STATISTICAL SEMINVARIANTS AND THEIR SETIMATES WITH PARTICULAR EMPHASIS ON THEIR RELATION TO ALGEBRAIC INVARIANTS

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INTRODUCTION

An important portion of algebraic invariant theory has been that devoted to a certain class of invariants called seminvariants, semi-invariants, or more rarely, half-invariants. Of these terms, "seminvariant" seems to be the one now commonly accepted. The same three terms have been applied at various times and by various writers to a system of moment functions of importance in statistical theory. The statistician using these terms has frequently done so with an apology for appropriating a term of the algebraist. As a portion of this paper we shall show that the moment functions of this system are actually algebraic seminvariants, and that there are other systems of moment functions which are equally entitled to the name seminvariant.

The study of the statistical seminvariants of a population leads naturally to consideration of the problem of obtaining from a sample unbiased estimates of the value of these seminvariants. Estimates of this kind have been defined and computed by previous authors, but no simple method of obtaining the estimates has been given. In this paper a simple procedure for calculation is given and it is furthermore demonstrated that these estimates form an important phase of statistical seminvariant theory.

The system of notation used for moment functions is that of R. A. Fisher, although the actual letters used in representing particular moment functions are not altogether the same as those used by Fisher. In general, a moment function of the population has been indicated by a Greek letter, the corresponding sample moment function by the corresponding English letter and the estimate by the corresponding capital English letter.

A list of references appears at the end of the paper. Each reference has been assigned a number and this number placed in square brackets is used in the body of the paper to indicate the reference. Pages of the reference are indicated by additional numbers inserted in the parentheses and separated from the reference number by a semicolon.

I. THE RELATION OF THE ALGEBRAIC SEMINVARIANT THEORY TO THE MOMENT FUNCTIONS OF STATISTICS

The purposes of this chapter are: (1) to review briefly and give adequate references to certain important phases of algebraic seminvariant theory, (2) to apply this material to the moment functions of statistics.

1. **Definitions.** Any function of the coefficients of the binary form

$$f = \sum_{i=0}^{n} \binom{n}{i} a_i X^{n-i} Y^i, \qquad a_0 \neq 0, \tag{1}$$

which is invariant under the transformation

$$X = \gamma_1 \xi + \gamma_2 \eta, \qquad Y = \delta_1 \xi + \delta_2 \eta, \qquad \Delta = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{vmatrix} \neq 0, \tag{2}$$

is called an invariant of the form f. See Dickson [1; 31–36].

Any function of the coefficients of f which is invariant under the transformation

$$X = \xi + \gamma \eta, \qquad Y = \eta, \tag{3}$$

is called a seminvariant of f.

The two operators

$$\Omega = \sum_{i=1}^{n} i a_{i-1} \frac{\partial}{\partial a_i}, \qquad O = \sum_{i=1}^{n} (n - i + 1) a_i \frac{\partial}{\partial a_{i-1}}, \tag{4}$$

are of fundamental importance in the theory of algebraic invariants and seminvariants and, indeed, invariants and seminvariants may be defined by means of these operators. A necessary and sufficient condition that an homogeneous isobaric function of the coefficients of f be an invariant is that it be annihilated by both Ω and O. See Elliott [2; 113, 124]. The necessary and sufficient condition that an homogeneous isobaric function of the coefficients of f be a seminvariant is that it be annihilated by Ω. See Elliott [2; 127].

It should be noted that there is nothing in the definitions above which requires that invariants or seminvariants be integral, although usually only this type is discussed. In what follows we shall find it more profitable to discuss homogeneous isobaric fractional seminvariants, the fractional quality resulting from the appearance of a_0 in the denominator.

2. Complete Systems of Seminvariants. By direct application of the transformation (3) to f the system of seminvariants [1; 47]

$$A_r = \sum_{i=0}^{r} \binom{r}{i} \left(-\frac{a_1}{a_0} \right)^i \frac{a_{r-i}}{a_0}, \qquad r \leq n, \tag{5}$$

is obtained. This system is a complete system, [2; 44, 205, 206], in the sense that all other seminvariants fractional in a_0 and of degree 0 are expressible rationally and integrally in terms of this system.

Other such systems can be defined. The system of minimum degree seminvariants, the seminvariants of even weight being of degree 2 and those of odd weight being of degree 3, has played an important role in the algebraic seminvariant theory. Elliott [2; 207–209] discusses this system and gives the general formula for the even weight seminvariants of the system. So far as the present writer has been able to discover the general formula for the odd weight seminvariants has never been published, although Hammond [3] may have obtained it. After some lengthy but not difficult computation the result has been obtained, so that the last mentioned system of seminvariants is completely defined by

$$\begin{aligned} C_{2r} &= \frac{1}{2} \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} \frac{a_i a_{2r-i}}{a_0^2}, \\ C_{2r+1} &= \sum_{i=0}^{r} (-1)^{i+r} \binom{2r}{i+r} \frac{2i+1}{i+r+1} \frac{a_{r-i} a_{r+i+1}}{a_0^2} \\ &\qquad + \sum_{i=0}^{2r} (-1)^{i+1} \binom{2r}{i} \frac{a_1 a_i a_{2r-i}}{a_0^3}. \end{aligned} \tag{6}$$

It is easily demonstrated that for each of the above seminvariants, and in fact for any seminvariant, the sum of the numerical coefficients is zero. Dickson [1; 55] gives a suggestion leading to a very simple proof.

3. The MacMahon Non-Unitary Symmetric Function Principle. Denoting the roots of $\sum_{i=0}^{n} \binom{n}{i} a_i X^{n-i} = 0$ by $\alpha_1, \alpha_2, \cdots \alpha_n$, the r-th power sum of these roots is defined by

$$s_r = \sum_{i=1}^{n} \alpha_i^r. \tag{7}$$

The form f may be written $\prod_{i=1}^{n} (X - \alpha_i Y)$.

By a result due to MacMahon [4; 131] the seminvariants of the form f are identical, except for numerical factors, with those symmetric functions of the roots of

$$g = \sum_{i=0}^{n} \frac{a_i}{i!} X^{n-i} = 0 \tag{8}$$

which when expressed in terms of sums of powers of these roots do not contain s_1. MacMahon called such symmetric functions "non-unitary."

As a result of this theorem, MacMahon was able to discuss the seminvariants of a binary form of infinite order by discussing the non-unitary symmetric functions of the roots of $\sum_{i=0}^{\infty} \frac{a_i}{i!} Y^i = 0$.

4. **A Third Complete System of Seminvariants.** By application of the result stated in the previous section, a third complete system of seminvariants can be immediately obtained. Obviously the power sums s_r, $r > 1$, are independent of s_1. By the Waring formula, Burnside and Panton [5; 91–92], if

$$\sum_{i=0}^{n} c_i Y^i = c_0 \prod_{i=0}^{n} (1 - \alpha_i Y)$$

then

$$s_r = \Sigma \frac{(-1)^\rho r(\rho - 1)!}{\pi_1!\pi_2! \cdots \pi_n!} \left(\frac{c_1}{c_0}\right)^{\pi_1} \left(\frac{c_2}{c_0}\right)^{\pi_2} \cdots \left(\frac{c_n}{c_0}\right)^{\pi_n}, \tag{9}$$

wherein

$$\rho = \sum_{i=1}^{n} \pi_i, \qquad r = \sum_{i=1}^{n} i\pi_i.$$

Then for

$$g = \sum_{i=0}^{n} \frac{a_i}{i!} X^{n-i},$$

$$-(r-1)!s_r = \Sigma \frac{(-1)^{\rho-1} r!(\rho - 1)! \left(\frac{a_1}{a_0}\right)^{\pi_1} \left(\frac{a_2}{a_0}\right)^{\pi_2} \cdots \left(\frac{a_n}{a_0}\right)^{\pi_n}}{\pi_1!\pi_2! \cdots \pi_n! \, (2!)^{\pi_2} \cdots (n!)^{\pi_n}}. \tag{10}$$

Placing $B_r = -(r - 1)!s_r$ the B's form a complete system of seminvariants. This result has some interesting statistical connections which will be mentioned later.

5. **Linearly Independent Seminvariants.** It follows from the MacMahon non-unitary symmetric function principle, or it can be proved easily in other ways, that the number of linearly independent seminvariants of a given weight r is

equal to the number of partitions of r which contain no unit part. Furthermore we have at our disposal a simple method for obtaining a set of linearly independent seminvariants of any given weight.

For many purposes the power product defined by Dwyer [6; 13] is more useful than the customary monomial symmetric function. The power product is defined by the right hand member and indicated by the left hand member of

$$(11) \qquad (q_1 \cdots q_r) = \sum_{i_1 \neq i_2 \neq \cdots \neq i_r} \alpha_{i_1}^{q_1} \alpha_{i_2}^{q_2} \cdots \alpha_{i_r}^{q_r},$$

where, for convenience, $q_1 \geq q_2 \geq \cdots \geq q_r$. The monomial symmetric function which will be denoted by $M(q_1 \cdots q_r)$ is related to the power product by the identity

$$(12) \qquad \pi_1! \cdots \pi_t! M(q_1^{\pi_1} q_2^{\pi_2} \cdots q_t^{\pi_t}) = (q_1^{\pi_1} q_2^{\pi_2} \cdots q_t^{\pi_t}),$$

so that a distinction occurs only when there are repeated exponents in the summation of (11).

If we desire a system of linearly independent seminvariants of weight 6, by the MacMahon principle we need only to compute the values of the power products (6), (42), (33), (222) in terms of the a's. In a somewhat different form these will be presented later.

6. The Roberts Theorem. Roberts, see [2; 231] and [5; 108], demonstrated the existence of a duality relationship between power sums, s's, and coefficients, a's such that corresponding to any seminvariant in terms of a's there exists a seminvariant in terms of s's obtained by replacing a_i by s_i. The proof consists of showing that the annihilator for seminvariants in terms of power sums is identical in form with Ω, a_i being replaced by s_i.

As a result of this duality, each of the systems of seminvariants which have been obtained yields, upon replacement of a_i by s_i, another system of seminvariants. In particular cases it may happen that the systems are identical when the identities connecting the a_i and s_i are taken into consideration.

We next wish to show that the systems of power sum seminvariants thus obtained either are identical with certain well known statistical moment functions or lead to new ones.

7. Statistical Distributions Represented by Binary Forms. The fact that statistical distributions may be represented by polynomials has long been recognized by statisticians, see Thiele [7; 24–26] and Bertilsen [8]. Indeed it was this fact which led Thiele to the definition of the seminvariants now called by his name. If we have given n observations $\alpha_1, \alpha_2, \cdots \alpha_n$, form the polynomial.

$$(13) \qquad F = \prod_{i=1}^{n} (X - \alpha_i) = \sum_{i=0}^{n} \binom{n}{i} \frac{a_i}{a_0} X^{n-i}.$$

F is not a binary form, but the seminvariant theory of binary forms is applicable since seminvariants are functions of the differences of the roots and are independent of the X and Y, which appear merely as convenient symbols to indicate the various terms of the algebraic form.

For distributions containing an infinite number of items the form F is of infinite order, but discussion of its seminvariants may be carried on by use of the MacMahon principle given in section 3.

8. **Three Systems of Statistical Seminvariants.** Before exhibiting some systems of statistical seminvariants it may be well to consider the meaning of "statistical seminvariant," for this phrase has been undefined. In fact the use of the phrase is merely a matter of convenience in that it emphasizes the fact that seminvariant moment functions have not previously been regarded as algebraic seminvariants. As used here a statistical seminvariant is an algebraic seminvariant which has some application in statistical theory.

The system of seminvariants (5) yields by application of the Roberts' Theorem the well known system of statistical seminvariants usually called central moments. If $\mu_r' = \frac{s_r}{n} = \frac{s_r}{s_0}$, the general formula may be written

$$\mu_r = \sum_{i=0}^{r} \binom{r}{i} \mu_{r-i}'(-\mu_1')^i. \tag{14}$$

The system of seminvariants (6) likewise leads to

$$\kappa_{2r} = \frac{1}{2} \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} \mu_i' \mu_{2r-i}',$$

$$\kappa_{2r+1} = \sum_{i=0}^{r} (-1)^{i+r} \binom{2r}{i+r} \frac{2i+1}{i+r+1} \mu_{r-i}' \mu_{r+i+1}' + \sum_{i=0}^{2r} (-1)^{i+1} \binom{2r}{i} \mu_1' \mu_i' \mu_{2r-i}', \tag{15}$$

a system which seems never to have been used by statisticians.

The system (10) leads to the well known Thiele seminvariants

$$\lambda_r = \Sigma \frac{(-1)^{\rho-1} r!(\rho-1)!(\mu_1')^{\pi_1}(\mu_2')^{\pi_2} \cdots (\mu_r')^{\pi_r}}{\pi_1!\pi_2! \cdots \pi_r!(2!)^{\pi_2} \cdots (r!)^{\pi_r}}. \tag{16}$$

From sections 3 and 4 it is apparent that the general formula for the Thiele seminvariants is a special case of the Waring formula for power sums in terms of coefficients. It does not seem that this fact has been previously recognized. An equivalent way of stating this idea is to say that the Thiele seminvariant λ_r is, except for the factor $-(r-1)!$, the sum of the r-th powers of the roots of the equations obtained by setting the moment generating function,

$$M_x(Y) = \sum_{i=0}^{\infty} \mu_i' \frac{Y^i}{i!},$$

equal to zero.

It is of historical interest to note that MacMahon published his non-unitary function principle and the resulting set of seminvariants in 1884. Cayley [8] published an article in 1885 dealing with this same system. Roberts' Theorem having been known for some time (probably about 20 years), it seems probable that MacMahon and Cayley were aware of the Thiele seminvariants four to five years before Thiele's definition [9] by an entirely different method.

9. Linearly Independent Statistical Seminvariants. At the end of section 5 a method was indicated whereby a complete set of linearly independent seminvariants of a given weight r could be obtained. It has been noted previously that the one part symmetric function s_r or (r) leads to the Thiele seminvariant λ_r. As a further illustration consider the power product (22). From a table of symmetric functions we find that

$$(22) = \frac{2a_4}{4!a_0} - \frac{2a_3a_1}{3!a_0^2} + \frac{a_2^2}{2!2!a_0^2}$$
$$= \frac{2}{4!}\left(\frac{a_4}{a_0} - \frac{4a_3a_1}{a_0^2} + \frac{3a_2^2}{a_0^2}\right),$$

and by the Roberts' Theorem the statistical seminvariant

$$\frac{2}{4!}(\mu_4' - 4\mu_3'\mu_1' + 3\mu_2'^2)$$

is obtained. In similar fashion a system of linearly independent seminvariants of weight $\leqq 8$ have been computed and are given in Table I. For the sake of brevity they are expressed in terms of central moments. Hence the degree, by which is meant the maximum degree in the μ''s, is not apparent in the table. This definition of degree associates with the statistical seminvariant the degree (in the usual sense) of the corresponding homogeneous integral seminvariant.

10. Statistical Invariants. If the transformation

$$(17) \qquad x = \xi + mk\eta, \qquad y = m\eta$$

is applied to the binary form f and, if, in particular

$$k = -\frac{a_1}{a_0} \qquad \text{and} \qquad m = \left[\frac{a_2}{a_0} - \frac{a_1^2}{a_0^2}\right]^{-\frac{1}{2}}$$

one system of invariants of f under this transformation is found to be

$$(18) \qquad D_r = A_r/A_2^{\frac{1}{2}r}, \qquad r \leq n,$$

where A_r is defined in (5). By the Roberts Theorem we obtain the fact that the standard moment $\mu_r/\mu_2^{\frac{1}{2}r}$ is an invariant of f under this transformation. Thus the standard moments, or standard seminvariants in general, have also an algebraic connection. The effect of the transformation (17) on the roots of f is indicated by

$$x - \alpha_i y = \xi + mk\eta - m\alpha_i\eta = \xi - m(\alpha_i - k)\eta.$$

If m and k are defined as above, the result is the equivalent of measuring in standard units denoted by $\frac{\alpha_i - \mu_1'}{\sqrt{\mu_2}}$.

The system (18) is not a system of algebraic invariants, for algebraic invariants must be invariant under rotation, translation and change of scale, or stretching. The component parts of the above system are invariant only under the last two

TABLE I

Linearly Independent Seminvariants of Weight $\leqq 8$

Weight	Degree	Seminvariants
0	1	μ_0
2	2	μ_2
3	3	μ_3
4	4	$\mu_4 - 3\mu_2^2$
	2	$\mu_4 + 3\mu_2^2$
5	5	$\mu_5 - 10\mu_3\mu_2$
	3	$\mu_5 + 2\mu_3\mu_2$
6	6	$\mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3$
	4	$\mu_6 + 5\mu_4\mu_2 - 10\mu_3^2 - 30\mu_2^3$
	3	$\mu_6 - 15\mu_4\mu_2 + 20\mu_3^2 + 30\mu_2^3$
	2	$\mu_6 + 15\mu_4\mu_2 - 10\mu_3^2$
7	7	$\mu_7 - 21\mu_5\mu_2 - 35\mu_4\mu_3 + 56\mu_3\mu_2^2$
	5	$\mu_7 + 9\mu_5\mu_2 - 35\mu_4\mu_3 - 90\mu_3\mu_2^2$
	4	$\mu_7 - 21\mu_5\mu_2 + 25\mu_4\mu_3 + 30\mu_3\mu_2^2$
	3	$\mu_7 + 9\mu_5\mu_2 - 5\mu_4\mu_3$
8	8	$\mu_8 - 28\mu_6\mu_2 - 56\mu_5\mu_3 - 70\mu_4^2 + 210\mu_4\mu_2^2 + 280\mu_3^2\mu_2 - 105\mu_2^4$
	6	$\mu_8 + 14\mu_6\mu_2 - 56\mu_5\mu_3 - 35\mu_4^2 - 210\mu_4\mu_2^2 + 140\mu_3^2\mu_2 + 630\mu_2^4$
	5	$\mu_8 - 28\mu_6\mu_2 + 49\mu_5\mu_3 - 35\mu_4^2 + 420\mu_4\mu_2^2 - 490\mu_3^2\mu_2 - 630\mu_2^4$
	4	$\mu_8 - 28\mu_6\mu_2 - 56\mu_5\mu_3 + 105\mu_4^2 - 420\mu_4\mu_2^2 + 560\mu_3^2\mu_2 + 630\mu_2^4$
	4	$\mu_8 + 14\mu_6\mu_2 - 56\mu_5\mu_3 + 35\mu_4^2 - 210\mu_4\mu_2^2 + 140\mu_3^2\mu_2$
	3	$\mu_8 - 7\mu_6\mu_2 + 49\mu_5\mu_3 - 35\mu_4^2 + 105\mu_4\mu_2^2 - 70\mu_3^2\mu_2$
	2	$\mu_8 + 28\mu_6\mu_2 - 56\mu_5\mu_3 + 35\mu_4^2$

types of transformation. In statistics translation and change of scale ordinarily constitute the only desired transformations so that the standard seminvariants $\frac{\mu_r}{\mu_2^{\frac{1}{2}r}}, \frac{\lambda_r}{\lambda_2^{\frac{1}{2}r}}, \frac{\kappa_r}{\kappa_2^{\frac{1}{2}r}}, \cdots$ might well be called statistical invariants.

11. **Seminvariants and Invariants of Samples.** Consideration of the definition of seminvariants and invariants shows that:

1. A seminvariant is a seminvariant not because it is a function of deviations from the mean, but because it is a function of the differences of the observations;
2. An invariant is an invariant not because it is a seminvariant divided by the standard deviation raised to the proper power, but because it is a ratio of two seminvariants which are of the same order in powers of the observations.

These facts are important from the statistics viewpoint because they show that seminvariants and invariants of samples are also seminvariants and invariants of the population from which the samples are drawn.

II. ESTIMATES

1. **Power Product Seminvariants.** The Roberts Theorem set up a duality relationship between seminvariants expressed in terms of coefficients and seminvariants in terms of power sums. It can be shown that corresponding to each pair thus determined there exists a third seminvariant expressed in terms of power products. This leads to what may be called a triple system of seminvariants, the interrelationships being most apparent when all three seminvariants are expressed in terms of the notation defined by (11). The seminvariant $\frac{a_3}{a_0} - \frac{3a_2a_1}{a_0^2} + \frac{2a_1^3}{a_0^3}$ becomes in this notation

$$\frac{(111)}{n^{(3)}} - \frac{3(11)(1)}{n^{(2)}n} + \frac{2(1)^3}{n^3}.$$

The corresponding power sum seminvariant is

$$\frac{(3)}{n} - \frac{3(2)(1)}{n^2} + \frac{2(1)^3}{n^3},$$

while the power product seminvariant just mentioned is

$$\frac{(3)}{n} - \frac{3(21)}{n^{(2)}} + \frac{2(111)}{n^{(3)}}$$

The value of the power product notation lies in the fact that the numerical coefficients of the three seminvariants are then identical, while this is not the case when monomial and elementary symmetric functions are used.

Perhaps a few remarks are in order in regard to the proof of the relationship above expressed. The annihilator, corresponding to Ω, for seminvariants in terms of roots is, see [2; 230–31],

$$-D = \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_i}.$$

It is easy to see that

$$-D\left[\frac{(p_1^{\pi_1}p_2^{\pi_2}\cdots p_s^{\pi_s})}{n^{(\rho)}}\right] = \frac{1}{n^{(\rho)}}\sum_{i=1}^{s}\pi_i p_i(p_1^{\pi_1}p_2^{\pi_2}\cdots p_i^{\pi_i-1}, p_i - 1, \cdots p_s^{\pi_s}),$$

and also that,

$$\frac{(p_1^{\pi_1}p_2^{\pi_2}\cdots p_{s-1}^{\pi_{s-1}}0)}{n^{(\rho)}} = \frac{(n-\rho+1)(p_1^{\pi_1}\cdots p_{s-1}^{\pi_{s-1}})}{n^{(\rho)}} = \frac{(p_1^{\pi_1}\cdots p_{s-1}^{\pi_{s-1}})}{n^{(\rho-1)}}.$$

Since

$$\Omega\left[\frac{(p_1)^{\pi_1}(p_2)^{\pi_2}\cdots(p_s)^{\pi_s}}{n^{\rho}}\right] = \frac{1}{n^{\rho}}\sum_{i=1}^{s}\pi_i p_i(p_1)^{\pi_1}(p_2)^{\pi_2}\cdots(p_i)^{\pi_i-1}(p_i-1)\cdots(p_s)^{\pi_s},$$

and

$$\frac{(p_1)^{\pi_1}(p_2)^{\pi_2}\cdots(p_{s-1})^{\pi_{s-1}}(0)}{n^p}=\frac{n(p_1)^{\pi_1}(p_2)^{\pi_2}\cdots(p_{s-1})^{\pi_{s-1}}}{n^p}=\frac{(p_1)^{\pi_1}\cdots(p_{s-1})^{\pi_{s-1}}}{n^{p-1}},$$

it becomes evident that corresponding to any power sum seminvariant there exists a power product seminvariant with the same numerical coefficients. The converse is also true.

2. **Unbiased Estimates of Rational Integral Moment Functions.** If τ represents a population parameter, and if t represents such a function of n observations that the expected value of t is equal to τ; then t is said to be an unbiased estimate of τ. See Tschuprow [11; 74–75], Bertilsen [8; 144], and Fisher [12].

Let $(p_1p_2\cdots p_s)$ denote a power product computed from a sample, the sample being from an infinite population. Then it is well known that

$$E\left[\frac{(p_1p_2\cdots p_s)}{n^{(s)}}\right]=\mu'_{p_1}\mu'_{p_2}\cdots\mu'_{p_s},$$

n being the number of items in the sample. If E^{-1} be interpreted as "unbiased estimate of," the above relation may also be written

$$E^{-1}[\mu'_{p_1}\mu'_{p_2}\cdots\mu'_{p_s}]=\frac{(p_1p_2\cdots p_s)}{n^{(s)}},\tag{19}$$

and it is seen at once that the power product seminvariants defined in section 1, if computed from a sample of n observations, are the unbiased estimates of the corresponding power sum seminvariants of the infinite population from which the sample is drawn.

This provides an algebraic interpretation as well as a different approach to a topic which has already aroused considerable interest among statisticians. In 1927 Bertilsen [8; 144] gave the estimates of the first four Thiele seminvariants of the population in terms of Thiele seminvariants of the sample. In 1929 R. A. Fisher [12] also obtained these results and gave in addition the estimates of the fifth and sixth Thiele seminvariants. His results are in terms of sample moments. In 1937, P. S. Dwyer [13; 26] gave the estimates of the first five population central moments and indicated also means for obtaining the estimate of any rational integral isobaric moment function.

In the remainder of this chapter

(1) Dwyer's method will be extended and perhaps somewhat simplified,

(2) certain properties of this type of estimate will be pointed out,

(3) estimates of all seminvariants of weight $\leqq 8$ will be made available.

3. **Computation of Estimates.** From the relationship (19) it is possible to write down immediately in a simple, although not immediately useful, form the estimate of any rational integral moment function. Thus the fourth Thiele seminvariant λ_4 is given by

$$\lambda_4=\mu'_4-4\mu'_3\mu'_1-3\mu'^2_2+12\mu'_2\mu'^2_1-6\mu'^4_1,$$

so that the estimate of λ_4 is

$$L_4 = \frac{(4)}{n} - \frac{4(31)}{n^{(2)}} - \frac{3(22)}{n^{(2)}} + \frac{12(211)}{n^{(3)}} - \frac{6(1111)}{n^{(4)}}.$$

Since power products are difficult to compute directly, it is necessary to express the estimates in terms of power sums. Dwyer [6; 30–33] gave a complete discussion of the problem of expanding power products in terms of power sums and also gave tables of power products in terms of power sums for weights $\leqq 6$. By use of (12) it is also possible to use tables giving monomial symmetric functions in terms of power sums. One table by J. R. Roe [14; plate 18] includes all cases of weight $\leqq 10$.

By use of such a table we find

$$\begin{aligned}(31) &= -(4) + (3)(1),\\ (22) &= -(4) + (2)(2),\\ (211) &= 2(4) - 2(3)(1) - (2)(2) + (2)(1)^2,\\ (1111) &= -6(4) + 8(3)(1) + 3(2)(2) - 6(2)(1)^2 + (1)^4.\end{aligned}$$

If these results are substituted in L_4 above and like terms are collected, it is found that

$$n^{(4)}L_4 = n^2(n+1)(4) - 4n(n+1)(3)(1) - 3n(n-1)(2)^2 + 12n(2)(1)^2 - 6(1)^4,$$

a result which agrees with that given by R. A. Fisher [12].

4. **The Dwyer Double Expansion Theorem.** The Dwyer double expansion theorem, [6; 34] and [11; 37–39], states that if any isobaric sum of power products of weight r indicated by

$$(20) \qquad \Sigma \frac{r!}{(q_1!)^{\pi_1} \cdots (q_t!)^{\pi_t} \pi_1! \cdots \pi_t!} b_{q_1^{\pi_1} \cdots q_t^{\pi_t}} (q_1^{\pi_1} \cdots q_t^{\pi_t})$$

be expanded in terms of power sums in a form indicated by

$$(21) \qquad \Sigma \frac{r!}{(p_1!)^{\pi_1} \cdots (p_s!)^{\pi_s} \pi_1! \cdots \pi_s!} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s},$$

then the coefficient a_r of the power sum (r) is given by

$$(22) \qquad a_r = \Sigma(-1)^{\rho-1} \frac{(\rho-1)!\,r!}{(p_1!)^{\pi_1} \cdots (p_s!)^{\pi_s} \pi_1! \cdots \pi_s!} b_{p_1^{\pi_1}} \cdots b_{p_s^{\pi_s}},$$

and that the coefficient $a_{r_1 \cdots r_m}$ of $(r_1)(r_2) \cdots (r_m)$ is

$$(23) \qquad a_{r_1 \cdots r_m} = \overline{a_{r_1} a_{r_2} \cdots a_{r_m}}.$$

The barred product indicates a symbolic multiplication by suffixing of subscripts which is exemplified by

$$\overline{a_3 a_2} = \overline{(b_3 - 3b_{21} + 2b_{111})(b_2 - b_{11})} = b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111} = a_{32}.$$

The application of this theorem to the present problem eliminates the use of tables and permits the independent computation of the coefficient of any particular products of power sums in the expansion in terms of power sums of any given estimate. The illustration given by Dwyer [13; 39, 40] exemplifies both of these points very well.

5. **Estimates of all Seminvariants of Weight** $\leqq 8$. If the estimates of any complete system of seminvariants and all products of these seminvariants up to and including weight r are known, then the estimates of all seminvariants of weight $\leqq r$ are obtainable as a linear combination of these known estimates. For example, suppose that we know the estimates of all Thiele seminvariants of weight $\leqq 5$ and wish to find the estimate of μ_5. Since $\mu_5 = \lambda_5 + 10\lambda_3\lambda_2$,

$$E^{-1}[\mu_5] = M_5 = E^{-1}[\lambda_5] + 10E^{-1}[\lambda_3\lambda_2] = L_5 + 10L_{32}.$$

In table II are given the estimates of all Thiele seminvariants and all products of Thiele seminvariants of weight $\leqq 8$. From this table the expressions for L_5 and L_{32} are obtained and, by taking the combination indicated above, it is seen that

$$\begin{aligned} n^{(5)}M_5 = (n^4 - 5n^3 + 10n^2)(5) - 5(n^3 - 5n^2 + 10n)(4)(1) \\ - 10(n^2 - n)(3)(2) + 10(n^2 - 4n + 8)(3)(1)^2 \\ + 30(n - 2)(2)^2(1) - 10n(2)(1)^3 + 4(1)^5, \end{aligned}$$

a result which checks with that given by Dwyer [13; 27]. In similar fashion the estimate of any other seminvariant of weight $\leqq 8$ can be obtained by use of table II.

6. **Computation Checks.** There are a number of checks which can be applied to the entries in table II. These may be of interest simply as properties of the estimates, and they may be of use in correcting errors which may possibly have crept into the tables.

When any power product of more than one part is expanded into power sums, the sum of the numerical coefficients of the expansion is zero. To prove this we need only to consider a set of observations of which one observation is unity and the rest are all zero. Then any power product of two or more parts is necessarily zero and all power sums are equal to unity. Hence the initial statement of the paragraph follows immediately.

From this fact it is apparent that the sum of the coefficients of L_r is $\frac{1}{n}$, and the sum of the coefficients of $L_{r_1 r_2 \cdots r_s}$ is zero. Thus for L_4 we have $\frac{n^3 + n^2 - 4(n^2 + n) - 3(n^2 - n) + 12n - 6}{n^{(4)}} = \frac{1}{n}$, and for L_{22} the sum of the coefficients is

$$\frac{1}{n^{(4)}}[-n^2 + n + 4n - 4 + n^2 - 3n + 3 - 2n + 1] = 0.$$

TABLE II

Estimates of All Thiele Seminvariants and Their Products of Weight ≤ 8

$w = 1$	nL_1
(1)	1

$w = 2$	$n^{(2)}L_2$
(2)	n
$(1)^2$	-1

$w = 3$	$n^{(3)}L_3$
(3)	n^2
(2)(1)	$-3n$
$(1)^3$	2

$w = 4$	$n^{(4)}L_4$	$n^{(4)}L_{22}$
(4)	$n^3 + n^2$	$-(n^2 - n)$
(3)(1)	$-4(n^2 + n)$	$4(n - 1)$
$(2)^2$	$-3(n^2 - n)$	$n^2 - 3n + 3$
$(2)(1)^2$	$12n$	$-2n$
$(1)^4$	-6	1

$w = 5$	$n^{(5)}L_5$	$n^{(5)}L_{32}$
(5)	$n^4 + 5n^3$	$-(n^3 - n^2)$
(4)(1)	$-5(n^3 + 5n^2)$	$5(n^2 - n)$
(3)(2)	$-10(n^3 - n^2)$	$n^3 - 2n^2 + 2n$
$(3)(1)^2$	$20(n^2 + 2n)$	$-(n^2 + 8n - 8)$
$(2)^2(1)$	$30(n^2 - n)$	$-3(n^2 - 2n + 2)$
$(2)(1)^3$	$-60n$	$+5n$
$(1)^5$	24	-2

$w = 6$	$n^{(6)}L_6$	* $n^{(6)}L_{42}$	$n^{(6)}L_{33}$	$n^{(6)}L_{222}$
(6)	$n^5 + 16n^4 + 11n^3 - 4n^2$	$-(n^4 + 2n^3 - 7n^2 + 4n)$	$-(n^4 - 2n^3 + 5n^2 - 4n)$	$2(n^3 - 3n^2 + 2n)$
(5)(1)	$-6(n^4 + 16n^3 + 11n^2 - 4n)$	$6(n^3 + 2n^2 - 7n + 4)$	$6(n^3 - 2n^2 + 5n - 4)$	$-12(n^2 - 3n + 2)$
(4)(2)	$-15(n^4 - 4n^3 - n^2 + 4n)$	$n^4 - 10n^2 + 45n - 60$	$3(2n^3 - 5n^2 - 5n + 20)$	$-3(n^3 - 7n^2 + 20n - 20)$
$(3)^2$	$-10(n^4 - 2n^3 + 5n^2 - 4n)$	$2(2n^3 - 5n^2 - 5n + 20)$	$n^4 - 8n^3 + 25n^2 - 10n - 40$	$-2(3n^2 - 15n + 20)$
$(4)(1)^2$	$30(n^3 + 9n^2 + 2n)$	$-(n^3 + 15n^2 + 20n - 60)$	$-3(7n^2 - 15n + 20)$	$3(n^2 + 3n - 10)$
(3)(2)(1)	$120(n^3 - n)$	$-4(n^3 + 6n^2 - 25n + 30)$	$-6(n^3 - 4n^2 + 15n - 20)$	$12(n^2 - 4n + 5)$
$(2)^3$	$30(n^3 - 3n^2 + 2n)$	$-3(n^3 - 7n^2 + 20n - 20)$	$-3(3n^2 - 15n + 20)$	$n^3 - 9n^2 + 29n - 30$
$(3)(1)^3$	$-120(n^2 + 3n)$	$4(n^2 + 9n - 10)$	$4(n^2 + 3n)$	$-4(3n - 5)$
$(2)^2(1)^2$	$-270(n^2 - n)$	$3(5n^2 - 9n + 10)$	$9(n^2 - n)$	$-3(n^2 - 3n + 5)$
$(2)(1)^4$	$360n$	$-18n$	$-12n$	$3n$
$(1)^6$	-120	6	4	-1

TABLE II—*Continued*

$w = 7$	$n^{(7)}L_7$	$n^{(7)}L_{52}$	$n^{(7)}L_{43}$	$n^{(7)}L_{322}$
(7)	$n^6 + 42n^5 + 119n^4 - 42n^3$	$-(n^5 + 12n^4 - 31n^3 + 18n^2)$	$-(n^5 + 5n^3 - 6n^2)$	$2(n^4 - 3n^3 + 2n^2)$
(6)(1)	$-7(n^5 + 42n^4 + 119n^3 - 42n^2)$	$7(n^4 + 12n^3 - 31n^2 + 18n)$	$7(n^4 + 5n^2 - 6n)$	$-14(n^3 - 3n^2 + 2n)$
(5)(2)	$-21(n^5 + 12n^4 - 31n^3 + 18n^2)$	$n^5 + 6n^4 - 28n^3 + 99n^2 - 162n$	$3(3n^4 - 10n^3 + 5n^2 + 18n)$	$-2(n^4 - 6n^3 + 17n^2 - 18n)$
(4)(3)	$-35(n^5 + 5n^3 - 6n^2)$	$5(3n^4 - 10n^3 + 5n^2 + 18n)$	$n^5 - 6n^4 + 30n^3 - 35n^2 - 30n$	$-(n^4 - n^3 - 10n^2 + 20n)$
(5)(1)^2	$42(n^4 + 27n^3 + 44n^2 - 12n)$	$-(n^4 + 27n^3 + 224n^2 - 552n + 216)$	$-6(5n^3 - 5n^2 + 20n - 12)$	$2(n^3 + 15n^2 - 46n + 24)$
(4)(2)(1)	$210(n^4 + 6n^3 - 13n^2 + 6n)$	$-5(n^4 + 15n^3 - 58n^2 + 114n - 108)$	$-3(n^4 + 9n^3 - 20n^2 - 10n + 60)$	$13n^3 - 63n^2 + 140n - 120$
(3)^2(1)	$140(n^4 + 5n^2 - 6n)$	$-20(3n^3 - 10n^2 + 5n + 18)$	$-4(n^4 - 6n^3 + 30n^2 - 35n - 30)$	$4(n^3 - n^2 -$ [illegible] $+ 20)$
(3)(2)^2	$210(n^4 - 3n^3 + 2n^2)$	$-10(n^4 - 6n^3 + 17n^2 - 18n)$	$-3(n^4 - n^3 - 10n^2 + 20n)$	$n^4 - 7n^3 + 21n^2 - 20n$
(4)(1)^3	$-210(n^3 + 13n^2 + 6n)$	$5(n^3 + 19n^2 + 36n - 108)$	$2(n^3 + 34n^2 - 45n + 90)$	$-4(3n^2 + 2n - 15)$
(3)(2)(1)^2	$-1260(n^3 + n^2 - 2n)$	$30(n^3 + 7n^2 - 28n + 36)$	$12(2n^3 - 2n^2 + 15n - 30)$	$-2(n^3 + 12n^2 - 48n + 60)$
(2)^3(1)	$-630(n^3 - 3n^2 + 2n)$	$30(n^3 - 6n^2 + 17n - 18)$	$9(n^3 - n^2 - 10n + 20)$	$-3(n^3 - 7n^2 + 21n - 20)$
(3)(1)^4	$840(n^2 + 4n)$	$-20(n^2 + 10n - 12)$	$-14(n^2 + 4n)$	$n^2 + 24n - 40$
(2)^2(1)^3	$2520(n^2 - n)$	$-30(3n^2 - 5n + 6)$	$-42(n^2 - n)$	$2(4n^2 - 9n + 15)$
(2)(1)^5	$-2520n$	$84n$	$42n$	$-7n$
(1)^7	720	-24	-12	2

TABLE II—*Continued*

$w = 8$	$n^{(8)}L_8$	$n^{(8)}L_{62}$	$n^{(8)}L_{63}$
(8)	$n^7 + 99n^6 + 757n^5 + 141n^4 - 398n^3 + 120n^2$	$-(n^6 + 37n^5 - 39n^4 - 157n^3 + 278n^2 - 120n)$	$-(n^6 + 9n^5 - 23n^4 + 111n^3 - 218n^2 + 120n)$
(7)(1)	$-8(n^6 + 99n^5 + 757n^4 + 141n^3 - 398n^2 + 120n)$	$8(n^5 + 37n^4 - 39n^3 - 157n^2 + 278n - 120)$	$8(n^5 + 9n^4 - 23n^3 + 111n^2 - 218n + 120)$
(6)(2)	$-28(n^6 + 37n^5 - 39n^4 - 157n^3 + 278n^2 - 120n)$	$n^6 + 20n^5 + 3n^4 - 336n^3 + 1736n^2 - 4424n + 3360$	$13n^5 - 14n^4 - 57n^3 - 406n^2 + 2744n - 3360$
(5)(3)	$-56(n^6 + 9n^5 - 23n^4 + 111n^3 - 218n^2 + 120n)$	$2(13n^5 - 14n^4 - 57n^3 - 406n^2 + 2744n - 3360)$	$n^6 + n^5 - 51n^4 + 527n^3 - 1134n^2 - 2128n + 6720$
$(4)^2$	$-35(n^6 + n^5 + 33n^4 - 121n^3 + 206n^2 - 120n)$	$5(3n^5 - 11n^4 + 11n^3 + 119n^2 - 602n + 840)$	$5(n^5 + n^4 - 39n^3 + 119n^2 + 182n - 840)$
$(6)(1)^2$	$56(n^5 + 68n^4 + 359n^3 - 8n^2 - 60n)$	$-(n^5 + 48n^4 + 1039n^3 - 1428n^2 - 2660n + 3360)$	$-(41n^4 + 238n^3 - 701n^2 + 2702n - 3360)$
(5)(2)(1)	$336(n^5 + 23n^4 - 31n^3 - 23n^2 + 30n)$	$-6(n^5 + 33n^4 - 11n^3 - 393n^2 + 1330n - 1680)$	$-3(n^5 + 27n^4 - 79n^3 + 413n^2 - 1946n + 3360)$
(4)(3)(1)	$560(n^5 + 5n^4 + 5n^3 - 5n^2 - 6n)$	$-10(25n^4 - 58n^3 - 13n^2 + 70n + 336)$	$-5(n^5 + 9n^4 - 43n^3 + 215n^2 - 182n - 672)$
$(4)(2)^2$	$420(n^5 + 2n^4 - 25n^3 + 46n^2 - 24n)$	$-15(n^5 - 2n^4 - 27n^3 + 236n^2 - 700n + 672)$	$-15(5n^4 - 26n^3 - 25n^2 + 406n - 672)$
$(3)^2(2)$	$560(n^5 - 4n^4 + 11n^3 - 20n^2 + 12n)$	$-10(n^5 - 5n^3 - 96n^2 + 532n - 672)$	$-10(n^5 - 11n^4 + 61n^3 - 101n^2 - 238n + 672)$
$(5)(1)^3$	$-336(n^4 + 38n^3 + 99n^2 - 18n)$	$6(n^4 + 38n^3 + 339n^2 - 738n)$	$2(n^4 + 68n^3 + 159n^2 - 288n + 756)$
$(4)(2)(1)^2$	$-2520(n^4 + 10n^3 - 17n^2 + 6n)$	$45(n^4 + 18n^3 - 41n^2 + 22n)$	$15(n^4 + 28n^3 - 113n^2 + 264n - 252)$
$(3)^2(1)^2$	$-1680(n^4 + 2n^3 + 7n^2 - 10n)$	$10(n^4 + 50n^3 - 121n^2 - 122n + 672)$	$20(n^4 - n^3 + 9n^2 + 57n - 210)$
$(3)(2)^2(1)$	$-5040(n^4 - 2n^3 - n^2 + 2n)$	$120(n^4 - n^3 - 16n^2 + 70n - 84)$	$60(n^4 - 6n^3 + 35n^2 - 126n + 168)$
$(2)^4$	$-630(n^4 - 6n^3 + 11n^2 - 6n)$	$30(n^4 - 12n^3 + 62n^2 - 147n + 126)$	$90(n^3 - 10n^2 + 35n - 42)$
$(4)(1)^4$	$1680(n^3 + 17n^2 + 12n)$	$-30(n^3 + 23n^2 + 54n - 168)$	$-10(n^3 + 38n^2 - 45n + 126)$
$(3)(2)(1)^3$	$13440(n^3 + 2n^2 - 3n)$	$-240(n^3 + 8n^2 - 31n + 42)$	$-20(7n^3 + 2n^2 + 45n - 126)$
$(2)^3(1)^2$	$10080(n^3 - 3n^2 + 2n)$	$-60(5n^3 - 27n^2 + 76n - 84)$	$-90(n^3 - 2n^2 - 5n + 14)$
$(3)(1)^5$	$-6720(n^2 + 5n)$	$120(n^2 + 11n - 14)$	$64(n^2 + 5n)$
$(2)^2(1)^4$	$-25200(n^2 - n)$	$90(7n^2 - 11n + 14)$	$240(n^2 - n)$
$(2)(1)^6$	$20160n$	$-480n$	$-192n$

TABLE II—*Continued*

$w = 8$	$n^{(8)}L_{44}$	$n^{(8)}L_{422}$	$n^{(8)}L_{332}$	$n^{(8)}L_{2222}$
(8)	$-(n^6 + n^5 + 33n^4 - 121n^3 + 206n^2 - 120n)$	$2(n^5 + 2n^4 - 25n^3 + 46n^2 - 24n)$	$2(n^5 - 4n^4 + 11n^3 - 20n^2 + 12n)$	$-6(n^4 - 6n^3 + 11n^2 - 6n)$
(7)(1)	$8(n^5 + n^4 + 33n^3 - 121n^2 + 206n - 120)$	$-16(n^4 + 2n^3 - 25n^2 + 46n - 24)$	$-16(n^4 - 4n^3 + 11n^2 - 20n + 12)$	$48(n^3 - 6n^2 + 11n - 6)$
(6)(2)	$4(3n^5 - 11n^4 + 11n^3 + 119n^2 - 602n + 840)$	$-2(n^5 - 2n^4 - 27n^3 + 236n^2 - 700n + 672)$	$-(n^5 - 5n^3 - 96n^2 + 532n - 672)$	$8(n^4 - 12n^3 + 62n^2 - 147n + 126)$
(5)(3)	$8(n^5 + n^4 - 39n^3 + 119n^2 + 182n - 840)$	$-4(5n^4 - 26n^3 - 25n^2 + 406n - 672)$	$-2(n^5 - 11n^4 + 61n^3 - 101n^2 - 238n + 672)$	$48(n^3 - 10n^2 + 35n - 42)$
$(4)^2$	$n^6 - 13n^5 + 76n^4 - 37n^3 - 497n^2 - 490n + 4200$	$(-n^5 - 8n^4 + 33n^3 + 62n^2 - 868n + 1680)$	$-2(4n^4 - 31n^3 + 65n^2 + 112n - 420)$	3($n^4 - 14n^3$ + [illegible]5$n^2 - 322n + 420$)
$(6)(1)^2$	$-8(5n^4 - 2n^3 + 121n^2 - 364n + 420)$	$2(n^4 + 26n^3 + 29n^2 - 464n + 588)$	$n^4 + 56n^3 - 229n^2 + 520n - 588$	$-8(n^3 + 9n^2 - 64n + 84)$
(5)(2)(1)	$-48(2n^4 - 5n^3 - 14n^2 + 119n - 210)$	$12(n^4 + 3n^3 - 53n^2 + 211n - 294)$	$6(2n^4 - 11n^3 + 56n^2 - 197n + 294)$	$-48(n^3 - 9n^2 + 32n - 42)$
(4)(3)(1)	$-8(n^5 - 8n^4 + 81n^3 - 232n^2 + 98n + 420)$	$4(2n^4 + 9n^3 - 64n^2 - n + 294)$	$2(5n^4 - 23n^3 + 57n^2 + 15n - 294)$	$-24(n^3 - 4n^2 - 5n + 28)$
$(4)(2)^2$	$-6(n^5 - 8n^4 + 33n^3 + 62n^2 - 868n + 1680)$	$n^5 - n^4 - 94n^3 + 868n^2 - 2952n + 3528$	$3(2n^4 - 7n^3 - 57n^2 + 380n - 588)$	$-6(n^4 - 16n^3 + 104n^2 - 305n + 336)$
$(3)^2(2)$	$-16(4n^4 - 31n^3 + 65n^2 + 112n + 420)$	$4(2n^4 - 7n^3 - 57n^2 + 380n - 588)$	$n^5 - 12n^4 + 63n^3 - 52n^2 - 536n + 1176$	$-8(3n^3 - 33n^2 + 128n - 168)$
$(5)(1)^3$	$48(3n^3 - 4n^2 + 31n - 42)$	$-4(3n^3 + 32n^2 - 77n - 42)$	$-2(5n^3 + 34n^2 - 117n + 126)$	$48(n^2 - 3n)$
$(4)(2)(1)^2$	$24(n^4 + 2n^3 + 32n^2 - 155n + 210)$	$-2(n^4 + 20n^3 - 22n^2 - 119n + 210)$	$-3(19n^3 - 92n^2 + 223n - 210)$	$12(n^3 - 3n^2 + 2n)$
$(3)^2(1)^2$	$16(n^4 - 4n^3 + 50n^2 - 167n + 210)$	$-4(6n^3 + 11n^2 - 185n + 378)$	$-(n^4 + 8n^3 - 29n^2 + 176n - 476)$	$8(9n^2 - 57n + 98)$
$(3)(2)^2(1)$	$24(n^4 + 8n^3 - 91n^2 + 322n - 420)$	$-4(n^4 + 11n^3 - 136n^2 + 526n - 672)$	$-6(n^4 - 8n^3 + 49n^2 - 166n + 224)$	$24(n^3 - 10n^2 + 38n - 49)$

$(2)^4$	$9(n^4 - 14n^3 + 95n^2 - 322n + 420)$	$-3(n^4 - 16n^3 + 104n^2 - 305n + 336)$	$-3(3n^3 - 33n^2 + 128n - 168)$	$n^4 - 18n^3 + 125n^2 - 384n + 441$
$(4)(1)^4$	$-12(n^3 + 17n^2 + 12n)$	$n^3 + 35n^2 + 138n - 504$	$41n^2 - 53n + 42$	$-6(n^2 + 7n - 28)$
$(3)(2)(1)^3$	$-96(n^3 + 2n^2 - 3n)$	$8(n^3 + 20n^2 - 87n + 126)$	$2(5n^3 + 14n^2 - 17n - 42)$	$-16(3n^2 - 14n + 21)$
$(2)^3(1)^2$	$-72(n^3 - 3n^2 + 2n)$	$18(n^3 - 7n^2 + 24n - 28)$	$3(3n^3 - 12n^2 + 15n + 14)$	$-4(n^3 - 9n^2 + 35n - 42)$
$(3)(1)^5$	$48(n^2 + 5n)$	$-4(n^2 + 23n - 42)$	$-4(n^2 + 11n - 14)$	$8(3n - 7)$
$(2)^2(1)^4$	$180(n^2 - n)$	$-3(11n^2 - 23n + 42)$	$-3(7n^2 - 11n + 14)$	$6(n^2 - 3n + 7)$
$(2)(1)^6$	$-144n$	$24n$	$16n$	$-4n$
$(1)^8$	36	-6	-4	1

A condition satisfied by the coefficients of any seminvariant is that their sum is equal to zero (See section 2). This provides another check on the entries of table II, although the seminvariant must be written in homogeneous form before the check is applied. Thus we may write

$$L_4 = \frac{n^3}{n^{(4)}}\left[(n+1)\frac{(4)}{n} - 4(n+1)\frac{(3)(1)}{n^2} - 3(n-1)\frac{(2)^2}{n^2} + 12n\frac{(2)(1)^2}{n^3} - 6n\frac{(1)^4}{n^4}\right],$$

and the sum of coefficients is

$$(n+1) - 4(n+1) - 3(n-1) + 12n - 6n = 0.$$

Several checks arise from the fact (see section 6) that every seminvariant must be annihilated by the operator

$$\Omega' = \sum_{i=1}^{n} i s_{i-1} \frac{\partial}{\partial s_i}. \tag{24}$$

Another check results from the discussion of the next section and is so apparent as to need no comment.

All the checks mentioned in this section are applicable to the estimate of any seminvariant.

7. **Estimates as Sums of Simple Seminvariants.** A seminvariant such as L_4 in which the coefficients of the m''s are functions of n will be called a composite seminvariant, while a seminvariant in which the coefficients of the m''s are purely numerical will be called simple. The fact that is to be established in this section is that every composite seminvariant is the sum of simple seminvariants. As an illustration consider L_4. It is apparent that

$$L_4 = \frac{n^4}{n^{(4)}} l_4 + \frac{n^3}{n^{(4)}} k_4,$$

where l_4 and k_4 are seminvariants of the sample corresponding to λ_4 and κ_4. Both l_4 and k_4 are simple seminvariants.

That a composite seminvariant may always be expressed as a sum of simple seminvariants can be demonstrated by considering the effect of Ω', (24), on a composite seminvariant. The coefficients are polynomials in n and are unaffected by the operator. The expression resulting from application of the operator can vanish only if the coefficient of n^r vanishes for every r. Thus a composite seminvariant which has r different powers of n appearing in its coefficients is expressible as the sum of r simple seminvariants, which are not necessarily distinct. Table III exhibits the estimates of Thiele seminvariants of weight $\leqq 6$ as sums of simple seminvariants.

Since the factors, appearing in front of each of the simple seminvariants in the expression resulting from breaking down a composite seminvariant, are of

TABLE III

Estimates as Sums of Seminvariants

L_2	$\frac{n^2}{n^{(2)}}$
$\frac{(2)}{n}$	1
$\frac{(1)^2}{n^2}$	−1

L_3	$\frac{n^3}{n^{(3)}}$
$\frac{(3)}{n}$	1
$\frac{(2)(1)}{n^2}$	−3
$\frac{(1)^3}{n^3}$	2

L^4	$\frac{n^4}{n^{(4)}}$	$\frac{n^3}{n^{(4)}}$
$\frac{(4)}{n}$	1	1
$\frac{(3)(1)}{n^2}$	−4	−4
$\frac{(2)^2}{n^2}$	−3	3
$\frac{(2)(1)^2}{n^3}$	12	
$\frac{(1)^4}{n^4}$	−6	

L_{22}	$\frac{n^4}{n^{(4)}}$	$-\frac{n^3}{n^{(4)}}$	$\frac{n^2}{n^{(4)}}$
$\frac{(4)}{n}$		1	1
$\frac{(3)(1)}{n^2}$		−4	−4
$\frac{(2)^2}{n^2}$	1	3	3
$\frac{(2)(1)^2}{n^3}$	−2		
$\frac{(1)^4}{n^4}$	1		

L_{32}	$\frac{n^5}{n^{(5)}}$	$-\frac{n^4}{n^{(5)}}$	$\frac{n^3}{n^{(5)}}$
$\frac{(5)}{n}$		1	1
$\frac{(4)(1)}{n^2}$		−5	−5
$\frac{(3)(2)}{n^2}$	1	2	2
$\frac{(3)(1)^2}{n^3}$	−1	8	8
$\frac{(2)^2(1)}{n^3}$	−3	−6	−6
$\frac{(2)(1)^3}{n^4}$	5		
$\frac{(1)^5}{n^5}$	−2		

L_6	$\frac{n^6}{n^{(6)}}$	$\frac{2n^5}{n^{(6)}}$	$\frac{n^4}{n^{(6)}}$	$-\frac{4n^3}{n^{(6)}}$
$\frac{(6)}{n}$	1	8	11	1
$\frac{(5)(1)}{n^2}$	−6	−48	−66	−6
$\frac{(4)(2)}{n^2}$	−15	−15	105	15
$\frac{(3)^2}{n^2}$	−10	10	−50	−10
$\frac{(4)(1)^2}{n^3}$	30	135	60	
$\frac{(3)(2)(1)}{n^3}$	120		−120	
$\frac{(2)^3}{n^3}$	30	−45	60	
$\frac{(3)(1)^3}{n^4}$	−120	−180		
$\frac{(2)^2(1)^2}{n^4}$	−270	135		
$\frac{(2)(1)^4}{n^5}$	360			
$\frac{(1)^6}{n^6}$	−120			

L_5	$\frac{n^5}{n^{(5)}}$	$\frac{5n^4}{n^{(5)}}$
$\frac{(5)}{n}$	1	1
$\frac{(4)(1)}{n^2}$	−5	−5
$\frac{(3)(2)}{n^2}$	−10	2
$\frac{(3)(1)^2}{n^3}$	20	8
$\frac{(2)^2(1)}{n^3}$	30	−6
$\frac{(2)(1)^3}{n^4}$	−60	
$\frac{(1)^5}{n^5}$	24	

TABLE III—*Continued*

L_{42}	$\frac{n^6}{n^{(6)}}$	$-\frac{n^5}{n^{(6)}}$	$-\frac{2n^4}{n^{(6)}}$	$\frac{n^3}{n^{(6)}}$	$-\frac{4n^2}{n^{(6)}}$
$\frac{(6)}{n}$		1	1	7	1
$\frac{(5)(1)}{n^2}$		−6	−6	−42	−6
$\frac{(4)(2)}{n^2}$	1	15	5	45	15
$\frac{(3)^2}{n^2}$		−4	5	−10	−10
$\frac{(4)(1)^2}{n^3}$	−1		10	60	
$\frac{(3)(2)(1)}{n^3}$	−4	24	−50	−120	
$\frac{(2)^3}{n^3}$	−3	−21	30	60	
$\frac{(3)(1)^3}{n^4}$	4	−36	20		
$\frac{(2)^2(1)^2}{n^4}$	15	27	−15		
$\frac{(2)(1)^4}{n^5}$	−18				
$\frac{(1)^6}{n^6}$	6				

L_{33}	$\frac{n^6}{n^{(6)}}$	$-\frac{n^5}{n^{(6)}}$	$\frac{n^4}{n^{(6)}}$	$-\frac{5n^3}{n^{(6)}}$	$\frac{4n^2}{n^{(6)}}$
$\frac{(6)}{n}$		1	2	1	1
$\frac{(5)(1)}{n^2}$		−6	−12	−6	−6
$\frac{(4)(2)}{n^2}$		−6	−15	3	15
$\frac{(3)^2}{n^2}$	1	8	25	2	−10
$\frac{(4)(1)^2}{n^3}$		21	45	12	
$\frac{(3)(2)(1)}{n^3}$	−6	−24	−90	−24	
$\frac{(2)^3}{n^3}$		9	45	12	
$\frac{(3)(1)^3}{n^4}$	4	−12			
$\frac{(2)^2(1)^2}{n^4}$	9	9			
$\frac{(2)(1)^4}{n^5}$	−12				
$\frac{(1)^6}{n^6}$	4				

L_{222}	$\frac{n^6}{n^{(6)}}$	$-\frac{3n^5}{n^{(6)}}$	$\frac{n^4}{n^{(6)}}$	$-\frac{6n^3}{n^{(6)}}$	$\frac{4n^2}{n^{(6)}}$
$\frac{(6)}{n}$			2	1	1
$\frac{(5)(1)}{n^2}$			−12	−6	−6
$\frac{(4)(2)}{n^2}$		1	21	10	15
$\frac{(3)^2}{n^2}$			−6	−5	−10
$\frac{(4)(1)^2}{n^3}$		−1	9	5	
$\frac{(3)(2)(1)}{n^3}$		−4	−48	−10	
$\frac{(2)^3}{n^3}$	1	3	29	5	
$\frac{(3)(1)^3}{n^4}$		4	20		
$\frac{(2)^2(1)^2}{n^4}$	−3	−3	−15		
$\frac{(2)(1)^4}{n^5}$	3				
$\frac{(1)^6}{n^6}$	−1				

successively lower order with respect to n; it is possible to obtain approximations of various orders to the value of an estimate by using the appropriate portion of the expression given in the table.

8. **The Estimates of the κ's.** The seminvariant κ_r possesses an interesting property which will be called invariance under estimate. By this is meant that the estimate of κ_r is k_r multiplied by a suitable factor. In particular, $\kappa_2 = \mu_2$ and $\kappa_3 = \mu_3$ and it is well known that

$$E^{-1}[\mu_2] = \frac{n^2}{n^{(2)}} m_2, \qquad E^{-1}[\mu_3] = \frac{n^3}{n^{(3)}} m_3$$

so that the κ_r certainly possesses the property for $r = 2$ and 3. It can be shown, however, that

$$K_{2r} = \frac{n^2}{n^{(2)}} K_{2r}, \qquad K_{2r+1} = \frac{n^3}{n^{(3)}} K_{2r+1}. \tag{25}$$

From (15)

$$\kappa_{2r} = \frac{1}{2} \sum_{i=0}^{2r} \binom{2r}{i} \mu'_i \mu'_{2r-i}$$

so that

$$K_{2r} = \frac{1}{2} \sum_{i=1}^{2r-1} (-1)^i \frac{(i, 2r - i)}{n^{(2)}} + \frac{(2r)}{n}.$$

By the Binet-Waring identities [15; 6–7]

$$(a \cdot b) = (a)(b) - (a + b) \tag{26}$$

and this holds for power products regardless of the values of a and b. Hence

$$K_{2r} = \frac{(2r)}{n} + \frac{1}{2} \sum_{i=1}^{2r-1} (-1)^i \binom{2r}{i} \frac{(i)(2r - i) - (2r)}{n^{(2)}}$$

$$= \frac{(2r)}{n} \left[1 - \frac{1}{2} \sum_{i=1}^{2r-1} \frac{(-1)^i \binom{2r}{i}}{n - 1} \right] + \frac{1}{2} \sum_{i=1}^{2r-1} (-1)^i \binom{2r}{i} \frac{(i)(2r - i)}{n^{(2)}}.$$

Since

$$\sum_{i=0}^{2r} (-1)^i \binom{2r}{i} = 0 = 2 + \sum_{i=1}^{2r-1} (-1)^i \binom{2r}{i},$$

the coefficient of $\frac{(2r)}{n}$ above is $\frac{n}{n - 1}$ and it follows immediately that

$$K_{2r} = \frac{1}{2} \sum_{i=0}^{2r} (-1)^i \binom{2r}{i} \frac{(i)(2r - i)}{n^{(2)}} = \frac{n^2}{n^{(2)}} K_{2r}.$$

This proves the first half of (25) and the second half can be proved in similar fashion, although with considerably more difficulty.

9. **Other Simple Seminvariants which are Invariant under Estimate.** It has been previously remarked (Chapter I, section 2) that the κ system of seminvariants are the seminvariants of minimum degree, those of even weight being of second degree and those of odd weight being of third degree. The κ_{2r}'s are the only seminvariants of degree 2, but for odd weights greater than 7, there exist more than one seminvariant of degree 3. It is not difficult to show that these additional minimum degree seminvariants are also invariant under estimate. The type of proof used could have been applied equally well to obtain the results of the preceding section and indicates that the property of invariance under estimate which is possessed by the κ's is a direct result of their minimum degree property.

Consider the estimate in power product form of any seminvariant of degree 3 and odd weight. Power products of 1, 2 and 3 parts will appear. By the Binet-Waring identities each three part power product (abc) yields a third degree power sum product $(a)(b)(c)$ plus other products of lower degree. Since $(a)(b)(c)$ comes only from (abc) its coefficient must be identical with that of (abc) and will therefore be a constant divided by $n^{(3)}$. The coefficient of each second degree product of power sums will be a sum of terms, the first of which comes from the corresponding two part power product with a coefficient identical with that of the power product, and the others come from the three part power products. Then the coefficient of a second degree product of power sums must be of the form

$$\frac{c_1}{n^{(2)}} + \frac{c_2 + c_3 + \cdots + c_t}{n^{(3)}} = \frac{c_1 n + c_2'}{n^{(3)}}.$$

Similarly the coefficient of the first degree power sum term will be of the form

$$\frac{d_1 n^2 + d_2 n + d_3}{n^{(3)}}.$$

Since the estimate of a seminvariant is a seminvariant, it follows that $d_3 \equiv 0$. This is true because the coefficient of $\frac{(r-1)(1)}{n^2}$ must be the coefficient of $\frac{(r)}{n}$ multiplied by $-r$. Furthermore $c_2' = d_2 = 0$ for if the contrary be assumed it is immediately possible to break the composite seminvariant into two simple seminvariants, the first being of degree 3 (the original seminvariant) and the second of degree 2. Since for odd weights no seminvariant of degree 2 exists, it follows that any seminvariant of degree 3 and odd weight is invariant under estimate. It is also apparent that the factor $n^3/n^{(3)}$ must appear in the estimate.

10. **Composite Seminvariants which are Invariant under Estimate.** For each weight $r \geqq 4$ there exists a composite seminvariant which is invariant under estimate. For weights 4 and 5 this seminvariant is easily obtained by use

of Table III. Thus for weight 4, form the seminvariant $\lambda_4 + c_{22}\lambda_2^2$. From the table we find that

$$E^{-1}[\lambda_4 + c_{22}\lambda_2^2] = \frac{n^4}{n^{(4)}} l_4 + \frac{n^3}{n^{(4)}} k_4 + c_{22}\frac{n^4}{n^{(4)}} l_2^2 - c_{22}\frac{n}{(n-2)^{(2)}} k_4$$

$$= \frac{n^4}{n^{(4)}}(l_4 + c_{22}l_2^2) + \frac{n}{n^{(4)}}(n^2 - n^{(2)}c_{22})k_4.$$

If $c_{22} = n^2/n^{(2)}$ the seminvariant is invariant under estimate. This seminvariant is

$$\psi_4 = \lambda_4 + \frac{n^2}{n^{(2)}}\lambda_2^2. \tag{27}$$

In similar fashion we find for weight 5

$$\psi_5 = \lambda_5 + \frac{5n^2}{n^{(2)}}\lambda_3\lambda_2. \tag{28}$$

For weights > 5 considerably more difficulty is encountered. For weight 6, for example, we consider the seminvariant

$$\lambda_6 + c_{42}\lambda_4\lambda_2 + c_{33}\lambda_3^2 + c_{222}\lambda_2^3.$$

By use of table III we obtain

$$E^{-1}[\lambda_6 + c_{42}\lambda_4\lambda_2 + c_{33}\lambda_3^2 + c_{222}\lambda_2^3] = \frac{n^6}{n^{(6)}}(l_6 + c_{42}l_4l_2 + c_{33}l_3^2 + c_{222}l_2^3) + \Phi,$$

where Φ is a sum of other seminvariants with coefficients which are functions of n and c_{42}, c_{33}, c_{222}. Now there are only four linearly independent seminvariants of weight 6 and it is necessary that one of these involve the term $(1)^6/n^6$. By an argument analogous to that of the previous section this term cannot appear in Φ and therefore Φ is expressible in terms of three or fewer seminvariants. Actually three are necessary and equating the coefficients of these to zero the values of c_{42}, c_{33} and c_{222} are uniquely determined. The result is somewhat lengthy and scarcely of sufficient interest to record here.

The same sort of procedure can be used for determining seminvariants of higher order which are invariant under estimate, but the labor of computation becomes very great.

It is possible to obtain moment functions which are invariant under estimate by means of a set of equations given by Dwyer [13; 38–39]. These equations connect the coefficients of a general isobaric moment function and the coefficients of the expected value of that function. In his notation if, for example,

$$f_4 = a_4(4) + 4a_{31}(3)(1) + 3a_{22}(2)^2 + 6a_{211}(2)(1)^2 + a_{1111}(1)^4,$$

then

$$E[f_4] = b_4 n\mu_4' + 4b_{31}n^{(2)}\mu_3'\mu_1' + 3b_{22}n^{(2)}\mu_2'^2 + 6b_{211}n^{(3)}\mu_2'\mu_1'^2 + n^{(4)}b_{1111}\mu_1'^4,$$

wherein:

$$
(29)\qquad \begin{aligned}
a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111} &= b_4 \ , \\
a_{31} \qquad + 3a_{211} + a_{1111} &= b_{31} \ , \\
a_{22} + 2a_{211} + a_{1111} &= b_{22} \ , \\
a_{211} + a_{1111} &= b_{211} \ , \\
a_{1111} &= b_{1111} \ .
\end{aligned}
$$

The problem at hand demands that

$$
E\left[na_4 \frac{(4)}{n} + 4n^2 a_{31} \frac{(3)(1)}{n^2} + 3n^2 a_{22} \frac{(2)^2}{n^2} + 6n^3 a_{211} \frac{(2)(1)^2}{n^3} + n^4 a_{1111} \frac{(1)^4}{n^4} \right]
$$
$$
= \lambda [na_4 \mu_4' + 4n^{(2)} a_{31} \mu_3' \mu_1' + 3n^{(2)} a_{22} \mu_2'^2 + 6n^{(3)} a_{211} \mu_2' \mu_1'^2 + n^{(4)} a_{1111} \mu_1'^4]
$$

so that the equations (29) become

$$
\begin{aligned}
n^4 a_{1111} &= \lambda n^{(4)} a_{1111} \ , \\
n^3 a_{211} &= \lambda n^{(3)} (a_{211} + a_{1111}), \\
n^2 a_{22} &= \lambda n^{(2)} (a_{22} + 2a_{211} + a_{1111}), \\
n^2 a_{31} &= \lambda n^{(2)} (a_{31} + 3a_{211} + a_{1111}), \\
na_4 &= \lambda n (a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111}),
\end{aligned}
$$

and from these equations a_4, a_{31}, a_{22}, a_{211} can be found in terms of a_{1111}. Obviously there is only one solution if none of the a's are zero. In general, for any weight r, a similar system of equations can be found and they determine the coefficients of a moment function of weight r which is invariant under estimate. It appears that this moment function is always a seminvariant although no proof of the fact has been found. The moment functions of weight 4, 5 and 6 obtained by this method are identical with ψ_4, ψ_5 and ψ_6 defined above.

Conclusion. The results of this paper include:

1. A demonstration of the fact that the theory of statistical seminvariants is identical with the theory of algebraic seminvariants.
2. The introduction of new statistical seminvariants.
3. Simplification of the computation of estimates.
4. Proof that the estimate of any seminvariant is also a seminvariant.
5. Proof of the existence of a trio of seminvariants with the same numerical coefficients.
6. A discussion of seminvariants which are invariant under estimate.

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