phi' - \Phi'_n\| \tag{16}$$

from this it follows that

$$\lim_{n \rightarrow \infty} P_\Lambda(\Phi^{(1)}, \Phi'_n) = P_\Lambda(\Phi) \tag{17}$$

uniformly in  $\Lambda$ . On the other hand by proposition 5

$$\lim_{\Lambda \rightarrow \infty} P_\Lambda(\Phi^{(1)}, \Phi'_n) = P(\Phi^{(1)}, \Phi'_n). \tag{18}$$

The existence of the limits (17) and (18) and the uniformity of (17) imply the existence of

$$\lim_{\Lambda \rightarrow \infty} P_\Lambda(\Phi) = \lim_{\Lambda \rightarrow \infty} \lim_{n \rightarrow \infty} P_\Lambda(\Phi^{(1)}, \Phi'_n). \tag{19}$$

This proves (11), (i) follows then from proposition 1; (ii) from proposition 2; the convexity of  $P(\cdot)$  implies its continuity in  $\Phi^{(1)}$  and then by (i) its continuity in  $\Phi$  follows.

*Remark.* From the above theorem we have the existence of

$$\beta p(\beta, \mu, \Phi') = \lim_{\Lambda \rightarrow \infty} V(\Lambda)^{-1} \log \Xi(\beta, \mu, \Phi') \tag{20}$$

where  $p$  is the thermodynamique pressure:

$$p(\beta, \mu, \Phi') = \beta^{-1} P(\beta(-\mu, \Phi')) . \tag{21}$$

### § 4. Existence of Correlation Functions

Let  $\Lambda$  be a finite subset of  $Z^v$ ,  $\Phi \in \mathcal{B}$ , the correlation function is defined by:

$$\varrho_{\Phi, \Lambda}(X) = Z_{\Lambda}(\Phi)^{-1} \sum_{\substack{Y \subseteq \Lambda \\ Y \cap X = \emptyset}} e^{-U(X \cup Y)} \tag{22}$$

if  $X \subseteq \Lambda$  and  $\varrho_{\Phi, \Lambda}(X) = 0$  otherwise. By averaging over translations we get

$$\bar{\varrho}_{\Phi, \Lambda}(\{x_1 \dots x_n\}) = V(\Lambda)^{-1} \sum_{X \in Z^v} \varrho_{\Phi, \Lambda}(x_1 + x, \dots, x_n + x) \tag{23}$$

so that if  $\Psi \in \mathcal{B}$  with corresponding potential energy  $V$ :

$$\begin{aligned} & \sum_{n \geq 1} \frac{1}{n!} \sum_{0 + x_1 \dots x_n \in Z^v} \bar{\varrho}_{\Phi, \Lambda}(0, x_2 \dots x_n) \Phi^{(n)}(0, x_2 \dots x_n) \\ &= V(\Lambda)^{-1} \sum_{X \neq \emptyset} \varrho_{\Phi, \Lambda}(X) \Psi(X) = Z_{\Lambda}(\Phi)^{-1} V(\Lambda)^{-1} \sum_{\substack{\Phi + X; Y \subseteq \Lambda \\ X \cap Y = \emptyset}} e^{-U(X \cup Y)} \Psi(X) \\ &= Z_{\Lambda}(\Phi)^{-1} V(\Lambda)^{-1} \sum_{X \subseteq \Lambda} e^{-U(X)} V(X) . \end{aligned} \tag{24}$$

Let  $T \subset \mathcal{B}$  be the set of  $\Phi$  such that the graph of  $P$  has a unique tangent plane at  $\Phi$ , i. e. there exists a unique  $\alpha_{\Phi}$  in the dual  $\mathcal{B}^*$  of  $\mathcal{B}$  such that for all  $\Psi \in \mathcal{B}$

$$P(\Phi + \Psi) \geq P(\Phi) - \alpha_{\Phi}(\Psi) \tag{25}$$

we note that  $\alpha_{\Phi}(\Psi)$  can be interpreted as the functional derivative of  $P(\Phi)$  in the direction  $\Psi$  [8].

**Theorem 2.** *If  $\Phi \in T$  then if  $V$  is the potential energy associated with any  $\Psi \in \mathcal{B}$  the limit*

$$\lim_{\Lambda \rightarrow \infty} Z_{\Lambda}(\Phi)^{-1} V(\Lambda)^{-1} \sum_{X \subseteq \Lambda} e^{-U(X)} V(X) = \alpha_{\Phi}(\Psi) \tag{26}$$

*exists and defines an element  $\alpha_{\Phi} \in \mathcal{B}^*$ ; the following limit therefore exists:*

$$\lim_{\Lambda \rightarrow \infty} \bar{\varrho}_{\Phi, \Lambda}(X) = \bar{\varrho}_{\Phi}(X) \tag{27}$$

*and defines the infinite volume correlation function  $\bar{\varrho}_{\Phi}$ .*

For finite  $\Lambda$ , the function  $P_{\Lambda}(\cdot)$  has a unique tangent plane at any  $\Phi \in \mathcal{B}$  corresponding to  $\alpha_{\Phi, \Lambda} \in \mathcal{B}^*$ :

$$\alpha_{\Phi, \Lambda}(\Psi) = Z_{\Lambda}(\Phi)^{-1} V(\Lambda)^{-1} \sum_{X \subseteq \Lambda} e^{-U(X)} V(X) . \tag{28}$$

From (3) it is clear that  $|\alpha_{\Phi}^{(\Psi)}| \leq \|\Psi\|$ , i. e.  $\|\alpha_{\Phi A}\| \leq 1$ . Let  $A$  be a total sequence in  $\mathcal{B}$  ( $\mathcal{B}$  is separable), one can choose a sequence  $A_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $\alpha_{\Phi, A_n}(\Psi)$  converges for every  $\Psi \in A$ . Since  $\|\alpha_{\Phi, A_n}\| \leq 1$ ,  $\alpha_{\Phi, A_n}$  converges weakly.

Let  $(\Phi + \Psi, \xi)$  be a point strictly above the graph of  $P$  in  $\mathcal{B} \times R$ , then for large  $A$ ,  $(\Phi + \Psi, \xi)$  is above the graph of  $P_A$  and therefore of  $\alpha_{\Phi A}$ : in particular if  $\alpha_{\Phi}$  is the limit of  $\alpha_{\Phi, A_n}$

$$\xi = P(\Phi) - \alpha_{\Phi}(\Psi) + \alpha_{\Phi}(\Phi) \tag{29}$$

is the equation of a tangent plane to  $P$  at  $\Phi$ . If  $\Phi \in T$ , the tangent plane is unique, therefore

$$\text{weak } \lim_{A \rightarrow \infty} \alpha_{\Phi A} = \alpha_{\Phi} . \tag{30}$$

*Remark.* If  $-\frac{dP(\Phi + \lambda\Psi)}{d\lambda} \Big|_{\lambda=0} = \alpha_{\Phi}(\Psi)$  exists for a certain  $\Psi$  then

$$\lim_{A \rightarrow \infty} \alpha_{\Phi A}(\Psi) = \alpha_{\Phi}(\Psi) . \tag{31}$$

We note also that the existence of  $\frac{dP(\Phi + \lambda\Psi)}{d\lambda} \Big|_{\lambda=0}$  for  $\Psi$  in a total set is a necessary and sufficient condition for the existence of a unique tangent plane at  $\Phi$ .

These results follow by inspection of the proof of the above theorem. We conclude with the following:

**Theorem 3.** i) *The set  $T$  contains a countable intersection of dense open subsets of  $\mathcal{B}$  and therefore is dense (Baire theorem [9]).*

ii) *There exists a dense subset  $T'$  of  $\mathcal{B}'$  such that for  $\Phi' \in T'$  and almost every  $(\beta, \mu) \in R_+ \times R$  the point  $\beta(-\mu, \Phi') \in T$ .*

i) follows by inspection of the proof of reference [10].

Let  $e_n$  be a base of normalized vectors on the space  $\mathcal{B}'$  [11]. Let  $\Phi'_{(0)}$  be an arbitrary point of  $\mathcal{B}'$  and let  $\{C_n\}$  be a sequence of positive numbers such that  $\sum_{n=1}^{\infty} C_n < +\infty$ . Let  $K$  be the set

$$K = \{\Phi' \in \mathcal{B}' : |\Phi'_n - \Phi'_{(0)n}| < C_n\} \tag{32}$$

where  $\Phi'_n, \Phi'_{(0)n}$  are the components of  $\Phi', \Phi'_{(0)}$  along  $e_n$ .

Let us consider the space  $R_+ \times R \times K$  of the variables  $(\beta, \Phi^{(1)}, \Phi')$  as a topological space with the topology product of the natural topologies on  $R_+$  and  $R$  and the relative topology on  $K$  as a subset of  $\mathcal{B}'$  (it is easy to see that this topology on  $K$  is identical with the product topology on  $K$  considered as  $\prod_{n=1}^{\infty} I_n$  where  $I_n = (-C_n, +C_n)$ ).

Let us introduce on  $R_+ \times R$  a normalized measure  $g(d\beta d\Phi^{(1)})$  equivalent to the Lebesgue measure and on  $K$  the measure  $\gamma(d\Phi')$

$$= \prod_{n=1}^{\infty} \frac{d\Phi'_n}{2C_n} .$$

Let  $\mu = g \times \gamma$  be the product measure of  $g$  and  $\gamma$  defined

on (the Borel sets of)  $R_+ \times R \times K$ . It is convenient to introduce the vector  $e_0 = (1, 0) \in \mathcal{B}$ .

Now the set  $B_n$  of points  $(\beta, \Phi^{(1)}, \Phi') \in R_+ \times R \times K$  where the derivative  $\frac{d}{d\lambda} P(\beta(\Phi + \lambda e_n))$  does not exist is a Borel set of  $R_+ \times R \times K$  since

$$B_n = \bigcup_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{|s|, |t| > N} C_{kst}^n, \quad n = 0, 1, \dots \tag{33}$$

where  $k, N$  are positive integers and  $s, t$  are integers and

$$C_{kst}^n = \left\{ (\beta, \Phi^{(1)}, \Phi') \in R_+ \times R \times K : \left| \frac{P(\beta(\Phi + 1/t e_n)) - P(\beta\Phi)}{1/t} - \frac{P(\beta(\Phi + 1/s e_n)) - P(\beta\Phi)}{1/s} \right| \geq \frac{1}{k} \right\} \tag{34}$$

Applying the pointwise Fubini-Jessen theorem [12] we get

$$\begin{aligned} \mu(B_n) &= \int_{R_+ \times R \times P} \chi_{B_n}(s) \mu(ds) \\ &= \lim_{M \rightarrow \infty} \int \cdots \int \chi_{B_n}(\beta, \Phi^{(1)}, \Phi'_1, \dots, \Phi'_M, \bar{\Phi}'_{M+1}, \dots) \mu_M(d\beta d\Phi^{(1)} \dots d\Phi'_M) \end{aligned} \tag{35}$$

where  $\chi_{B_n}$  is the characteristic function of  $B_n$ ,  $\mu_M(d\beta \dots d\Phi'_M) = g(d\beta d\Phi^{(1)}) \times \prod_{m=1}^M \left( \frac{d\Phi'_m}{2C_m} \right)$  and  $\bar{\Phi}' = \sum_{m=1}^{\infty} \bar{\Phi}'_m e_m$  is a suitable point of  $K$ .

But as soon as  $M > n$  the integral in the r. h. s. of (35) is zero because of the ordinary Fubini theorem and the well known fact that a convex function depending on one variable is differentiable except for a denumerable set of points. Hence  $\mu(B_n) = 0$  and then  $\mu\left(\bigcup_{n=0}^{\infty} B_n\right) = 0$ . Let  $D$  be the complement in  $K$  of  $\bigcup_{n=0}^{\infty} B_n$  then, as a consequence of the definition of  $B_n$  and of the remark following theorem 2, at every point of  $D$  there is a unique tangent plane. Furthermore  $\mu(D) = 1$ .

From

$$\begin{aligned} 1 = \mu(D) &= \int_{R_+ \times R \times K} \chi_D(\beta, \Phi^{(1)}, \Phi') g(d\beta d\Phi^{(1)}) \gamma(d\Phi') \\ &= \int_K \gamma(d\Phi') \int_{R_+ \times R} \chi_D(\beta, \Phi^{(1)}, \Phi') g(d\beta d\Phi^{(1)}) \end{aligned} \tag{36}$$

and from the fact that all measures are normalized we get

$$\int_{R_+ \times R} \chi_D(\beta, \Phi^{(1)}, \Phi') g(d\beta d\Phi^{(1)}) = 1 \tag{37}$$

for  $\Phi'$   $\gamma$ -almost everywhere in  $K$ . Then (ii) follows from the equivalence to the Lebesgue measure of  $g$  and from the arbitrariness of the "center"  $\Phi'_0$  of  $K$  and of the dimensions  $\{C_n\}$  of  $K$ .

Theorems 2 and 3 specify in which sense the set of  $\Phi \in \mathcal{B}$  and  $\beta > 0$  for which the infinite volume correlations functions exist and are unique is large.

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