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STATISTICAL PROPERTIES OF CHAOTIC DYNAMICAL SYSTEMS WHICH EXHIBIT STRANGE ATTRACTORS

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Statistical Properties of

Chaotic Dynamical Systems which Exhibit Strange Attractors

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A path integral method is developed for the calculation of the statistical properties of turbulent dynamical systems. The method is applicable to conservative systems which exhibit a transition to stochasticity as well as dissipative systems which exhibit strange attractors. A specific dissipative mapping is considered in detail which models the dynamics of a Brownian particle in a wave field with a broad frequency spectrum. Results are presented for the low order statistical moments for three turbulent regimes which exhibit strange attractors corresponding to strong, intermediate, and weak collisional damping. In the dissipationless limit this map is equivalent to the conservative Chirikov-Taylor mapping. The turbulent behavior of the Chirikov-Taylor mapping is shown to be diffusive due to the intrinsic stochasticity; and the stochastic diffusion coefficient derived by Rechester and White is recovered. The statistical dynamics are significantly altered by the inclusion of damping which restricts the chaotic motion to a

strange attractor which is bounded in phase space. Specifically, for weak and intermediate damping the long-time dynamics are no longer diffusive; and the corrections to the random phase results for the low order moments decay away for long times. In addition the "accelerator modes" which are exhibited by the Chirikov-Taylor mapping, are destroyed by the drag. These results provide a new description of the effects of collisional damping on the stochastic heating of plasmas by electrostatic waves.

1. Introduction

Classical dynamical systems which exhibit a transition to chaos arise in a variety of physical, chemical, and biological problems.[1] Two different kinds of chaotic or turbulent behavior have been observed. For conservative or Hamiltonian systems the phase space orbits appear to wander ergodically. This motion is called stochastic.[2] For dissipative systems the trajectories are attracted to a very complicated manifold in phase space. This complex structure is called a strange attractor.[3] The chaotic dynamics of both systems are characterized by a sensitivity to initial conditions, the divergence of nearby orbits, and a positive Kolmogorof-Sinai entropy.[2,3,4]

The mechanisms for the onset of both types of turbulent behavior have been examined extensively.[5,6,7,8] It is currently believed that the onset of chaos follows a sequence of bifurcations of stable periodic orbits. These studies are important because they provide criteria for the appearance of chaotic motion.

The complexity of the dynamics after the transition to chaos suggests that a statistical description is appropriate. Although considerable work has been devoted to proving that a statistical description is valid,[4] less analytic progress has been made on the calculation of the probability distribution which describes the turbulent dynamics.

A complete theory of turbulence must provide both the conditions for onset and the means for predicting observable properties of the turbulent dynamics. The purpose of this paper is to present a path integral method for analytically calculating the statistical properties of turbulent dynamical systems. Our method is based on a very powerful functional integral approach to classical statistical dynamics[9,10] which has been discussed in an earlier publication.[11] This formalism is applicable to a wide class of dynamical systems described by differential equations as well as maps. In our initial investigations we have restricted our attention to maps because they are easier to treat. Although systems with continuous time such as the Lorenz[12] model pose a number of technical difficulties,[13,14] we hope to consider them in future work.

We have previously applied this approach to a class of dissipative mappings which exhibit strange attractors.[15] Here we will consider a more general mapping which encompasses conservative as well as dissipative systems.

The following two-dimensional map was proposed by Zaslavskii[16,17] to model the effects of a periodic perturbation with a broad frequency spectrum on a nonlinear oscillator with a stable limit cycle

$$x_n = x_{n-1} + y_n \qquad \text{mod } 1 \tag{1}$$

$$y_n = \lambda y_{n-1} + k \sin 2\pi x_{n-1} \tag{2}$$

where $\lambda < 1$. In terms of action-angle variables, x is proportional to the angle, y is related to the action, and k is the magnitude of the perturbation.

Eqs. (1) and (2) can also be used to describe the motion of charged particles in a field of electrostatic waves. Consider the interaction of a test particle with an electric field composed of a broad spectrum of plane waves with equally spaced phase velocities.[18] The equations of motion in one dimension can be written

$$\frac{dx}{dt} = v \tag{3}$$

$$\frac{dv}{dt} = -\nu v + k \sum_{n=-N}^{N} \sin 2\pi(x - nt) + r \tag{4}$$

where k is proportional to the amplitude of the waves.

Weak collisions with background particles give rise to the drag ν and the source of random noise r . The drag due to collisions with neutrals can be assumed to be independent of v; however, for Coulomb collisions with charged particles a more complicated velocity dependence is required for ν. The collisional noise is assumed to have Gaussian statistics with zero mean and short correlation time

$$\langle r(t)r(t')\rangle = 2D\delta(t-t') \quad . \tag{5}$$

For a neutral background at temperature Θ , ν and D are related by[19]

$$D = \Theta\nu \tag{6}$$

in the absence of other dissipative forces.

If we let $N\to\infty$ Eqs. (3) and (4) can be reduced to the mapping[18]

$$x_n = x_{n-1} + v_n \tag{7}$$

$$v_n = \lambda v_{n-1} + k \sin 2\pi x_{n-1} + r_{n-1} \tag{8}$$

where $\lambda \equiv 1 - \nu$. For sufficiently large k we can neglect r_{n-1} compared with the nonlinear term; and we recover the Zaslavskii map.

The Zaslavskii map is dissipative for $\lambda < 1$. In numerical studies of Eqs. (1) and (2), Zaslavskii[16,17] found that for $k \ll 1$ all orbits are attracted to stable fixed points; however, for $k \gtrsim 1$ a strange attractor appears. In Figs. 1 and 2 we have advanced the mapping for 10^4 time steps with $\lambda = .1$ for $k = 1.4$ and $k = 11.4$, respectively. Figure 3 shows a magnified view of Fig. 1 which reveals the complex structure of the attractor.

In terms of our physical model, Brownian particles in a wave field wander randomly along a strange attractor in a bounded region of phase space for $k \gtrsim 1$ and $\lambda < 1$. The primary effect of the fluctuating part of the collisional noise, r_{n-1}, in Eq. (8) is to wash out the fine structure of the attractor.[20]

Using the path integral method we calculate new, analytic results for the first few statistical moments of the dynamics of Eqs. (1) and (2) as functions of time for three turbulent regimes corresponding to strong, intermediate, and weak damping. These results are significant because they show explicitly the effects of dissipation on the turbulent dynamics.

In the dissipationless limit, $\lambda \equiv 1$, Eqs. (1) and (2) reduce to the Chirikov-Taylor mapping which has been used extensively to model the dynamics of Hamiltonian systems.[2,5,18] For $2\pi k < .97$ the orbits in phase space are confined by preserved KAM surfaces.[6] For $2\pi k > .97$ a transition to global stochasticity occurs. In this turbulent regime the orbits of the particles in our physical model diffuse in velocity.

Rechester and White[21] have recently calculated the stochastic diffusion coefficient for the Chirikov-Taylor mapping using an analytic method similar to ours which, however, required the introduction of a small random velocity field with zero mean in Eq. (1). In the nondissipative limit $\lambda=1$, our result for the second moment of y_T averaged over initial x_0 reproduces the leading terms of their asymptotic expression for the diffusion coefficient.

Furthermore, in order to characterize the statistical dynamics by a diffusion coefficient the statistics must be Gaussian. Although Rechester and White did not calculate the higher moments, our results show that these statistical moments are approximately Gaussian for $\lambda \equiv 1$.

Recently, several papers have appeared which use similar path integral methods to study the effect of Gaussian random noise with zero mean on Hamiltonian systems.[22,23] However, these investigations neglect the drag which is generally associated with collisional noise. Our results for the Zaslavskii map show that the long-time statistical properties of the strange attractor, which characterizes the chaotic behavior of the dissipative systems, can be very different from the statistical properties of systems which include only the fluctuating part of the collisional noise. In particular, since the dynamics lie on a strange attractor which is bounded in phase space, the statistics are no longer Gaussian. Moreover, the damping destroys the "accelerator modes" which can dominate the statistical dynamics of doubly periodic, conservative systems with noise.[22]

Our calculations for the dissipative cases are not significantly changed by the inclusion of a small amount of random noise. Although our formalism can be applied directly to problems with random forces, as shown in Section 2, we have chosen to neglect the sources of random noise since they only obscure the chaotic behavior induced by the nonlinearities alone.

This work has direct applications to problems in stochastic heating of plasmas by electrostatic waves. In the long-time limit the statistical moments describe the particle distribution function which results from the balancing of the collisional drag with the nonlinear acceleration. Previous studies of the steady-state distribution function have ignored the strange attractor which characterizes the dynamics. Our results for the statistical properties of the strange attractor provide a new qualitative and quantitative description of the effects of collisional damping on the stochastic wave heating.

The path integral formalism is developed for a very general class of mappings in Section 2. In Section 3 we present our calculations for the Zaslavskii map; we compare the results for the conservative and dissipative cases; and we discuss the applications to problems in stochastic wave heating. In Section 4 we summarize our contributions.

2. Path Integral Formalism

Consider a broad class of dynamical systems defined by mappings of the form

$$\vec{x}_i = \vec{f}(\vec{x}_{i-1}), \qquad i = 1,\ 2 \ldots \qquad . \tag{9}$$

We are interested in calculating observable properties $F\left(\{\vec{x}_k\}_{k\,=\,0,\ 1,\ldots T}\right)$ of the dynamics. If we could solve Eq. (9) analytically it would be a simple matter to evaluate any functional of the dynamics $F\left(\{\vec{x}_k\}_k\right)$. However, for nonlinear maps which exhibit strange attractors or stochasticity this can only be done numerically in most cases.

The path integral formalism provides an alternative approach. Any functional of the dynamics $F\{\vec{x}_k\}_k$ can be represented by a path integral using the following identity

$$F\left(\{\vec{x}_k\}_k\right) = \prod_{i=0}^{T} \int d\vec{x}_i{}' \ \delta(\vec{x}_i{}' - \vec{x}_i) F\left(\{\vec{x}_k{}'\}_k\right) \qquad . \tag{10}$$

Since the integrands only have support for $\vec{x}_i{}'$ which are solutions to Eq. (9) with a given initial condition $\vec{x}_0$, Eq. (10) can be rewritten

$$F\left(\{\vec{x}_k\}_k\right) = \prod_{i=0}^{T} \int d\vec{x}_i{}' \ \delta\left(\vec{x}_i{}' - f(\vec{x}_{i-1}{}')\right)\delta(\vec{x}_0{}' - x_0) F\{\vec{x}_k{}'\}_k \quad . \tag{11}$$

For example, one functional of the dynamics of considerable interest is the conditional probability distribution for arriving at a point $\vec{x}$ in phase space at time T given the initial position $\vec{x}_0$

$$P(\vec{x},T|\vec{x}_0) = \delta(\vec{x} - \vec{x}_T) \quad . \tag{12}$$

Here $\vec{x}_T$ is the solution of Eq. (9) at time T. Using Eq. (11) the conditional probability can be written

$$P(\vec{x},T|\vec{x}_0) = \prod_{i=1}^{T-1} \int d\vec{x}_i \left\{ \delta\left(\vec{x} - \vec{x}_{T-1} - \vec{f}(\vec{x}_{T-1})\right) \right.$$

$$\left. \times\ \delta\left(\vec{x}_{T-1} - \vec{x}_{T-2} - f(\vec{x}_{T-2})\right) \times \ldots \times\ \delta(\vec{x}_1 - \vec{x}_0 - \vec{f}(\vec{x}_0)) \right\} \quad . \tag{13}$$

Eq. (13) can also be derived using the semi-group property of the transition probability[13,15]

$$P(\vec{x}_n|\vec{x}_k) = \int d\vec{x}_m\ P(\vec{x}_n|\vec{x}_m) P(\vec{x}_m|\vec{x}_k) \quad . \tag{14}$$

If we replace the Dirac δ functions by their Fourier transforms, Eq. (11) can be written in a form reminiscent of the path integrals which arise in quantum theories. If the range of any component x_i^α is $(-\infty,\infty)$, then

$$\delta\left(x_i^\alpha - f(x_{i-1}^\alpha)\right) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_i^\alpha\ e^{ip_i^\alpha\left[x_i^\alpha - f(x_{i-1}^\alpha)\right]} \quad ,\alpha=1,2\ldots \quad . \tag{15}$$

If the range of x_i^α is periodic or is bounded on a finite interval, for example $(0,2\pi)$, or is periodic, then the corresponding $\delta\left(x_i^\alpha - f(x_{i-1}^\alpha)\right)$ can be replaced by

$$\sum_{n=-\infty}^{\infty} \delta\left(x_i^\alpha - f(x_{i-1}^\alpha) - 2\pi n\right) \cong \frac{1}{2\pi} \sum_{p_i^\alpha=-\infty}^{\infty} e^{ip_i^\alpha\left[x_i^\alpha - f(x_{i-1}^\alpha)\right]} \quad . \tag{16}$$

Using either Eq. (15) or (16) we can then express any functional of the dynamics as

$$F\left(\{\vec{x}_k\}_k\right) = \prod_{j=0}^{T} \int d\vec{x}\,'_j \left\{ \prod_{i=1}^{T} \int \frac{d\vec{p}_i}{V} e^{i\vec{p}_i \cdot \left[\vec{x}\,'_i - \vec{f}(\vec{x}\,'_{i-1})\right]} \right.$$

$$\left. \times\, \delta(\vec{x}_0\,' - \vec{x}_0) \right\} F\left(\{\vec{x}_k\,'\}_k\right) \tag{17}$$

where for notational convenience we use $\int d\vec{p}_i$ to represent both the integral and the sum. The normalization constant is $V = (2\pi)^d$ where d is the number of components of $\vec{x}$.

In particular, the conditional probability is given by

$$P(\vec{x},T|\vec{x}_0) = \prod_{j=1}^{T-1} \int d\vec{x}_j \left\{ \prod_{i=1}^{T} \int \frac{d\vec{p}_i}{V} e^{i\vec{p}_T \cdot \left[\vec{x} - \vec{f}(\vec{x}_{T-1})\right]} \right.$$

$$\left. \times\, e^{i \sum_{i=1}^{T-1} \vec{p}_i \cdot \left[\vec{x}_i - \vec{f}(\vec{x}_{i-1})\right]} \right\} \quad . \tag{18}$$

Reordering the indices and interchanging the order of integration, Eq. (18) can be rewritten as

$$P(\vec{x},T|\vec{x}_0) = \int \frac{d\vec{p}_T}{V} e^{i\vec{p}_T \cdot \vec{x}} \, C(\vec{p}_T|\vec{x}_0) \tag{19}$$

where the characteristic function is

$$C(\vec{p}_T|\vec{x}_0) \equiv \prod_{i=1}^{T-1} \int \frac{d\vec{p}_i}{V} \left\{ \int d\vec{x}_i \; e^{i\vec{p}_i\cdot\vec{x}_i - i\vec{p}_{i+1}\cdot\vec{f}(\vec{x}_i)} \right\} \times e^{-i\vec{p}_1\cdot\vec{f}(\vec{x}_0)} \quad . \tag{20}$$

In Reference 15 the characteristic function, Eq. (20), served as the starting point for our calculation of the statistical properties of a strange attractor.

One advantage of the path integral representation of the characteristic function lies in the fact that averages over random forces, interactions, or initial conditions can be easily performed. For example, averages with respect to initial conditions contribute a factor of $\int dx_0 \exp\{-i\vec{p}_1\cdot \vec{f}(\vec{x}_0)\}$ to the integrand of the characteristic function, Eq. (20). Then the statistical moments $\langle\vec{x}_T{}^n\rangle$ are derived by differentiating the averaged characteristic function n times with respect to $\vec{p}_T$ and setting $\vec{p}_T = 0$.

Furthermore, if we add a random force $\vec{r}_i$ to Eq. (9), then the integrand of the characteristic function is modified by a factor of $\exp\{-i\vec{p}_i\cdot\vec{r}_i\}$. Averages of the characteristic function over the distribution of $\vec{r}_i$ act only on this factor. If the statistics are Gaussian with zero mean and covariance $\overleftrightarrow{\sigma}_i$, then this average gives

$$\langle e^{-i\vec{p}_i\cdot\vec{r}_i}\rangle = e^{-\frac{1}{2}\vec{p}_i\cdot\overleftrightarrow{\sigma}_i\cdot\vec{p}_i} \quad . \tag{21}$$

The expression for the characteristic function for the Chirikov-Taylor mapping with noise, which was derived in Reference 21, follows directly from Eqs. (20) and (21).

The most important advantage of the path integral method is revealed if we perform the $\vec{x}_i$ integrations in Eq. (20). Then the characteristic function for the statistical dynamics of the mapping, Eq. (9), is expressed entirely in terms of paths in the Fourier transform space spanned by $\vec{p}_i$. The distribution of these paths in Fourier space is sharply peaked in Fourier space near $\vec{p}_i = 0$ when the dynamics are turbulent and the trajectories in real space wander chaotically over the entire range of $\vec{x}_i$. This greatly simplifies the evaluation of the characteristic function and its derivatives. We have exploited this feature of the path integral representation of turbulent dynamics to explicitly calculate the moments of the conditional probability distribution in Section 3.

If the trajectories are confined by KAM surfaces or attracted to periodic orbits, then the dynamics on these localized structures are poorly approximated by the assumption that only $\vec{p}_i$ near 0 contribute to the characteristic function. In these cases the calculation of the characteristic function requires summations or integrations over a broad range of $\vec{p}_i$, which, in general, are no longer analytically tractable. However, if random noise with zero mean is added to the mapping, numerical evaluation of the characteristic function is possible since Eq. (21) cuts off the contributions from large $\vec{p}_i$. This is the basis of the numerical path-diagram method of Rechester, Rosenbluth, and White[23] which has been used to study the turbulent diffusion coefficient in the vicinities of the stochastic transition point and the localized "accelerator islands."[22]

Our path integral formalism provides a much more general foundation for studies of the statistical properties of dynamical systems than other approaches which have only considered the conditional probability distribution.[13,21] For example, the path integral representation of functionals of the dynamics can also be used to calculate joint probabilities and multi-time correlation functions. We can define a characteristic functional $Z(\vec{\eta},\vec{\zeta})$ which provides a complete statistical dynamical description of these systems

$$Z(\{\vec{\eta},\vec{\zeta}\}) = \prod_{j=0}^{T} \int d\vec{x}_j \int \frac{d\vec{p}_j}{V} e^{i\sum_{i=1}^{T} \vec{p}_i\cdot[\vec{x}_i - f(\vec{x}_{i-1})]}$$

$$\times e^{i\sum_{i=0}^{T} [\vec{\eta}_i\cdot\vec{x}_i + \vec{\zeta}_i\cdot\vec{p}_i]} \quad . \tag{22}$$

All correlation and response functions are generated by derivatives of Z with respect to $\vec{\eta}_m$ and $\vec{\zeta}_n$. The properties of this characteristic functional, as well as the extension of the path integral formalism to dynamical systems defined by differential equations, is discussed in detail in Reference 11. In this paper we will restrict our attention to the conditional probability distribution, Eq. (18), and the corresponding characteristic function Eq. (20).

3. Statistical Properties of the Zaslavskii Map

To illustrate the utility of our formalism we calculate some of the statistical properties of the Zaslavskii map.

For Eqs. (1) and (2) the characteristic function, Eq. (20), is

$$C(\hat{x}\ \hat{y}\ |x_0 y_0) = \prod_{i=1}^{T-1} \sum_{\hat{x}_i=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\hat{y}_i}{2\pi}$$

$$\times \left\{ \int_0^1 dx_i e^{i2\pi x_i[\hat{x}_i - \hat{x}_{i+1}] - i\hat{y}_{i+1} k \sin 2\pi x_i} \right\}$$

$$\times \left\{ \int_{-\infty}^{\infty} dy_i e^{iy_i[\hat{y}_i - \lambda\hat{y}_{i+1} - 2\pi\hat{x}_i]} \right\}$$

$$\times e^{-i\hat{x}_1 x_0 - \hat{y}_1 k \sin 2\pi x_0} e^{-i\lambda\hat{y}_1 y_0} \tag{23}$$

where we have defined $\vec{x}_i \equiv (x_i, y_i)$ and $\vec{p}_i \equiv (\hat{x}_i, \hat{y}_i)$ for $i < T-1$. Since Eq. (1) gives x_n in terms of x_{n-1} and y_n, the arguments of the characteristic function are $\vec{p}_T \equiv (\hat{x}, \hat{y}) = (\hat{x}_T, \hat{y}_T - \hat{x}_T)$. The x_i integrations give ordinary Bessel functions of integer order and the y_i integrations provide δ functions which are used to eliminate the $\hat{y}_i$ integrals. These manipulations leave

$$C(\hat{x}\ \hat{y}\ |x_0 y_0) = \prod_{i=1}^{T-1} \sum_{\hat{x}_i=-\infty}^{\infty} J_{\hat{x}_i - \hat{x}_{i+1}}(k\hat{y}_{i+1})$$

$$\times e^{-2\pi\hat{x}_1 x_0 - \hat{y}_1 k \sin 2\pi x_0} e^{-i\lambda\hat{y}_1 y_0} \tag{24}$$

where the δ functions from the y_i integrations require that

$$\hat{y}_{i+1} \equiv \left[\sum_{j=i+1}^{T-1} \lambda^{j-i-1} 2\pi\hat{x}_j + \lambda^{T-i-1} \hat{y}_T \right], \quad i < T-1 . \tag{25}$$

The last two factors in Eq. (24) contain the initial conditions. If we average over a uniform distribution of x_0 on $(0,1)$,* we arrive at our final expression for the characteristic function

$$C(\hat{x}\,\hat{y}\,|y_0) = \prod_{i=1}^{T-1} \sum_{\hat{x}_i=-\infty}^{\infty} J_{\hat{x}_i-\hat{x}_{i+1}}(k\hat{y}_{i+1})$$

$$\times J_{-\hat{x}_1}(k\hat{y}_1)\, e^{-i\lambda \hat{y}_1 y_0} \quad . \tag{26}$$

The statistical moments of y_T averaged over initial x_0 are derived from Eq. (26) by differentiating with respect to $\hat{y}$ and setting $\hat{y}_T, \hat{x}_T = 0$. For example, the second moment is given by

$$\langle y_T{}^2 \rangle = \frac{-\partial^2}{\partial \hat{y}^2} C(\hat{x}_T \hat{y}_T | y_0) \Big|_{\hat{x}_T, \hat{y}_T = 0} \quad . \tag{27}$$

The derivatives of Eq. (26) are easily evaluated using the Bessel function identity

$$\frac{dJ_n(ax)}{dx} = \frac{a}{2}\left[J_{n-1}(ax) - J_{n+1}(ax)\right] \quad . \tag{28}$$

Differentiating Eq. (26) twice with respect to $\hat{y}$ and setting $\hat{x}_T, \hat{y}_T = 0$ gives

$$\langle y_T{}^2 \rangle = - \left\{ \sum_{j=0}^{T-1} \frac{k^2 \lambda^{2(T-j-1)}}{4} \prod_{i \neq j} \sum_{\hat{x}_i=-\infty}^{\infty} \sum_{\hat{x}_j=-\infty}^{\infty} \right.$$

*We remark that the method does not require that the distribution be uniform. In fact an average with respect to a distribution with support on any open set will do. However, since the initial conditions decay away rapidly for turbulent systems, the form of the distribution of initial conditions is not very important.

$$\left[J_{\hat{x}_j-\hat{x}_{j+1}-2}(z_{j+1}) - 2J_{\hat{x}_j-\hat{x}_{j+1}}(z_{j+1})\right.$$

$$\left. + J_{\hat{x}_j-\hat{x}_{j+1}+2}(z_{j+1})\right] J_{\hat{x}_i-\hat{x}_{i+1}}(z_{i+1})$$

$$+ k^2 \sum_{\substack{j=0 \\ j\neq l}}^{T-1} \sum_{l=0}^{T-1} \frac{\lambda^{T-j-1}\lambda^{T-l-1}}{4} \prod_{i\neq j,l} \sum_{\hat{x}_i=-\infty}^{\infty} \sum_{\hat{x}_j=-\infty}^{\infty} \sum_{\hat{x}_l=-\infty}^{\infty}$$

$$\times \left[J_{\hat{x}_j-\hat{x}_{j+1}-1}(z_{j+1}) - J_{\hat{x}_j-\hat{x}_{j+1}+1}(z_{j+1})\right]$$

$$\left.\times \left[J_{\hat{x}_l-\hat{x}_{l+1}-1}(z_{l+1}) - J_{\hat{x}_l-\hat{x}_{l+1}+1}(z_{l+1})\right] J_{\hat{x}_i-\hat{x}_{i+1}}(z_{i+1}) \right\}$$

$$\times e^{-i\lambda z_1 y_0}$$

$$+ \lambda^{2T} y_0^{\,2} \tag{29}$$

where we have defined

$$z_{i+1} = 2\pi k \sum_{m=i+1}^{T-1} \lambda^{m-i-1} \hat{x}_m \quad . \tag{30}$$

The first group of terms in Eq. (29) result from the application of both derivatives on the j^{th} Bessel function in Eq. (26). The second group of terms arise from the action of one derivative on the j^{th} Bessel function and one on the l^{th}. Finally, the last term which exhibits the decay of the initial conditions results from the application of both derivatives on the factor $\exp\{-i\lambda^T \hat{y}_T y_0\}$. The contribution resulting from the application of one derivative on a Bessel function and the second on $\exp\{-i\lambda^T \hat{y}_T y_0\}$. vanishes identically due to the symmetry of the mapping in y.

In order to calculate $\langle y_T{}^2 \rangle$ we must perform the sums over the different combinations of $\hat{x}_i$s . In Reference 15 the mapping was simpler and we were able to evaluate the sums over $\hat{x}_i$ exactly. However, because of the coupling between x_i and y_i in Eq. (1), we can only calculate approximate results for the statistical moments of the Zaslavskii mapping. In Reference 21 Rechester and White observed that in the turbulent regime, $k \gtrsim 1$, for the Chirikov-Taylor mapping, the asymptotic form of the Bessel functions

$$J_n(z) \sim \left(\frac{2}{\pi z} \right)^{1/2} \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \tag{31}$$

can be used to expand the characteristic function in powers of $\frac{1}{\sqrt{2\pi k}}$. The same approach can be used to evaluate Eq. (29).

Unless the arguments of the Bessel functions are very small, each Bessel function in Eq. (29) contributes a factor of $\frac{1}{\sqrt{2\pi k}}$. Therefore, the dominant contributions for $k \gtrsim 1$ arise from the terms of Eq. (29) with the least number of Bessel functions with nonvanishing arguments.

For example, if all $\hat{x}_i = 0$, then the arguments of all the Bessel functions are identically zero. Consequently, all of the Bessel functions of nonzero order vanish and we are left with

$$\langle y_T^2\rangle_0 = \frac{k^2}{2}\sum_{j=0}^{T-1} \lambda^{2(T-j-1)} + \lambda^{2T} y_0^2$$

$$= \frac{k^2}{2}\left(\frac{1-\lambda^{2T}}{1-\lambda^2}\right) + \lambda^{2T} y_0^2 \quad . \tag{32}$$

This is the dominant contribution to $\langle y_T^2\rangle$ since the terms corresponding to other combinations of $\{\hat{x}_i\}_{i=1,\dots,T-1}$ will contain at least one Bessel function with large argument, which is small of order $\frac{1}{\sqrt{2\pi k}}$.

As in Reference 15 the combination of $\{\hat{x}_i\}_i = 0$ is equivalent to a random phase approximation. If we neglect Eq. (1) and assume that the x_is are independent random variables, uniformly distributed on (0,1), then averaging Eq. (2) over x_i we recover Eq. (32)

$$\langle y_T^2\rangle_{R.P.A.} = \langle y_T^2\rangle_0 \quad . \tag{33}$$

The corrections to Eq. (32) can be determined by calculating the contributions from the combinations of $\{\hat{x}_i\}_i$ which give rise to terms with one or more Bessel functions with large arguments. Physically, these corrections to the random phase approximation correspond to the persistence of correlations between successive time steps. These correlations diminish with increasing k.

Since $k \gtrsim 1$, the arguments, z_{i+1}, defined by Eq. (30) will be large unless

$$\sum_{l=i+1}^{T-1} \lambda^{l-i-1}\hat{x}_l < \frac{1}{2\pi k} \quad . \tag{34}$$

The first order correction in $\frac{1}{\sqrt{2\pi k}}$ to $\langle y_T{}^2 \rangle$ results from combinations of $\{\hat{x}_i\}_{i=1,\ldots,T}$ for which only one z_{i+1} is large.

In addition to the constraints on the arguments, Eq. (34), a condition on the order of the Bessel functions must also be satisfied. The order n of any Bessel function with small argument $z_i \ll 1$ must be zero, otherwise a factor of $\left(\frac{z_i}{2}\right)^n \ll 1$ arises.

A careful study of the orders of the Bessel functions in Eq. (29) and the constraints on the arguments shows that first order corrections in $\frac{1}{\sqrt{2\pi k}}$ result only in the limits of strong damping, $\lambda \ll 1$, and weak damping, $(1 - \lambda) \ll 1$. For intermediate damping, the first corrections to the random phase approximation are much higher order.

We will explicitly calculate the first order corrections to the random phase result in the strong and weak damping limits. For intermediate damping our analysis indicates that the corrections are very small; and that the random phase approximation corresponding to taking all $\{\hat{x}_i\}_i = 0$ is very good. For long times this conclusion has been verified by numerically advancing the mapping. Figure 4 shows a comparison of the analytic theory, Eq. (32), and the numerical calculations of $\langle y_T{}^2 \rangle$ as a function of k for intermediate values of the damping, $\lambda = .1$ and $.5$.

A. Strong Damping

Consider the strong damping limit $\lambda \ll \frac{1}{2\pi k}$. We can satisfy the first requirement that only one z_j be large if we take $\hat{x}_j = \{\text{integer} \ll \frac{1}{2\pi k\lambda}\}$ and $\{\hat{x}_i\}_{i\neq j} \equiv 0$. Then $z_{m>j} = 0$, $z_j = 2\pi k\hat{x}_j$, and $z_{m<j} = 2\pi k\lambda^{n-m}\hat{x}_j \ll 1$. An examination of the terms of Eq. (29) shows that the second condition, that the order of Bessel functions with vanishing

arguments must be zero, can only be satisfied if $\hat{x}_j = \pm 2$ or ± 1 . If $\hat{x}_j = \pm 2$, the first group of terms in Eq. (29) gives a contribution of

$$\frac{-k^2}{4} \sum_{j=0}^{T-1} \lambda^{2(T-j-1)} \left[J_2(4\pi k) + J_{-2}(-4\pi k)\right] \prod_{i<j} J_0\left(4\pi k \lambda^{j-i}\right)$$

$$\simeq \frac{-k^2}{4} \left(\frac{1 - \lambda^{2T}}{1 - \lambda^2} \right) \left[J_2(4\pi k)\right] \qquad (35)$$

where we have used the symmetries of the even ordered Bessel functions. For $\hat{x}_j = \pm 1$, the leading contribution from the second group of terms in Eq. (29) is smaller than Eq. (35) by a factor of λ ; and it can be neglected. Therefore, combining Eqs. (32) and (35), $\langle y_T^2 \rangle$ in the strong damping limit is given to first order in $\frac{1}{\sqrt{2\pi k}}$ by

$$\langle y_T^2 \rangle \simeq \frac{k^2}{2} \left[1 - J_2(4\pi k)\right] \quad . \qquad (36)$$

The terms that we have neglected in writing Eq. (36) are smaller by factors of $\left(\frac{1}{\sqrt{2\pi k}} \right)$ or $(2\pi k \lambda)$.

In Figure 5 we show a comparison of the analytic result for $\langle y_T^2 \rangle$ as a function of k, with values obtained by numerically advancing the mapping with $\lambda = 10^{-2}$ and 10^{-3} for 10^4 time steps. Eq. (5) agrees very well with the numerical results except for { k= integer or half integer }. The discrepancies are due to the appearance of attracting periodic orbits in the vicinity of integer and half integer values of k. These attracting periodic points have also been observed by Zaslavskii.[16,24] Since the periodic points are highly localized in phase space and the trajectories no longer wander ergodically, it is expected that our method which keeps only a few Fourier components $\hat{x}_i$ near 0 should fail. Although the terms we have neglected in our asymptotic result, Eq. (36), are small, they are numerous. For integer and half integer values of k the contribution of these other combinations of Fourier components $\{\hat{x}_i\}_i$ can no longer be ignored.

These periodic points are not related to the "accelerator islands"[22] which modify the diffusion for the Chirikov-Taylor mapping with noise. The attracting periodic points of the Zaslavskii map can be studied analytically in the strong damping limit, $\lambda \to 0$, by reducing Eqs. (1) and (2) to a one dimensional map[24]

$$x_i = x_{i-1} + k \sin 2\pi x_{i-1} \quad . \tag{37}$$

Moreover, since Eq. (36) is independent of λ, it also describes the second moment of $y_T \equiv k \sin x_{T-1}$ for the one-dimensional map, Eq. (37). The results of these investigations will be discussed in future work.

B. Weak Damping

Consider now the limit of weak damping, $(1 - \lambda) \equiv \nu \ll \frac{1}{2\pi k}$, which is, physically, the most important case. The z_{i+1}, defined by Eq. (30), can only be small if terms in the sum, $\sum_{l=i+1}^{T} \lambda^{l-i-1} \hat{x}_l$, cancel. Therefore, the first order correction to the random phase result arises from terms in which two $\hat{x}_i$s are nonzero. The $\hat{x}_i$s are restricted even further by the second requirement that the orders of Bessel functions with vanishing arguments be zero.

The detailed calculations of the leading corrections to the random phase result for $\langle y_T^2 \rangle$ for weak damping are relegated to the Appendix. Combining Eq. (32), (A4), and (A5), the final expression for short times is

$$\langle y_T^{\,2}\rangle = \frac{k^2}{2}\left[\left(\frac{1-\lambda^{2T}}{1-\lambda^2}\right) - \lambda^{2T-4} J_2(4\pi k)\cos(4\pi k\lambda y_0) \times \left\{\begin{matrix}0, & T<2\\ 1, & T>2\end{matrix}\right\}\right.$$

$$- 2\lambda^{2T-3}\left[J_2(2\pi k) - J_0(2\pi k)\right]\cos(2\pi k\lambda y_0) \times \left\{\begin{matrix}0, & T<2\\ 1, & T>2\end{matrix}\right\}$$

$$- 2(T-2)\,\lambda^2 J_2(2\pi k) \times \left\{\begin{matrix}0, & T<3\\ 1, & T>3\end{matrix}\right\}$$

$$\left. - 2(T-2)\,\lambda\pi k\nu\left[J_1(2\pi k) - J_3(2\pi k)\right] \times \left\{\begin{matrix}0, & T<3\\ 1, & T>3\end{matrix}\right\}\right]$$

$$+ \lambda^T y_0^{\,2} \quad . \tag{38}$$

This result is valid for

1) large k $\quad \frac{1}{\sqrt{2\pi k}} << 1 \quad ,$ (39)

2) weak damping $\quad 2\pi k\nu << 1 \quad$, and (40)

3) short times $\quad T << \gamma^{-1} \equiv \{\mathrm{Min}\,[2\nu,\ (\pi k\nu)^2]\}^{-1} \quad .$ (41)

Eq. (38) has been verified by numerically advancing the mapping T time steps for a uniform distribution of x_0s on the interval (0,1). A comparison of the numerical and analytical results for k = 10.-12., ν = .01, and T = 5, which satisfy the conditions (39)-(41), is shown in Figure 6.

However, for long times $T > \gamma^{-1}$ the corrections to the random phase result decay away exponentially with time. The third and fourth terms decay as λ^{2T}, and the remaining terms decay as $\exp\{-\gamma T\}$ as shown in the Appendix. This result indicates that for long times the random phase approximation provides a good description of the statistical dynamics for weak damping. The convergence to the random phase result for $\langle y_T^2\rangle$ for large T is illustrated in Figure 7.

C. No Damping (Chirikov-Taylor Map)

In the dissipationless limit $\nu = 0$, Eq. (38) reduces to

$$\langle y_T^2\rangle = \frac{k^2}{2} \quad T - J_2(4\pi k)\cos(4\pi k y_0) \times \left\{ \begin{matrix} 0, & T<2 \\ 1, & T>2 \end{matrix} \right\}$$

$$- 2\left[J_2(2\pi k) - J_0(2\pi k)\right]\cos(2\pi k y_0) \times \left\{ \begin{matrix} 0, & T<2 \\ 1, & T>2 \end{matrix} \right\}$$

$$- 2(T-2)J_2(2\pi k) \times \left\{ \begin{matrix} 0, & T<3 \\ 1, & T>3 \end{matrix} \right\} + y_0^2 \qquad (42)$$

Eq. (42) is valid for all T. For large T >> 1 we recover the first order results of Rechester and White for the turbulent diffusion coefficient of the Chirikov-Taylor mapping[21]

$$D_s = \frac{\langle y_T^2\rangle}{2T} = \frac{k^2}{4}\left[1 - 2J_2(2\pi k)\right] \quad . \qquad (43)$$

The diffusion coefficient, Eq. (43), characterizes the turbulent dynamics only if the asymptotic long-time statistics are Gaussian. A straightforward calculation shows that for $\lambda \equiv 1$ the long-time moments are Gaussian to first order in $\frac{1}{\sqrt{2\pi k}}$.

For example, the fourth moment of y_T averaged over x_0 is defined by

$$\langle y_T{}^4\rangle = \frac{\partial^4 C}{\partial \hat{y}^4}\left(\hat{x}_T \hat{y}_T | y_0\right)\Bigg|_{\hat{x}_T, \hat{y}_T = 0} \tag{44}$$

For any $0 < \lambda < 1$ the contribution from all $\{\hat{x}_i\}_i = 0$ gives the random phase result

$$\langle y_T{}^4\rangle = \frac{3k^4}{4}\left(\frac{1-\lambda^{2T}}{1-\lambda^2}\right)^2 - \frac{3}{8}k^4\left(\frac{1-\lambda^{4T}}{1-\lambda^4}\right)$$
$$+ 3k^4\left(\frac{1-\lambda^{2T}}{1-\lambda^2}\right)(\lambda^T y_0)^2 + k^4(\lambda^T y_0)^4 \quad . \tag{45}$$

In the limit $\lambda \to 1$, $T \to \infty$ the first term in Eq. (45) dominates. Finally, the inclusion of the first order corrections in $\frac{1}{\sqrt{2\pi k}}$ due to other combinations of $\{\hat{x}_i\}_i$ gives

$$\langle y_T{}^4\rangle \simeq \frac{3}{4} k^2 T^2 \left[1 - 4J_2(2\pi k)\right]$$

$$\simeq 3\langle y_T{}^2\rangle \tag{46}$$

where we have neglected terms of order $\frac{1}{T}$ and $\frac{1}{2\pi k}$.

The long time statistical dynamics are significantly changed by the inclusion of any amount of damping, $\nu > 0$. First, the corrections to the random phase result for the statistical moments are negligibly small for large $T > \gamma^{-1}$. Second, since the dynamics are restricted to a bounded attractor, these moments do not describe a Gaussian distribution. For example, in the limit $T \to \infty$, Eqs. (32) and (45) reduce to

$$\langle y_\infty^2\rangle \sim \frac{k^2}{2}\left(\frac{1}{1-\lambda^2}\right) \tag{47}$$

$$\langle y_\infty^4 \rangle \sim \frac{3k^4}{4} \left(\frac{1}{1 - \lambda^2} \right) - \frac{3k^4}{8} \left(\frac{1}{1 - \lambda^4} \right) \quad . \tag{48}$$

The second term in Eq. (48) is a non-Gaussian contribution to the fourth moment. Third, since the dissipation destroys the periodicity of the map in the y direction, the "accelerator modes"[22] are eliminated.

D. Stochastic Wave Heating

The Zaslavskii map has important applications in the study of stochastic wave heating of plasmas by electrostatic waves.[18] If we neglect collisions, charged particles diffuse in velocity for sufficiently large wave amplitudes. Then the heating of the plasma distribution function can be approximately described by a Fokker-Planck equation with a stochastic diffusion coefficient D_s determined by the single particle Hamiltonian dynamics, Eq. (43).

Previous efforts to include the effects of collisional damping have simply added a viscous drag, ν, to the Fokker-Planck equation.[25] In this case a steady state is achieved where the collisional drag, ν, balances the collisionless diffusion D_s. The resulting distribution is Maxwellian with a temperature

$$\Theta = \langle y_\infty^2 \rangle = \frac{D_s}{\nu} \tag{49}$$

Unfortunately, this approach completely ignores the strange attractor which is embedded in the real dissipative system. In the long time limit, $T\to\infty$, the leading term for the second and fourth velocity moments on the strange attractor are given by Eqs. (47) and (48). These moments do not describe a Maxwellian distribution function. Furthermore, using Eqs. (43) and (47), the average particle energy can be written to lowest order as

$$\langle y_\infty^2 \rangle = \frac{D_s}{\nu\left(1 - \frac{\nu}{2}\right)} \tag{50}$$

where $\nu \equiv 1 - \lambda$. These conclusions differ qualitatively and quantitatively from those which ignore the strange attractor. However, in the limit of weak damping, $\nu \ll 1$, Eq. (50) reduces to Eq. (49) and the higher moments become approximately Maxwellian. For example, as $\nu \to 0$, we can neglect the second term in Eq. (48) compared with the first which gives

$$\langle y_\infty^4 \rangle \simeq 3\langle y_\infty^2 \rangle^2 \tag{51}$$

Consequently, the Fokker-Planck approach proves adequate in the limit of weak damping. But for moderate damping the physical picture and the quantitative results for the particle heating can be significantly different.

4. Conclusion

We have discussed a very powerful path integral method for the calculation of the statistical properties of turbulent dynamical systems which is applicable to systems described by differential equations as well as maps. Since the effects of random forces, interactions, and initial conditions are easily included, the method provides a foundation for similar approaches to the calculation of the statistical dynamics of conservative[21,22] and dissipative systems[20] which introduce a source of random noise.

The path integral method is illustrated by the calculation of the low order statistical moments of the Zaslavskii map[17], Eqs. (1) and (2), which can be used to model the dynamics of a Brownian particle in a wave field.[18] We calculate new results for three turbulent regimes which exhibit strange attractors corresponding to strong, intermediate, and weak collisional damping.

In the collisionless limit this mapping reduces to the conservative Chirikov-Taylor map which exhibits a transition to stochasticity rather than a strange attractor. Our results show that this system is diffusive and our expression for the second moment (average particle energy) recovers the leading terms of the turbulent diffusion coefficient derived by Rechester and White.[21]

However, the inclusion of any amount of damping has a significant effect on the statistical dynamics. The long time statistical properties of the dissipative system are determined by an invariant distribution on the strange attractor. Since the attractor is bounded in phase space this distribution is not Gaussian. Moreover, our calculations for the low order statistical moments for weak and intermediate damping indicate that they are well described by the random phase approximation in the long-time limit. The corrections decay away exponentially for times longer than a damping time γ^{-1}.

These results provide a new qualitative and quantitative description of the effects of damping on the stochastic heating of plasmas by electrostatic waves. Dissipation should also have a significant effect on the statistical dynamics of the Fermi map[26] which has been used to model cyclotron heating in magnetically confined plasmas and the Karney map[25] which has been used to study lower hybrid wave heating[25] and the loss of high energy particles from tokamaks.[27] These problems will be considered in subsequent publications.

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Appendix

We calculate the leading corrections to the random phase result for $\langle y_T^2 \rangle$ for weak damping.

In order for $z_i = \sum_{l=i+1}^{T} \lambda^{l-i-1} \hat{x}_l$ to be small at least two $\hat{x}_l$s must be nonzero. For example, if $\hat{x}_n \neq 0$, $\hat{x}_{n-1} = -\hat{x}_n$ and $\{\hat{x}_i\}_{i\neq n,n-1} \equiv 0$, then $z_{m>n} = 0$, $z_n = 2\pi k\hat{x}_n$, $z_{n-1} = 2\pi k\nu\hat{x}_n$, and $z_{m<n} = 2\pi k\lambda^{n-m}\nu\hat{x}_n$. In this case only one z_n is large for $\hat{x}_n = \{ \text{integer} \ll \frac{1}{2\pi k\nu} \}$. An additional contribution with only one large z_n arises if $\hat{x}_1 \neq 0$ but all $\hat{x}_{i>1} \equiv 0$. Then $z_{l>1} \equiv 0$ and $z_1 = 2\pi k\hat{x}_1$. These choices of $\{\hat{x}_i\}_i$ provide the only first order corrections to the random phase result. All other combinations of $\{\hat{x}_i\}_i$ give contributions which are higher order in $\frac{1}{\sqrt{2\pi k}}$.

The $\hat{x}_i$s are further restricted, as in the case of strong damping, by the requirement that the order of Bessel functions with vanishing argument be zero. Again a careful examination of the terms in Eq. (29) shows that this condition can only be satisfied for $\hat{x}_j = \pm 1$ or ± 2.

If $\hat{x}_j = \pm 1$ and $\hat{x}_{j-1} = \mp 1$, then the second group of terms in Eq. (29) provides two first order contributions. First, for $l \equiv j-1$ we get

$$C_1 = -k^2 \lambda \sum_{l=2}^{T-1} \lambda^{2(T-l-1)} J_0(0)[J_1(2\pi k) - J_3(2\pi k)]$$

$$\times J_1(2\pi k\nu) \prod_{i<l} J_0(2\pi k\lambda^{l-i}\nu) \qquad \text{(A1)}$$

where we have used the odd symmetries of the Bessel functions in order and argument to add the contributions from the combinations $\hat{x}_j = +1$, $\hat{x}_{j-1} = -1$ and $\hat{x}_j = -1$, $\hat{x}_{j-1} = +1$. An additional factor of 2 results from the interchange of j and l in the double sum in Eq. (29). Second, similar arguments for l=j-2 give

$$C_2 = -k^2\ \lambda^2 \sum_{l=2}^{T-1} \lambda^{2(T-l-1)}\ J_0(0)J_2(2\pi k)$$

$$\times \left[J_0(2\pi k\nu) - J_2(2\pi k\nu)\right] \times \prod_{i<l} J_0(2\pi k\lambda^{l-i}\nu) \quad . \qquad (A2)$$

Since we assumed $2\pi k\nu << 1$, Eqs. (A1) and (A2) can be simplified further by expanding some of the Bessel functions in their small arguments and using the approximation

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \ldots \simeq e^{-\left(\frac{x}{2}\right)^2} \sim 1\ ,\ \text{for } x << 1 \quad . \qquad (A3)$$

We also expand $\lambda^2 \simeq 1 - 2\nu \simeq e^{-2\nu}$ for $\nu << 1$. Then, for $T << \gamma^{-1} \equiv \{\text{Min}\left[2\nu,\ (\pi k\nu)^2\right]\}^{-1}$,Eqs. (A1) and (A2) reduce to

$$C_1 + C_2 \simeq -k^2(T-2)\ \{\lambda^2 J_2(2\pi k) + \lambda\pi k\nu\left[J_1(2\pi k) - J_3(2\pi k)\right]\} \quad . \qquad (A4)$$

For large $T > \gamma^{-1}$ these corrections decay away as $e^{-\gamma T}$. (For T < 3 these corrections do not appear.) For times shorter than a damping time, γ^{-1} , the system appears to be diffusive; however, for long times the strange attractor dominates the dynamics.

Since we assumed that $k\nu$ is small the first term in the brackets dominates. The terms that we have neglected in writing Eq. (41) are much smaller of order $(2\pi k\nu)^2$.

Similar arguments show that for $\hat{x}_j = \pm 2$ the first group of terms in Eq. (29) give a small contribution of order $(2\pi k\nu)^2$; and the combinations

$\hat{x}_1 = \pm 1,\ \pm 2$ give a first order contribution which is significant only for small $T > 2$

$$C_3 = -k^2 \left\{ \lambda^{2T-3} \left[J_2(2\pi k) - J_0(2\pi k) \right] \cos(2\pi k \lambda y_0) + \frac{\lambda^{2T-4}}{2} J_2(4\pi k) \cos(4\pi k \lambda y_0) \right\} \qquad \text{(A5)}$$

where the cosines of y_0 arise from the factor $\exp(-i\lambda z_1 y_0)$ in Eq. (29).

Equations (A4) and (A5) give the leading corrections to $\langle y_T^{\,2} \rangle$ for $\frac{1}{\sqrt{2\pi k}} \ll 1$, $2\pi k \nu \ll 1$, and $T \ll \gamma^{-1}$.

FIGURE CAPTIONS

Fig. 1 The strange attractor for $\lambda = .1$ and $k = 1.4$.

Fig. 2 The strange attractor for $\lambda = .1$ and $k = 11.4$.

Fig. 3 A magnified view of the strange attractor in Fig. 1 which shows the complex structure of the attractor.

Fig. 4 Comparison of the analytical and numerical values for $\langle y_T^2 \rangle$ for intermediate damping, $\lambda = .1$ and $\lambda = .5$. The mapping was advanced 10^4 time steps for the numerical calculations.

Fig. 5 Comparison of the analytical and numerical values of $\langle y_T^2 \rangle$ for strong damping, $\lambda = .01$ and $\lambda = .001$. The numerical points for $\lambda = .001$ and $k = 10.0$ and 11.0 lie outside of the range of the figure because of the presence of attracting periodic points.

Fig. 6 Comparison of the short-time analytic results for $\langle y_T^2 \rangle$ with the numerical calculations for weak damping, $\nu \equiv (1-\lambda) = .01$, and $T = 5$.

Fig. 7 Comparison of the short and long-time analytic predictions for $\langle y_T^2 \rangle$ with numerical calculations for weak damping, $\nu = .01$, and long time, $T = 30$.

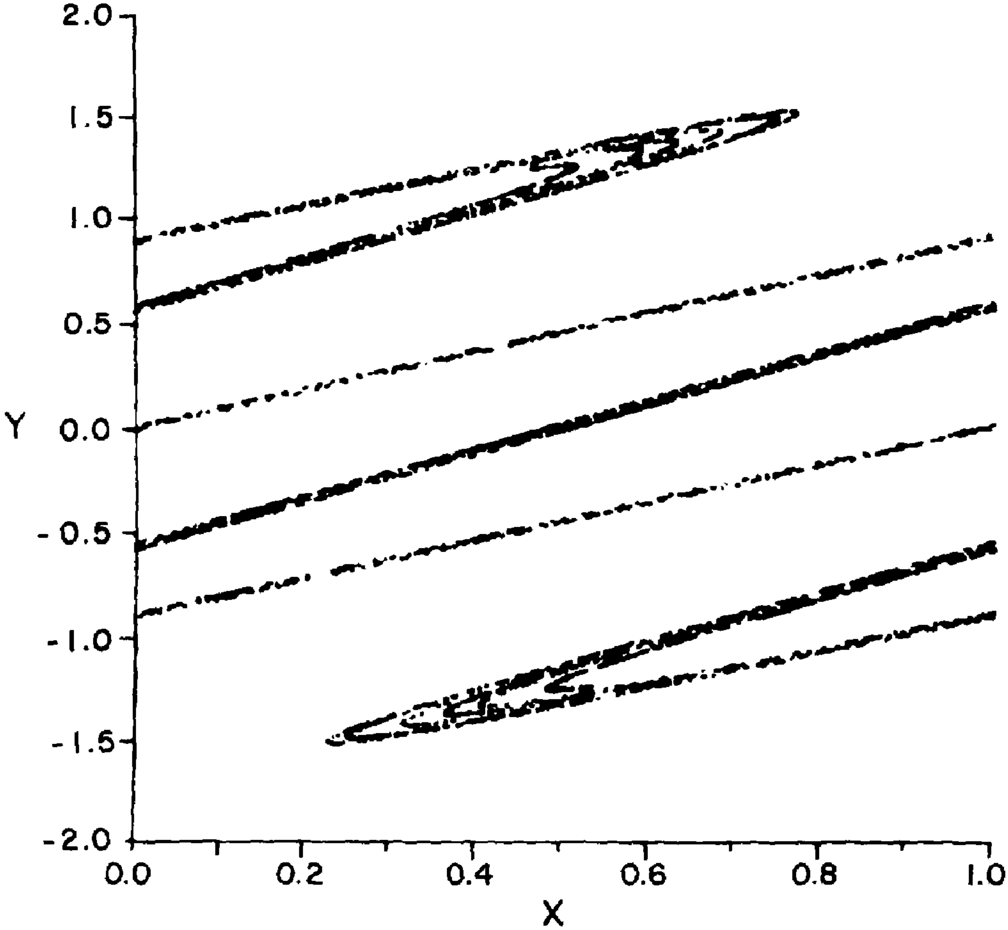


Fig. 1

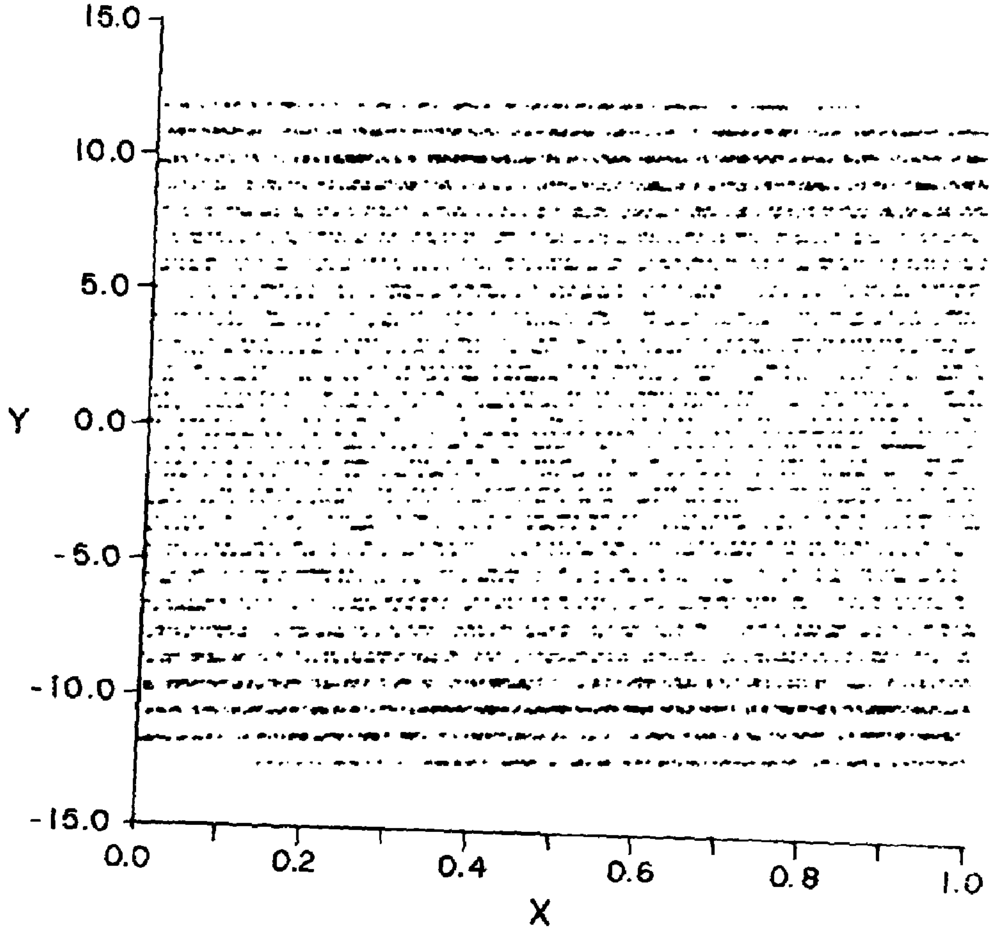


Fig. 2

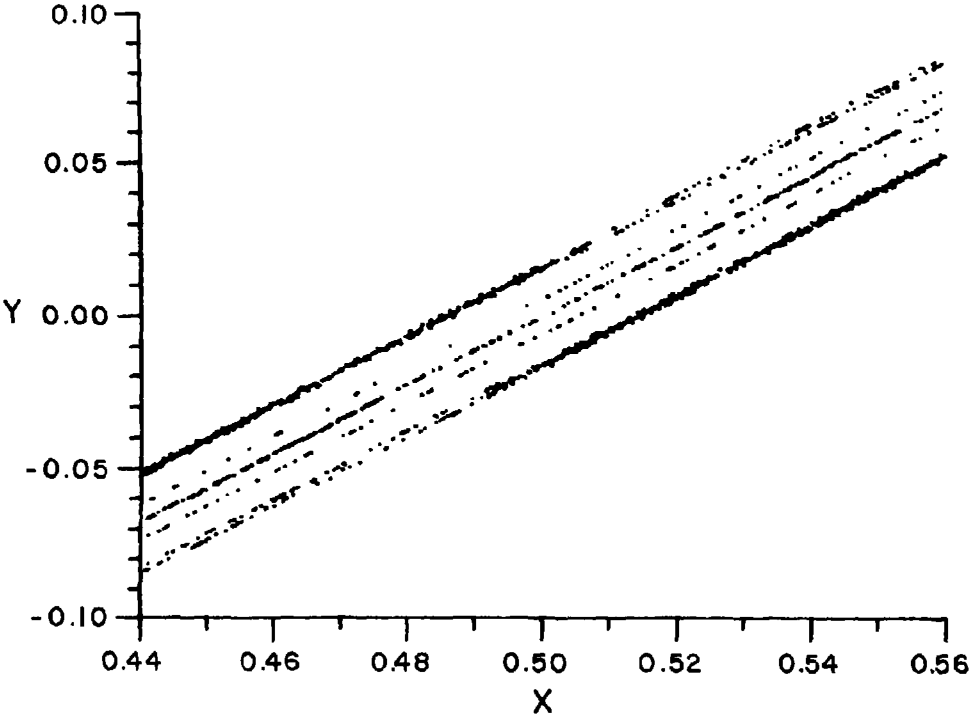
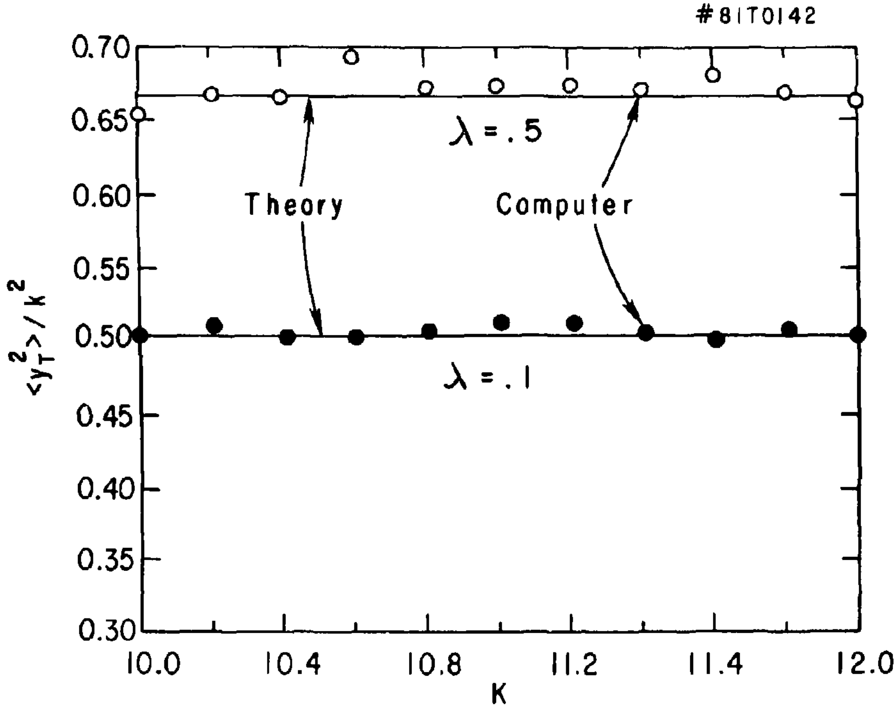


Fig. 3



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0.60
0.55
0.50
0.45
0.40
0.35
0.30
$\langle y_T^2 \rangle / k^2$
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Theory
Computer
λ = .1
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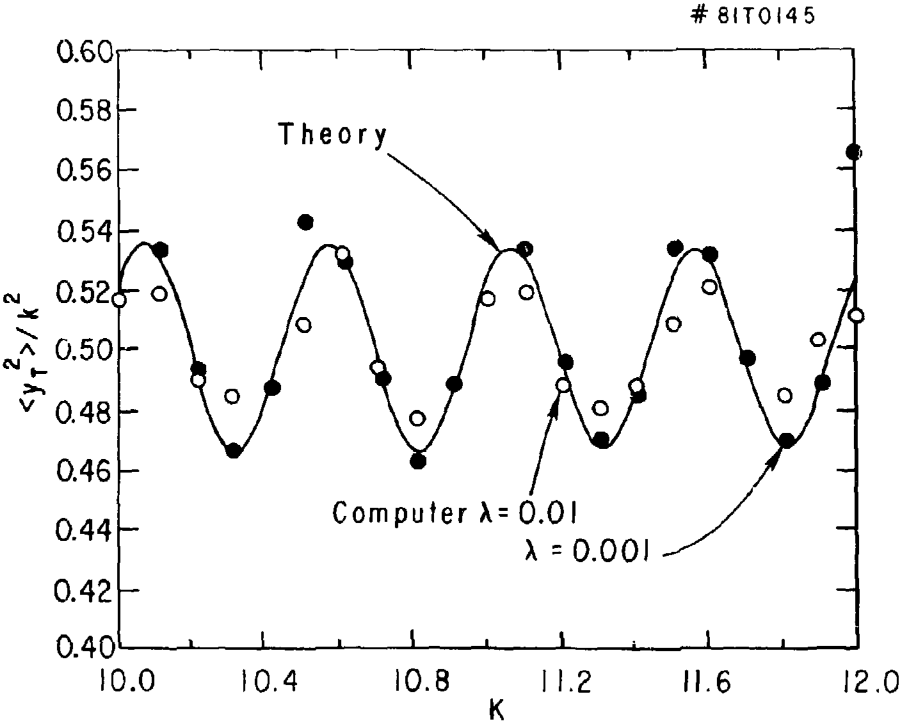
10.4
10.8
11.2
11.4
12.0
K

Fig. 4



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Theory
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$\langle y_T^2 \rangle / k^2$
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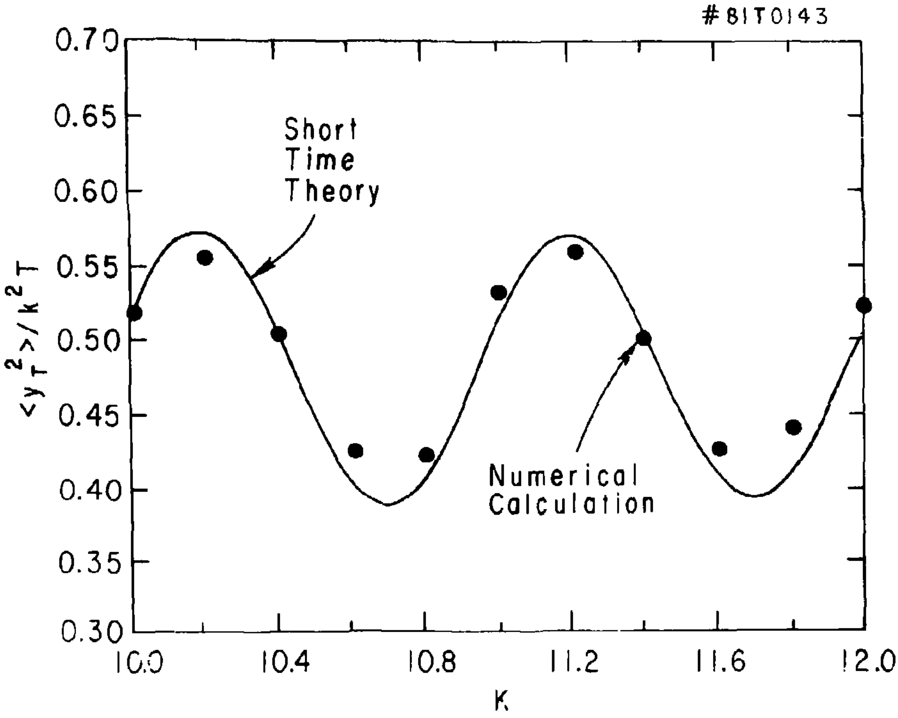
K

Fig. 5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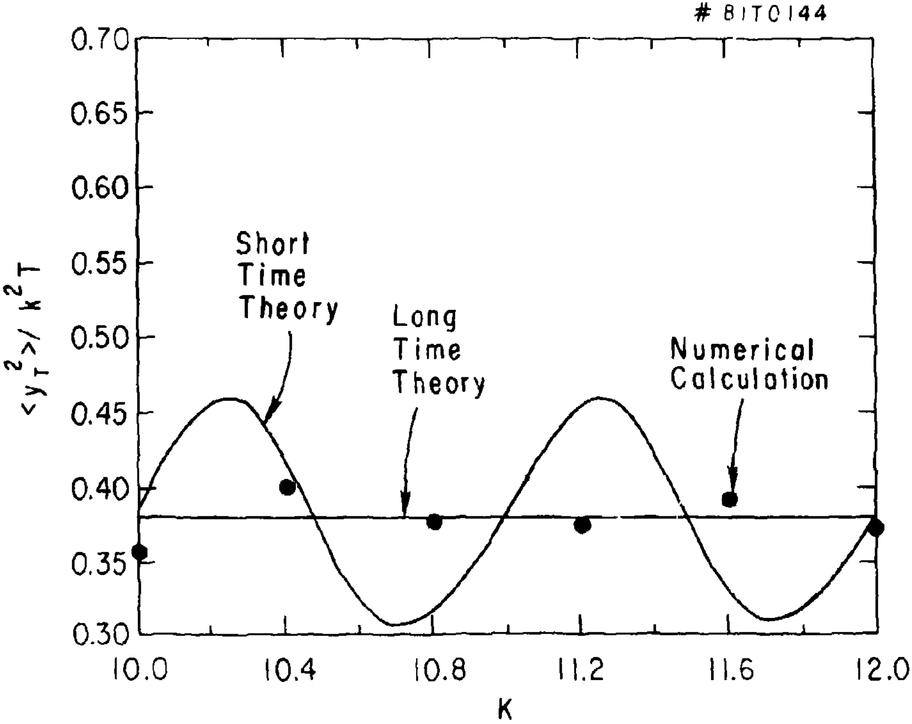
$\langle y_T^2 \rangle / k^2 T$

Fig. 1



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0.30
<y_T^2>/ k^2 T
Short Time Theory
Long Time Theory
Numerical Calculation
10.0
10.4
10.8
11.2
11.6
12.0
K

Fig. 7