CERN-TH.3889

STATISTICAL THERMODYNAMICS IN RELATIVISTIC PARTICLE AND ION PHYSICS: CANONICAL OR GRAND CANONICAL?

R. Hagedorn and K. Redlich*)

CERN - Geneva

ABSTRACT

We consider relativistic statistical thermodynamics of an ideal Boltzmann gas consisting of the particles K, N, Λ, Σ and their antiparticles. Baryon number (B) and strangeness (S) are conserved. While any relativistic gas is necessarily grand canonical with respect to particle numbers, conservation laws can be treated canonically or grand canonically. We construct the partition function for canonical B×S conservation and compare it with the grand canonical one. It is found that the grand canonical partition function is equivalent to a large B approximation of the canonical one. The relative difference between canonical and grand canonical quantities seems to decrease like const/B (two numerical examples) and from this a simple thumb rule for computing canonical quantities from grand canonical ones is guessed. For precise calculations, an integral representation is given.

*)On leave of absence from: Institute for Theoretical Physics, Wroclaw, Poland. Present address: Institut für Theoretiscne Physik, Technische Hochschule Darmstadt, F.R.G.

CERN-TH.3889

May 1984

1. INTRODUCTION

Relativistic statistical thermodynamics is applied in some parts of high-energy physics ranging from cosmology to particle collisions in the laboratory. Of present interest is the phase transition from hadronic matter to a quark gluon plasma as well as the description of hadronic matter and the quark-gluon plasma as such [1].

While cosmological applications deal with large amounts of matter/radiation in large volumes, where the grand canonical description is the adequate tool, the laboratory situation is much more complicated: not only do we deal with very small amounts of matter in tiny volumes, but even the applicability of equilibrium statistical thermodynamics is questionable in processes taking place at times of the order of 10^{-23} s over space dimensions of a few fermi and in the presence of collective motions involving relative velocities reaching the order of the velocity of light. Experience in high-energy physics accumulated over almost three decades shows, however, that it often does make sense to assume approximate *local* thermal equilibrium in such processes: models combining collective motions with equilibrium thermodynamics in local rest frames often describe large amounts of experimental data in good overall approximation.

Therefore, it seems reasonable to further develop equilibrium thermodynamics (of relativistic gases) because it is one of the ingredients of a full -- if approximate -- phenomenological description of processes taking place in high-energy particle collisions, in particular in relativistic heavy ion collisions, where the space-time regions and numbers of particles involved are larger than in collisions of elementary particles and where it is hoped that the phase transition to a quark-gluon plasma might be observed experimentally.

We shall, therefore, study in this paper relativistic statistical thermodynamics as if there were no doubts about its usefulness. In fact, we shall address ourselves to one particular question which seems not yet to have been dealt with in the otherwise vast literature:

What are the differences between the canonical and the grand canonical treatment of conservation laws?

Let us illustrate the situation in a simple example making clear one of the essential distinctions between non-relativistic and relativistic statistical thermodynamics. In non-relativistic statistical thermodynamics particle numbers are -- in the absense of chemical reactions -- conserved. Therefore the canonical N-particle partition function is a senseful quantity; for an ideal gas it reads:

$$Z_N(T,V) = \frac{1}{N!} Z_1(T,V)^N \tag{1.1}$$

$$Z_1(T,V) = \text{"one particle partition function".}$$

One can go over to the grand canonical description by introducing the fugacity $\lambda = \exp(\mu/T)$ and defining*) the grand canonical partition function $Z(T, V, \lambda)$:

$$Z(T,V,\lambda) := \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} Z_1(T,V)^N = \exp\left[\lambda Z_1(T,V)\right] \quad , \tag{1.2}$$

Now not N but only the average particle number $\langle N\rangle$ is determined:

$$\langle N\rangle := \frac{\sum N \frac{\lambda^N}{N!} Z_1^N}{\sum \frac{\lambda^N}{N!} Z_1^N} = \lambda \frac{\partial}{\partial\lambda} \ln Z(T,V,\lambda) = \lambda Z_1(T,V) \tag{1.3}$$

We only have to choose $\lambda = \langle N\rangle / Z_1$ to obtain a prescribed mean particle number.

The relativistic situation is fundamentally different: the particle number N is not conserved, because particles can be created from kinetic energy. Therefore, the canonical partition function for N particles makes no sense. The adequate partition function is the grand canonical one with $\lambda = 1$ (after differentiations). Any value of λ different from 1 is not permitted, since it would impose a constraint on particle production from energy.

If there are K kinds of particles, one introduces an extra λ_i for each kind i and has $(Z = Z_1 \cdot Z_2 \cdot \ldots \cdot Z_k)$:

$$\ln Z(T,V,\lambda_1,\lambda_2,\ldots,\lambda_K) = \sum_{i=1}^{K} \lambda_i Z_1^{(i)}(T,V) \tag{1.4}$$

$$\langle N_i\rangle = \lambda_i \frac{\partial}{\partial\lambda_i} \ln Z \Big|_{\lambda_1=\lambda_2=\cdots=\lambda_K=1}$$

*) We always employ the same letter, Z, for all sorts of partition functions.

We state the general rule:

With respect to particle numbers we are forced, in relativistic statistical thermodynamics, to use only the grand canonical situation with all $\lambda_i = 1$ *(after differentiations).*

There are, however, conservation laws which *do* impose constraints on particle production; here the rule is:

With respect to conservation laws we have, in relativistic statistical thermodynamics, the choice between the canonical and the grand canonical formalism.

We shall extensively discuss the canonical case; so let us illustrate the interplay of particle numbers and conservation laws in the grand canonical language.

Consider a gas of one sort of baryons and antibaryons with baryonic charges ±1 respectively. The baryon number B = N (baryons) - $\bar{N}$ (antibaryons) is conserved, while N and $\bar{N}$ individually are not. In order to control B we introduce a baryon fugacity λ_B exclusively related to the conservation law. As the one-particle partition function depends only on T, V, and the mass of the particle, it is the same for particle and antiparticle. Thus

$$\ln Z(T,V) = (\lambda_B + 1/\lambda_B)\, Z_1(T,V) \tag{1.5}$$

where $1/\lambda_B$ refers to antibaryons. Then

$$\langle B \rangle = \lambda_B \frac{\partial}{\partial \lambda_B} \ln Z = (\lambda_B - 1/\lambda_B)\, Z_1(T,V) \tag{1.6}$$

Mean particle numbers are now found by introducing an extra λ for each sort of particle; in the present example λ for baryons and $\bar{\lambda}$ for antibaryons. Thus:

$$\begin{aligned}
\ln Z(T,V,\lambda_B,\lambda,\bar{\lambda}) &= (\lambda\lambda_B + \bar{\lambda}/\lambda_B)\, Z_1(T,V) \\
\langle B \rangle &= \lambda_B \frac{\partial}{\partial \lambda_B} \ln Z(T,V,\lambda_B,1,1) \\
\langle N \rangle &= \lambda \frac{\partial}{\partial \lambda} \ln Z(T,V,\lambda_B,\lambda,1)\Big/_{\lambda=1} \\
\langle \bar{N} \rangle &= \bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} \ln Z(T,V,\lambda_B,1,\bar{\lambda})\Big/_{\bar{\lambda}=1}
\end{aligned} \tag{1.7}$$

where λ, $\bar{\lambda}$ *must* be set equal to one, while λ_B may be chosen to obtain a prescribed value of $\langle B \rangle$; $\lambda_B = 1$ gives $\langle B \rangle = 0$ as is obvious from (1.6), while $\langle N \rangle$ and $\langle \bar{N} \rangle \neq 0$.

Equations (1.5) to (1.7) are grand canonical with respect to particle numbers *and* with respect to the conservation law. As emphasized above, we must remain in the grand canonical description (with $\lambda = 1$) as far as particle numbers are concerned, but we may choose the canonical description with respect to conservation laws. The above recipe to calculate particle numbers remains the same in both cases, however.

As there are methods available to deal with conservation laws canonically, we have to decide which description to choose. This is a non-trivial problem.

Obviously, if both B and the volume V go to very large values with B/V → const., then the grand canonical method is adequate; this happens in cosmology and astrophysics. But what if B is of the order of 1 to 100 and V of the order of 1 to 100 nucleon volumes, as in particle (ion) collisions in the laboratory? In such collisions we know B to be exactly conserved and since its value is small we might doubt whether conservation on the average, i.e. the grand canonical $\langle B \rangle$, is adequate.

We shall consider this question in some detail and try to find where one can use the grand canonical and where one should use the canonical description of conservation laws.

We shall restrict ourselves to the simple cases of baryon number B and strangeness S conservation, separately as well as simultaneously; baryon number conservation $U(1)_B$ alone is almost trivial and serves here mainly as an illustration and check of the methods used; simultaneous B and S conservation $U(1)_B \times U(1)_S$ is less trivial since these charges are unsymmetrically distributed among the particles considered here: K, N, Λ, Σ.

The paper is organized as follows: In Section 2 we introduce the canonical description of $U(1)_B$ and $U(1)_B \times U(1)_S$. In Section 3 we consider the case $U(1)_B$, which can be solved analytically and here serves as a testing ground for the approximation to be used in the $U(1)_B \times U(1)_S$ case. In Section 4 we treat the case $U(1)_B \times U(1)_S$, show that the approximation used is exactly equivalent to the grand canonical description, and compute $\langle N\rangle/\langle\bar{N}\rangle$ numerically without approximation to compare it with the approximate result.

Remarks: - The words grand canonical (GC) and canonical (C) in the text refer exclusively to conservation laws; with respect to particle numbers we are always in the GC description ($\lambda = 1$);

- We neglect quantum statistics; i.e. we assume temperature and density regimes such that all particles can be treated as Boltzmann particles;
- We neglect interaction, thus dealing with ideal Boltzmann gases. This is not an essential restriction, because interaction can be introduced via a mass spectrum and proper volumes of particles [2a, d] leaving us with a mixture of infinitely many ideal Boltzmann gases. The following considerations apply then to the components N, $\bar{N}$, K, $\bar{K}$, Λ, $\bar{\Lambda}$, Σ, $\bar{\Sigma}$ of this mixture, but not to the (interacting) gas as a whole. For our present purpose, namely to compare critically the C with the GC description, this is sufficient.

2. CANONICAL DESCRIPTION OF SYMMETRIES

2.1 General method

Exact implementation of conservation laws in statistical physics has a long history, even if we only consider the use of group theoretical methods applied to internal symmetries [3]. At present the most adequate method is based on the use of group characters [4]; we will employ it here.

Let $\chi^\alpha(\phi_1, \ldots, \phi_r)$ be the character of the $d(\alpha)$-dimensional, irreducible representation $U^\alpha(\phi_1, \ldots, \phi_r)$ of the internal symmetry group of rank r.

The formalism now used resembles that of the GC description, the difference being that instead of defining a GC partition function by introducing fugacities we here define a generating function by introducing group characters [Note the formal similarity with Eqn. (1.2)]:

$$\tilde{Z}(T, V, \varphi_1 \ldots \varphi_r) := \sum_\alpha \frac{\chi^\alpha(\varphi_1 \ldots \varphi_r)}{d(\alpha)} Z_\alpha(T, V) \tag{2.1}$$

where

$$Z_\alpha(T, V) := \mathrm{Tr}_\alpha\left[\exp(-H/T)\right] \tag{2.2}$$

is the canonical partition function counting exactly those quantum states which transform under $U^\alpha(\phi_1, \ldots, \phi_r)$. Using the orthogonality of group characters

$$\int d\mu(\varphi_1 \ldots \varphi_r)\, \bar{\chi}^\alpha(\varphi_1 \ldots \varphi_r)\, \chi^{\alpha'}(\varphi_1 \ldots \varphi_r) = \delta_{\alpha\alpha'} \tag{2.3}$$

we solve Eq. (2.1) for the canonical partition function:

$$Z_\alpha(T, V) = d(\alpha) \int d\mu(\varphi_1 \ldots \varphi_r)\, \bar{\chi}^\alpha(\varphi_1 \ldots \varphi_r)\, \tilde{Z}(T, V, \varphi_1 \ldots \varphi_r) \tag{2.4}$$

where $d\mu(\phi_1, \ldots, \phi_r)$ is the usual Haar measure on the group.

These formal manipulations will allow us to calculate the wanted canonical partition function $Z_\alpha(T, V)$, if we succeed in specifying $\tilde{Z}(T, V, \phi_1, \ldots, \phi_r)$ and if we can calculate the integral (2.4).

It has been shown in Ref [4] that

$$\tilde{Z}(T, V, \varphi_1 \ldots \varphi_r) = \exp\left[\sum_\alpha \frac{\chi^\alpha(\varphi_1 \ldots \varphi_r)}{d(\alpha)} Z^1_\alpha(T, V)\right] \tag{2.5}$$

with Z^1_α being the relativistic one-particle partition function belonging to the mass m_α (of the multiplet transforming under the α representation):

$$Z^1_\alpha(T, V) = \frac{VT^3}{2\pi^2}\left(\frac{m_\alpha}{T}\right)^2 K_2\left(\frac{m_\alpha}{T}\right) \tag{2.6}$$

For a derivation of Eq. (2.6) see, for example, Ref. [2].

Note again the formal similarity of Eq. (2.5) with what Eq. (1.2) becomes when several sorts of particles are involved.

While Eq. (2.6) holds for Boltzmann statistics, it can be generalized to include Bose and Fermi statistics [4], which, however, does not interest us here.

The method just sketched can be used for Abelian as well as non-Abelian symmetries.

2.2 Restriction to $U(1)_B$ and $U(1)_B \times U(1)_S$

We restrict all further considerations to Boltzmann statistics with baryon and baryon × strangeness conservation. In this case the characters are trivial:

$$\chi^B_{U(1)_B}(\varphi) = e^{iB\varphi}$$

$$\chi^S_{U(1)_S}(\psi) = e^{iS\psi} \tag{2.7}$$

$$\chi_{U(1)_B \times U(1)_S} = \chi_{U(1)_B} \cdot \chi_{U(1)_S} = e^{i(B\varphi + S\psi)}$$

so that

$$Z_B(T,V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iB\varphi}\,\tilde{Z}(T,V,\varphi)\,d\varphi$$

$$Z_{B,S}(T,V) = \frac{1}{4\pi^2}\int_0^{2\pi} e^{-iB\varphi}\,d\varphi \int_0^{2\pi} e^{-iS\psi}\,\tilde{Z}(T,V,\varphi,\psi)\,d\psi \tag{2.8}$$

For the discussion of baryon-number conservation we consider only N and $\bar{N}$, while for combined baryon-number and strangeness conservation we use K, $\bar{K}$, N, $\bar{N}$, Λ, $\bar{\Lambda}$, Σ, $\bar{\Sigma}$ as input particles. Then from Eqs. (2.5) to (2.7), with appropriate statistical factors g_i for spin and isospin ($g_N = 4$):

$$\ln \tilde{Z}_B(T,V,\varphi) = g_N Z^1_N(T,V)\left(e^{i\varphi} + e^{-i\varphi}\right) = 2g_N Z^1_N(T,V)\cos\varphi \tag{2.9}$$

similarly ($g_K = 2$; $g_\Lambda = 2$; $g_\Sigma = 6$):

$$\ln \tilde{Z}_{B,S}(T,V,\varphi,\psi) = 2g_K Z^1_K \cos\psi + 2g_N Z^1_N \cos\varphi + 2g_Y Z^1_Y \cos(\varphi - \psi)$$

$$g_Y Z^1_Y(T,V) := g_\Lambda Z^1_\Lambda(T,V) + g_\Sigma Z^1_\Sigma(T,V) \tag{2.10}$$

Inserting these in Eqs. (2.8) yields the wanted canonical partition functions. The only problem is the evaluation of the integrals, to which we return in Sections 3 and 4.

2.3 Mean particle numbers

It is now easy to see how to calculate mean particle numbers following the rule given in the Introduction: we write Eq. (2.9) more explicitly by separating the particle terms from the antiparticle terms, e.g.

$$\ln \tilde{Z}_B(T,V,\varphi) = g_N Z^1_N e^{i\varphi} + g_N Z^1_{\bar{N}} e^{-i\varphi} \tag{2.11}$$

and correspondingly in Eq. (2.10). Of course $Z^1_N = Z^1_{\bar{N}}$, etc. Now we introduce the fugacities for particle numbers λ_N, $\lambda_{\bar{N}}$, λ_K, ..., $\lambda_{\bar{\Sigma}}$, replace

$$\begin{aligned} Z^1_N &\to \lambda_N Z^1_N \\ Z^1_{\bar{N}} &\to \lambda_{\bar{N}} Z^1_{\bar{N}} \\ &\vdots \\ Z^1_{\bar{\Sigma}} &\to \lambda_{\bar{\Sigma}} Z^1_{\bar{\Sigma}} \end{aligned} \tag{2.12}$$

write down the canonical partition functions in the form of the integrals (2.8), and differentiate:

$$\langle \bar{K} \rangle = \lambda_{\bar{K}} \frac{\partial}{\partial \lambda_{\bar{K}}} \ln Z_{B,S}(T,V) \Big|_{\text{all } \lambda_i = 1} \tag{2.13}$$

Similarly for all other particles and antiparticles. Since all integrals converge, we may differentiate under the integral; this simplifies the calculation. Ratios of particle numbers are easier to calculate, since $Z_{B,S}$ drops out.

2.4 Useful variables; Compression

We are mainly interested in situations where the baryon number density is of the order of nuclear density. In this case it is useful to replace the pair of variables, {B, V}, by another one {B, c}, where the "compression" c is defined by the deviation from "normal" nuclear density: let

$$V_0 := 4\pi/(3m_\pi^3) \qquad \text{(nucleon volume)}$$

$$V_B := BV_0 \qquad \text{(nuclear volume of B nucleons)} \tag{2.14}$$

$$c := \frac{V_B}{V} = \frac{BV_0}{V} \qquad \text{(compression)}$$

The formal choice of V_0 implies that in real nuclei c is not equal to one (but near to it)*).

Obviously the pair of variables {B, c} should only be used if $B \geq 1$; then, however, it has the advantage that one may keep c onstant while varying B from 1 to ∞, which would correspond to going from laboratory physics to astrophysics. [In the case B = 0 one goes back to the pair {B, V}].

Our main question can now be put in the form:

Assuming a reasonable value of the compression c, which are the values B of the baryon number requiring the canonical formalism?

To simplify the situation we observe that according to Eqs. (2.9) and (2.10) the one-particle partition function appears always multiplied by its statistical factor g; furthermore, they are all proportional to the volume V which we wish to replace by the compression. Therefore, we write, using Eq. (2.6),

$$g_i Z_i^1 = \frac{V}{V_0}\left[\frac{g_i V_0 T^3}{2\pi^2}\left(\frac{m_i}{T}\right)^2 K_2\left(\frac{m_i}{T}\right)\right]$$

$$=: \frac{V}{V_0} z_i = \frac{B}{c} z_i \qquad ; \tag{2.15}$$

with $V_0 = 4\pi/(3m_\pi^3)$ we find

$$z_i := \frac{2g_i}{3\pi}\left(\frac{m_i}{T}\right)^2 K_2\left(\frac{m_i}{T}\right) \quad . \tag{2.16}$$

z_i is the one-particle partition function of particle i in the volume V_0 and with the spin-isospin factor g_i included.

It will prove useful to introduce, by way of abbreviation, one more quantity a_i:

$$a_i := \frac{2z_i}{c} = \frac{2V}{BV_0} z_i = \frac{2}{B} g_i Z_i^1 \tag{2.17}$$

Note that for c = 2 (which seems not unreasonable a value in ion collisions) $a_i = z_i$.

In Fig. 2.1 we plot z_i (T) for K, N, Λ, Σ as a function of the temperature for $\{0.1 \leq T \leq 0.2$ GeV$\}$ (below 0.1 GeV nothing interesting happens and for T > 0.2 GeV hadrons presumably cease to exist).

---

*) Since the density of a proton is different than that of nuclei which again differ among themselves, we prefer this formal definition.

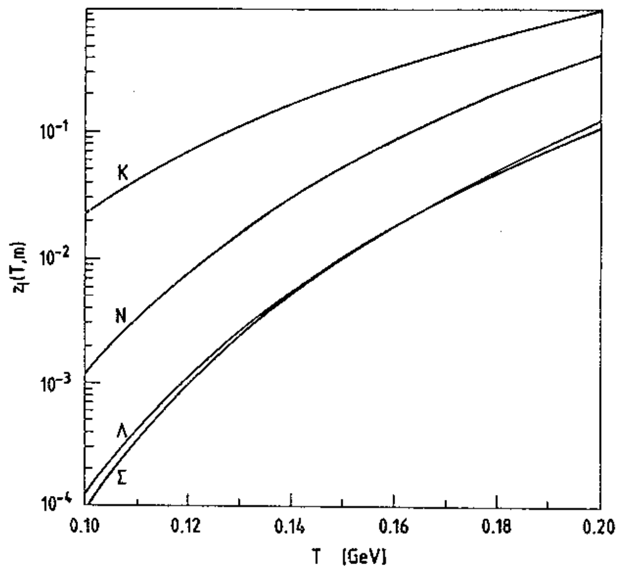

Fig. 2.1 The one-particle partition functions (including statistical weight factors g_i).

3. BARYON NUMBER CONSERVATION $U(1)_B$

This case can be solved exactly and will serve as a testing ground for the approximations to which we shall be forced in the $U(1)_B \times U(1)_S$ case.

3.1 Exact solution

From Eqs. (2.8) and (2.9) and with (2.15) we have

$$Z_B(T,V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iB\varphi} \exp\left[\frac{2B}{c} z_N \cos\varphi\right] d\varphi$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos B\varphi \exp\left[\frac{2B}{c} z_N \cos\varphi\right] d\varphi \tag{3.1}$$

$$= I_B\left(\frac{2B}{c} z_N\right) = I_B(B a_N)$$

with [see Eq. (2.17)]

$$a_N = \frac{2z_N}{c} = \frac{2V}{BV_0} z_N = \frac{2}{B} g_N Z_N^1 \tag{3.2}$$

In Eq. (3.1) we used Ref. [5a, (8. 431.5)].

For $B = 0$ one must replace B/c by V/V_0, since then expression (3.1) is misleading.

For small $B \geq 1$ Eq. (3.1) can be evaluated using tables or computer library programs. For large B we find [5b, (9.7.7)].

$$Z_B(T,c) \underset{B\to\infty}{\sim} \frac{\exp\left(B\sqrt{1+a_N^2}\right)}{\sqrt{2\pi B}} \left(\frac{a_N}{1+\sqrt{1+a_N^2}}\right)^B \left(\frac{1}{1+a_N^2}\right)^{1/4} \tag{3.3}$$

3.2 Baryon Conservation only; Approximations

Anticipating the $U(1)_B \times U(1)_S$ case in which the integrals cannot be solved analytically (though they are of a similar general structure) we ask: suppose we had not been able to solve the integral (3.1) how would we approximate it?

3.2.1 "Gaussian" approximation

We go back to Eq (3.1) and write it using Eq. (3.2)

$$Z_B(T,c) = \frac{1}{\pi}\int_0^{\pi} \cos B\varphi \, \exp\left[Ba_N \cos\varphi\right] d\varphi \tag{3.4}$$

For $B \to \infty$ one expects that only small ϕ contribute, so that, with $\cos\phi \simeq 1 - \phi^2/2$ [5a, (3.896.4)]

$$\begin{aligned} Z_B(T,c) &\cong e^{Ba_N} \frac{1}{\pi}\int_0^{\pi} \cos B\varphi \, \exp\left[-Ba_N \varphi^2/2\right] d\varphi \\ &= \exp\left[B\left(a_N - \frac{1}{2a_N}\right)\right] \Big/ \sqrt{2\pi B a_N} \end{aligned} \tag{3.5}$$

As one sees at once, this expression coincides with (3.3) only if not only $B \to \infty$ but also $a_N \gg 1$, which with regard to Fig. 2.1 has to be discarded as physically unrealistic. We thus do not further consider the Gaussian approximation.

3.2.2 "Chebyshev" approximation (CA)

The method for this approximation was invented [6] to calculate coefficients in Chebyshev series approximations*). We therefore use this name although we shall not explicitly introduce Chebyshev polynomials $T_B(\cos\phi) \equiv \cos B\phi$ to replace $\cos B\phi$ under the integral.

*) Indeed the wanted Z_B is the coefficient in the Chebyshev expansion $\tilde{Z}(T, V, \cos\phi) = \Sigma\, T_B(\cos\phi) Z_B$.

Going back to Eq. (3.4) we introduce the identity

$$\exp\left[Ba_N \cos\varphi\right] = \frac{1}{2\pi i}\oint \frac{dw}{w - \cos\varphi} \exp\left[Ba_N w\right] \tag{3.6}$$

interchange integrations, and use [5a, (3.613.2)] with $|w| > 1$;

$$\frac{1}{\pi}\int_0^{\pi} \frac{\cos B\varphi}{w - \cos\varphi} d\varphi = \left[\sqrt{w^2-1}\left(w + \sqrt{w^2-1}\right)^B\right]^{-1} \tag{3.7}$$

to arrive at

$$Z_B(T,c) = \frac{1}{2\pi i}\oint \frac{\exp\left[Ba_N w\right] dw}{\sqrt{w^2-1}\left(w+\sqrt{w^2-1}\right)^B} \tag{3.8}$$

where the contour C has to lie outside the unit circle to satisfy $|w| > 1$. Inspection of Eq. (3.8) reveals:

- the integrand has, on the real axis, a single minimum at x_0, which for B and $Ba_N \gg 1$ becomes very sharp; note that a_N need not be large if B is sufficiently large;
- the contour can be deformed into a large rectangle such that its right-most side goes from $x_0 - i\infty$ to $x_0 + i\infty$ through the saddle point x_0, while the other sides go to ∞, where their contributions to the integral vanish on account of the oscillating exponential and the high powers of w in the denominator;
- the only remaining contribution comes from the saddle point, since also on the line from $x_0 - i\infty$ to $x_0 + i\infty$ the rest is negligible.

By logarithmic differentiation of the integrand one finds for $B \gg 1$ the position x_0 of the minimum

$$x_0 = \sqrt{1+a_N^2} \,/\, a_N \quad ; \tag{3.9}$$

then follows the usual saddle-point integration: for the remaining integral from $x_0 - i\infty$ to $x_0 + i\infty$ we write

$$Z_B(T,c) \cong \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[f(x_0) - \frac{1}{2} f''(x_0) y^2\right] dy = \exp[f(x_0)] \sqrt{\frac{1}{2\pi f''(x_0)}}$$

$$f(x) := B a_N x - B \ln\left(x + \sqrt{x^2 - 1}\right) - \frac{1}{2} \ln(x^2 - 1) \tag{3.10}$$

yielding

$$Z_B(T,c) \underset{B\to\infty}{\sim} \frac{\exp\left(B\sqrt{1+a_N^2}\right)}{\sqrt{2\pi B}} \left(\frac{a_N}{1+\sqrt{1+a_N^2}}\right)^B \left(\frac{1}{1+a_N^2}\right)^{1/4} \tag{3.11}$$

which is identical with the asymptotic expression for the exact result [Eqs. (3.1) and (3.3)].

3.2.3 Numerical check

We have compared numerically the exact expression with the Chebyshev approximation (3.11) and found that the relative error decreases as const/B. For the typical value $a_N = 0.1$ we found the relative error in per cent to be*)

$$\text{Error}\,(a_N = .1) = \frac{8.08}{B}\ \% \tag{3.12}$$

so that even at B = 1 the error is only ∿ 8%.

3.2.4 The grand canonical limit

We shall now show that for $B \to \infty$ the expressions (3.3) and (3.11) correspond to the grand canonical limit.

The grand canonical partition function is here

$$\ln Z_{GC} = \left(\lambda + \frac{1}{\lambda}\right) g_N Z_N \tag{3.13}$$

We put $\lambda = \exp(\alpha)$ and use Eq. (2.15) to obtain

$$\ln Z_{GC} = 2\,\frac{V z_N}{V_0} \cosh\alpha \tag{3.14}$$

while from expressions (3.3) and (3.11), dropping terms which grow less than proportional to B,

$$\ln Z_B \cong B\left[\sqrt{1+a_N^2} + \ln\frac{a_N}{1+\sqrt{1+a_N^2}}\right] \tag{3.15}$$

---

*) Which can be explained using [5b, 9.7.7 and 9.3.9].

Equations (3.15) and (3.14) cannot be compared directly [neither can Eqs. (1.1) and (1.2)]: while $\ln Z_{GC}$ is proportional to V, $\ln Z_B$ depends on V via a_N [which itself is proportional to V; see Eq. (2.17)] in a somewhat complicated way. Thus only physical quantities of the density type should be compared and should be the same in both descriptions when $B \to \infty$. We choose here the simplest one: $P/T = (\partial/\partial V) \ln Z$.

From Eq. (3.14)

$$\frac{\partial \ln Z_{GC}}{\partial V} = \frac{2 z_N}{V_0} \cosh\alpha \tag{3.16}$$

cosh α is determined by prescribing ⟨B⟩/V:

$$\frac{\langle B\rangle}{V} = \frac{2 z_N}{V_0} \sinh\alpha \tag{3.17}$$

From Eqs. (3.16) and (3.17) it follows that

$$\frac{\partial \ln Z_{GC}}{\partial V} = \frac{\langle B\rangle}{V}\sqrt{1 + 1/\sinh^2\alpha} = \frac{\langle B\rangle}{V}\sqrt{1 + \left(\frac{B}{\langle B\rangle} a_N\right)^2} \tag{3.18}$$

where Eq. (3.17) together with (3.2) was used to eliminate sinh α.

On the other hand, from Eqs. (3.15) and (3.2)

$$\frac{\partial \ln Z_B}{\partial V} = \frac{\partial \ln Z_B}{\partial a_N}\frac{\partial a_N}{\partial V} = \frac{B}{V}\sqrt{1 + a_N^2} \tag{3.19}$$

Thus Eq. (3.19) [large B limit] and Eq. (3.18) [grand canonical limit] become the same if one identifies B with ⟨B⟩.

4. SIMULTANEOUS BARYON NUMBER AND STRANGENESS CONSERVATION

4.1 The integrals

4.1.1 The partition function

For reasons which will become obvious, we shall temporarily use the pair of variables {B, V} instead of {B, c} and explicitly exhibit this choice in the arguments of the partition function: $Z_{B,S}(T, V)$.

From Eqs. (2.8), (2.10), and (2.15) we have

$$Z_{B,S}(T,V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iB\varphi} \exp\left[2\frac{V}{V_0} z_N \cos\varphi\right] \times$$
$$\times \frac{1}{2\pi}\int_0^{2\pi} e^{-iS\psi} \exp\left[2\frac{V}{V_0}\left(z_K\cos\psi + z_Y\cos(\varphi-\psi)\right)\right] d\varphi\, d\psi \tag{4.1}$$

with $z_Y := z_\Lambda + z_\Sigma$. In the exponent of the ψ integral we write

$$z_K\cos\psi + z_Y\cos(\varphi-\psi) =: z(\varphi)\cos(\psi-\alpha) \tag{4.2}$$

where

$$z(\varphi) := \left(z_K^2 + 2z_K z_Y\cos\varphi + z_Y^2\right)^{1/2}$$
$$e^{i\alpha(\varphi)} := \frac{z_K}{z(\varphi)} + \frac{z_Y}{z(\varphi)} e^{i\varphi} \tag{4.3}$$
$$\alpha(-\varphi) = -\alpha(\varphi)$$

Since the ψ integration goes over a whole period, we may shift the integration by α and obtain

$$Z_{B,S}(T,V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iB\varphi} e^{-iS\alpha(\varphi)} \exp\left(2\frac{V}{V_0} z_N\cos\varphi\right) \times$$
$$\times \frac{1}{2\pi}\int_0^{2\pi} e^{-iSt} \exp\left(\frac{2V}{V_0} z\cos t\right) dt\, d\varphi \tag{4.4}$$

The last integral is [see Eq. (3.1)] equal to $I_S\left[(2V/V_0)z\right]$, hence

$$Z_{B,S}(T,V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i[B\varphi + S\alpha(\varphi)]} \exp\left(2\frac{V}{V_0} z_N\cos\varphi\right) I_S\left(2\frac{V}{V_0} z(\varphi)\right) d\varphi\,. \tag{4.5}$$

Writing $\int_0^{2\pi} = \int_0^{\pi} + \int_\pi^{2\pi}$ and manipulating the second integral while using the oddness of ϕ and $\alpha(\phi)$ and the evenness of all other functions, we find

$$Z_{B,S}(T,V) = \frac{1}{\pi}\int_0^{\pi} \cos(B\varphi + S\alpha(\varphi)) \exp\left(\frac{2V}{V_0} z_N\cos\varphi\right) I_S\left(\frac{2V}{V_0} z(\varphi)\right) d\varphi \tag{4.6}$$

This integral cannot be solved analytically.

4.1.2 Mean particle numbers

We shall use the recipe given in Section 2.3. If we wish to calculate the mean multiplicity of particle i we simply separate for this species the particle and antiparticle term, multiply the relevant one by λ, differentiate and put $\lambda = 1$ afterwards. Since all other particles are not involved we leave their terms as they were.

The recipe is the same for all particles; hence we illustrate it for one only, say Λ. From Eq. (4.1) then

$$Z_{B,S}(T,V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iB\varphi} \exp\left(2\frac{V}{V_0}\cos\varphi\right) \times$$
$$\times \frac{1}{2\pi}\int_0^{2\pi} e^{-iS\psi} \exp\left[\frac{2V}{V_0}\left(z_K\cos\psi + z_\Sigma\cos(\varphi-\psi)\right) + \right.$$
$$\left. + \frac{V}{V_0} z_\Lambda\left(\lambda e^{i(\varphi-\psi)} + e^{-i(\varphi-\psi)}\right)\right] d\varphi\, d\psi \tag{4.7}$$

Thus

$$\langle\Lambda\rangle_{B,S} = \lambda\frac{\partial \ln Z_{B,S}(T,V)}{\partial\lambda}\Big|_{\lambda=1} =$$
$$= \frac{V z_\Lambda}{Z_{B,S}(T,V)\, V_0}\; \frac{1}{2\pi}\int_0^{2\pi} e^{-i(B-1)\varphi} \exp\left(2\frac{V}{V_0}\cos\varphi\right) \times$$
$$\times \frac{1}{2\pi}\int_0^{2\pi} e^{-i(S+1)\psi} \exp\left[2\frac{V}{V_0}\left(z_K\cos\psi + z_Y\cos(\varphi-\psi)\right)\right] d\varphi\, d\psi \tag{4.8}$$

Comparison with Eq. (4.1) reveals

$$\langle\Lambda\rangle_{B,S} = z_\Lambda\frac{V}{V_0}\frac{Z_{B-1,S+1}(T,V)}{Z_{B,S}(T,V)} \tag{4.9}$$

Note that the appearance of B − 1, S + 1 is purely formal; we deal with the situation where B and S are fixed. In particular, if at any time we wish to re-introduce {B, c} instead of {B, V}, the factor V/V_0 equals B/c and not (B − 1)/c.

It is now obvious how all other multiplicities will come out: let B_i and S_i be baryon number and strangeness of particle i, then

$$\langle n_i \rangle_{B,S} = z_i(m_i, T)\, \frac{V}{V_0}\, \frac{Z_{B-B_i,\, S-S_i}(T,V)}{Z_{B,S}(T,V)} \tag{4.10}$$

Particularly simple becomes the ratio of the multiplicities (antiparticle)/(particle)

$$\frac{\langle n_i \rangle_{B,S}}{\langle n_{\bar{i}} \rangle_{B,S}} = \frac{Z_{B-B_i,\, S-S_i}(T,V)}{Z_{B+B_i,\, S+S_i}(T,V)} \tag{4.11}$$

In view of possible applications in heavy ion collisions we restrict the further discussion to non-strange systems: S = 0.

4.2 Chebyshev approximation to $Z_{B,S}$ (T, V)

From Eqs. (4.7) to (4.11) we see that for the partition function and for particle numbers we need $Z_{B,S}$ with arbitrary B and S = - 1, 0, 1.

We insert $e^{i\alpha}$ from Eq. (4.3) into Eq. (4.5) and obtain

$$Z_{B,S=\pm 1} = \frac{z_K}{\pi} \int_0^\pi \cos B\varphi \cdot F_1(\cos\varphi)\, d\varphi + \frac{z_Y}{\pi} \int_0^\pi \cos[(B\pm 1)\varphi] \cdot F_1(\cos\varphi)\, d\varphi \tag{4.12}$$

$$Z_{B,S=0} = \frac{1}{\pi} \int_0^\pi \cos B\varphi \cdot F_0(\cos\varphi)\, d\varphi$$

where

$$F_1(\cos\varphi) := \exp\left(\frac{2V}{V_0} z_N \cos\varphi\right) \cdot \frac{I_1\left(\frac{2V}{V_0} z(\varphi)\right)}{z(\varphi)} \tag{4.13}$$

$$F_0(\cos\varphi) := \exp\left(\frac{2V}{V_0} z_N \cos\varphi\right) \cdot I_0\left(\frac{2V}{V_0} z(\varphi)\right)$$

We thus have to compute integrals of the type (σ = 0, 1):

$$J_B^{(\sigma)} := \frac{1}{\pi} \int_0^\pi \cos B\varphi \cdot F_\sigma(\cos\varphi)\, d\varphi \tag{4.14}$$

As in Section 3 we write

$$F_\sigma(\cos\varphi) = \frac{1}{2\pi i} \oint_C \frac{F_\sigma(w)\, dw}{w - \cos\varphi} \tag{4.15}$$

insert it into Eq. (4.14), integrate over ϕ, and obtain

$$J_B^{(\sigma)} = \frac{1}{2\pi i} \oint_C \frac{F_\sigma(w)\, dw}{\sqrt{w^2-1} \cdot \left(w + \sqrt{w^2-1}\right)^B} \; . \tag{4.16}$$

We introduce a few abbreviations:

$$w =: \cosh\gamma \tag{4.17}$$

gives

$$\frac{1}{\sqrt{w^2-1}\left(w+\sqrt{w^2-1}\right)^B} = \frac{e^{-B\gamma}}{\sinh\gamma} \tag{4.18}$$

The arguments of the Bessel functions are called x:

$$x := \frac{2V}{V_0} z(\varphi) = \frac{2V}{V_0} \sqrt{z_K^2 + 2 z_K z_Y w + z_Y^2} \; . \tag{4.19}$$

With

$$z_S := \sqrt{2 z_K z_Y} \qquad q := \frac{z_K^2 + z_Y^2}{2 z_K z_Y} \tag{4.20}$$

we have

$$x = \frac{2V}{V_0} z_S \sqrt{q + w} \tag{4.21}$$

and finally, in view of the saddle-point evaluation

$$J_B^{(\sigma)} = \frac{1}{2\pi i} \oint_C \exp[f_\sigma(w)]\, dw$$

$$f_0(w) := \frac{2V}{V_0} z_N w + \ln I_0(x) - B\gamma - \ln(\sinh\gamma)$$

$$f_1(w) := \frac{2V}{V_0} z_N w + \ln I_1(x) - B\gamma - \ln\left(z_S \sqrt{q+w}\right) - \ln(\sinh\gamma) \tag{4.22}$$

As in Section 3, the integrands have again a minimum on the positive real axis (at some w_0), which becomes very steep for large B and V. Contrary to the previous case we encounter here a square root cut [Eq. (4.21)] along the negative real axis from $w = -q$ to $-\infty$. Blowing up the contour C to an infinite rectangle whose right vertical side passes through w_0, we obtain (in addition to the vanishing contributions from the other three sides at ∞) a contribution from the integral over the discontinuity along the cut. This also vanishes, since there x is purely imaginary so that $I_{0,1} \to J_{0,1}$ (real argument), which are oscillating with decreasing amplitude when the absolute value of their argument grows. The other strongly decreasing factor

$$\exp\left(\frac{2V}{V_0} z_N w\right) \Big/ \left(\sqrt{w^2-1}\left(w + \sqrt{w^2-1}\right)^B\right) \approx \frac{e^{\frac{2V}{V_0} z_N w}}{w^{B+1}}$$

suppresses this contribution when $V, B \to \infty$. We therefore need only consider the saddle-point contribution at w_0:

$$J_B^{(\sigma)} \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[f_\sigma(w_0) - \frac{y^2}{2} f_\sigma''(w_0)\right] dy = e^{f(w_0)} \sqrt{\frac{1}{2\pi f_\sigma''(w_0)}} \quad . \tag{4.23}$$

While in the exponential factor we use Eq. (4.22), where for V/V_0 fixed the value of B may vary by ± 1 [see Eq. (4.12)], we need for*) $f_\sigma'(w)$ and $f_\sigma''(w)$ only consider terms which go to ∞ proportional to B. For these manipulations, we thus replace V/V_0 by B/c [see Eq. (2.15)]; further we note that x, x', x'' are proportional to B. We then rewrite Eq. (4.22) dropping all terms not containing B:

$$\begin{aligned} f_0(w) &\cong \frac{2Bz_N}{c} w + \ln I_0(x) - B\gamma \\ f_1(w) &\cong \frac{2Bz_N}{c} w + \ln I_1(x) - B\gamma \end{aligned} \tag{4.22'}$$

For the following we use

*) The prime means d/dw.

$$\begin{aligned} \frac{dI_0(x)}{dx} &= I_1(x) \\ \frac{dI_1}{dx} &= I_0(x) - I_1(x)/x \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} \frac{I_0}{I_1} &\xrightarrow[x\to\infty]{} 1 + \frac{1}{2x} + O(1/x^2) \\ \frac{I_1}{I_0} &\xrightarrow[x\to\infty]{} 1 - \frac{1}{2x} + O(1/x^2) \end{aligned} \tag{4.25}$$

further

$$\begin{aligned} \gamma' &= \frac{1}{\sinh x} = \frac{1}{\sqrt{w^2-1}} \\ x' &= \frac{Bzs}{c} \frac{1}{\sqrt{q+w}} \\ x'' &= -\frac{Bzs}{c} \frac{1}{\sqrt{q+w}^{\,3}} \end{aligned} \tag{4.26}$$

From the condition $f_{0,1}'(w_0) = 0$ we obtain in the large B limit for both functions the same equation for w_0:

$$\sqrt{w_0^2 - 1}\left(2z_N + \frac{2s}{\sqrt{q + w_0}}\right) = c \tag{4.27}$$

This sixth order equation can be solved easily by iteration: put

$$w_{n+1} = \sqrt{1 + \left(\frac{c}{2z_N + \frac{2s}{\sqrt{q + w_n}}}\right)^2} \quad . \tag{4.28}$$

Then, starting with $w_1 = 1$, each iteration improves the result by roughly one decimal place. We checked this for $0.1 \le T \le 0.2$ GeV and $1 \le c \le 10$, where over this whole range 15 iterations led to the limit of computer precision (10^{-14}). In Fig. 4.1 we plot the solution w_0 (T, c) of Eq. (4.27) for {c = 1, 2, 4, 8, 16 ; $0.15 \le T \le 0.2$ GeV}. To obtain a rather precise value for w_0 one takes a rough value from Fig. 4.1 and enters it as w_1 in Eq. (4.28); a few iterations suffice.

We thus consider the solution w_0 of Eq. (4.27) as known [for physical reasons the iteration solution is the interesting one].

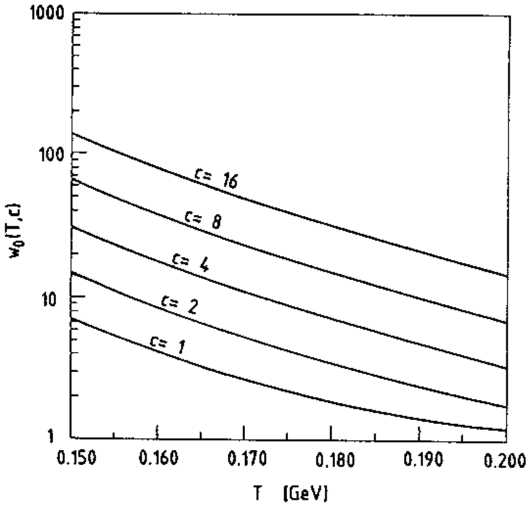

Fig. 4.1 The physical root w_0 of Eq. (4.27) for various compressions c as a function of the temperature.

Furthermore, the second derivative is, in the large B limit, also the same for both functions:

$$f_c''(w_0) \approx f_1''(w_0) \underset{B\to\infty}{\sim} B\left(\frac{w_0}{(w_0^2-1)^{3/2}} - \frac{z_S/2c}{(q+w_0)^{3/2}} \right). \tag{4.29}$$

Thus finally

$$J_B^{(0)} = \frac{\exp\left(2\frac{Vz_N}{V_0} w_0\right) I_0\left(2\frac{Vz_S}{V_0}\sqrt{q+w_0}\right)}{\sqrt{w_0^2-1}\left(w_0+\sqrt{w_0^2-1}\right)^B \sqrt{2\pi f_0''(w_0)}} \tag{4.30}$$

$$J_B^{(1)} = \frac{\exp\left(2\frac{Vz_N}{V_0} w_0\right) I_1\left(2\frac{Vz_S}{V_0}\sqrt{q+w_0}\right)}{\sqrt{w_0^2-1}\left(w_0+\sqrt{w_0^2-1}\right)^B z_S\sqrt{q+w_0}\,\sqrt{2\pi f_1''(w_0)}} \tag{4.31}$$

$$Z_{B,\pm 1}(T,V) = z_K J_B^{(1)} + z_Y J_{B\pm 1}^{(1)} \tag{4.32}$$

$$Z_{B,0}(T,V) = J_B^{(0)} \tag{4.33}$$

4.3 Particle multiplicities

Collecting the results (4.32), (4.33) and (4.10), (4.11), and with Eq. (4.17) we have

$$e^{\gamma_0} = w_0 + \sqrt{w_0^2-1} \tag{4.34}$$

where w_0 (T, c) is the solution of Eq. (4.27).

So we obtain

$$\left\langle \begin{Bmatrix} N \\ \bar{N} \end{Bmatrix} \right\rangle = z_N \frac{V}{V_0} e^{\pm\gamma_0} \tag{4.35}$$

$$\left\langle \begin{Bmatrix} K \\ \bar{K} \end{Bmatrix} \right\rangle = z_K \frac{V}{V_0} \cdot \frac{z_K + z_Y e^{\pm\gamma_0}}{z_S\sqrt{q+w_0}} \tag{4.36}$$

$$\left\langle \begin{Bmatrix} \Lambda \\ \bar{\Lambda} \end{Bmatrix} \right\rangle = z_\Lambda \frac{V}{V_0} \cdot \frac{z_Y + z_K e^{\pm\gamma_0}}{z_S\sqrt{q+w_0}} \tag{4.37}$$

$$\left\langle \begin{Bmatrix} \Sigma \\ \bar{\Sigma} \end{Bmatrix} \right\rangle = z_\Sigma \frac{V}{V_0} \cdot \frac{z_Y + z_K e^{\pm\gamma_0}}{z_S\sqrt{q+w_0}} = \left\langle \begin{Bmatrix} \Lambda \\ \bar{\Lambda} \end{Bmatrix} \right\rangle \cdot \frac{z_\Sigma}{z_\Lambda} \tag{4.38}$$

$$\frac{\langle N \rangle}{\langle \bar{N} \rangle} = e^{2\gamma_0} \tag{4.39}$$

$$\frac{\langle K \rangle}{\langle \bar{K} \rangle} = \frac{z_K + z_Y e^{\gamma_0}}{z_K + z_Y e^{-\gamma_0}} \tag{4.40}$$

$$\frac{\langle \Lambda \rangle}{\langle \bar{\Lambda} \rangle} = \frac{\langle \Sigma \rangle}{\langle \bar{\Sigma} \rangle} = \frac{z_Y + z_K e^{\gamma_0}}{z_Y + z_K e^{-\gamma_0}} \tag{4.41}$$

In this large B limit particle number ratios do not depend on B and should therefore be the same as in the grand canonical description. We now show that this is so.

4.4 Comparision with the grand canonical result

Following the recipe (1.5) generalized to the present case we find [7] the grand canonical partition function [using the notation (2.15)]:

$$\ln Z(T,V,\lambda_B,\lambda_S) = \frac{V}{V_0}\left[z_N\left(\lambda_B + 1/\lambda_B\right) + z_K\left(\lambda_S + 1/\lambda_S\right) + z_Y\left(\frac{\lambda_B}{\lambda_S} + \frac{\lambda_S}{\lambda_B}\right)\right]. \tag{4.42}$$

From this we derive

$$\langle B\rangle = \lambda_B \frac{\partial \ln Z}{\partial \lambda_B} = \frac{V}{V_0}\left[z_N\left(\lambda_B - 1/\lambda_B\right) + z_Y\left(\frac{\lambda_B}{\lambda_S} - \frac{\lambda_S}{\lambda_B}\right)\right] \tag{4.43}$$

$$\langle S\rangle = \lambda_S \frac{\partial \ln Z}{\partial \lambda_S} = \frac{V}{V_0}\left[z_K\left(\lambda_S - 1/\lambda_S\right) + z_Y\left(\frac{\lambda_S}{\lambda_B} - \frac{\lambda_B}{\lambda_S}\right)\right] \tag{4.44}$$

The condition $\langle S\rangle = 0$ leads to $\lambda_S \neq 0$:

$$\lambda_S = \sqrt{\frac{z_K + z_Y \cdot \lambda_B}{z_K + z_Y/\lambda_B}} \tag{4.45}$$

Then, applying the rules (2.12), (2.13):

$$\frac{\langle N\rangle}{\langle \bar{N}\rangle} = \lambda_B^2 \tag{4.46}$$

$$\frac{\langle K\rangle}{\langle \bar{K}\rangle} = \lambda_S^2 = \frac{z_K + z_Y \cdot \lambda_B}{z_K + z_Y/\lambda_B} \tag{4.47}$$

$$\frac{\langle \Lambda\rangle}{\langle \bar{\Lambda}\rangle} \approx \frac{\langle \Sigma\rangle}{\langle \bar{\Sigma}\rangle} = \left(\frac{\lambda_B}{\lambda_S}\right)^2 = \frac{z_Y + z_K \cdot \lambda_B}{z_Y + z_K/\lambda_B} \tag{4.48}$$

From this follows the general rule [valid only grand canonically: see (4.11)]

$$\frac{\langle N\rangle}{\langle \bar{N}\rangle} \cdot \frac{\langle \bar{K}\rangle}{\langle K\rangle} = \frac{\langle \Lambda\rangle}{\langle \bar{\Lambda}\rangle} = \frac{\langle \Sigma\rangle}{\langle \bar{\Sigma}\rangle} \tag{4.49}$$

Comparing Eqs. (4.46) - (4.48) with their large-B canonical counter-parts (4.39)-(4.41), we observe that they are identical provided that $\lambda_B \equiv \exp(\gamma_0)$. It is

easy to prove that this is true: in Eq. (4.43) we replace V/V_0 by B/c [see Eq. (2.15)] and identify ⟨B⟩ with B, thus obtaining, with λ_S taken from Eq. (4.45) and after some trivial manipulations:

$$c = z_N\left(\lambda_B - 1/\lambda_B\right) + z_Y z_K\left(\lambda_B - 1/\lambda_B\right)\frac{1}{\sqrt{z_K^2 + z_Y^2 + z_K z_Y\left(\lambda_B + 1/\lambda_B\right)}} \tag{4.50}$$

This is a sixth-order equation for λ_B. Turning now to Eq. (4.27), which is a sixth-order equation for w_0, we insert there

$$w_0 = \frac{1}{2}\left(\lambda_B + \frac{1}{\lambda_B}\right) \tag{4.51}$$

which by Eq. (4.17) is equivalent to $\lambda_B = \exp(\gamma_0)$. Using Eqs. (4.20) we then discover that Eq. (4.27) becomes identical to (4.50). This proves that

$$\lambda_B = e^{\gamma_0} = w_0 + \sqrt{w_0^2 - 1}$$
$$w_0 = \frac{1}{2}\left(\lambda_B + \frac{1}{\lambda_B}\right) \tag{4.52}$$

and that the Chebyshev approximation to the canonical description is equivalent to the grand canonical description.

4.5 When can the grand canonical description be used?

Having established the equivalence of the grand canonical description and the Chebyshev-Saddle-point-approximation of the canonical one, we now know that we may indiscriminately use the one or the other -- when and where they are valid. Incidentally, one might think that there is a practical difference in that the canonical procedure deals directly with the physically obvious varibles B and c, while the grand canonical one uses the less obvious λ_B; there remains, however, in the canonical case the sixth-order equation for w_0 to be solved [Eq. (4.27)], and for the same price [solving the identical Eq. (4.50) for λ_B] we can also relate the grand canonical description to obvious variables B and c. Thus there is no difference whatsoever - except when comparing two phases: equal λ's (equilibrium) and equal compressions (B-densities) refer to different physical situations.

But when are the Chebyshev approximation and the grand canonical formalism valid? Unfortunately it seems that this has to be checked in each case (given compression c and temperature T) individually. We provide an order-of-magnitude guess by calculating an example: $c = 2$; $T = 0.15$ GeV; we compute the ratio $R_N :$ $= \langle N\rangle/\langle\bar{N}\rangle$ by numerical integration from Eqs. (4.11) and (4.6) for B = 1, 2, ..., 10 and compare it with the B independent ($B \to \infty$) approximation (4.39). It turns out that -- as in the case of only B conservation -- the relative difference between the two decreases like const/B but with a much larger proportionality constant:

$$\frac{R_N(\text{canonical}) - R_N(\text{Chebyshev})}{R_N(\text{Chebyshev})} \simeq \frac{200}{B}\ \% \tag{4.53}$$

$$R_N := \langle N\rangle/\langle\bar{N}\rangle$$

This result might (wrongly) suggest that for small B -- in particular for B = 0 -- the grand canonical description could never be used; one must remember, however, our particular choice of variables: namely, {B, c = const.}. Since $B/c = V/V_0$ small B then means in fact small V. If one wishes to consider B = 0 one must use V/V_0 and B as variables.

From Eqs. (4.53) and (3.12) we may infer a thumb rule for calculating approximately the canonical values from the grand canonical ones: without trying to prove anything here we simply *guess* from the two numerical examples that quite generally *presumably* the relative difference between the two decreases as f (T, c)/B. *Suppose this to be true,* then any quantity F_i can be calculated canonically for any B from the grand canonical result and two canonical test calculations at low B (where they are simple):

$$F_i^{(can)}(T,c,B) = F_i^{(Grd.can)}(T,c)\left(1 + \frac{f_i(c,T)}{B}\right) \tag{4.54}$$

Here f_i (c, T) can be determined at, say, B = 1 and 2:

$$f_i(c,T) = B\left[\frac{F_i^{(can)}(T,c,B) - F_i^{(Grd.can)}(T,c)}{F_i^{(Grd.can)}(T,c)}\right] \tag{4.55}$$

Where $F_i^{(can)}$ (T, c, B) has to be evaluated numerically from the integral representation (4.6); for B = 1, 2 this is a very simple matter (not so for large B). If then f_i (c, T) is roughly the same for B = 1 and B = 2, Eq. (4.53) may be used with confidence at any B.

5. CONCLUSIONS

We have constructed a description of an ideal relativistic Boltzmann gas which is grand canonical in particle numbers but canonical in B and/or S conservation.

We have solved the case of B conservation analytically, the B × S case by an approximation ("Chebyshev") which turns out to be exactly equivalent to the grand canonical description of B × S conservation.

Thus the Chebyshev approximation to the canonical formalism and the grand canonical one may be used alternatively when they are valid. Where and when they are valid may be easily found out by two test calculations (at B = 1 and 2) and the thumb rule (4.54), (4.55). If this does not work, one can still go back to the explicit integral representation given in the text and evaluate it numerically.

We finish with a few further remarks:

- We have stressed that particle numbers *must* be treated grand canonically, while for conservation laws the choice is open. Therefore, one can also have mixed representations of conservation laws: if several charges are conserved; one may treat some of them canonically, some grand canonically.

- There is a very different way of accounting for conservation laws canonically: namely, by explicitly constructing the partition function such that all states violating them are excluded "by hand" (probably only applicable for Abelian charges) without using group theoretical projection techniques; this technique was applied to estimate pair creation rates [8], to treat "brought-in baryons" and production of strange particles [9] (all in a first approximation), and later carried through exactly [10] to show that multiplicity ratios may significantly depend on the interaction volume. In this last paper strangeness was treated exactly (i.e. canonically), while the baryon number was represented grand canonically by a chemical potential.

- Numerical calculations were done [11] with canonical B and S conservation for $0 \le B \le 12$ and $0 \le |S| \le 3$ with the same group theoretical projection technique as here but with another method to do the integral (4.6) (transforming it into an infinite sum, while we integrated it on a computer).

Acknowledgement

One of us (K.R.) would like to thank J. Rafelski for discussions and the Theory Division of CERN for kind hospitality.

REFERENCES

1. For a review see for example,
   a) H. Satz (Ed.), Statistical Mechanics of Quarks and Hadrons, Proc. Int. Symposium, Bielefeld, 1980 (North Holland, Amsterdam, 1981).
   b) M. Jacob and H. Satz (Eds.), Quark Matter Formation and Heavy Ion Collisions, Proc. Workshop, Bielefeld, 1982 (World Scientific, Singapore, 1982).
   c) M. Jacob and J. Tran Tanh Van, Phys. Rep. 88 (1982) 321.
2. a) R. Hagedorn and J. Rafelski in Ref. [1a].
   b) R. Hagedorn and J. Rafelski, Phys. Lett. 97B (1980) 136.
   c) R. Hagedorn, I. Montvay and J. Rafelski in Erice Workshop "Hadronic Matter at Extreme Energy Density" (Eds. N. Cabibbo and L. Sertorio; Plenum Press, New York, 1980).
   d) R. Hagedorn, Z. Phys. C17 (1983) 265.
3. A few of the more important earlier papers are:
   a) F. Cerulus, Nuovo Cimento 22 (1961) 958.
   b) K. Zalewski, Acta Phys. Pol. 28 (1965) 933.
   c) E.M. Ilgenfritz and J. Kripfganz, Nucl. Phys. B62 (1973) 141.
4. a) K. Redlich and L. Turko, Z. Phys. C5 (1980) 201.
   b) L. Turko, Phys. Lett. 104 B (1981) 153.
5. For integrals and function definitions we use:
   a) I.S. Gradshteyn and I.M. Ryshik, Tables of Integrals, Series and Products (Academic Press, New York, 1965).
   b) Milton Abramowitz and Irene A. Stegun (Eds.), Handbook of Mathematical Functions (Dover, New York, 1970).
6. a) D. Elliott, Math. Comput. 18 (1964) 274.
   b) D. Elliott and G. Szekeres, Math. Comput. 19 (1965) 25.
7. Compare:
   P. Koch, W. Greiner and J. Rafelski, preprint CERN-TH-3781 (1983), to be published in Phys. Lett. B, where a more general case (B, S, and electric charge conserved) is discussed.
8. R. Hagedorn, Suppl. Nuovo Cimento 6 (1968) 311.

9. R. Hagedorn and J. Ranft, Suppl. Nuovo Cimento 6 (1968) 169.

10. J. Rafelski and M. Danos, Phys. Lett. 97 B (1980) 279.

11. Ch. Derret, Diplomarbeit, Inst. Theor. Phys. Frankfurt, Germany, (1983).