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J. R. HERRING

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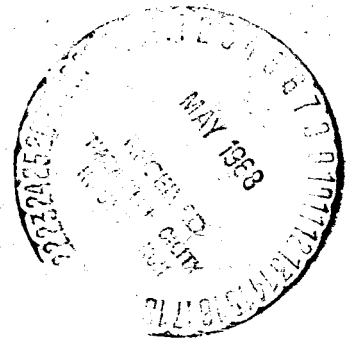
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THE STATISTICAL THEORY OF THERMAL  
CONVECTION AT LARGE PRANDTL NUMBER

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## ABSTRACT

Three statistical theories of turbulence are applied to thermal convection between infinite slippery plates in the limit of infinite Prandtl number. The range of Rayleigh number  $R$  investigated is  $657 \leq R \leq 1.25 \times 10^4$ . The theories, which are compared in a series of numerical calculations, are the direct interaction approximation, the quasi-linear approximation, and the quasi-normal approximation. The direct interaction approximation gives results for the evolution Nusselt number in good agreement with some simple exact solutions to the problem. The flow predicted by this method is very persistent. Turbulence first appears just at critical  $R$  and its intensity gradually increases with increasing  $R$ . The quasi-normal approximation gives satisfactory results for  $R \lesssim 2000$ , but some of the initial value problems lead to an unphysical negative temperature autocorrelation spectrum for larger  $R$ . In none of the initial value problems for  $R > 2 \times 10^3$  did the quasi-normal procedure yield a sensible stationary state. The quasi-linear approximation appears to upperbound the heat transport given by both the other approximations predicting about 10% larger heat transport than the direct interaction approximations.

## I. INTRODUCTION

This paper presents results of an application of the direct interaction (DI) approximation<sup>1</sup> to thermal convection at infinite Prandtl number, in a fluid confined by infinite, slippery, infinitely conductive horizontal boundaries. Results are presented for initial value problems in which the temperature and velocity covariances are allowed to evolve to their steady state values. The infinite Prandtl number ( $\sigma \rightarrow \infty$ ) régime is particularly simple because the non-linear terms in the momentum equation vanish, so that one has to study a single non-linear, scalar equation for the covariance of the temperature field. Since the direct interaction equations are formidable, it is attractive to study first a relatively tractable situation to assess the method's realism before addressing the more interesting and difficult case of convection at finite Prandtl number  $\sigma$ .

There is also presented a comparison of the DI approximation with the "quasi-linear" approximation,<sup>2</sup> the "quasi-normal" approximation,<sup>3</sup> and a direct integration of the amplitude equations for a reasonably large ensemble of initial data. The first three approaches are methods of closing the hierarchy of moment equations.

The quasi-linear or "mean-field" approximation, discards third order cumulants. This is equivalent to neglecting deviations of the bilinear terms in the Navier Stokes equations from their horizontal averages. The method appears to have some quantitative validity, for high Prandtl number fluids.<sup>2, 4</sup> It is known to have several defects, including the inability to assess the importance of the

terms it omits. For systems which have no mean fields, this method gives the trivial answer of dissipative decay of the initial flow and temperature fields.

The quasi-normal approximation<sup>3</sup> evaluates fourth order moments as if the flow and temperature fields were normally distributed. This approach has been investigated for isotropic turbulence by Ogura,<sup>5</sup> who solved some of the initial value problems and found the unphysical result of a partly negative kinetic energy spectrum. However, failure for the isotropic problem does not preclude success in problems in which there is a mean field present. Interaction of fluctuations with the mean field may stabilize the system.

The direct interaction approximation may be viewed as a logical improvement on the quasi-normal approximation. Both approximations give quantitative values for an effective dissipation acting on each degree of freedom of the fluid. The expressions for this dissipation involve functions which represent the average response of a mode to an infinitesimal initial perturbation. The quasi-normal approximation yields response functions determined wholly by molecular dissipation. The direct interaction approximates the effects of eddy viscosity, in a self consistent fashion.

The direct numerical integrations of the equations of motion serve to assess the validity of the statistical methods described in the preceding paragraphs, especially in lieu of any experimental data for slip boundary conditions. These results were obtained by constructing a random set of initial data, allowing each realization of the flow to evolve in three dimensions, and then ensemble-averaging the results, thereby obtaining values for the heat transport, and correlation functions.

The numerical results suggest that the direct interaction approximation may be satisfactory for thermal turbulence. The convective heat transport predicted is in good agreement with the direct integrations of the Boussinesq equations. The auto-correlation functions have very large relaxation times, suggesting an almost static flow for the range of Rayleigh numbers investigated ( $R < 1.25 \times 10^4$ ). The behavior of the steady-state Green's functions indicate that the system is close to being marginally stable. That is to say, a large-scale temperature perturbation decays very slowly under the joint action of the mean field and the eddy-dissipation processes. The fact makes the predictions of the direct interaction approximation agree closely with those of the quasilinear method with regard to the steady state value of the Nusselt number. The Green's functions predicted by the two methods are completely different.

The quasi-normal approximation appears to be a reasonable approximation only for  $R < 2000$ . For  $R > 2000$ , the approximation predicts the development of a negative value for the square of the temperature fluctuation field for some initial data investigated. It thus appears that the stabilizing influence of the fluctuation-field-mean-field interaction is very limited, and the eddy-type terms introduced by the direct interaction represent an essential feature. We should point out, however, that our results for the initial value problem do not preclude the validity of the quasi-normal approach for steady-state turbulence at any value of  $R$ .

## II. THE EQUATIONS OF MOTION

Under the stated boundary conditions, the infinite  $-\sigma$  Boussinesq equations for the horizontally averaged temperature field  $\bar{T}(z, t)$  and deviation  $\theta(\vec{r}, t)$  of temperature  $T(\vec{r}, t)$  from  $\bar{T}(z, t)$  may be reduced to

$$\left(\frac{d}{dt} + \nu_n^\alpha\right) \theta_n^\alpha = \sum_m J_m^\alpha B_{nm} \theta_m^\alpha + \frac{1}{2} \sum_{\substack{p, q \\ \alpha = \alpha' + \alpha''}} M_{npq}^{\alpha\alpha'\alpha''} \theta_p^{\alpha'} \theta_q^{\alpha''}, \quad (1)$$

$$\left(\frac{d}{dt} + n^2\right) \bar{T}_n = -\frac{\pi n}{2} \sum_{n, \alpha} J_m^\alpha \theta_m^\alpha \left(\theta_{n+m}^{-\alpha} - \sigma(n-m) \theta_{|n-m|}^{-\alpha}\right), \quad (2)$$

where;

$$J_n^\alpha = (R/\pi^4) \alpha^2 / (n^2 + \alpha^2)^2$$

$$B_{nm} = \frac{1}{2} (\beta_{|n-m|} - \beta_{n+n})$$

$$\beta_n = n\pi\bar{T}_n$$

and

$$M_{npq}^{\alpha\alpha'\alpha''} = E_{pq}^{\alpha\alpha''\alpha'} \delta_{n, p+q} + F_{pq}^{\alpha\alpha'\alpha''} (\delta_{p, n+q} + \delta_{q, n+p})$$

with

$$E_{pq}^{\alpha\alpha'\alpha''} = J_p^{\alpha'} \left( q + \frac{p}{2} - \frac{p}{2} \left( \frac{\alpha^2 - \alpha''^2}{\alpha'^2} \right) \right) \frac{\pi}{2}$$

$$F_{pq}^{\alpha\alpha'\alpha''} = J_p^{\alpha'} \left( q - \frac{p}{2} + \frac{p}{2} \left( \frac{\alpha^2 - \alpha''^2}{\alpha'^2} \right) \right) \frac{\pi}{2} \sigma(p - q)$$

In Eqs. (1) and (2),  $\theta_n^\alpha$  is the component of  $\theta(\vec{r}, t)$  proportional to  $e^{i\vec{\alpha}\cdot\vec{x}} \sin n\pi z$ , and  $\bar{T}_n$  is the component of  $\bar{T}(z, t)$  proportional to  $\sin \pi z$ . A brief derivation of Eqs. (1) and (2) from the Boussinesq form of the Navier Stokes equations is given in Appendix A.

The nonlinear terms in (1) are sorted into two groups: the terms proportional to  $B_{nm}$ , called the mean field terms, and the terms proportional to  $M$ , called the fluctuating self interactions. The  $M$  coefficients satisfy

$$M_{npq}^{\alpha\alpha'\alpha''} + M_{pnq}^{\alpha'a-\alpha''} + M_{qnp}^{\alpha''\alpha-\alpha'} = 0,$$

from which it follows that the volume-averaged square of the temperature field satisfies

$$\begin{aligned} \frac{d}{dt} \sum_{n,\alpha} |\theta_n^\alpha|^2 + \sum_{n,\alpha} \nu_n^\alpha |\theta_n^\alpha|^2 + \sum_n \nu_n^0 |\bar{T}_n|^2 \\ = 1 + \pi \sum_n n \bar{T}_n \end{aligned} \quad (3)$$

Equation (3) states that the flux of heat into the system (the right hand side) is equal to the time rate of accumulation of the heat energy plus the rate of molecular dissipation of macroscopic motion into heat.



### III. STATISTICAL THEORIES OF TURBULENT CONVECTION

Our primary goal is to determine the temperature autocorrelation function matrix,

$$\psi_{nm}^{\alpha}(t | t') = \langle \theta_n^{\alpha}(t) \theta_m^{-\alpha}(t') \rangle, \quad (4)$$

The average  $\langle \rangle$  is over an initial ensemble, in which the  $\theta_n^{\alpha}$  are assigned normally distributed values, and have a covariance matrix  $\psi_{nm}^{\alpha}(0 | 0)$ . Statistical homogeneity in the horizontal, and identity of horizontal averages to ensemble averages are assumed. Homogeneity in the horizontal implies

$$\langle \theta_n^{\alpha}(t) \theta_m^{\alpha'}(t') \rangle = 0, \quad \text{unless} \quad \vec{a} + \vec{a}' = 0.$$

The heat transport per unit area through the convecting layer is determined by (2) and the assumed equivalence between horizontal and ensemble averages.

An equation for  $\psi$  may be obtained by multiplying (1) by  $\theta_m^{-\alpha}(t')$  and averaging:

$$\left( \frac{d}{dt} + \nu_n^{\alpha} \right) \psi_{nm}^{\alpha}(t | t') = \sum_p J_p^{\alpha} B_{np} \psi_{pm}^{\alpha}(t | t') + \sum_{\substack{p, q \\ \vec{a} = \vec{a}' + \vec{a}''}} M_{npq}^{\alpha\alpha'\alpha''} \mathcal{G}_{npq}^{\alpha\alpha'\alpha''}(t | t'), \quad (5)$$

where

$$\mathcal{G}_{mpq}^{\alpha\alpha'\alpha''}(t | t') = \langle \theta_m^{-\alpha}(t') \theta_p^{\alpha'}(t) \theta_q^{\alpha''}(t) \rangle.$$

The remainder of this section outlines several theoretical approaches for obtaining closed equations for expressing the triple moment  $\mathfrak{J}$  as a functional of  $\psi$  and  $B$ . The particular closure approximations to be studied are the quasi-linear approximation (in which  $\mathfrak{J} = 0$ ), quasi-normal approximation, and the direct interaction approximation. All three of these approximations have been discussed in some detail elsewhere,<sup>1,2,3</sup> and the discussion to follow is intended to form a unified reference frame for the remainder of the paper. The only new possible ingredient is in the discussion of the quasi-normal approximation. Here, some new non-diagonal correlation functions  $\psi_{nm}$  occur, which are zero in isotropic turbulence.

#### a. The Quasi-Linear Approximation

If the horizontal exchange of heat and energy is very weak compared to the vertical exchange induced by the term  $B\psi$  it may be argued that  $\mathfrak{J} \simeq 0$  is a good approximation. This procedure — called hereafter the quasi-linear approximation — has been investigated in some detail.<sup>2,4</sup> It appears to be a fairly accurate and consistent method for large Prandtl number fluids. The mean-field terms, in an initial value problem, inhibit the continued growth of an initial perturbation by modifying the mean field through (2). This feedback mechanism stabilizes the system, yielding an eventual steady state. It is not clear that once the  $\mathfrak{J}$  term is reintroduced, the mean-field terms still play this same role.

b. The Quasi-Normal Approximation

A systematic formal procedure for including approximations to the triple-moment transfer term is discarded of cumulants procedure.<sup>6</sup> An hierarchy of equations for all the moments with simultaneous time arguments is first obtained. An equation for  $\mathfrak{J}$  is obtained by multiplying (1), by  $\theta_m^{-\alpha'}(t)\theta_p^{-\alpha''}(t)$ , ensemble averaging, and adding the result to similar expressions obtained from (1) by permuting the indices n, p, and q. The equation for  $\mathfrak{J}$  thereby obtained is

$$\left(\frac{d}{dt} + \nu_m^{\alpha} + \nu_p^{\alpha'} + \nu_q^{\alpha''}\right) \mathfrak{J}_{mpq}^{\alpha\alpha'\alpha''} = \sum_r \left( B_{mr}^{\alpha} \mathfrak{J}_{rpq}^{\alpha\alpha'\alpha''} + B_{pr}^{\alpha'} \mathfrak{J}_{mrq}^{\alpha\alpha'\alpha''} + B_{qr}^{\alpha''} \mathfrak{J}_{mpr}^{\alpha\alpha'\alpha''} \right) + \sum_{r, \nu, \alpha''', \alpha'v} \left\{ M_{mrv}^{-\alpha-\alpha''-\alpha'v} Q_{pqr}^{\alpha'\alpha''-\alpha''-\alpha'v} + M_{prv}^{\alpha'\alpha''\alpha'v} Q_{mqr}^{-\alpha\alpha''\alpha''\alpha'v} + M_{qrv}^{\alpha''\alpha''\alpha'v} Q_{mpr}^{-\alpha\alpha'\alpha''\alpha'v} \right\}, (6)$$

where

$$Q_{nmp}^{\alpha\alpha'\alpha''\alpha'''} = \langle \theta_n^{\alpha} \theta_m^{\alpha'} \theta_p^{\alpha''} \theta_r^{\alpha'''} \rangle$$

is a fourth order moment. An equation for it may be obtained by an algorithm like that used to obtain the  $\mathfrak{J}$  equation.

By proceeding in this way, a hierarchy of equations is obtained which relate the moments of order n to those of order n + 1. To obtain closed equations of motion for a given moment (for example  $\mathfrak{J}$ ) the higher-order moment (Q) is

eliminated from the equation of motion by assuming that the form of the higher order moment is the same as it would be if the  $\theta_n^a(t)$  were Gaussianly distributed. Thus, in the fourth order closure,

$$Q_{nmpq}^{aa'a''a'''} = \psi_{nm}^a \psi_{pr}^{a''} \delta_{a,-a'} \delta_{a'',-a'''} + \psi_{np}^a \psi_{mr}^{a'} \delta_{a,-a''} \delta_{a',-a'''} \\ + \psi_{nr}^a \psi_{mp}^{a'} \delta_{a,-a'''} \delta_{a',-a''} \quad (7)$$

Equations (5), (6), and (7) now comprise a complete set of equations for the determination of  $\psi(t, t)$ .

The closure procedure using (6) and (7) is a generalization of the quasi-normal approximation — originally proposed for isotropic turbulence by Tatsumi and by Proudman and Reid.<sup>3</sup> Ogura<sup>5</sup> has shown numerically that the method is not satisfactory for isotropic turbulence; an integration of some sample initial data developed an energy spectrum with negative regions after a short time, for moderate Reynolds numbers.

Higher-order cumulant discard approximations may in principle be an improvement over the Tatsumi-Proudman-Reid quasi-normal approximation, but to integrate numerically any such higher order scheme appears prohibitively difficult.

The quasi-normal equation for  $\psi$  may be put into more convenient form by introducing the Green's function  $\hat{G}_{nm}^{\circ}(t, t')$ , which satisfies the following

equations:

$$\left(\frac{d}{dt} + \nu_n^a\right) \mathring{G}_{nm}^a(t|t') = \sum_p B_{np}^a \mathring{G}_{pm}^a(t|t'),$$

with

$$\mathring{G}_{nm}^a(t|t) = \delta_{nm} \quad (8)$$

Using (6), (7), and (8),  $\mathfrak{J}$  may be evaluated as

$$\mathfrak{J}_{npq}^{aa'a''} = \sum_{abc} \int_0^t dt' \mathring{G}_{na}^a(t|t') \mathring{G}_{pb}^{a'}(t|t') \mathring{G}_{qc}^{a''}(t|t') S_{abc}^{aa'a''}(t')$$

where;

$$S_{abc}^{aa'a''} = \sum_{rv} \left( M_{arv}^{-aa'a''} + M_{avv}^{-aa'a''} \right) \psi_{br}^{a'} \psi_{cv}^{a''} + 2 \sum_{rv} \left( M_{brv}^{a'-aa''} + M_{bvr}^{a'a''-a} \right) \psi_{ar}^a \psi_{cv}^{a''} \quad (9)$$

### c. The Direct Interaction Approximation

In the DI approximation, the evaluation of  $\mathfrak{J}$  is accomplished through the intermediary of a Green's function,  $g_{nm}^a(t|t')$ , whose role in the theory is similar to the  $\mathring{G}$  function introduced in the quasi-normal theory. This function is defined as the ensemble-averaged response of mode  $(l)_n^a$  at time  $t$  to an infinitesimal

impulsive perturbation in mode  $\theta_m^\alpha$  at time  $t'$ . In terms of the  $g$ 's, the direct interaction approximation for  $\mathfrak{J}$  is<sup>7</sup>

$$\begin{aligned}
\mathfrak{J}_{mpq}^{\alpha\alpha'\alpha''}(t|t') &= \sum_{s,r,v} M_{srv}^{-\alpha-\alpha'-\alpha''} \int_0^{t'} dt'' g_{ms}^\alpha(t'|t'') \psi_{pr}^{\alpha'}(t|t'') \psi_{qv}^{\alpha''}(t|t'') \\
&+ \sum_{s,r,v} M_{srv}^{\alpha'\alpha-\alpha''} \int_0^t dt'' g_{ps}^{\alpha'}(t|t'') \psi_{mr}^\alpha(t'|t'') \psi_{qv}^{\alpha''}(t|t'') \\
&+ \sum_{s,r,v} M_{srv}^{\alpha''\alpha-\alpha'} \int_0^t dt'' g_{qv}^{\alpha''}(t|t'') \psi_{mr}^\alpha(t'|t'') \psi_{pv}^{\alpha'}(t|t'') \quad (10a)
\end{aligned}$$

The DI approximation for  $g_{nm}^\alpha(t|t')$  is<sup>8</sup>

$$\begin{aligned}
\left(\frac{d}{dt} + \nu_n^\alpha\right) g_{nm}^\alpha(t|t') &= \sum_p B_{np}^\alpha g_{pm}^\alpha(t|t') \\
&+ \sum_{\substack{s,r,v \\ \alpha=\alpha'+\alpha''}} M_{npq}^{\alpha\alpha'\alpha''} M_{srv}^{\alpha''-\alpha'\alpha} \int_{t'}^t g_{vm}^\alpha(t''|t') g_{qs}^{\alpha''}(t|t'') \psi_{pr}^{\alpha'}(t|t'') \quad (10b)
\end{aligned}$$

with  $g_{nm}^\alpha(t|t) = \delta_{nm}$ .

Equations (10a) and (10b) determine  $\psi_{nm}^\alpha(t|t')$  given the initial values of  $\psi_{nm}^\alpha(0|0)$ .

The DI equation for  $g_{nm}^\alpha$  bear some similarity to that for  $\overset{\circ}{G}_{nm}^\alpha$ . Equation (10b) contains an additional convolutional term which represents the relaxational effect of eddy dissipation. In the limit  $R \rightarrow 0$ ,  $g_{nm}^\alpha$  and  $\overset{\circ}{G}_{nm}^\alpha$  become identical.

The structures of the quasi-normal and direct interaction approximations are similar with regard to  $\mathfrak{B}(t | t')$  also. The quasi-normal equations may be obtained from the DI equations by making on the latter the following alterations:

1. Change  $g_{nm}^\alpha(t | t')$  to  $\overset{\circ}{G}_{nm}^\alpha(t | t')$  by deleting the last term of (10b). Then,
2. Replace  $\psi_{nm}^\alpha(t | t')$  in (10a) according to

$$\psi_{nm}^\alpha(t | t') \rightarrow \sum_a \overset{\circ}{G}_{na}^\alpha(t | t') \psi_{am}^\alpha(t') \quad (11)$$

In the direct interaction equations, (5), (10a), and (10b), three basic terms determine the behavior of  $\psi$  and  $g$ : (1) the dissipation terms, proportional to  $\nu_n^\alpha$ ; (2) the mean-field terms — the first terms on the right hand side of all the equations, and (3) the fluctuating self-interactions, which are the last terms in the equations. Only the first two of these terms enter the net entropy balance.

A physical interpretation of the direct-interaction treatment of the  $\mathfrak{B}$ -self-interactions may be given in terms of a generalization of eddy-dissipation concepts. The right hand side of (10a) consists of basically two types of terms:

- (1) an input term to the mode  $\alpha$ , and two drain terms from  $\alpha$ . The input terms

contain the factor

$$\int g_{ms}^{\alpha} (t' | t'') dt'' ,$$

and the drain terms contain the factors

$$\int g_{ps}^{\alpha'} (t | t'') dt''$$

The conservative nature of all three statistical procedures discussed here may be verified directly from (5), (9) or (10a), on using the identity the M coefficients satisfy. For the direct-interaction method the conservation law is independent of the values of  $g_{nm}^{\alpha}$ .

#### IV. SOME GENERAL COMMENTS ON THE STATISTICAL THEORIES

Before proceeding to the numerical results, it is worthwhile to consider first certain consistency properties and some general properties of the solutions for the statistical approximations.

The most obvious consistency property to examine is the conservation of entropy (3). All the approximation considered here satisfy this equation, as stated previously. A second, and perhaps less obvious consistency condition on



the methods has to do with bounds on the  $\psi_{nm}(t | t')$ , which arises from the fact that these quantities are ensemble averages of products of real temperature fluctuation fields,  $\theta(r, t)$ . From this fact and by Schwartz<sup>1</sup> inequality it follows that

$$\psi_{nm}(t | t') \leq \sqrt{\psi_{nn}(t | t) \psi_{mm}(t' | t')} ,$$

or defining the correlation coefficient,

$$R_{nm}(t | t') \equiv \frac{\psi_{nm}(t | t')}{\sqrt{\psi_{nn}(t | t) \psi_{mm}(t' | t')}} ,$$

$$|R_{nm}(t | t')| \leq 1 .$$

For the quasi-linear approximation,  $R_{nm}(t | t') = 1$  because  $\theta$  is an exact solution of amplitude equations, without phase mixing (fluctuating self interactions are discarded).

The quasi-normal approximation deals only with the time-diagonal terms  $\psi_{nm}(t | t)$ . It does not assure  $R_{nm}(t | t) \leq 1$ .

In the direct interaction approximation,  $|R_{nm}(t | t')| \leq 1$  for all  $t, t'$ . This follows from the fact that the approximation corresponds to an exact solution for a set of model amplitude equations. This point has been extensively discussed by Kraichnan.<sup>9</sup>

The coefficients  $R_{nm}(t | t')$  provide a measure of the rate of change of the temperature and velocity patterns in a typical realization. Consider a statistically

stationary situation, for which  $R_{nm}(t|t')$  depends only on the time difference  $\tau = |t - t'|$ . If  $R_{nm}(\tau)$  decays rapidly as  $\tau$  increases, this rate of change is large. On the other hand, if  $R_{nm}(\tau)$  decays very slowly, there is nearly steady cellular convection. If  $R_{nm}$  is independent of  $\tau$ , the system is said to be static. A static form for  $R_{nm}$  need not preclude a random irregularity of the spatial pattern of the flow.

The direct interaction approximation may admit of static solutions, as has been pointed out by Kraichnan.<sup>10</sup> To see that the assumption of  $t$ - and  $t'$ -independent  $\psi(t|t')$  is consistent with the direct interaction equations in the limit  $t \rightarrow \infty$ , it is only necessary to examine (5), (10a), and (10b) and recall that as  $t \rightarrow \infty$  both  $\psi$  and  $g$  should relax to functions of argument  $\tau = t - t'$  only. Then (5) and (10a) imply that the static solution for  $\psi$  is a function of the integral of  $g$ ,

$$\mathcal{G}_{nm} = \int_0^{\infty} ds g_{nm}(s) ds .$$

An equation for  $\mathcal{G}_{nm}$  may be obtained by integrating (10b) over  $t$ , and using the boundary conditions on  $g$ . This equation is

$$\begin{aligned} \nu_{nm}^{\alpha} \mathcal{G}_{nm}^{\alpha} = & -\delta_{nm} + \sum_p B_{np}^{\alpha} \mathcal{G}_{pm}^{\alpha} \\ & + \sum M_{nps} M_{srv} \mathcal{G}_{vm}^{\alpha} \mathcal{G}_{qs}^{\alpha'} \psi_{pr}^{\alpha''} , \end{aligned} \quad (12)$$

assuming, of course that the integral of  $g$  exists. The static solution is thereby reduced to a set of algebraic equations. The question of whether these solutions are stable as compared to the dynamic solutions is unanswered. Conceivably the static solutions are stable below a certain critical Rayleigh number while the dynamic solutions are stable above.

The quasi-normal approximation cannot distinguish static and dynamic statistically stationary flows because it deals with only simultaneous correlation coefficients  $R_{nm}(t | t)$ .

A third consistency requirement comes from the requirement that the temperature field  $T(\vec{r}, t)$  be limited by the boundary temperatures  $T(0, t) = 0$ , and  $T(1, t) = -1.0$ , provided the initial temperature field is so limited. This comes about because there are no heat sources in the fluid. This constraint on  $T(\vec{r}, t)$  implies

$$0 \leq \langle [-T(\vec{r}, t)]^n \rangle \leq 1 (n = 1, 2, \dots)$$

If the constraints are not imposed at  $t = 0$ , they are applicable only as  $t \rightarrow \infty$ , in which limit the smoothing action of the diffusivity erases transient behavior making the statistical moments settle down to final state values.

None of the statistical approximations discussed in this paper give a priori assurance of conforming to all the moment bounds on  $T$ . The only method for which results are available is the quasi-linear approximation, and they indicate violations of the bounds for certain problems. Indeed, the contrary would be

surprising, since this approximation omits some of the terms responsible for the bounds. Recently, Durney<sup>11</sup> has applied the quasi-linear approximation to a study of convection in spherical shells. He finds that for  $R \gtrsim 2.4 \times 10^4$  the odd-ordered moment bounds are violated in a thin region in near the boundary layer. The violation is slight but still none the less present.

For the problem of plane parallel convection the quasi-linear approximation does not lead to violation of the bound on T for either free or rigid boundaries, or for any value of R, provided the system has reached the steady state. This statement is in apparent contradiction to conclusions reached by Deardorff,<sup>12</sup> who reported violations of the bound conditions near the boundary, for a particular initial value integration.

## V. RESULTS

### a. Numerical Procedure

The numerical procedure for integrating the statistical equations for  $\psi_{nm}(t | t')$  is described in detail in Appendix C. The complexity of the equations imposes severe restrictions on the number of modes explicitly treated, and on the range of Rayleigh number, R. The present paper restricts itself to only five vertical wave numbers, and up to three horizontal wave numbers to describe the  $\psi_{nm}^{\alpha}(t | t')$  field. The results of the next two sub-sections are obtained by approximating the  $\alpha$ -dependence of  $\psi(\alpha)$  by a constant value for  $1/\sqrt{2} \leq \alpha \leq \sqrt{2}$  and zero elsewhere. In section Vc, in which greater accuracy is desired three values of  $\alpha$  are included to represent  $\psi(\alpha)$ . Details of both methods of numerical approximation are given in appendix C.

to difference the equations, and the trapizoidal rule to evaluate the convolution integrals in (10a), and (10b). The procedures guarantee exact conservation properties for the system.

b. Results for the Direct Interaction Method

The first point to establish concerning the validity of the direct interaction method in whether or not it faithfully reproduces the onset of convection just above the critical Rayleigh number,  $R_c = 657.71$ . To test this point, three runs were made at  $R = 700, 830, 1000$ , using only two vertical modes and a single  $\alpha$ . These runs used unaveraged  $J_n^\alpha$ , and  $\nu_n^\alpha$  coefficients with  $\alpha = 1/\sqrt{2}$ , in order to facilitate a comparison of the numerical results with results known from perturbation theory. The parameters of these runs were  $\alpha = 1/\sqrt{2}$ ,  $\alpha_1 = \sqrt{2}$ , and  $\Delta = 0.20$ . The initial value the  $\psi_{nm}$  spectrum was

$$\psi_{11} = \frac{2}{\pi^2 J} (1 - R_c/R) \quad (13)$$

$$\psi_{nm} = 0, \quad \text{for } n \neq m.$$

The initial  $\bar{T}_n$  field was such that  $(\partial \bar{T}_n / \partial t) = 0$  for a  $\psi_{nm}$  field given by (13). The above  $\psi$  spectrum is that implied by perturbation theory, as for example that of Malkus and Veronis.<sup>13</sup> The values of  $N_u$  for  $t = 8.0$  are shown in Fig. 1, and are compared with results of other statistical theories. This value of  $t$  is sufficient in these cases to assure  $|\psi_{11}/\psi_{11}| \leq 0.006$ .

The values of  $R_{11}(t | t')$  and  $R_{22}(t | t')$  for the  $R = 10^3$  run is shown in Fig 2. These curves indicate very long correlation times, which implies a very

persistent flow. The Green's functions  $g_{11}(t | t')$  and  $g_{22}(t | t')$  are shown in Fig. 3. These results are in good agreement with the analysis of the two mode static system analyzed in Appendix B (cf. B12 and B13), which indicate an extremely slow decay of small perturbations introduced in the flow. Results for the  $R(t | t')$  and  $g(t | t')$  at the other values of  $R$  show an entirely similar behavior.

Some results for  $R = 4 \times 10^3$  are shown in Figs. 4-7. The parameters, for this case are,  $M = 5$ ,  $\alpha_0 = 1/\sqrt{2}$ ,  $\alpha_1 = \sqrt{2}$ , and  $\Delta = 0.05$ . The initial value for  $\psi_{nm}$  is that obtained from the symmetrized single -  $\alpha$  hexagon solution. Figures 4, and 5 gives the correlation functions  $R_{ij}(t | t')$ , for  $t = 1.5$ . We surmise from these figures that at  $R = 4 \times 10^3$ , the solutions are no longer nearly static in character. This point is not entirely clear from the numerical analysis since the solutions have not entirely reached steady state at  $t = 1.5$ , and continuing the calculation beyond  $t \cong 2$  by the time forward integration technique becomes prohibitive in terms of machine time.

The Green's functions are given in Figs. 6, and 7. Again, note the slow relaxation of the large scale mode,  $g_{11}$ . A striking feature seen in these curves is the qualitatively different behavior between the even-even modes and the odd-odd modes. All the odd-odd Green's functions are "locked in" to the behavior of  $G_{11}$ , and have quite long relaxation times. None of them develop a negative value. The even-even modes are also "locked in" to the behavior of  $G_{22}$ . They all develop negative values, and their time integral is quite small compared, say,

to  $\bar{\nu}_2^{-1}$ . This behavior is reminiscent of that predicted by (B12) and (B13), and makes the even-even  $\psi_{nm}(t | t)$  correlation matrix elements very small.

Figures 8 and 9 give some results for  $R = 10^4$ . These results are similar to the corresponding ones found at  $R = 4 \times 10^3$ , but the system appears to be less static. This is suggested by comparing the correlation coefficients as given in Figs. 4 and 9.

### c. Results for the Quasi-Normal Method

The equations for this method are obtained from (5), (10a), and (10b) by eliminating  $\psi(t | t')$  using (11), and replacing the G equations by (8). Numerical results are first presented for the two mode system described at the beginning of Section V (b).

For  $R \gtrsim 2000$  the two-mode quasi-normal results appear to be satisfactory. The evolution of the system from the initial state prescribed in (12) is reasonable and closely resembles the direct interaction results. Values for the Nusselt number,  $N_u$ , are given in Fig. 1, and are compared to results for the other methods investigated. The steady state values of  $N_u$  are consistently lower for this method than for the DI or quasi-linear method.

For  $R \gtrsim 2000$  the quasi-normal procedure does not give sensible results. The predicted behavior of  $\psi_{11}(t)$  is shown in Fig. 10 for  $R = 2000$ . It is seen that  $\psi_{11}(t)$  evolves into an unphysical negative region after  $t \simeq 2.0$ . The numerical accuracy of this curve is not too great especially near the point at

which  $\psi_{11} \rightarrow 0$ . The time step size used in this graph is  $\Delta = 0.15$ . It was found that decreasing the step size made the behavior of  $\psi$  near  $t = 2.0$  more precipitous, so that the actual long time behavior of  $\psi_{11}(t)$  is slightly more singular than is shown in Fig. 10.

One source of this trouble is not difficult to trace. Recall first that the value of  $N_u$  for this method is below the quasi-linear result, for which  $\nu_1 = B_{11}$ . This means that for the steady state  $\overset{\circ}{G}_{11}(t)$ , as given by (8) has an exponential growth rate  $\lambda = -\nu_1 + B_{11} > 0$ . The numerical results show, reasonably enough that  $\lambda$  increases with  $R$ . Now look at (B1) as it specializes to the quasi-normal case. The net contribution of the fluctuating terms to  $\psi_{11}$  is,

$$2M_{112}^2 \int_0^t \overset{\circ}{g}_1(t|s) \overset{\circ}{g}_1(t|s) \overset{\circ}{g}_2(t|s) (\psi_1(s)\psi_2(s) - \psi_1(s)\psi_1(s))$$

As the steady state is approached, the Green's function product becomes  $e^{(2\lambda - \nu_2)(t-s)}$ . If now  $\psi_2 > \psi_1$ , as is only sensible at small  $R$ , the above term will remain finite as  $t \rightarrow \infty$  only if

$$\lambda < \nu_2/2 .$$



Thus, as  $R$  is increased above the value for which  $\lambda = \nu_2/2$ , either  $\psi_{11}(t | t)$  will eventually become negative as  $t$  increases or  $\psi_2 > \psi_1$ . The numerical results indicate this value of  $R$  to be  $\simeq 2,300$ .

These simple results suggest that the quasi-normal method suffers yet another defect than that pointed out by Ogura;<sup>5</sup> an inability to cope with systems having driving forces produced by an instability in the flow field. Of course, it may be argued that these results do not survive if a more realistic flow field consisting of many Fourier modes is studied. This does not appear to be the case, as is discussed in section Vd.

#### d. Numerical Integration of the Amplitude Equations

Although experimental results are not yet available for free boundaries, the results of the statistical methods can be compared to the numerical integrations of the equations for  $T(\vec{r}, t)$ .

The problem to be considered is: given an initial ensemble of amplitudes  $T_{n,i}^\alpha(0)$  (where  $i = (1, \dots, N)$  labels the ensemble member) determine the ensemble average covariance,

$$\psi_{n,m}^\alpha(t | t') = \frac{1}{N} \sum_{i=1}^N \theta_{n,i}^\alpha(t) \theta_{n,i}^{\alpha*}(t').$$

Here  $\theta$  is the fluctuation of the temperature field  $T$  from the horizontally averaged value,  $\bar{T}_{n,i}$ . The equations of motion for  $\theta_{n,i}^\alpha$  and  $\bar{T}_{n,i}$  are (1) and (2). The initial data for the complex amplitude  $\theta_{n,i}^\alpha$  are Gaussianly distributed in the index (i)

with a specified covariance matrix  $\psi_{nm}^{\alpha}(0)$ . The initial values for  $\bar{T}_{n,i}$  is such that  $(dT_{n,i}/dt) = 0$ . These initial data are consistent with those used for  $\theta$  and  $T$  in the statistical treatment of the problem.

The correspondance between amplitude and statistical calculations is complete only if the ensemble is homogenous and isotropic in the horizontal. This condition permits a replacement of horizontal averages by ensemble averages, and is useful in closing the statistical hierarchy of moment equations. By appealing to this equivalence  $\theta$  may be interpreted as a fluctuation from an ensemble average.

The  $\theta_{n,i}^{\alpha}$  were selected from a Gaussianly distributed complex set of numbers (with real covariance  $\psi_{nm}^{\alpha}(0)$ ) constructed by first generating a uniform distribution of numbers, and using this uniform set in connection with the central limit theorem to obtain the Gaussian distribution. Care was taken to assure statistical independence of real and imaginary parts of  $\theta$ .

The condition of isotropy in the horizontal may be realized in a numerical experiment only if the number of  $\alpha_{-}$ 's included is very large. Since machine calculations necessarily treat a finite number of  $\alpha_{-}$ 's it is important to have some measure of the degree of isotropy of the experiment. We shall measure this by a parameter  $p$  defined by,

$$p = 1 - (\lambda_{+} - \lambda_{-}) / (\lambda_{+} + \lambda_{-})$$

Here  $\lambda_+$  and  $\lambda_-$  are the eigenvalues of the matrix,

$$U_{xy} = \frac{\sum_i (\alpha_x^i \alpha_y^i / \alpha_i^2) |\theta^{\alpha i}|^2}{\sum_i |\theta^{\alpha i}|^2} .$$

For cellular two-dimensional flow  $p = 0$  and for three dimensional isotropic flow  $p = 1$ .

The horizontal spectrum is generated by the following formula;

$$\alpha_x^N = N\delta , \quad N = 1, 2, \dots$$

$$\alpha_y^M = M\delta , \quad M = 1, 2, \dots$$

Here  $\alpha_x^N$  and  $\alpha_y^M$  are the x and y components of  $\vec{\alpha}$ . Truncation is achieved simply by discarding any  $\vec{\alpha}$  for  $|a| < a_0$  or  $|a| > a_1$ . In the present experiment  $a_0 = 1/\sqrt{2}$ ,  $a_1 = \sqrt{2}$  and  $\delta = a_0/4$ . The number of  $\alpha_-$ 's generated this way is 76 and the number of fluctuating-self-interactions treated is 6360 per vertical mode.

In the present calculation the Rayleigh number is 3000, and 3 vertical wave numbers are used to describe the flow. This modest value of R, together with the small number of vertical wave numbers is dictated by machine time considerations. It may be objected at this point that R is too small to assess the accuracy of a statistical theory, since the flow is likely to be non-turbulent. Our point of view here is that even at these relatively small values of R a meaningful assessment of a statistical theory may be obtained by studying the relaxation

of the system from random initial conditions. A proper statistical theory should accurately treat the build up of correlations from an initial value of zero. We shall therefore insist that an acceptable statistical procedure should accurately treat the transient behavior of the system.

In Fig. 11 and 12 we present results of the present numerical experiment for  $N_u(t)$  and  $\psi_{11}(\alpha, t)$ . These results are an ensemble averaged over 10 experiments. The initial value for the isotropy parameter is  $p = 0.88$ ; at  $t = 2.0$ ,  $p = 0.80$ . The results are compared in the figures to results for the statistical theories. For the latter we have not used the single band approximation to treat the horizontal wave number spectrum, but instead have used a more accurate treatment involving three  $\alpha$ 's. The procedure treats the  $\alpha$ -integrations in (10a) and (10b) by the mid-point rule, and the  $\alpha$ -integration in (2) by a Lagrange interpolation formula. The procedure is described in more detail in appendix C.

The results for the direct interaction approximation for both  $N(t)$  and  $\psi_{11}(\alpha, t)$  appear to be satisfactory. The slight overestimation of  $N_u$  at large  $t$ , together with a slight under estimation of  $\psi_{22}(\alpha, t)$  may be traceable to the fact that the discretizing procedure for  $\alpha$  omits an adequate representation of the linear harmonics (those terms for which  $\vec{\alpha}_i = 2\vec{\alpha}_j$ ). These terms may play an important role in the energy transfer between  $\psi_{11}$  and  $\psi_{22}$ , because these modes interact directly only through the mean field term.

The quasi-linear approximation for  $N_u(t)$  is also fairly good, especially considering the simplicity of this procedure. Its results for  $\psi_{11}(\alpha)$  for short

times is satisfactory but for long times it gives a  $\psi_{11}$  which sharpens into a delta function. Also, it predicts a  $\psi_{22}(\alpha)$  which tends to zero very rapidly as time increases (at  $t = 1.0$ ,  $\psi_{22} \sim 10^{-6}$ , while  $\psi_{22} \simeq 10^{-3}$  for either the numerical experiment or the direct interaction).

The quasi-normal approximation does not appear to behave satisfactorily;  $\psi_{11}(\alpha_3, t)$  becomes negative for  $t \geq 0.3$ .

The direct interaction results are sufficiently close to those of the numerical experiment to make it of interest to inquire as to what significance should be attached to their difference. Some assessment of the numerical accuracy of the direct interaction integrations may be made by varying the method of integration in the following two respects: (1) decrease the time step  $\Delta t$ , (2) use a different integration method for the  $\alpha$ -spectrum. With regard to the first point it appears that the time step used in the calculation is sufficiently small to guarantee good accuracy for the  $N_u(t)$  curve: decreasing from 0.05 to 0.025 produces only a 0.01% difference in  $N_u(t)$ . Two tests were made to assess the accuracy of the  $\alpha$ -integrations; first, the  $\alpha$ -integration of Eq. (2) was modified from the Lagrange interpolation method to the mid point rule. This change produced a maximum difference in  $N_u(t)$  of 0.26%. Next, the treatment of the fluctuating terms was changed by using a  $\rho$  (see Appendix C) averaged over  $\Delta\alpha$  instead of an un-averaged one. The combined change produced by both modifications is at most 0.4%.

The point to test in the numerical experiment is whether as the number of  $\alpha$ -modes is systematically increased a significant change in any of the ensemble

quantities results. In order to test this point, a calculation was made using  $\delta = 1/5$ , which gives 124  $\bar{\alpha}$ 's distributed on  $(\alpha_0 \leq \alpha \leq 2\alpha_0)$ . Some values of  $N_u$  are indicated in Fig. 11 at the time for which the deviation of the 124 mode from the 76 mode experiment is maximum. The difference is at most 1.6%. However, it is not certain that this deviation is statistically significant since the standard deviation of  $N_u$  in the 76 mode experiment is 2.2%.

From the preceding paragraphs it appears that the maximum error for the direct interaction at  $R = 3 \times 10^3$  is at most  $\sim 3\%$ . The maximum occurs at  $t = 0.5$ , and the approximation appears better at larger times.

## VI. DISCUSSION OF RESULTS AND CONCLUDING COMMENTS

A comparison of the DI results with the numerical experiments described in the last section suggests that the DI procedure is in good quantitative agreement with the exact solution to the thermal convection problem at large Prandtl number. The accuracy of the method on this point appears good for the transient behavior, as shown by Fig. 11 and 12, as well as for the steady state value. At  $R = 10^4$ , the value of the Nusselt number predicted by the DI method is 5.03, as compared to 5.45 for the quasi-linear method, and 5.20 for the roll solution with  $\alpha_0 = 1/\sqrt{2}$ .

Although  $R = 10^4$  is not considered a large value of Rayleigh number, it should be remembered that for the quasi-linear method it is sufficiently large for free boundaries for the dependence of  $N_u$  on  $R$  to reach its asymptotic form.

This is also true for the two dimensional calculations of Fromm<sup>14</sup> at a Prandtl number of unity.

However  $R = 10^4$  is not sufficiently large to draw conclusions about the asymptotic shape of either the mean temperature field, or the temperature covariance field. The computed temperature profiles for  $R \leq 10^4$  are properly bounded in the sense described in Section IV b, but violations of this bound if they occur at all would probably not appear until  $R \cong 10^5$ .

The question of whether or not the numerical solutions of the Direct Interaction equations are indeed static solutions is not entirely satisfactorily settled by the present calculation. It would be difficult to decide this issue entirely on the basis of numerical studies in any case, and the fact that the system evolves very slowly makes the problem that much more difficult. What is needed here is a stability analysis of the steady state solution at large Rayleigh numbers. Such a calculation seems rather formidable. Despite these uncertainties, we believe the numerical results suggest a non-static solution for the convection. The non-static character appears just above critical Rayleigh numbers, and becomes more pronounced as the Rayleigh number is increased. The occurrence of the time dependence apparently does not commence suddenly at a certain critical Rayleigh number as for example would occur in shear flow problems, if the Reynolds number is increased beyond a certain critical value.

In Appendix B, it is pointed out that the DI approximation could give (static) results identical to the quasi-linear procedure provided  $R \lesssim 1.18 \times 10^4$ .

The  $\psi$ -field would in these circumstances be static and non-statistical. Above  $R = 1.18 \times 10^4$  such a solution cannot be obtained, as is shown in Appendix B. The numerical results of the preceding section are again not conclusive, but they indicate that the final asymptotic state is not the quasi-linear type solution at any  $R$ . This conclusion is partly suggested by the shape of the correlation functions as given in Figs. 4, and 9. A comparison of 4 with 9, suggests that as  $R$  increases, the solutions become less static in character. The conclusion is reinforced by the fact that the numerical solution for  $R = 1.25 \times 10^4$ , which is above the critical value  $R = 1.12 \times 10^4$  beyond which the static quasi-linear solutions cannot exist show the same qualitative behavior as solutions below  $R = 1.12 \times 10^4$ . We therefore tentatively conclude that the quasi-linear-type solutions are not stable.

The preceding paragraph tacitly assumes that the static solutions are the quasi-linear solutions, an assumption which is not necessarily true. In this connection, it may be pointed out that all efforts to find static solutions by iteration techniques always converged to the quasi-linear results. The attempt to find static solutions mentioned here used a vertical modes, one value of  $\alpha$ , and  $R = 4 \times 10^3$ .

Our investigation suggests that the infinite Prandtl number thermal convection problem is a peculiar limit case with some features are probably not to be found in the finite Prandtl number system at high Rayleigh numbers: namely, the extremely slow approach of the system to the steady state, and



the fact that the steady state solutions are nearly static. These features are not expected at finite Prandtl numbers, since the eddy viscosity then plays an important role, and presumably introduces major decorrelation effects. The near static nature of the solutions suggests that the infinite Prandtl numbers regime may be an inhospitable one for statistical approximations. By the same token, it may therefore subject a statistical theory to a severe test.

The analysis of the quasi-normal procedure indicates that this procedure is not an acceptable one to treat the evolution of nonstationary turbulence to the steady state for  $R \gtrsim 2 \times 10^3$ . The procedure may be an acceptable one to treat stationary turbulence for larger  $R$ , but the present results strongly suggest that if this is so, the method is very unstable to small perturbations and could not be used to treat small departures from equilibrium.

## Appendix A

### Equations of Motion at Infinite Prandtl

### Number and Their Fourier Transform

The equations determining the velocity field,  $\vec{v}$  and temperature field  $T$  written in a convenient non-dimensional form are;

$$\nabla \cdot \vec{v} = 0 \quad (\text{A1})$$

$$\left( \frac{1}{\sigma} \frac{\partial}{\partial t} - \nabla^2 \right) \vec{v} = - \nabla p - \frac{1}{\sigma} (\vec{v} \cdot \nabla) \vec{v} - \hat{k} R T \quad (\text{A2})$$

$$\left( \frac{\partial}{\partial t} - \nabla^2 \right) T = - \vec{v} \cdot \nabla T \quad (\text{A3})$$

These are the Boussinesq approximation to the Navier-Stokes equations. In this non-dimensional form the only constants which appear in these equations are the Prandtl number  $\sigma = \nu/\kappa$  and the Rayleigh number  $R = g\alpha D^3 \Delta T/\kappa\nu$ . Here  $\alpha$  is the thermometric coefficient of expansion,  $\nu$  is the kinematic viscosity,  $\kappa$  is the coefficient of thermometric diffusivity,  $\Delta T$  is the temperature excess of the lower plate above the top one,  $D$  is the distance between the plates, and  $g$  is the gravitational acceleration. The non-dimensional variables  $\vec{v}$ ,  $T$ ,  $\vec{r}$ ,  $t$  are related to the dimensional ones  $v'$ ,  $T'$ ,  $r'$ , and  $t'$  by;

$$\vec{v} = \frac{k}{D} \vec{v}'$$

$$T' = \frac{T}{T_0}$$

$$\vec{r}' = D\vec{r}$$

$$t' = \frac{D}{k} t$$

In the limit  $\sigma \rightarrow \infty$ , Eq. (A2) simplifies to a linear relationship between the velocity field and the temperature field, and thus in this limit  $\vec{v}$  may be eliminated from the system. The elimination of the velocity field is facilitated by taking the curl of the curl of (A2). This leads to,

$$\nabla^4 \vec{v} = R\nabla \left( \frac{\partial T}{\partial z} \right) - R\hat{k} \nabla^2 T \quad (\text{A4})$$

Slip boundary conditions for  $\vec{v}$  are assumed on the conducting plates. Thus, if  $w$  is the vertical component of  $\vec{v}$ :

$$w = \frac{\partial^2 w}{\partial z^2} = 0$$

for

$$\vec{r} = (x, y, 0)$$

and

$$\vec{r} = (x, y, 1)$$

The boundary conditions on the temperature field assume the boundary plates are infinitely conducting compared to the fluid they bound

$$T(x, y, 0; t) = 0$$

$$T(x, y, 1; t) = -1$$

Next, the temperature field  $T$  is split into its horizontal averages plus deviation from this average,

$$T = \bar{T} + \theta$$

The velocity field has zero horizontal average. An average over the horizontal is indicated by the over bars. On using the above definition of  $\theta$ , Eq. (A3) splits into two components;

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = \beta w - \nabla \cdot (\vec{v}\theta - \overline{v\theta}) \quad (\text{A3a})$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right)\bar{T} = -\frac{\partial}{\partial z} \overline{w\theta} \quad (\text{A3b})$$

where;

$$\beta \equiv -\frac{d\bar{T}}{dz}$$

Now let the Fourier transform coefficients,  $w_n^\alpha$ ,  $\theta_n^\alpha$ , and  $\bar{T}_n$  be defined by

$$w(\vec{r}, t) = \sum_{\alpha, n} w_n^\alpha(t) e^{i\pi\vec{\alpha}\cdot\vec{r}} \sin n\pi z, \quad (\text{A4a})$$

$$\theta(\vec{r}, t) = \sum_{\alpha, n} \theta_n^\alpha(t) e^{i\pi\vec{\alpha}\cdot\vec{r}} \sin n\pi z, \quad (\text{A5b})$$

and

$$\bar{T}(z, t) = \sum_n \bar{T}_n(t) \sin n\pi z - z. \quad (\text{A5c})$$

Here  $w(\vec{r}, t)$  is the vertical component of the velocity field. The horizontal component of  $\vec{v}$  may be obtained in terms of  $\theta$  by using (A4) and (A5c). The reality of  $\vec{v}(\vec{r}, t)$  and  $T(\vec{r}, t)$  implies that

$$(\vec{v}_n^\alpha)^* = \vec{v}_n^{-\alpha},$$

and

$$(T_n^\alpha)^* = T_n^{-\alpha}.$$

Here the asterisk denotes complex conjugation. The equations for  $w_n^\alpha$ ,  $\theta_n^\alpha$ , and  $\bar{T}_n$  may now be obtained by introducing (A5a), (A5b), and (A5c) into (A3a), and (A3b). The results are Eqs. (1), and (2) of the text.

## Appendix B

### Connection Between Quasilinear and Direct Interaction Approximations

#### a. Considerations Based on a Simple Model

This section describes an analytic approach to the direct interaction equations for a simple model system obtained by deleting all but the first two vertical Fourier modes and all but a single horizontal wave number  $\alpha$ . This highly truncated system is investigated in the hope of gaining some insight into the structure of the full direct interaction equations. A direct analytic attack on either the quasi-normal or direct interaction equations is too difficult. The structure of the quasi-linear equations is sufficiently simple to yield to analytic methods.<sup>4</sup> The result of the present analysis is that for the simple two-mode model the static DI results for  $\psi_{nm}$  are identical to the quasi-linear results. However, the two methods predict different results for  $g_{nm}$ .

Consider then the simple system for which the only vertical modes are  $n = 1, 2$ , so that the system (5), (10a), and (10b) consist of the eight functions  $\psi_{11}(t | t')$ ,  $\psi_{12}(t | t')$ ,  $\psi_{22}(t | t')$ ,  $G_{11}(t | t')$ ,  $G_{12}(t | t')$ ,  $G_{21}(t | t')$ , and  $G_{22}(t | t')$ . The four off diagonal terms  $\psi_{12}$ ,  $\psi_{21}$ ,  $G_{12}$ , and  $G_{21}$  are put equal to zero, since the statistical fields  $\psi(\vec{r} | \vec{r}')$  may be assumed to be symmetric about the mid point,  $z = 1/2$ . The equations are,

$$\begin{aligned} \left( \frac{d}{dt} + \nu_1^\alpha \right) \psi_1^\alpha(t | t') &= B_{11}^\alpha \psi_1(t | t') + M^2 \int_0^{t_1} ds g_1^\alpha(t' | s) \psi_2^{\alpha'}(t | s) \psi_1^{\alpha''}(t | s) \\ &+ M^2 \int_0^t ds g_1^{\alpha'}(t | s) \psi_1^\alpha(t' | s) \psi_2^{\alpha''}(t | s) - M^2 \int_0^t ds g_2^{\alpha'}(t | s) \psi_1^\alpha(t' | s) \psi_1^{\alpha''}(t | s) \quad (B1) \end{aligned}$$

$$\begin{aligned}
\left(\frac{d}{dt} + \nu_2^a\right) \psi_2^a(t | t') &= B_{22}^a \psi_2^a(t | t') \\
&+ 2M^2 \int_0^{t_1} ds g_2^a(t' | s) \psi_1^{a'}(t | s) \psi_2^{a''}(t | s) \\
&- 2M^2 \int_0^t ds g_1^{a'}(t | s) \psi_2^a(t' | s) \psi_1^{a''}(t | s) \quad (B2)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d}{dt} + \nu_1^a\right) g_1^a(t | t') &= B_{11}^a g_1^a(t | t') \\
&- 2M^2 \int_{t'}^t g_1^a(s | t') ds \left( g_2^{a'}(t | s) \psi_1^{a''}(t | s) \right. \\
&\quad \left. + g_1^{a'}(t | s) \psi_2^{a''}(t | s) \right) \quad (B3)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{d}{dt} + \nu_2^a\right) g_2^a(t | t') &= B_{22}^a g_2^a(t | t') \\
&- 2M^2 \int_{t'}^t ds g_2^a(s | t') g_1^{a'}(t | s) \psi_1^{a''}(t | s) \quad (B4)
\end{aligned}$$

where,  $M = M_{112}$ . In writing (B1) - (B4), the condition  $M_{121} + M_{211} + M_{212} = 0$  has been used. The  $\alpha$  index is suppressed and only one vertical index is used to specify the G and  $\psi$  fields. The equations for  $\psi(t | t)$  are not recorded here; they are identical with (B1) and (B2) with  $t = t'$  except that the term  $(\partial\psi/\partial t)$  is replaced by  $(1/2) (d\psi/dt)$ .

It is now shown that in the limit  $t \rightarrow \infty$ , a particular solution to (B1) - (B4) consists of a static quasi-linear  $\psi$ -field, for which  $\psi_2 = 0$ , and  $\psi_1$  is specified by

$$\nu_1^\alpha = B_{11}^\alpha$$

or

$$\psi_1 = \frac{2}{\pi^2} (1 - R_c/R), \quad R_c = 657.7 . \quad (B5)$$

Examination of the steady state form of (B1) - (B4) indicates that to prove this assertion it is necessary that

$$\psi_2 = 0 \quad (B6)$$

and

$$\int_0^\infty ds g_2(s) = 0 . \quad (B7)$$



Equation (B7) is a consequence of (B6) as may be seen by examining (B7) and using the assumption that as  $t \rightarrow \infty$ ,  $\psi$  becomes a static field.

To prove (B7) write the steady state form of (B3) and (B4), and make a change of time scale,  $\tau = c^2 (t - t')$ ,  $c^2 = 2M^2 \psi_1$ ;

$$\frac{d}{d\tau} g_1(\tau) = - \int_0^\tau ds g_1(\tau - s) g_2(s)$$

$$\left( \frac{d}{d\tau} + \mu \right) g_2(\tau) = - \int_0^\tau dr g_2(\tau - s) g_1(s)$$

where,

$$\mu = (\nu_2 - B_{22})/c^2 .$$

These equations may be solved by using Laplace transform theory. Thus, introducing

$$\hat{g}_1(s) = \int_0^\infty g_1(t) e^{-st} dt \quad (B8)$$

$$\hat{g}_2(s) = \int_0^\infty g_2(t) e^{-st} dt \quad (B9)$$

algebraic equations for  $g_1(s)$  and  $g_2(s)$  may be obtained, whose solutions are

$$\hat{g}_1(s) = \frac{2(s + \mu)}{\sqrt{s^2 (s + \mu)^2 + 4s(s + \mu) + s(s + \mu)}} \quad (\text{B10})$$

$$\hat{g}_2(s) = s g_1(s) / (s + \mu) . \quad (\text{B11})$$

From (B10) it follows that in the limit  $s \rightarrow 0$ ,  $\hat{g}_2(s) \xrightarrow{s \rightarrow 0} \sqrt{s}$ , and hence, from (B9), (B7) is valid provided the limit process and Laplace transformation are interchangeable.

Equations (B10) and (B11) may be inverted to give  $g_1(t)$ , and  $g_2(t)$ . It is of interest to have their asymptotic forms:

$$g_1(\tau) \xrightarrow{\tau \rightarrow \infty} \sqrt{\mu/\pi} \tau^{-1/2} \quad (\text{B12})$$

$$g_2(\tau) \xrightarrow{\tau \rightarrow \infty} - (1/2 \sqrt{\mu\tau}) \tau^{-3/2} . \quad (\text{B13})$$

This analysis assumes the stationary state system is static. As pointed out before, this assumption is mathematically consistent, but the static solutions may be unstable to small perturbations, which will induce the system to revert to a unstable state.

The question now arises as to whether these simple results are generalizable to the complete multi-mode system. They may be. To see why, consider the DI equations assuming that the  $\psi$ -field is that found from

quasi-linear theory,  $\psi_L$ . It is known<sup>4</sup> that this field has only a single  $\alpha$  (say  $\alpha_0$ ), and that it produces (through Eq. (2)) a mean temperature distribution upon which any temperature fluctuation field other than  $\psi_L$  will decay. We shall now argue that in the limit  $t \rightarrow \infty$  the assumption  $\psi \rightarrow \psi_L$  is self consistent; that is to say, if  $\psi_L$  is inserted into the right hand side of the DI equations no new terms other than  $\psi_L$  are generated at  $t \rightarrow \infty$ .

To establish this point, we need certain properties of the Green's functions for  $\psi = \psi_L$ . The particular properties of  $G_{nm}^{\alpha_0}(t|t')$  necessary are generalizations of the integrals of  $G_1$  and  $G_2$  established for the simple model. These are,

$$\lim_{t \rightarrow \infty} \int_0^t g_{ij}^{\alpha_0}(t') dt' \rightarrow +\infty, \quad \text{for } (i, j) = \text{odd}, \quad (\text{B14})$$

$$\lim_{t \rightarrow \infty} \int_0^t g_{ij}^{\alpha_0}(t') dt' \rightarrow -0, \quad \text{for } (i, j) = \text{odd}. \quad (\text{B15})$$

The even-odd components of  $g_{ij}^{\alpha}$  are zero, since  $\psi_L$  has no even-odd components. For  $\alpha \neq \alpha_0$ , the integrals are finite.

The conditions under which (B14) and (B15) are valid are discussed in section b of the present appendix. The results derived there indicate that these equations may be valid for  $R \lesssim 1.1 \times 10^4$ ; but cannot be correct for  $R \gtrsim 1.1 \times 10^4$ .

The consistency of the assertion that the DI solutions are  $\psi_L$  is now examined. To do this, it will suffice to examine (B1) - (B4) provided we recall that

these equations should have appropriate  $\alpha$  integrals as given in (10a) and (10b). Note that in (B1, B4) the fluctuating self interactions make two types of contributions to  $\psi^\alpha$ ; the input term, containing the factor  $g^\alpha$ , and the drain term, containing the factor  $\psi^\alpha (t' | s)$ . For example, in (B1) the second term on the right hand side is an input term, and the last two terms are classified as drain terms. The drain terms may be positive (as is the third term on the right hand side of (B1)) but in the main, they are negative. Similarly, the input terms are mainly positive (they are strictly positive if the non-diagonal elements of  $\psi$  and  $g$  are suppressed). Consider first the odd-odd  $\psi^{\alpha_0}$  equations. The input terms all vanish, since these contain even-even indexed  $\psi$  fields, and  $\psi_L$  contains no such components. The drain term vanishes because of (B15). Next consider the even-even  $\psi^{\alpha_0}$  equation. The input term vanishes because of (B15), and the drain term vanishes because there is no even-even component in  $\psi^{\alpha_0}$ .

For  $\alpha \neq \alpha_0$ , the odd-odd components receive zero input, and a finite drain. Hence, they will vanish in the limit  $t \rightarrow \infty$ . The even-even components do receive an input, however, the drain term (the last term in (B2)) becomes indefinitely large because of (B14): hence the even-even components  $\rightarrow 0$  as  $t \rightarrow \infty$ . The input terms cannot become indefinitely large because if  $\alpha \neq \alpha_0$  the time integrals of  $G^\alpha$  are finite.

The above discussion suggests that there may exist static solutions to the DI equations which in the limit  $t \rightarrow \infty$  which are identical with the quasi-linear solutions. This does not imply that if  $\psi_L$  is introduced as initial data the solution

will remain equal to  $\psi_L$ . This is because the DI equations involve the history of the system. The present discussion suggests that in this case,  $\psi$  will develop transient values  $\neq \psi_L$  which will decay as  $t \rightarrow \infty$ .

Nothing has been said in the above discussion about the stability of the  $\psi_L$  field, or indeed of any static solution to the DI equations. Such information would be very valuable.

b. Asymptotic Form of Green's Function for  $\psi_L$

This section examines the form of the DI Green's function  $G_{ij}(t - t')$ , for the quasi-linear intensities,  $\psi_L$ . The goal is to seek conditions under which (B14) and (B15) are valid. It is first shown that if  $G_{ij}(t)$  tends to a finite limit as  $t \rightarrow \infty$ , then

$$G_{ij}(t) \rightarrow A_{ij}/\sqrt{t}, \quad A_{ij} > 0$$

for

$$(i, j) = \text{odd}, \quad (\text{B16})$$

and

$$\int_0^{\infty} G_{ij}(t) dt = 0$$

for

$$(i, j) = \text{even}. \quad (\text{B17})$$

This is done by examining the Laplace transforms of  $G_{ij}(t)$ :

$$G_{ij}(s) = \int_0^{\infty} e^{-st} dt G_{ij}(t) \quad (\text{B18})$$

The behavior of  $G_{ij}(t)$  for  $t \rightarrow \infty$  may then be inferred by means of the Abel theorem,<sup>19</sup>

$$\lim_{t \rightarrow +\infty} \frac{G(t)}{t^\nu} \rightarrow \lim_{s \rightarrow +0} [G(s) S^{\nu+1} \Gamma(\nu+1)] \quad (\text{B19})$$

if  $\nu > -1.0$ , and

$$\lim_{t \rightarrow \infty} t [G(t) - G(t-1)] \rightarrow 0 ,$$

and

$$\lim_{t \rightarrow \infty} e^{-pt} G(t) \rightarrow 0 ,$$

for

$$\text{Real}(p) > 0$$

The Laplace transform of the direct interaction Green's function Eq. (10b) for  $\psi = \psi_L$ , may be written in matrix form as,

$$(s + \mu)G = I + CGCG \quad (\text{B20})$$

where;

$$\mu_{nm} = -\nu_n \delta_{nm} + B_{nm}$$

$$I_{nm} = \delta_{nm}$$

$$C_{nm} = \sum_p M_{npm} \theta_p$$

here  $\theta_p$  is the static quasi-linear temperature fluctuation amplitude. In deriving (B20) use of the condition  $\psi_{nm} = \theta_n \theta_m$  for the quasi-linear system has been made. The mean field for the quasi-linear interaction,  $B_{nm}$  is also made up of the  $\theta$ -field in accordance with (2). Recall now that  $\theta_n$  has only odd components. Then by using the definition of  $M_{npm}$  (see Eq. (2)) it may be shown that  $C_{nm} \neq 0$  only if  $n + m = \text{odd}$ . This permits (B18) to be split apart into even and odd components. Let  $P_a$  be a projection matrix which deletes from any vector the even-Fourier components, and let  $P_b$  be the compliment of  $P_a$ ,  $P_a + P_b = I$ . Then two equations for the even-even and odd-odd components of  $G$  may be found by pre and post-multiplying (B20) by first  $P_a$  and then by  $P_b$ :

$$(sI_a + \bar{\mu}_a) \bar{G}_a = I_a + \bar{C}_{ab} \bar{G}_b \bar{C}_{ba} \bar{G}_a \quad (\text{B21})$$

$$(sI_b - \bar{\mu}_b) \bar{G}_b = I_b + \bar{C}_{ba} \bar{G}_a \bar{C}_{ab} \bar{G}_b \quad (\text{B22})$$

where

$$\bar{G}_a = P_a G P_a ,$$

$$I_a = P_a ,$$

$$\bar{C}_{ab} = P_a C P_b ,$$

$$\bar{C}_{ba} = P_b C P_a , \quad \text{etc.}$$

The derivation of (B21) and (B22) uses the fact that  $P_a C P_a = P_b C P_b = 0$ . Equations (B21) and (B22) may be collapsed by systematically deleting the zero interstitial rows and columns. The collapsed equations are identical in form to (B21 and (B22), with  $I_a$  and  $I_b$  replaced by  $I$ :

$$(s + \mu_a) G_a = 1 + C_{ab} G_b C_{ba} G_a \quad (\text{B23})$$

$$(s + \mu_b) G_b = 1 + C_{ba} G_a C_{ab} G_b \quad (\text{B24})$$

The collapsed form of C's are

$$C_{ab} = \begin{pmatrix} C_{12} & C_{14} & \text{----} \\ C_{32} & C_{34} & \text{----} \end{pmatrix}$$

and

$$C_{ba} = \begin{pmatrix} C_{21} & C_{23} & \text{----} \\ C_{41} & C_{43} & \text{----} \end{pmatrix}$$

The solution of (B23) and (B24) for  $G_a$  is;

$$G_a = (s + \mu_a)^{-1} Q \quad (\text{B25})$$

where

$$Q^2 - \Gamma Q = -\Gamma$$

and

$$\Gamma = (s + \mu_a) C_{ba}^{-1} (s + \mu_b) C_{ab}^{-1} .$$



It is convenient to write the matrix elements of  $G_a$  in the laminar eigenmode representation. The elements are

$$\langle k_a | G_a | j_a \rangle = \frac{1}{2(s + \mu_a^k)} \sum_i (\gamma_i \pm \sqrt{\gamma_i^2 - 4\gamma_i}) \langle k_a | \gamma_i \rangle \langle \gamma_i | j_a \rangle \quad (\text{B26})$$

In Eq. (B26),  $\mu_a^k$  is the  $k^{\text{th}}$  eigenvalue of the odd-odd part of  $\mu$ ,  $|j_a\rangle$  is its associated eigenvector, and  $\langle k_a |$  denotes the eigenvector to the adjoint matrix,  $\tilde{\mu}$ . The  $\gamma_i$ 's are eigenvalues of the matrix  $\Gamma$ ,  $|\gamma_i\rangle$  denotes the corresponding eigenvector, and  $\langle \gamma_i |$  the eigenvector of the adjoint matrix  $\tilde{\Gamma}$ .

Equation (B24) may be cast in alternate forms by using the properties of  $\Gamma$ :

$$\langle k_a | G_a | j_a \rangle = \frac{1}{2(s + \mu_a^j)} \sum_i (\gamma_i \pm \sqrt{\gamma_i^2 - 4\gamma_i}) \langle k_a | E | \gamma_i \rangle \langle \gamma_i | E^{-1} | j_a \rangle \quad (\text{B27})$$

$$= \frac{1}{2} \sum_i \frac{1}{\gamma_i} (\gamma_i \pm \sqrt{\gamma_i^2 - 4\gamma_i}) \langle k_a | E | \gamma_i \rangle \langle \gamma_i | j_a \rangle \quad (\text{B28})$$

where;

$$E = C_{ba}^{-1} (s + \mu_b) C_{ab}^{-1}$$

The sign of the radicals in (B26), and (B28) is to be chosen so that

$$\lim_{s \rightarrow 0} (\langle k_a | G_a | j_a \rangle) \rightarrow \delta_{kj} ,$$

which corresponds to the condition initial condition on  $G_{ij}(t)$ ,  $G_{ij}(0) = \delta_{ij}$ .

Equations (B27) and (B28) may be established from (B26) by using the definition of  $\Gamma$  and the relations,

$$\langle k_a | \gamma_i \rangle = \frac{(s + \mu_a^k)}{\gamma_i} \langle k_a | \mathbf{E} | \gamma_i \rangle$$

$$\langle \gamma_i | j_a \rangle = \frac{\gamma_i}{s + \mu_a^j} \langle \gamma_i | \mathbf{E}^{-1} | j_a \rangle$$

We are now in a position to examine the limit of  $G_a(s)$  as  $s \rightarrow 0$ . First, note from (B26) and (B27) that if  $\gamma_i$ ,  $\langle k_a | \mathbf{E} | \gamma_i \rangle$ ,  $\langle \gamma_i | \mathbf{E}^{-1} | j_a \rangle$ ,  $\langle k_a | \gamma_i \rangle$ , and  $\langle \gamma_i | j_a \rangle$  remain finite as  $s \rightarrow 0$ , so does  $\langle k_a | G_a | j_a \rangle$  for either  $\mu_a^j$  or  $\mu_a^k \neq 0$ . We assume temporarily that the above is so and shall return later to a (numerical) verification. Thus, under the above conditions all the laminar eigenmode matrix elements of  $G$  will remain finite as  $s \rightarrow 0$  with the exception of  $\langle \mu_k^a = 0 | G_a | \mu_k^a = 0 \rangle$ . Furthermore, from (B25), it is expected that at least one of the  $\gamma_i$ 's, say  $\gamma_1$  becomes linear in  $s$  as  $s \rightarrow 0$ , since the first factor in  $\Gamma$ ,  $(\mu_a + s)$  has one zero eigenvalue as  $s \rightarrow 0$ , namely that corresponding to  $\mu_k^a = 0$ . We assume temporarily that  $\gamma_1 \rightarrow -as$  as  $s \rightarrow 0$ , where  $a > 0$ , and suppose that  $\gamma_1(s)$  remains negative and real for sufficiently small  $s$ . Then according to (B25) and (B26),

$$\langle \mu_a^1 | G_a | \mu_a^1 \rangle \xrightarrow{s \rightarrow 0} \sqrt{\frac{a}{s}} \langle \mu_a^1 | \mathbf{E} | \gamma_1 \rangle \langle \gamma_1 | \mathbf{E}^{-1} | \mu_a^1 \rangle \quad (\text{B29})$$

Here (B28) is used only to exclude the possibility that  $G \rightarrow 1/s$  as  $s \rightarrow \infty$ .

Equations (B29) and (B19) may now be used to infer the behavior of  $G(t)$  as  $t \rightarrow \infty$ :

$$\langle \mu_a^1 | G_a | \mu_a^1 \rangle \xrightarrow{t \rightarrow \infty} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{t}} \langle \mu_a^1 | E | \gamma_1 \rangle \langle \gamma_1 | E^{-1} | \mu_a^1 \rangle \quad (\text{B30})$$

Or, in terms of the Fourier representation of  $G_a$ ,

$$G_{nm}^a(t) \xrightarrow{t \rightarrow \infty} \frac{1}{\sqrt{\pi t}} A_{nm} \quad (\text{B31})$$

where

$$A_{nm} = \langle n | \mu_a^1 \rangle \langle \mu_a^1 | 1 \rangle \langle \mu_a^1 | E | \mu_a^1 \rangle \langle \mu_a^1 | E^{-1} | \mu_a^1 \rangle$$

Equations (B30) and (B31) are valid provided the conditions of the Abel theorem are met; i.e., provided the Laplace transform integral of  $G_{nm}^{(a)}(t)$  exists in the right half  $t$ -plane, exclusive of the imaginary axis. Examination of (B26) shows that a necessary condition for this to be so is that the quantity  $\gamma_i(s) \{ \gamma_i(s) - 4 \}$  does not become negative for any  $s > 0$  and any  $i$ . If the above conditions are not satisfied, Eq. (B26) implies that  $G(s)$  has a branch cut on the real  $s$ -axis which must intrude into the right half plane. The branch cut will make a contribution to  $G(t)$  which will behave at large  $t$  like  $\sqrt{t} e^{at}$ . In order to obtain sufficient conditions for (B31) to be valid, it must be shown that none of the possible singularities of  $G(s)$  in the right hand plane contribute to the inverse Laplace transform.

Equation (B2) for the even-even components of  $G_{ij}$  may now be verified by inserting (B15) into (B9) and taking the limit as  $s \rightarrow 0$ . From this it follows that

$$G_{ij}(s) \rightarrow \sqrt{s} B_{ij} ,$$

for

$$(i, j) = \text{even} .$$

(B2) follows immediately by using (B3) in the limit  $s \rightarrow 0$ .

To complete the discussion of (B1) and (B2) we must clear up several assumptions made in the discussion relating to the matrices A, E, and  $\Gamma$ . In order for (B1) and (B2) to be valid,  $A_{ij} > 0$ , E must be non-singular, and the eigenvalue spectrum of  $\Gamma(s)$  must be such as to make  $\sqrt{\gamma_i^2(s) - 4\gamma_i(s)}$  real for  $s > 0$ . These points may be explored numerically in lieu of an analytic procedure.

The numerical results on this point indicate that for  $R < 1.18 \times 10^4$  the assumptions enumerated in the above paragraph are valid. The computed  $\gamma_i$  spectra consists of negative real numbers, for sufficiently small  $s > 0$ , and explicit calculation shows that  $A_{ij} > 0$ . For  $R > 1.18 \times 10^4$  the behavior of  $\gamma_1(s)$  is not consistent with the above condition on the  $\gamma_i$ 's. For  $R > 1.18 \times 10^4$ ,  $\gamma_1$  is real and positive at small values of  $s$  and becomes real and negative as  $s \rightarrow +\infty$ . The behavior of  $\gamma_1(s)$  for  $s \rightarrow 0$  as a function of  $R$  is depicted in Fig. (B1).

The numerical calculations from which these conclusions were drawn consist of using an eight mode Fourier representation of the quasi-linear amplitude system,  $\theta_n$ . The relevant matrices  $C_{ab}$ ,  $C_{ba}$ ,  $\Gamma$ , etc. were then truncated  $4 \times 4$

matrices. Several runs using a four mode Fourier representation of  $\theta_n$  indicated that the eight mode representation is adequate.

## Appendix C

### Numerical Procedure

This section describes the procedure for numerically integrating the direct interaction Eqs. (5), (10a), and (10b), starting from arbitrary initial conditions for the covariance matrix  $\psi_{nm}(0|0)$ . Integration of the equations for the quasi-linear and quasi-normal approximations may be made by simply setting to zero certain of the DI terms (as in the quasi-linear approximation) or replacing the certain terms by simplified expressions (as for example, in the quasi-normal method Eq. (11) is used to approximate the non-simultaneous correlation coefficients in terms of simultaneous ones).

Equations (5), (10a), and (10b) are a set of integro-differential equations for  $\psi_{nm}^{\alpha}(t|t')$ . They are discrete in the vertical index  $(n, m)$  and continuous in the indices (or arguments)  $\alpha$ , and  $(t, t')$ . In treating these equations numerically, a suitable truncation procedure for the wave numbers  $(n, \vec{\alpha})$ , and a discretizing procedure for  $\alpha$ ,  $t$ , and  $t'$  must be prescribed. The procedure now described does this in a way so as to preserve the conservation properties of the flow.

The flow is assumed to be isotropic in the horizontal direction, so that  $\psi^{\alpha}$  and  $g^{\alpha}$  depend only on the magnitude of  $\vec{\alpha}$ .

To truncate the system in wave number  $(n, \alpha)$  discard any  $\psi_{nm}$  or  $g_{nm}$  function if either  $n$  or  $m > M$  or if  $|\vec{\alpha}| > \alpha_1$  or  $|\vec{\alpha}| < \alpha_0$ , where  $\alpha_0 < \alpha_1$ . In practice, the finite machine storage capacity impose a rather severe limit on the number of individual dependent variables treated. In the present analysis it was necessary

to take  $M \leq 5$ , to fit the program into the IBM 36065 fast storage. According to previous numerical results for the quasi-linear method,  $M = 5$  is sufficient to accurately describe the mean temperature field up to a Rayleigh number of  $\sim R = 10^4$ .

To proceed further, the basic Eqs. (5), (10a), and (10b) are reduced by using the assumption of horizontal isotropy. These are of the form

$$\begin{aligned} \left( \frac{d}{dt} + \nu^\alpha \right) \psi^\alpha &= B^\alpha \psi^\alpha + \int_0^{t'} ds \int_0^\infty \frac{da' da''}{a} \rho(\alpha, \alpha', \alpha'') A(\alpha, \alpha', \alpha'') g^\alpha \psi^{\alpha'} \psi^{\alpha''} \\ &+ \int_0^t ds \int_0^\infty \frac{da' da''}{a} \rho(\alpha, \alpha', \alpha'') C(\alpha, \alpha', \alpha'') \psi^\alpha g^{\alpha'} \psi^{\alpha''} \end{aligned} \quad (C1)$$

$$\left( \frac{d}{dt} + \nu^2 \right) g^2 = B^\alpha g^\alpha + \int_{t'}^t ds \int_0^\infty \frac{da' da''}{a} \rho(\alpha, \alpha', \alpha'') C(\alpha, \alpha', \alpha'') g^\alpha \psi^{\alpha'} g^{\alpha''} \quad (C2)$$

$$\left( \frac{d}{dt} + \nu_n^0 \right) \beta_n = \pi^2 \sum_{p=1}^{\alpha} \int_0^\infty (2\pi a a a) J_p^\alpha (\sigma(n-p) \psi_{p, n-p}^\alpha - \psi_{p, n+p}^\alpha) \quad (C3)$$

where;

$$\rho(\alpha, \alpha', \alpha'') = 4\alpha\alpha' \alpha'' / \sqrt{(\alpha + \alpha' + \alpha'') (\alpha + \alpha' - \alpha'') (\alpha + \alpha'' - \alpha') (\alpha' + \alpha'' - \alpha)}$$

Here, the vertical index  $(n, m)$  as well as the time arguments are suppressed.

The latter are  $(t' | s)$ ,  $(t | s)$ , and  $(t | s)$ , in the order prescribed by (C1), and (C2). The factor  $\rho(\alpha, \alpha', \alpha'')$  is the density of triangles whose sides are of

length  $\alpha$ ,  $\alpha'$ , and  $\alpha''$ . It must be taken to be zero if any of the factors in its denominator is zero.

a. Single Band Approximation to Horizontal Spectrum

The horizontal wave numbers spectrum may be most easily approximated by assuming for a simple square wave form;

$$\begin{aligned} \psi &= \left( \pi(\alpha_1^2 - \alpha_0^2) \right)^{-1} \psi, & \alpha_0 \leq \alpha \leq \alpha_1 \\ &= 0, \text{ otherwise.} \end{aligned} \tag{C4}$$

The procedure used in this paper is as follows:

Multiply (C1) and (C2) by  $(2\pi\alpha \, d\alpha)$  and integrate over  $\alpha$ , using (C4) to approximate the  $\psi$ -spectrum in the convolution terms.

Approximate  $g^\alpha(t | t')$  by a constant-in- $\alpha$ -value,  $g(t | t')$ . There results equations for  $\psi$  and  $g$  which differ from (C1) and (C2) by first deleting the  $\alpha$  integrations (with the  $\rho$ -factor) and then using averaged coefficients in place of  $\nu^\alpha$ ,  $B^\alpha$ ,  $A$  and  $C$ . The average coefficients are,

$$\bar{\nu}_n = \left( \pi(\alpha_1^2 - \alpha_0^2) \right)^{-1} \int_{\alpha_0}^{\alpha_1} (2n\alpha) \nu_n^\alpha = n^2 + \frac{1}{2} (\alpha_1^2 + \alpha_0^2)$$

$$\bar{B}_{nm} = \frac{1}{2} (\beta_{1n-m1} - \beta_{n+m}) \bar{J}_m$$

$$\bar{J}_n = (R/\pi^4) \left\{ \left[ \ln \left( \frac{n^2 + \alpha_1^2}{n^2 + \alpha_0^2} \right) \right] / (\alpha_1^2 - \alpha_0^2) - n^2 / [(n^2 + \alpha_0^2)(n^2 + \alpha_1^2)] \right\}$$

$$\bar{A} = \iiint d\alpha \, d\alpha' \, d\alpha'' \rho(\alpha, \alpha', \alpha'') A(\alpha, \alpha', \alpha'') / \iiint \rho(\alpha, \alpha', \alpha'') \, d\alpha \, d\alpha' \, d\alpha'' \dots$$



The mean field Eq. (C3) is similarly averaged with the result that  $\bar{J}_p$  replaces  $J_p^\alpha$ . It may be verified directly that the above averaging procedure preserves the conservation property given by (3).

b. Approximation of  $\alpha$ -Integration Using Several  $\alpha_i$ .

In this case it is necessary to work with  $\phi(\alpha)$  and  $\tilde{g}(\alpha)$  where,

$$\phi(\alpha) = 2\pi\alpha \psi(\alpha)$$

and

$$\tilde{g}(\alpha) = 2\pi\alpha g(\alpha)$$

These quantities satisfy equations like (C1) and (C2) except that  $\rho$  is replaced by  $\tilde{\rho} = \rho / (4\pi^2 \alpha \alpha' \alpha'')$ . The initial condition on  $\tilde{g}$  is  $\tilde{g}(\alpha, 0) = 2\pi\alpha$ . Next, (C1) and (C2) are evaluated at  $\alpha_k = \alpha_0 + (2k - 1)\Delta/N$  and the first integral in (C1) is approximated by,

$$\Delta^2 \sum_{j,k=1}^N \tilde{\rho}(\alpha_i, \alpha_j, \alpha_k) A(\alpha_i, \alpha_j, \alpha_k) \tilde{g}(\alpha_i) \phi(\alpha_j) \phi(\alpha_k)$$

Here  $\alpha_0$  and  $2\alpha_0$  are lower and upper limits beyond which the integrand is considered to make a negligible contribution to the integral. Similar approximations are made in the other integrals of (C1) and (C2).

The integration in (C3) is simple enough to make a more accurate integration formula practical. In this case we use the points  $\phi(\alpha_1)$ ,  $\phi(\alpha_2)$ , and  $\phi(\alpha_3)$  to make a quadratic interpolation of  $\phi(\alpha)$ . The effect of this is to change  $J_n(\alpha_i)$  to  $J_n(i)$

where,

$$J_n(i) = \int_{\alpha_0}^{2\alpha_0} \frac{(\alpha - \alpha_j)(\alpha - \alpha_k)}{(\alpha_i - \alpha_j)(\alpha_i - \alpha_k)} J_n(\alpha) \, d\alpha, \quad (j, k \neq i).$$

These integrals are of standard form.

### c. Time Integration.

The time integration of (C1) and (C2) is straightforward and has been described elsewhere in more detail by Kraichnan.<sup>19</sup> The domain of the integration is  $t \geq t'$ . For  $t = t'$ ,  $G(t | t') = 1$ , and  $\psi(t | t')$  is given by (C1) with the term  $(\partial\psi/\partial t)$  replaced by  $(1/2)(d\psi/dt)$ . Equation (C1) involves  $\psi_{nm}^\alpha(t | t')$ ,  $t' > t$  in the convolution integral in the first term on the right hand side. Such terms are evaluated by using the fact that  $\psi_{nm}(t | t') = \psi_{mn}(t' | t)$ , which follows from Eq. (4).

Next, a mesh of points  $(t_i, t_j)$  is placed in the  $(t, t')$  plane, and, using them, the time integrals are approximated by the trapezoidal rule. The mesh point spacing  $\Delta$  is taken to be uniform and the same in both time directions. To obtain  $\psi(t_i | t_j)$  from  $\psi(t_{i-1} | t_j)$ , formally integrate (C1) over the interval  $(t_{i-1}, t_i)$ , replacing the right hand side by its average value on  $(t_{i-1}, t_i)$ . The result is,

$$\psi(t_i | t_j) = e^{-\bar{v}_n \Delta} \psi(t_{i-1} | t_j) + \frac{(1 - e^{-\bar{v}_n \Delta})}{2\bar{v}_n} [F(t_i) + F(t_{i-1})],$$

where  $F(t_i)$  represents the right hand side of (C1). This is an implicit equation for  $\psi(t_i | t_j)$ , since  $F(t_i)$  involves  $\psi(t_i | t_j)$  itself. It must be solved by

iteration. The procedure used here is to approximate  $\psi(t_i | t_j)$  by the second order iteration using  $F(t_i) = F(t_{i-1})$  to initialize the right hand side. This guarantees accuracy of order  $\Delta^2$ . Aside from the exponential factors this scheme is the standard predictor - corrector procedure, in which the predictor is first order and the corrector is of second order. The presence of the exponential factors stabilizes the behavior of the large wave number modes.

The initial values of  $\psi_{nm}^\alpha(0 | 0)$  are arbitrary. For most of the calculations they are chosen to be those values obtained from exact solutions to the hexagon amplitude equations, in which only a single horizontal wave number is included. That is to say,

$$\psi_{nm}^\alpha(0 | 0) = \theta_n \theta_m \quad (C5)$$

This choice is a convenient one in that the system does not develop violent transient oscillations from these initial data. This permits a relatively large mesh size to be used.

It is known that the hexagon solutions to the convection problem  $\theta(\vec{r}, t)$ , are asymmetric about the midpoint  $z = 1/2$ , with the direction of asymmetry arbitrary. This asymmetry is eliminated by averaging (C5) over the direction of asymmetry, which is equivalent to setting to naught the even-odd components of (C5).

## FIGURE CAPTIONS

- Fig. 1 Nusselt number as a function of Rayleigh number  $R$  for: (a) the direct interaction, (b) quasi-linear, (c) quasi-normal methods. The calculations contain 2 vertical and one horizontal degrees of freedom.
- Fig. 2  $R_{11}(8 | t)$  and  $R_{22}(8 | t)$  for the direct interaction procedure for  $R = 10^3$ .
- Fig. 3 Green's functions for direct interaction method for  $R = 10^3$ . Parameters of the calculation are the same as in Fig. 2
- Fig. 4 Direct interaction correlation coefficients  $R_{11}(t | t - 1.5)$  and  $R_{22}(t | t - 1.5)$  for  $R = 4 \times 10^3$ .
- Fig. 5 Direct interaction correlation coefficients  $R_{31}(t | t - 1.5)$  and  $R_{42}(t | t - 1.5)$  for same initial conditions as on Fig. 3.
- Fig. 6 Direct interaction Green's functions  $G_{11}$ ,  $G_{13}$ , and  $G_{31}$  for  $R = 4 \times 10^3$ .
- Fig. 7 Direct interaction Green's functions  $G_{22}$ , and  $G_{24}$  for  $R = 4 \times 10^3$ .
- Fig. 8 Evolution of Nusselt number for  $R = 10^4$  according to Direct Interaction approximation.
- Fig. 9 Direct interaction correlation functions  $R_{11}(t | 0.75 - t)$  and  $R_{22}(t | 0.75 - t)$  for  $R = 10^4$ .
- Fig. 10 Evolution of  $\psi_{11}(t | t)$  according to the quasi-normal approximation at  $R = 2 \times 10^3$ . Run contains two vertical and one horizontal wave numbers.
- Fig. 11 Evolution of the Nusselt number for  $R = 3 \times 10^3$ , according to; (1) Numerical experiment (solid line), (2) The direct interaction (dashed

line), (3) Quasi-linear approximation points enclosed in triangles,  
(4) Quasi-normal approximation (dotted line). Points enclosed in  
squares give  $N_u(t)$  for 124  $\alpha$ -mode experiment.

Fig. 12 Evolution of  $\psi_{11}(\alpha, t)$  for  $R = 3 \times 10^4$ . Solid curve represents the  
numerical experiment and the encircled points are direct interaction  
results.

Fig. B1  $(\gamma_1(s)/s)$  for  $s = 10^{-4}$  as a function of  $R$ .

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