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Statistical-Thermodynamics Formalism of Self-Similarity

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The self-similarity in dynamical and stochastic systems is formulated from a statistical-thermodynamical standpoint. The global structures of the self-similarity are found to be determined by one generating function which plays a role similar to the Helmholtz free energy in the equilibrium statistical mechanics. The interrelations among fractal measure theories developed for, e.g., generalized fractal dimensions of strange sets and velocity structure functions in turbulence are clarified from a unified point of view.

§ 1. Introduction

The Mandelbrot fractal dimension[1] turns out to be a fundamental quantity to characterize strange sets. However, it does not describe non-uniformity structures of the strange set in the state space. Some of such aspects are explained by a set of dimensions $\{D_q\}$ introduced by Hentschel and Procaccia,[2] and by Grassberger.[3]

Recently a new set of quantities characterizing fluctuations of scaling indices for strange sets has been proposed by Halsey et al.[4] by employing a set of dimensions $\{D_q\}$. A similar fractal measure theory has been constructed in the context of velocity structure functions in developed turbulence.[5] After these works many studies are reported for, e.g., strange attractors,[6)~9)] diffusion-limited aggregations[10),11)] and turbulence.[12)] The fractal measure theory transforms the set of dimensions into the spectrum of scaling indices of the fractal measure. The key point of the spectrum of scaling indices in the above cases seems to be based on the fractal concept. However, as a more fundamental concept than fractals so as to construct the spectrum theory of scaling indices, we propose the importance of the *self-similarity* concept. The main aim of the present paper is to construct the spectrum theory of fluctuations obeying a self-similar statistics. This will turn out to lead to the statistical-thermodynamics formalism of the self-similarity.

The present paper is constructed as follows. In § 2, we give the characteristic function theory of the self-similarity in connection with the theory of similarity exponents.[13),14)] The statistical-thermodynamics formalism of the self-similarity is developed in § 3. In § 4, we will discuss the interrelations among the present formalism and others for a set of dimensions, similarity exponents and in developed turbulence. A summary and remarks are given in § 5.

§ 2. Self-similarity and characteristic function

Let $\{u_n;\ n=0, 1, 2, 3, \cdots\}$ be a one-dimensional sequence generated by x_n, underlying degrees of freedom, through $u_n=u(x_n)$, where x_n obeys a dynamical or a stochastic law. The u_n is statistically independent of n, i.e., steady in n, for a large n.

Without loss of generality, we hereafter assume that u_n is bounded, $u^{\min} < u_n < u^{\max}$ for any n. When a set $\{A_n\}$ is generated according to

$$\frac{A_{n+1}}{A_n} = e^{u_n}, \qquad (n=0, 1, 2, 3, \cdots) \tag{2·1}$$

the set is called *self-similar*.[13),14)] This is the explicit expression of the self-similarity. Equation (2·1) is solved as

$$A_n = A_0 \exp(n z_n), \qquad z_n \equiv \frac{1}{n}\sum_{j=0}^{n-1} u_j . \tag{2·2}$$

Let $\rho_n(\alpha')$ be the probability density that z_n takes the value between α' and $\alpha' + d\alpha'$. If z_n has no fluctuation, $\rho_n(\alpha')$ has a sharp peak. The fluctuation of z_n, on the other hand, produces a spectral structure of $\rho_n(\alpha')$. By noting that the fluctuation of z_n becomes small for $n \to \infty$, to study the global structures of the self-similarity is equivalent to studying how the fluctuation of z_n reduces as the averaging scale n increases. This will be carried out by considering the statistics of A_n instead of z_n itself. It is closely related to the magnification of the fluctuation of z_n for large n.[13)]

The order-q moment $M_n(q)$ of A_n is calculated by

$$M_n(q) \equiv \langle A_n{}^q \rangle \sim \int \rho_n(\alpha') e^{q\alpha' n} d\alpha' , \tag{2·3}$$

where q is real and $\langle \cdots \rangle$ is the ensemble average. Noting that $\ln A_n$ is the *extensive* variable in the sense that it is, roughly speaking, proportional to n, we assume that $M_n(q)$ asymptotically takes the form

$$M_n(q) \sim \exp(q \lambda_q n) \tag{2·4}$$

for a large n. Here the characteristic function λ_q has been introduced by[13),14)]

$$\lambda_q = \frac{1}{q} \lim_{n\to\infty} \frac{1}{n} \ln M_n(q) , \tag{2·5}$$

where we get

$$\frac{d\lambda_q}{dq} \geqq 0 . \tag{2·6}^{*)}$$

First we assume that λ_q is expanded as the cumulant expansion[13)]

$$\lambda_q = \sum_{m=1}^{\infty} c_m q^{m-1} , \tag{2·7}$$

where c_m is determined by the m-th order cumulant of $\ln A_n$. This expansion is meaningful only when $|q|$ is smaller than $\varkappa$, the radius of the convergence. Especially, for $|q| \ll \varkappa$, (2·7) is approximated as

$$\lambda_q = \lambda_0 + Dq . \tag{2·8}$$

Here $\lambda_0 = \langle u_n \rangle (\equiv c_1)$, and $D (\equiv c_2 > 0)$ is defined through $\langle (z_n - \langle z_n \rangle)^2 \rangle \simeq 2D/n$ for a large n. λ_0 and D have the meanings of the drift velocity and the diffusion coefficient,

*) This is easily shown with the inequality $x \ln x \geqq x - 1$.

respectively, provided that n is regarded as the time. On the other hand, for $|q|\gg\varkappa$, the above expansion does not converge. Instead of the cumulant expansion, λ_q is generally expanded as the following form:

$$\lambda_q=\lambda_{\varepsilon\infty}-\frac{1}{q}\left[\frac{1}{\tau_\varepsilon}+\mathrm{const}\cdot\exp(-\gamma_\varepsilon|q|)\right] \qquad (2\cdot9)^{*)}$$

for $\varepsilon q\gg\varkappa$, $(\varepsilon=\pm)$, where $\tau_\varepsilon>0$ and $\gamma_\varepsilon>0$. The statistical characteristics for $|q|\ll\varkappa$ is picturized as the diffusion process.[14] On the other hand, for $|q|\gg\varkappa$, the global statistics of $\{u_j\}$ is described as the *intermittency*[14] as follows. The most dominant contribution to $M_n(q)$ for, e.g., $q\gg\varkappa$ comes from a series S_l which gives the largest value λ_∞ of z_n. In this sense the series $S_l(u_0, u_1, u_2, \cdots, u_{n-1})$ is coherent and is called *laminar*. The laminar regions are interrupted by the insertion of non-laminar states (*bursts*). λ_∞ is nothing but the average of u_j over laminar regions, and τ_+ estimates the duration of one laminar region.[14] A similar argument holds for $\varepsilon=-$. The large difference between the diffusion and the intermittency is namely the physical reason of the existence of the radius of the convergence in $(2\cdot7)$, i.e., the difference of asymptotic forms of $(2\cdot8)$ and $(2\cdot9)$.

§ 3. Statistical-thermodynamics formalism of self-similarity

Now we assume that $\rho_n(\alpha')$ is asymptotically given by

$$\rho_n(\alpha')\sim e^{-n\sigma(\alpha')} \qquad (\sigma(\alpha')\geqq0) \qquad (3\cdot1)$$

for a large n. $\sigma(\alpha')$ estimates the rate of the decrease of the probability density that z_n takes the value α' as the step number proceeds. Inserting $(3\cdot1)$ into $(2\cdot3)$ and applying the saddle point method, one obtains

$$\lambda_q=-\frac{1}{q}\min_{\alpha'}[-q\alpha'+\sigma(\alpha')]\,. \qquad (3\cdot2)$$

This can be rewritten in a slightly different form as in Table I. The transformation from the set (q, λ_q) to $(\alpha, \sigma(\alpha))$ is identical to the Legendre transformation, λ_q being the generating function of α and $\sigma(\alpha)$. We note that α is a monotonic function of q,

$$\frac{d\alpha}{dq}\geqq0\,, \qquad (3\cdot3)^{**)}$$

$(\sigma''(\alpha)>0)$, and that $\sigma(\alpha)$ has only one minimum at $\alpha=\lambda_0$, $(q=0)$. One easily

Table I. Thermodynamic relations among variables obtained from the generating function λ_q. C was introduced in connection with the heat capacity in the equilibrium statistical mechanics.

$\alpha=\frac{d}{dq}(q\lambda_q)$
$\sigma(\alpha)=-\frac{d\lambda_q}{dq^{-1}}(\geqq0)$
$\frac{d\sigma(\alpha)}{d\alpha}=q$
$\lambda_q=\alpha-\frac{\sigma(\alpha)}{q}$
$C\equiv\frac{d\alpha}{dq^{-1}}(<0)$
$\sigma''(\alpha)>0$

*) One should note that $(2\cdot9)$ is derived from Eqs. $(3\cdot3)$ and $(3\cdot4)$ in Ref. 14) by including the next order term smaller than $O(q^{-1})$. Results $(2\cdot8)$ and $(2\cdot9)$ correspond to the diffusion-branch and the intermittency-branch results in Ref. 14), respectively.

**) This can be proved with the inequality $\langle x(\ln x)^2\rangle\geqq\langle x\ln x\rangle^2$ provided that $\langle x\rangle=1$.

sees that by regarding q, λ_q, α, $\sigma(\alpha)$ and C the inverse temperature ($=1/k_B T$ with the Boltzmann constant k_B and the temperature T of the system), the Helmholtz free energy, the internal energy, the entropy and the heat capacity, respectively, relations in Table I are equivalent to the thermodynamic relations in the equilibrium statistical mechanics.[15] Furthermore the quantity $\zeta_q \equiv -q\lambda_q$ corresponds to the Massieu function.

Depending on asymptotic laws of λ_q for $|q| \ll \varkappa$ and $|q| \gg \varkappa$, the q vs α and α vs $\sigma(\alpha)$ curves have different asymptotic forms. Namely, for $|q| \ll \varkappa$, one obtains

$$\alpha = \lambda_0 + 2Dq\,, \tag{3·4a}$$

$$\sigma(\alpha) = \frac{(\alpha-\lambda_0)^2}{4D}\,. \tag{3·4b}$$

The parabolicity of $\sigma(\alpha)$ near $\alpha=\lambda_0$ is known as the central limit theorem result.[13),14)] On the other hand, for $\varepsilon q \gg \varkappa$, (2·9) leads to

$$\alpha - \lambda_{\varepsilon\infty} \propto \exp(-\gamma_\varepsilon |q|)\,, \tag{3·5a}$$

$$\sigma(\alpha) - \frac{1}{\tau_\varepsilon} \propto |\alpha - \lambda_{\varepsilon\infty}| \ln |\alpha - \lambda_{\varepsilon\infty}|^{-1}\,. \tag{3·5b}$$

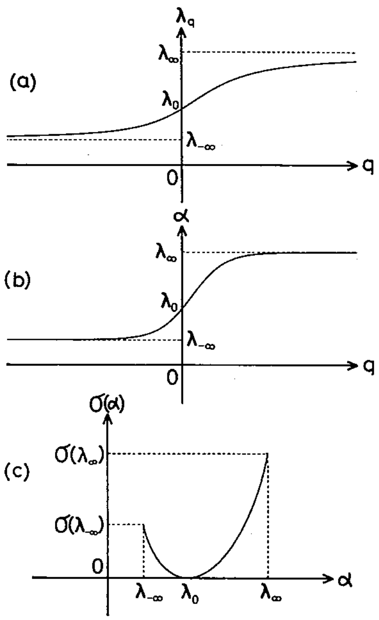

Fig. 1. Statistical-thermodynamic quantities drawn as functions of q and α for the two level process ($u_n = a_1$ and a_2 with the probabilities p_1 and $p_2(=1-p_1)$, respectively), i.e., $\lambda_q = q^{-1}\ln(p_1 e^{qa_1} + p_2 e^{qa_2})$, ($a_1=4.5$, $a_2=1$ and $p_1=2.5/7$). Although λ_q and α have similar q-dependence, the slope of α at $q=0$ is twice as steep as that for λ_q, and approaches $\lambda_\infty(\lambda_{-\infty})$ more rapidly than λ_q as $q\to\infty(-\infty)$.

These behaviors are extremely different from (3·4), and are not perturbationally connected with (3·4). As α gradually deviates from $\alpha=\lambda_0$, $\sigma(\alpha)$ changes its asymptotic form from (3·4b) to (3·5b), roughly speaking, at $\alpha(q=\varepsilon\varkappa)$. Note that $\alpha(\varepsilon\infty)=\lambda_{\varepsilon\infty}$, and $\sigma(\alpha(\varepsilon\infty))=1/\tau_\varepsilon$ which agrees with the previous result,[14),*)] and that the derivative $d\sigma(\alpha)/d\alpha$ logarithmically diverges as α approaches $\lambda_{\varepsilon\infty}$. Furthermore the boundedness of α, ($\lambda_{-\infty} \leqq \alpha \leqq \lambda_\infty$), implies $\rho_n(\alpha)=0$ for $\alpha > \lambda_\infty$ and $\alpha < \lambda_{-\infty}$, i.e., $\sigma(\alpha)=\infty$ for $\alpha > \lambda_\infty$ and $\alpha < \lambda_{-\infty}$. Schematic forms of thermodynamic variables as functions of q and α are drawn in Fig. 1.

Here we note that for the pure intermittency process[14)] which has $\lambda_q = -\varepsilon\infty$ ($\varepsilon q \leqq 0$), $\lambda_{\varepsilon\infty} - \tau_\varepsilon^{-1} q^{-1}$ ($\varepsilon q > 0$), where $\varepsilon = +$ or $-$, the Legendre transformation gives just one line in the α-σ plane, ($\alpha \leqq \lambda_{\varepsilon\infty}$, $\sigma(\alpha) = \tau_\varepsilon^{-1}$). The pure intermittency

*) The dominant contribution to $M_n(q)$ for $q \gg \varkappa$ comes from laminar regions giving the largest value λ_∞ of z_n for a large n. Since e^{-1/τ_+} is the probability that the laminar region continues further by one step, we asymptotically get $M_n(q) \sim (e^{-1/\tau_+})^n \cdot (e^{\lambda_\infty n})^q$, which gives $\lambda_q = \lambda_\infty - \tau_+^{-1} q^{-1}$.

process is therefore characterized by the absence of spectral structure in $\sigma(\alpha)$.

The probability distribution $\rho_n(\alpha')$ for a large n has a peak at $\alpha=\lambda_0$, whose width Γ_n is estimated as

$$\Gamma_n \sim \left(\frac{D}{n}\right)^{1/2}. \tag{3·6}$$

Accordingly, the peak height is evaluated as

$$\rho_n(\lambda_0) \sim \Gamma_n^{-1} \sim \left(\frac{n}{D}\right)^{1/2}. \tag{3·7}$$

One should remark that the central parts of $\rho_n(\alpha')$ are determined by the quantities near $q=0$ (Eqs. (3·4)). On the contrary, large $|q|$ characteristics are contained in tail regions of $\rho_n(\alpha')$. $\rho_n(\alpha')$ approaches the delta function $\delta(\alpha'-\lambda_0)$ as n proceeds. So if we first take the limit $n\to\infty$, α' takes only the value λ_0. To single out the global fluctuation statistics, one should first evaluate deviations from λ_0 for a finite n, and then take the limit $n\to\infty$.

As is shown in Table I, a set K of quantities q, λ_q, α and $\sigma(\alpha)$ constitutes a set of thermodynamic variables. Noting that λ_q and α are single-valued function of q, we can define another set $\hat{K}$ of thermodynamic variables $\hat{q}$, $\hat{\lambda}_{\hat{q}}$, $\hat{\alpha}$ and $\hat{\sigma}(\hat{\alpha})$, employing a one-to-one transformation $K\to\hat{K}$. The transformation is defined so that $\hat{\alpha}\equiv d[\hat{q}\hat{\lambda}_{\hat{q}}]/d\hat{q}$ and $\hat{\sigma}(\hat{\alpha})\equiv -d\hat{\lambda}_{\hat{q}}/d\hat{q}^{-1}$, and so we have $\hat{\lambda}_{\hat{q}}=\hat{\alpha}-\hat{\sigma}(\hat{\alpha})/\hat{q}$. This transformation is complete provided that two interrelations between variables in K and $\hat{K}$ are given. As an illustrative example, we introduce the transformation $T(a, b)$ by

$$\hat{q}=aq+b, \tag{3·8a}$$

$$\hat{\lambda}_{\hat{q}}=-(q/\hat{q})\lambda_q, \tag{3·8b}$$

where a and b are constants. Under $T(a, b)$, thermodynamic variables are transformed as

$$\hat{\alpha}=-\alpha/a, \qquad \hat{\sigma}(\hat{\alpha})=-b\alpha/a-\sigma(\alpha). \tag{3·9}$$

As will be seen later, $T(a, b)$ is the important transformation connecting the present formalism with others proposed in different situations.

§ 4. Comparisons with other theories

In the present section we discuss the interrelations among the present formalism and others introduced independently in different contexts of nonlinear dynamical systems.

(A) *Strange sets*

The first example is the spectrum of scaling indices of the fractal measure.[4)] Let N be the number of state points of the strange set, experimentally observed. The state space is divided into boxes each of which has the volume $\sim l^d$, where

$$l=l_0 e^{-n}, \qquad (n=0, 1, 2, 3, \cdots) \tag{4·1}$$

l_0 being the largest scale of a series of divisions, and d is the topological dimension of the state space where the strange set is embedded. By putting $p_i = \lim_{N\to\infty}(N_i/N)$, N_i being the number of state points being in the i-th box, the order-q dimension of the strange set is defined by[2),3)]

$$D_q = \lim_{l\to 0} \frac{1}{1-q} \frac{\ln\chi(q)}{\ln(1/l)} \tag{4·2}$$

with

$$\chi(q) = \sum_i p_i^q , \tag{4·3}$$

where the summation is over boxes with non-vanishing p_i. Let $\tilde{p}_j$ be the probability for the j-th box for the division $l_0 e^{-(n+1)} = l/e$. We write p_i^{-1} and $\tilde{p}_j^{-1}$ symbolically as

Table II. Correspondence of thermodynamic relations among the statistical-mechanics formalism and others in different systems. The minor differences are the sign of the heat capacity correspondence and the convexity of the entropy correspondence. $\Omega_N(E)$ is the state density function. The quantity e^{-E} should be read as the use of the total energy E appropriately made dimensionless.

equilibrium statistical mechanics	strange sets	time series	velocity structure functions in turbulence
N particle number	$\ln(1/l)$	n	$\ln(1/r)$
e^{-E} E: total energy	p_i^{-1}	e^{X_n} $(X_n \equiv \sum_{j=0}^{n-1} u_j)$	$\Delta V(r)$
$\beta(=1/k_B T)$ inverse temperature	$1-q$	q	p
$\int \Omega_N(E) e^{-\beta E} dE$ partition function	$\sum_i p_i^q$ $(\equiv\langle (p_i^{-1})^{1-q}\rangle)$	$\langle [e^{X_n}]^q \rangle$	$\langle [\Delta V(r)]^p \rangle$
f Helmholtz free energy	D_q	λ_q	$\frac{-\zeta_p}{p}$
$\frac{d}{d\beta}(\beta f) \equiv u$ internal energy	$\frac{d}{d(1-q)}[(1-q)D_q]$ $(\equiv\alpha)$	$\frac{d}{dq}(q\lambda_q)$ $(\equiv\alpha)$	$\frac{d}{dp}\left(p\cdot\frac{-\zeta_p}{p}\right)$ $(\equiv -h)$
$-\frac{df}{d\beta^{-1}} \equiv s(u) \geq 0$ entropy$/k_B$	$-\frac{dD_q}{d(1-q)^{-1}}$ $(\equiv \alpha - f(\alpha) \geq 0)$	$-\frac{d\lambda_q}{dq^{-1}}$ $(\equiv \sigma(\alpha) \geq 0)$	$-\frac{d}{dp^{-1}}\left(\frac{-\zeta_p}{p}\right)$ $(\equiv 3-d(h) \geq 0)$
$\frac{ds(u)}{du} = \beta$	$\frac{d(\alpha - f(\alpha))}{d\alpha} = 1-q$	$\frac{d\sigma(\alpha)}{d\alpha} = q$	$\frac{d(3-d(h))}{d(-h)} = p$
$f = u - \frac{s(u)}{\beta}$	$D_q = \alpha - \frac{\alpha - f(\alpha)}{1-q}$	$\lambda_q = \alpha - \frac{\sigma(\alpha)}{q}$	$\frac{-\zeta_p}{p} = -h - \frac{3-d(h)}{p}$
$\frac{du}{d\beta^{-1}} > 0$	$\frac{d\alpha}{d(1-q)^{-1}} < 0$	$\frac{d\alpha}{dq^{-1}} < 0$	$\frac{d(-h)}{dp^{-1}} < 0$
$\frac{d^2 s(u)}{du^2} < 0$	$\frac{d^2(\alpha - f(\alpha))}{d\alpha^2} > 0$	$\frac{d^2\sigma(\alpha)}{d\alpha^2} > 0$	$\frac{d^2(3-d(h))}{d(-h)^2} > 0$

A_n and A_{n+1}, respectively, which are fluctuating quantities because they depend on boxes. If the ratio A_{n+1}/A_n is assumed to be statistically independent of n, we can apply the formalism developed in the former half of this paper. In fact $\chi(q)$ can be identified with the moment as

$$\chi(q)=\sum_i p_i\cdot(p_i^{-1})^{1-q}\equiv\langle A_n^{1-q}\rangle=M_n(1-q)\,. \tag{4·4}$$

The definition (4·2) gives $\chi(q)\sim l^{-(1-q)D_q}\sim\exp[(1-q)D_q n]$. D_q is therefore nothing but λ_{1-q} (see Eq. (2·5)).[15] The correspondence are summarized in Table II in connection with the thermodynamic relations in the equilibrium statistical mechanics.

The essentially same formalism for strange sets has already been proposed in Ref. 4). The scaling index $f(\alpha)$ in Ref. 4) (see also Table II) is different from $\sigma(\alpha)$, the entropy correspondence. However, after the transformation $T(-1,1)$, i.e., from (q, $\lambda_q\equiv D_{1-q}$) to ($\hat{q}$, $\hat{\lambda}_{\hat{q}}=(1-1/\hat{q})D_{\hat{q}}$), we obtain $\hat{\alpha}=\alpha(\hat{q})$ and $\hat{\sigma}(\hat{\alpha})=f(\alpha)$. So $f(\alpha)$ plays the same role as the entropy in the ($\hat{q}$, $\hat{\lambda}_{\hat{q}}$) representation.

(B) *Time series*

The second example is similarity exponents[13),14)] introduced so as to single out global characteristics of steady time series. Let $x_n(\equiv x(t_n))$, underlying degrees of freedom, be chaotic, where $x(t)$ obeys a nonlinear dynamical law and t_n is the n-th discrete time appropriately chosen. In this case, the step number n is directly related to the time, and the characteristic function λ_q is called the order-q *similarity exponent.*[14)] The corresponding thermodynamic variables and relations are summarized in Table II.

An illustration of similarity exponents is the one for the fluctuation dynamics of local expansion rates of adjacent trajectories[16),17)] in a chaotic one-dimensional map $x_{n+1}=g(x_n)$, where u_n is given by[13)]

$$u_n\equiv\ln|g'(x_n)|\,. \tag{4·5}$$

In this case, λ_0 is just the usual Lyapunov exponent $\langle\ln|g'(x_n)|\rangle$, and λ_q for $q\neq0$ measures the fluctuation of u_n from its average λ_0. One should remark that since λ_q is invariant under a one-to-one transformation $x_n\to\tilde{x}_n(\equiv h(x_n))$,[13),16)] α and $\sigma(\alpha)$ are also invariant under the transformation.

Another approach to the fluctuation dynamics of local expansion rates has been proposed by Takahashi and Oono.[18)] Especially they have introduced the free energy $F(\beta)$ and obtained the thermodynamic relation $F(\beta)=U(\beta)-\beta^{-1}S(\beta)$, where $U(\beta)(\equiv d[\beta F(\beta)]/d\beta)$ and $S(\beta)(\equiv\beta^2 dF(\beta)/d\beta)$ were called the internal energy and the entropy, respectively.[18)] Recently Benzi et al.[17)] postulated the interrelation between λ_q and $F(\beta)$ as

$$F(\beta)=-\frac{q}{\beta}\lambda_q\,. \qquad (\beta=1-q) \tag{4·6}$$

This relation is supported by the Ledrappier condition $F(1)=0$.[19)] Equation (4·6) holds in fact for a simple model.[15)] One should note that (4·6) is equivalent to the transformation $T(-1,1)$ given in (3·8), and that resultant relations (3·9) yield $U(\beta)$

$\equiv \hat{\alpha}=\alpha$ and $S(\beta)\equiv\hat{\sigma}(\hat{\alpha})=\alpha-\sigma(\alpha)$. We note that a similar formalism for the similarity exponent associated with fluctuations of local expansion rates[13),16),17)] was recently given by Sano, Sato and Sawada.[20)]

The replacement (4·5) is just one choice in a time series. One can easily apply the present formalism to an arbitrary steady time series generated by chaotic[21)] and stochastic[22)] dynamics. Recently we have found scaling laws of λ_q near chaotic transition points, especially associated with the breakdown of the chaos symmetry, for time series appropriately obtained. The scaling laws are written as[21)]

$$\lambda_q=\hat{\varepsilon}^{\mu}\Lambda(q/\hat{\varepsilon}^{\nu}), \tag{4·7}$$

where $\hat{\varepsilon}$ denotes the difference of the control parameter from the transition point value, and μ and ν are constants. For explicit forms of the scaling function $\Lambda(x)$, see Ref. 21). The α vs $\sigma(\alpha)$ relation also satisfies the scaling form

$$\sigma(\alpha)=\hat{\varepsilon}^{\mu+\nu}\Sigma(\alpha/\hat{\varepsilon}^{\mu}) \tag{4·8}$$

with $\Sigma(y)=x^2 d\Lambda(x)/dx$, where x is the inverse function of $y=d[x\Lambda(x)]/dx$.

(C) *Velocity structure function in turbulence*

The last example is the intermittency effect to the K41 theory[23)] of velocity structure functions in fully developed turbulence, where the Kolmogorov micro-scale tends to zero. The intermittency effect is observed in the exponent ζ_p defined through

$$\langle[\Delta V(r)]^p\rangle\sim r^{\zeta_p}, \tag{4·9}$$

where $\Delta V(r)(>0)$ is the longitudinal velocity difference between two points separated by the distance r. One important contribution to this problem is known as the lognormal theory.[24)] An alternative approach to this problem was discussed by Mandelbrot,[25)] and Frisch, Sulem and Nelkin[26)] by adopting the so-called black and white model.

Recently Frisch and Parisi[5)] developed the multifractal theory of the intermittency effect to ζ_p on the basis of Refs. 25) and 26). Let r_0 be the energy-injection scale. Nonlinear interactions in the Navier-Stokes equation produce successively the excitation of smaller scale active eddies. The origin of the self-similarity is due to such eddy excitation mechanism. Introducing the scales

$$r_n=r_0 e^{-n}, \qquad (n=0,1,2,3,\cdots) \tag{4·10}$$

we postulate that the ratio

$$\frac{\Delta V(r_{n+1})}{\Delta V(r_n)}=e^{u_n} \tag{4·11}$$

is statistically independent of n. Solving (4·11) yields

$$\Delta V(r_n)\sim\exp(-nh')\sim r_n{}^{h'}, \tag{4·12}$$

where $h'\equiv-n^{-1}\sum_{j=0}^{n-1}u_j$ is a stochastic variable. The expression (4·12) is essentially equal to Eq. (A·1) in Ref. 5). After Frisch and Parisi let $d(h')$ be the Hausdorff dimension of the set of points satisfying (4·12). The exponent ζ_p is thus obtained as[5)]

$$\zeta_p = \min_{h'}[ph' + 3 - d(h')] . \tag{4·13}$$

This corresponds to (3·2) and has the same structure as in other systems discussed in the present paper (Table II). The function ζ_p corresponds to the Massieu function in the equilibrium statistical mechanics, and the Helmholtz free energy as the generating function corresponds to $-\zeta_p/p$.

§ 5. Summary and remarks

In the present paper, we have developed a statistical-thermodynamical approach to the self-similarity. We have found that its global aspects are characterized by the function λ_q and variables α and $\sigma(\alpha)$ derived from λ_q. Furthermore we discussed the fractal measure theories and thermodynamic theories of chaos so far proposed, in a unified way from the self-similarity viewpoint.

Once the generation law of u_n is known, one can calculate $\{\lambda_q\}$, at least in principle, from the law. Carrying out its Legendre transformation, one can study the spectral nature of $\sigma(\alpha)$. The other advantage of the present formalism is that when the underlying mechanism for the generation of $\{u_n\}$ is unknown, one can calculate $\{\lambda_q\}$ directly from the output $A_0, A_1, A_2, A_3, \cdots$ or equivalently from $u_0, u_1, u_2, u_3, \cdots$ through experiments. In this sense, the present formalism gives a new, practical and powerful tool for the analysis of the global aspects of any steady series $\{u_j\}$. Finally we add the following remark. When D, the diffusion coefficient, of $\{u_j\}$ is finite, $\sigma(\alpha)$ has the parabola (3·4b) around its minimum $\alpha=\lambda_0$ (the central limit theorem). This is a simple consequence of the statistical characteristics near $q=0$. As $|q|$ differs from $q=0$, the corresponding $\sigma(\alpha)$ tends to differ from (3·4b). Namely, the deviation of $\sigma(\alpha)$ from the parabola (3·4b) is remarkably observed in the α region $|\alpha-\lambda_0| \gtrsim 2D\varkappa$.

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References

1) B. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (Freeman, San Francisco, 1977).
2) H. G. E. Hentschel and I. Procaccia, Physica **8D** (1983), 435.
3) P. Grassberger, Phys. Lett. **97A** (1983), 227.
4) T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia and B. I. Schraiman, Phys. Rev. **A33** (1986), 1141.
5) U. Frisch and G. Parisi, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, International School of Physics, "Enrico Fermi", Course LXXXVIII, ed. M. Ghil, R. Benzi and G. Parisi (North-Holland, New York, 1985), p. 84.
6) M. H. Jensen, L. P. Kadanoff, A. Libchaber, I. Procaccia and J. Stavans, Phys. Rev. Lett. **55** (1985), 2798.
7) D. Bensimon, M. H. Jensen and L. P. Kadanoff, Phys. Rev. **A33** (1986), 3622.
8) J. P. Eckmann and I. Procaccia, Phys. Rev. **A34** (1986), 659.

9) M. Feigenbaum, M. H. Jensen and I. Procaccia, Phys. Rev. Lett. **57** (1986), 1503.
10) T. C. Halsey, P. Meakin and I. Procaccia, Phys. Rev. Lett. **56** (1986), 854.
11) C. Amitrano, A. Coniglio and F. di Liberto, Phys. Rev. Lett. **57** (1986), 1016.
12) R. Benzi, G. Paladin, G. Parisi and A. Vulpiani, J. of Phys. **A17** (1984), 3521.
13) H. Fujisaka, Prog. Theor. Phys. **71** (1984), 513.
14) H. Fujisaka and M. Inoue, Prog. Theor. Phys. **74** (1985), 20.
15) M. Inoue and H. Fujisaka, Prog. Theor. Phys. **77** (1987), 1077.
16) H. Fujisaka, Prog. Theor. Phys. **70** (1983), 1264.
17) R. Benzi, G. Paladin, G. Parisi and A. Vulpiani, J. of Phys. **A18** (1985), 2157.
18) Y. Takahashi and Y. Oono, Prog. Theor. Phys. **71** (1984), 851.
19) F. Ledrappier, Commun. Math. Phys. **81** (1981), 229.
20) M. Sano, S. Sato and Y. Sawada, Prog. Theor. Phys. **76** (1986), 945.
21) H. Fujisaka, M. Inoue and H. Uchimura, Prog. Theor. Phys. **72** (1984), 23.
M. Inoue and H. Fujisaka, Phys. Rev. **B32** (1985), 277.
M. Inoue, T. Kawaguchi and H. Fujisaka, Phys. Lett. **115A** (1986), 139.
H. Ishii, H. Fujisaka and M. Inoue, Phys. Lett. **116A** (1986), 257.
H. Uchimura, H. Fujisaka and M. Inoue, Prog. Theor. Phys. **77** (1987), 1344.
22) H. Fujisaka and T. Yamada, Prog. Theor. Phys. **77** (1987), 1045.
23) A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30** (1941), 301.
24) A. M. Obukhov, J. Fluid Mech. **13** (1962), 77.
A. N. Kolmogorov, J. Fluid Mech. **13** (1962), 82.
A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, Mass., 1975), vol. 2.
25) B. Mandelbrot, J. Fluid Mech. **62** (1974), 331.
26) U. Frisch, P. L. Sulem and M. Nelkin, J. Fluid Mech. **87** (1978), 719.