Statistics of the von Mises stress response for structures subjected to random excitations

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Finite element-based random vibration analysis is increasingly used in computer aided engineering software for computing statistics (e.g., root-mean-square value) of structural responses such as displacements, stresses and strains. However, these statistics can often be computed only for Cartesian responses. For the design of metal structures, a failure criterion based on an equivalent stress response, commonly known as the von Mises stress, is more appropriate and often used. This paper presents an approach for computing the statistics of the von Mises stress response for structures subjected to random excitations. Random vibration analysis is first performed to compute covariance matrices of Cartesian stress responses. Monte Carlo simulation is then used to perform scatter and failure analyses using the von Mises stress response.

Keywords: Random vibration, von Mises stress, Monte Carlo simulation, covariance matrix, failure analysis

1. Introduction

Many structural and mechanical systems are exposed to stochastic loads, e.g., aircrafts subjected to atmospheric turbulence, buildings and bridges subjected to earthquakes, vehicle components subjected to vibrations arising from rough roads, and ships and offshore platforms subjected to wind and wave loads. The theory of random vibration is central to the analysis and design of structures exposed to random excitations and is being increasingly used in practice for a variety of engineering problems (Chen [4]; Nigam [12]; Soong and Grigoriu [13]). In recent years, RVA in conjunction with the finite element (FE) method is available in several commercial software, such as ANSYS, NASTRAN, ABAQUS, I-DEAS, ALGOR and ADINA. Most of these software can be used to compute root-mean-square (r.m.s.) values of Cartesian displacements, stresses and strains. However, Cartesian responses alone are often inadequate for failure analyses in structural design. A failure criterion based on the maximum octahedral shear stress, often referred to as the von Mises theory, is usually used. This theory is widely regarded to be the most reliable for the design of ductile materials. The statistics of the von Mises stress cannot be readily computed in FE analysis software because it is nonlinearly related to the Cartesian stresses and its statistics cannot be easily related to those of the Cartesian stresses. Harichandran and Chen [7] explored the use of a first-order second-moment approach to approximately compute statistics of principal stress-related quantities, but calibrated correction factors were needed to obtain acceptable accuracy and the method may not be sufficiently robust.

In this paper, direct Monte Carlo simulation (MCS) is used to compute the statistical moments of the von Mises stress. Because an explicit formula relating the von Mises stress to the Cartesian stresses is available, the simulation requires little computer time. For 3D problems, correlations between the Cartesian stresses at each node must be taken into account, and hence the twenty-one components of a 6×6 variance–covariance matrix at each node must be computed from RVA prior to the MCS. Using closed-form solutions for stochastic responses from RVA (Chen and Ali [5]; Harichandran [6]) the variance–covariance matrix at each node can be computed quickly and efficiently for common types of excitations. As a result, computation of the statistics of the von Mises

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stress, and other quantities such as the stress intensity and maximum shear stress, becomes feasible. Scatter analysis based on the von Mises stress response of a structure subjected to random vibration is also presented in this study. Histograms approximating the probability distribution function (PDF) and cumulative distribution function (CDF) of the von Mises stress can be obtained, which can provide useful information for designing against yield and fatigue failures.

In failure analysis, a design strength taken as a deterministic value or a random variable (typically Gaussian or Weibull) may be specified such that probability of failure of a structure can be computed. The failure probability estimated by MCS is approximate, but the coefficient of variation of the failure probability also can be estimated to assess its accuracy. In addition, the required strength for engineering design corresponding to a desired probability of failure may be estimated easily through MCS. In this paper, a simple example consisting of a shaker table excited by random loads, is used to illustrate the various features of the analysis. By using the proposed technique, the von Mises failure criterion is applied for the reliability-based design of the shaker table.

2. Finite element-based RVA

The dynamic equations of motion of a finite element system may be expressed in the partitioned form (Harichandran and Wang [8]):

$$\begin{bmatrix} M_{\mathrm{FF}} & M_{\mathrm{FR}} \\ M_{\mathrm{RF}} & M_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \{\ddot{u}_{\mathrm{F}}\} \\ \{\ddot{u}_{\mathrm{R}}\} \end{bmatrix} + \begin{bmatrix} C_{\mathrm{FF}} & C_{\mathrm{FR}} \\ C_{\mathrm{RF}} & C_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \{\dot{u}_{\mathrm{F}}\} \\ \{\dot{u}_{\mathrm{R}}\} \end{bmatrix} + \begin{bmatrix} K_{\mathrm{FF}} & K_{\mathrm{FR}} \\ K_{\mathrm{RF}} & K_{\mathrm{RR}} \end{bmatrix} \begin{bmatrix} \{u_{\mathrm{F}}\} \\ \{u_{\mathrm{R}}\} \end{bmatrix} = \begin{bmatrix} \{P\} \\ \{R\} \end{bmatrix} \tag{1}$$

in which $\{u_{\mathrm{F}}\}$ are free degrees of freedom (DOFs), $\{u_{\mathrm{R}}\}$ are restrained DOFs, and $\{P\}$ and $\{R\}$ are the excitations at free nodes and reactions at the restrained nodes, respectively. The excitations $\{P\}$ are assumed to be zero-mean stationary stochastic processes. The free nodal displacement can be decomposed into pseudo-static and dynamic parts as $\{u_{\mathrm{F}}\} = \{u_{\mathrm{s}}\} + \{u_{\mathrm{d}}\}$. Assuming light damping and performing modal analysis and RVA, the mean-square displacement response can be expressed as

$$\sigma^2_{u_{\mathrm{F}}} = \sigma^2_{u_{\mathrm{s}}} + \sigma^2_{u_{\mathrm{d}}} + 2\,\mathrm{Cov}(u_{\mathrm{s}}, u_{\mathrm{d}}), \tag{2}$$

where the mean-square values of the i-th dynamic and pseudo-static displacements and their covariance can be respectively obtained as

$$\sigma^2_{u_{\mathrm{d}_i}} = \sum_{j=1}^{n}\sum_{k=1}^{n} \Phi_{ij}\Phi_{ik} \left[\sum_{l=1}^{t}\sum_{m=1}^{t} \frac{\Phi_{lj}\Phi_{mk}}{M_j M_k} \times \int_0^\infty \overline{H_j(\omega)} H_k(\omega) S_{P_l P_m}(\omega)\,\mathrm{d}\omega + \sum_{l=1}^{r}\sum_{m=1}^{r} \Gamma_{lj}\Gamma_{mk} \times \int_0^\infty \overline{H_j(\omega)} H_k(\omega) S_{\ddot{u}_{\mathrm{R}_l}\ddot{u}_{\mathrm{R}_m}}(\omega)\,\mathrm{d}\omega \right], \tag{3}$$

$$\sigma^2_{u_{\mathrm{s}_i}} = \sum_{l=1}^{r}\sum_{m=1}^{r} A_{il}A_{im} \int_0^\infty \omega^{-4} S_{\ddot{u}_{\mathrm{R}_l}\ddot{u}_{\mathrm{R}_m}}(\omega)\,\mathrm{d}\omega \tag{4}$$

and

$$\mathrm{Cov}(u_{\mathrm{s}_i}, u_{\mathrm{d}_i}) = -\sum_{j=1}^{n}\sum_{l=1}^{r}\sum_{m=1}^{r} \Phi_{ij}A_{il}\Gamma_{mj} \times \int_0^\infty \omega^{-2} H_j(\omega) S_{\ddot{u}_{\mathrm{R}_l}\ddot{u}_{\mathrm{R}_m}}(\omega)\,\mathrm{d}\omega, \tag{5}$$

where

- n = number of modes, t = number of free DOF with force excitations, and r = number of restrained DOF with acceleration excitations.
- Φ_{lj} = lj-th element of the displacement mode shape matrix, Γ_{lj} = l-th element of the participation vector for mode j, A_{il} = the i-th displacement response due to a unit displacement at restrained DOF l.
- $S_{P_l P_m}(\omega)$ and $S_{\ddot{u}_{\mathrm{R}_l}\ddot{u}_{\mathrm{R}_m}}(\omega)$ = power spectral density (PSD) functions of nodal force and base acceleration excitations, respectively; ω = circular frequency.
- $H_j(\omega)$ = j-th modal frequency response function given by

  $$H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + 2\mathrm{i}\omega_j\zeta_j\omega}, \tag{6}$$

  where ω_j and ζ_j are the modal frequency and damping ratio. An overbar denotes the complex conjugate.

3. Cartesian stress responses

As with the displacement response, any Cartesian stress can also be decomposed into pseudo-static and dynamic parts, $\{\tau_{\mathrm{s}}\}$ and $\{\tau_{\mathrm{d}}\}$, respectively. Consider two Cartesian stresses, τ_1 and τ_2, which can be expressed as $\tau_1 = \tau_{1\mathrm{s}} + \tau_{1\mathrm{d}}$ and $\tau_2 = \tau_{2\mathrm{s}} + \tau_{2\mathrm{d}}$, respectively. Thus,

$$\begin{aligned} E[\tau_1, \tau_2] &= E[(\tau_{1\mathrm{s}} + \tau_{1\mathrm{d}})(\tau_{2\mathrm{s}} + \tau_{2\mathrm{d}})] \\ &= E[\tau_{1\mathrm{d}}\tau_{2\mathrm{d}} + \tau_{1\mathrm{s}}\tau_{2\mathrm{s}} + \tau_{1\mathrm{d}}\tau_{2\mathrm{s}} + \tau_{1\mathrm{s}}\tau_{2\mathrm{d}}], \end{aligned} \tag{7}$$

where $E[\cdot]$ represents the expected value. Because all Cartesian stresses are assumed to be zero-mean Gaussian processes, Eq. (7) becomes

$$\begin{aligned} \mathrm{Cov}(\tau_1, \tau_2) &= \mathrm{Cov}(\tau_{1\mathrm{d}}, \tau_{2\mathrm{d}}) + \mathrm{Cov}(\tau_{1\mathrm{s}}, \tau_{2\mathrm{s}}) \\ &\quad + \mathrm{Cov}(\tau_{1\mathrm{d}}, \tau_{2\mathrm{s}}) + \mathrm{Cov}(\tau_{1\mathrm{s}}, \tau_{2\mathrm{d}}). \end{aligned} \tag{8}$$

As a result, the covariance matrix of the six Cartesian stresses at the i-th node may be expressed as

$$\begin{aligned} \left[\mathrm{Cov}(\tau_p, \tau_q)\right]_i &= \left[\mathrm{Cov}(\tau_{p\mathrm{d}}, \tau_{q\mathrm{d}})\right]_i + \left[\mathrm{Cov}(\tau_{p\mathrm{s}}, \tau_{q\mathrm{s}})\right]_i \\ &\quad + \left[\mathrm{Cov}(\tau_{p\mathrm{d}}, \tau_{q\mathrm{s}})\right]_i + \left[\mathrm{Cov}(\tau_{p\mathrm{s}}, \tau_{q\mathrm{d}})\right]_i, \end{aligned} \tag{9}$$

in which $[\mathrm{Cov}(\tau_{p\mathrm{s}}, \tau_{q\mathrm{s}})]_i$ and $[\mathrm{Cov}(\tau_{p\mathrm{d}}, \tau_{q\mathrm{d}})]_i$ are the variance–covariances matrix of pseudo-static and dynamic stresses at the i-th node; $[\mathrm{Cov}(\tau_{p\mathrm{d}}, \tau_{q\mathrm{s}})]_i$ and $[\mathrm{Cov}(\tau_{p\mathrm{s}}, \tau_{q\mathrm{d}})]_i$ are the variance–covariance matrix between static and dynamic stresses; and $p = 1, \ldots, 6$ and $q = 1, \ldots, 6$ are subscripts denoting the stress components, in which the six stress components $\tau_x, \tau_y, \tau_z, \tau_{xy}, \tau_{xz}$ and τ_{yz} are denoted by $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ and τ_6, respectively. In expanded form, the variance–covariance matrix is

$$\begin{aligned} &\left[\mathrm{Cov}(\tau_p, \tau_q)\right]_i \\ &= \begin{bmatrix} \mathrm{Var}(\tau_1) & \mathrm{Cov}(\tau_1, \tau_2) & \ldots & \mathrm{Cov}(\tau_1, \tau_6) \\ \mathrm{Cov}(\tau_2, \tau_1) & \mathrm{Var}(\tau_2) & \ldots & \mathrm{Cov}(\tau_2, \tau_6) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Cov}(\tau_6, \tau_1) & \mathrm{Cov}(\tau_6, \tau_2) & \ldots & \mathrm{Var}(\tau_6) \end{bmatrix}_i . \end{aligned} \tag{10}$$

Note that $[\mathrm{Cov}(\tau_p, \tau_q)]_i$ is a symmetric matrix, thus, only 21 components need be solved for 3D problems. Similar to the displacement responses in Eqs (3), (4) and (5), the covariance of the dynamic stresses at the i-th node can be expressed as

$$\begin{aligned} \left[\mathrm{Cov}(\tau_{p\mathrm{d}}, \tau_{q\mathrm{d}})\right]_i &= \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\left(\Psi_{uj}\Psi_{\nu k} + \Psi_{\nu j}\Psi_{uk}\right) I_{jk}, \\ p, q &= 1, \ldots, 6, \qquad i = 1, \ldots, N_n. \end{aligned} \tag{11}$$

where $[\Psi]$ are the stress mode shapes, the subscripts u and ν are $u = 6(i-1) + p$ and $\nu = 6(i-1) + q$, N_n is the total number of nodes, n is the number of modes used, and I_{jk} represents the terms within the square brackets in Eq. (3). The pseudo-static stresses may be expressed as

$$\{\tau_{\mathrm{s}}\} = [\eta]\{u_{\mathrm{R}}\}, \tag{12}$$

where the j-th column of $[\eta]$ are the stresses due to a unit displacement along the j-th restrained DOF. Consequently, the covariance matrix can be obtained as

$$\begin{aligned} &\left[\mathrm{Cov}(\tau_{p\mathrm{s}}, \tau_{q\mathrm{s}})\right]_i \\ &\quad = \sum_{l=1}^{r}\sum_{m=1}^{r} \frac{1}{2}\left(\eta_{ul}\eta_{\nu m} + \eta_{\nu l}\eta_{um}\right) \\ &\quad\quad \times \int_{-\infty}^{\infty} \frac{1}{\omega^4} S_{\ddot{u}_{\mathrm{R}_l}\ddot{u}_{\mathrm{R}_m}}(\omega)\,\mathrm{d}\omega, \\ &\quad\quad p, q = 1, \ldots, 6, \qquad i = 1, \ldots, N_n. \end{aligned} \tag{13}$$

Finally, the covariance matrix between the pseudo-static and dynamic stress responses at the i-th node can be expressed as

$$\begin{aligned} &\left[\mathrm{Cov}(\tau_{p\mathrm{d}}, \tau_{q\mathrm{s}}) + \mathrm{Cov}(\tau_{p\mathrm{s}}, \tau_{q\mathrm{d}})\right]_i \\ &\quad = \mathrm{Re}\left[\sum_{j=1}^{n}\sum_{l=1}^{r}\sum_{m=1}^{r}\left(\Psi_{uj}\eta_{\nu l} + \Psi_{\nu j}\eta_{ul}\right)\right. \\ &\quad\quad \left. \times \int_{-\infty}^{\infty} \Gamma_{mj} \frac{-1}{\omega^2} S_{\ddot{u}_{\mathrm{R}_l}\ddot{u}_{\mathrm{R}_m}}(\omega)\,\mathrm{d}\omega\right], \\ &\quad\quad p, q = 1, \ldots, 6, \qquad i = 1, \ldots, N_n, \end{aligned} \tag{14}$$

where Re$[\cdot]$ denotes the real part.

4. Von Mises stress simulation

MCS is a well-known and straightforward method for estimating the statistics of a random variable, such as the von Mises stress τ_{eq}. Numerous samples of Gaussian Cartesian stresses, consistent with the covariance matrix computed through RVA, are first simulated at each node. Morris [11] provides a convenient subroutine to generate random multivariate normal samples with a specified mean and covariance matrix. The density function of the multivariate Gaussian distribution for $\{\tau\} = [\tau_1, \ldots, \tau_m]^{\mathrm{T}}$ is given by

$$\begin{aligned} f(\tau) &= \frac{1}{\sqrt{(2\pi)^m \det([C])}} \\ &\quad \times \exp\left(-\frac{1}{2}\left(\{\tau\} - \{\mu\}\right)[C]^{-1}\right. \\ &\quad\quad \left. \times \left(\{\tau\} - \{\mu\}\right)^{\mathrm{T}}\right), \end{aligned} \tag{15}$$

in which $[C]$ is the covariance matrix of the variates $\{\tau\}$ and $\{\mu\}$ is the mean vector. In the case studied, $\{\mu\} = \{0\}$ and $m = 6$ for 3D problems.

Samples of τ_{eq} based on von Mises theory (Boresi and Sidebottom [3]) are determined from each set of Cartesian stresses through

$$\tau_{\text{eq}} = \left[\frac{(\tau_{xx}-\tau_{yy})^2+(\tau_{xx}-\tau_{zz})^2+(\tau_{yy}-\tau_{zz})^2}{2} + 3(\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2)\right]^{1/2}. \tag{16}$$

A sufficient number of samples must be generated to obtain acceptable accuracy in the computed statistics. About 10 000 samples was found to be adequate to perform scatter analysis for the mean (μ), standard deviation (σ), skewness coefficient (C_{s}) and kurtosis coefficient (C_{k}) of τ_{eq}, in which (Harr [9])

$$C_{\text{s}} = \frac{\sum_{i=1}((\tau_i)_{\text{eq}} - \mu_{\tau_{\text{eq}}})^3}{N\sigma^3_{\tau_{\text{eq}}}} \quad \text{and}$$

$$C_{\text{k}} = \frac{\sum_{i=1}((\tau_i)_{\text{eq}} - \mu_{\tau_{\text{eq}}})^4}{N\sigma^4_{\tau_{\text{eq}}}}, \tag{17}$$

where N is the total number of simulated values.

In failure analysis, a deterministic or random design strength, τ^*_{eq}, may be specified and statistics of the safety margin, $M = \tau^*_{\text{eq}} - \tau_{\text{eq}}$, can be computed. In this study, τ^*_{eq} and τ_{eq} are assumed to be statistically independent. In general, a comprehensive reliability analysis should compute the Hasofer–Lind reliability index, which is invariant with respect to different but mechanically equivalent formulations of the failure criterion (Madsen et al. [10]). Here, because M is a linear function of τ^*_{eq} and τ_{eq}, the Hasofer–Lind reliability index is identical to the Cornell reliability index, μ_M/σ_M, where μ_M and σ_M are the mean and standard deviation of M. The required number of simulations (N) for acceptable accuracy is dependent on the design failure probability, p_{f}, with the rule-of-thumb being:

$$N = \begin{cases} 10\,000 & \text{if } 0.01 < p_{\text{f}} < 1, \\ 100\,000 & \text{if } 0.001 < p_{\text{f}} \leqslant 0.01, \\ 1\,000\,000 & \text{if } 0.0001 \leqslant p_{\text{f}} \leqslant 0.001. \end{cases} \tag{18}$$

In general, $N \geqslant 100/p_{\text{f}}$. The probability of failure is determined by $p_{\text{f}} = N_{\text{f}}/N$ where N_{f} is the total number of failure events. For $N = 1\,000\,000$, the sample generation takes only about 20 seconds on a SGI IRIS4D Unix workstation. For $p_{\text{f}} < 0.01\%$, direct MCS is not recommended because of the very larger number of samples required. Alternatively, efficient MCS using importance sampling, the first order reliability method (FORM), or the second order reliability method (SORM) may be used (Ang and Tang [1]). In these methods the most probable failure point needs to be solved using an iterative optimization technique. On the other hand, for the many structures that are designed with $p_{\text{f}} > 0.01\%$, the proposed technique is quick and simple.

The accuracy of the estimated probability of failure also can be estimated using the MCS results. As N approaches infinity, p_{f} approaches the true value. Assume that each simulation cycle constitutes a Bernoulli trial, the variance of the estimated probability of failure can be computed approximately as (Ayyub and Mccuen [2])

$$\text{Var}(p_{\text{f}}) = \frac{(1-p_{\text{f}})p_{\text{f}}}{N}. \tag{19}$$

Consequently, the accuracy of the estimated p_{f} can be measured by its coefficient of variation

$$\text{COV}(p_{\text{f}}) = \frac{1}{p_{\text{f}}}\sqrt{\frac{(1-p_{\text{f}})p_{\text{f}}}{N}}. \tag{20}$$

The smaller the coefficient of variation, the better the accuracy of the estimated p_{f}. Equation (18) always yields a coefficient of variation of less than 10%, which is acceptable for most engineering problems.

Another feature of the proposed design technique is that the design strength, τ^*_{eq}, corresponding to a specified probability of failure can be computed. The result can be cross-examined with that in failure analysis. The array of N simulated samples are first sorted in decreasing order as $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_N$. The design strength corresponding to a probability of failure p_{f} is then the value of the i-th sample, where i = integer part of Np_{f}. For very small p_{f}, the arrangement of the large number of simulated samples in decreasing order would be very time-consuming and inefficient. An improved technique is to use a threshold stress, τ_{T}, to filter out the larger values of τ_{eq} which could possibly cause failure, and then rank order only the filtered values. A suitable threshold is

$$\tau_{\text{T}} = 0.8\left(\widehat{\mu} + \left(2.3 + 0.71\log_{10}\left(\frac{0.01}{p_{\text{f}}}\right) + C_{\text{s}}\right)\widehat{\sigma}\right) \geqslant \widehat{\mu},$$

$$0.01 \leqslant p_{\text{f}} \leqslant 0.0001, \tag{21}$$

in which $\widehat{\mu}$ and $\widehat{\sigma}$ are the mean and standard deviation of τ_{eq} estimated from a small number of τ_{eq} samples (say 2000 values). The coefficient of 0.8 is used to ensure that no probable failure samples are left out due to estimation errors in $\widehat{\mu}$ and $\widehat{\sigma}$.

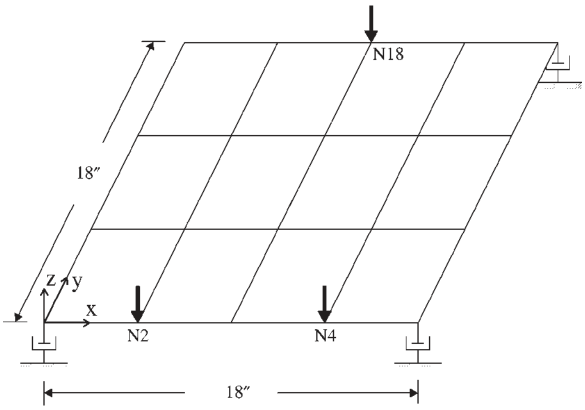


Fig. 1. Mesh and boundary conditions for shaker table.

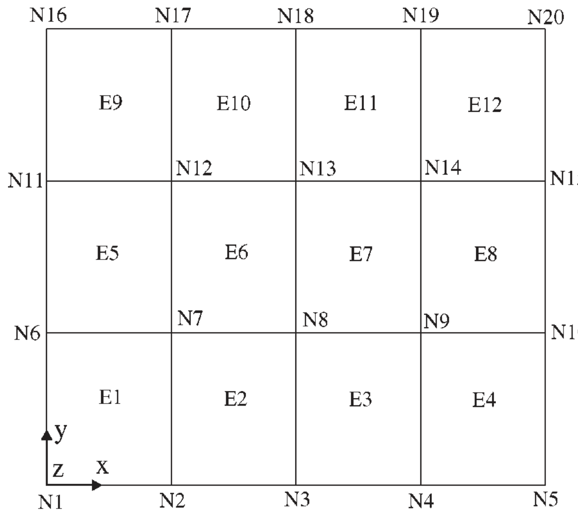


Fig. 2. Node and element numbers.

5. Example

An aluminum shaker table with a spring-damper for each leg is used to illustrate the proposed technique. The structure having a dimension of 0.457 m $\times$ 0.457 m (18″ $\times$ 18″) is represented by a finite element model as shown in Fig. 1. The model consists of 12 elastic shell elements (SHELL 63) and 4 spring-dampers (COMBINE 14). The four nodes at the base of the dampers are completely restrained and two of the nodes connecting the table and the dampers (nodes 1 and 5) are restrained in the X and Y directions. The node and element labels are given in Fig. 2. Random forces with the PSD shown in Fig. 3(a) are applied in the Z direction at nodes 2 and 4. Another random force with the PSD shown in Fig. 3(b) is applied in the Z direction at node 18.

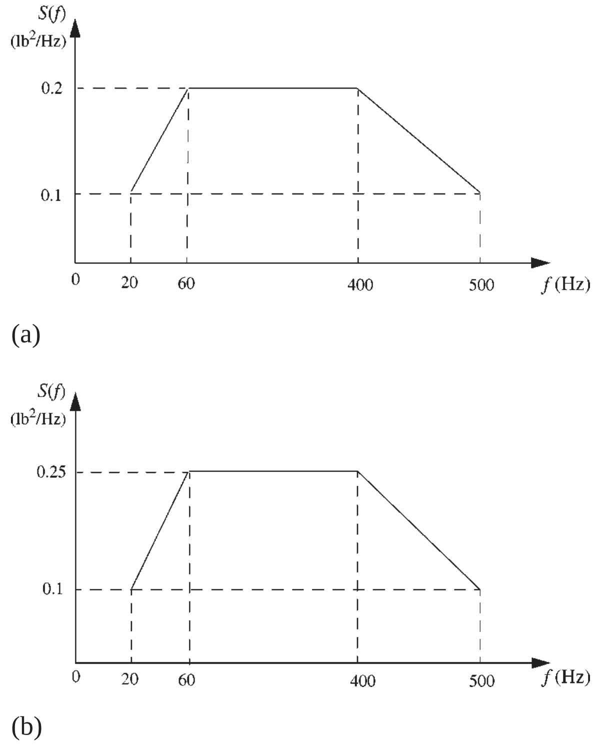


Fig. 3. Power spectral density function for out-of-plane excitations at: (a) nodes 2 and 4, and (b) node 18.

The two sets of forces are assumed to be statistically independent. Root-mean-square values of the nodal stresses, τ_x, τ_y and τ_{xy}, for each element are shown in Figs 4, 5 and 6, respectively. The largest responses in the figures are 37.2 MPa (5399 psi) for τ_x, 35.7 MPa (5177 psi) for τ_y and 10.6 MPa (1533 psi) for τ_{xy}, which occur at nodes 3, 18 and 15, respectively. The contours are plotted without averaging the responses on the boundary of each element.

In order to obtain the maximum von Mises stress for design, covariance matrices were computed for some critical points which had large Cartesian stresses. Since only τ_x, τ_y and τ_{xy} are available for the shell elements, only six elements of the covariance matrices are required for computing the von Mises stress, and the results are given in Table 1. The mean, standard deviation, skewness coefficient and kurtosis coefficient of the von Mises stress response obtained via MCS are listed in Table 2. It can be seen that node 3 has a maximum mean value of 33.9 MPa (4910 psi) and a maximum standard deviation of 17.6 MPa (2545 psi). Histograms for the PDF and CDF of the stress at node 3 are shown in Fig. 7 (a) and (b), respectively. Note that the von Mises stress response has a non-zero mean and is

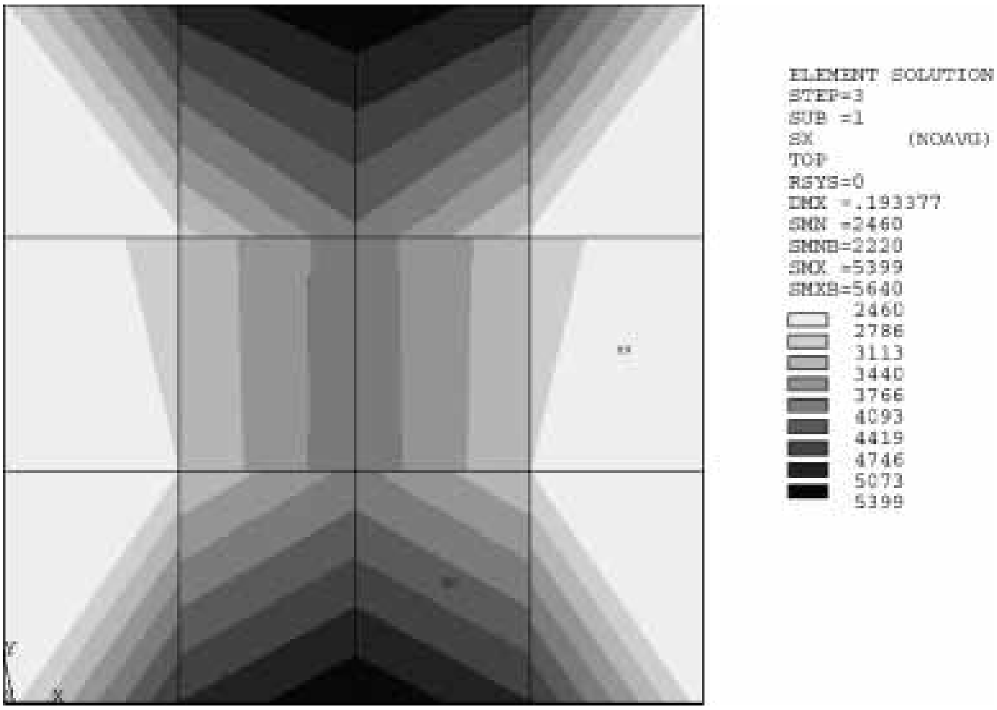


Fig. 4. Contours of root-mean-square response of X normal stress component.

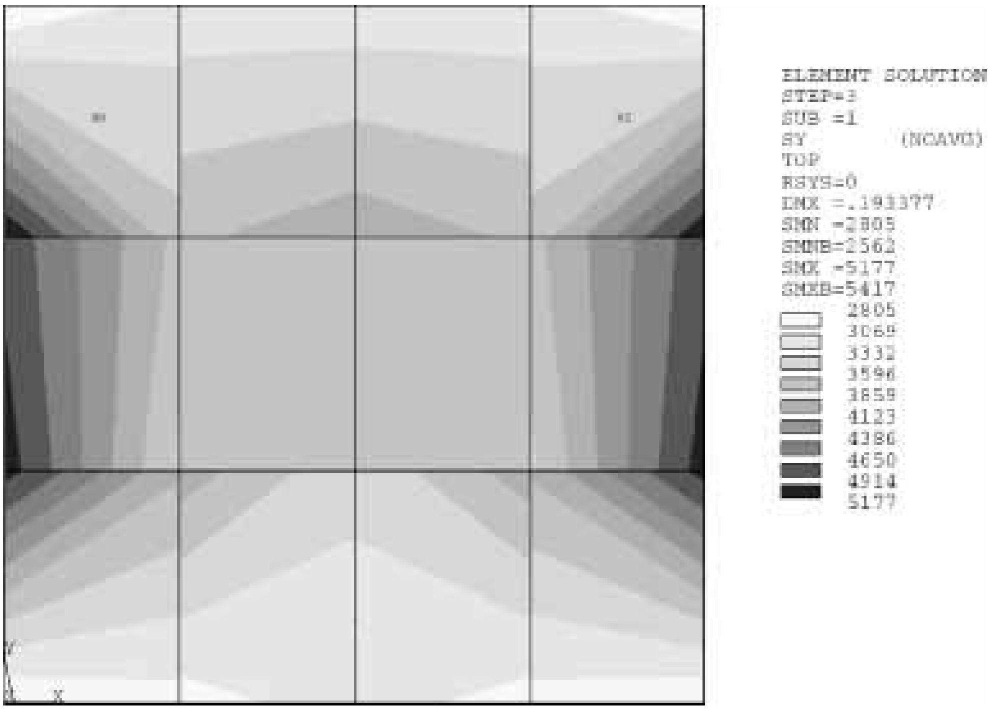


Fig. 5. Contours of root-mean-square response of Y normal stress component.

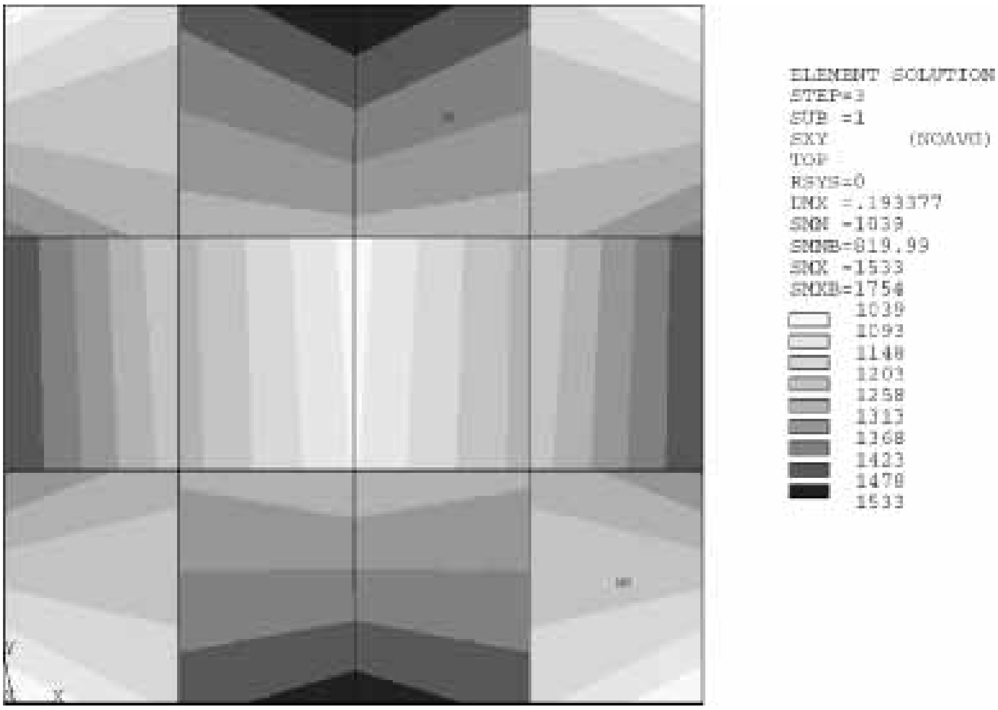


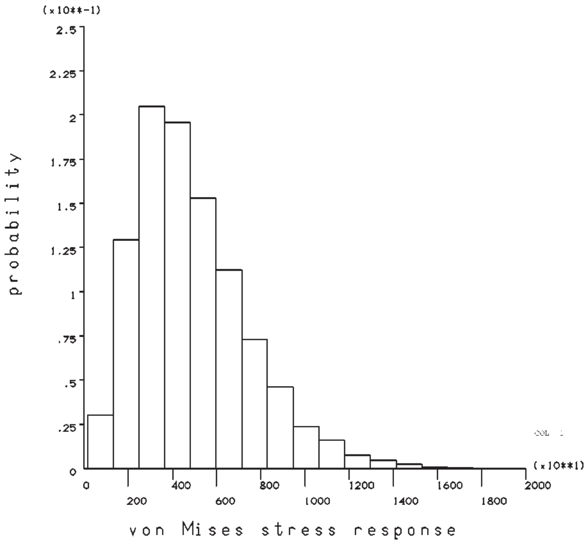
Fig. 6. Contours of root-mean-square response of XY shear stress component.

Table 1
Variance–covariance components of Cartesian stresses at the critical points MPa^2(lb^2/in^4)

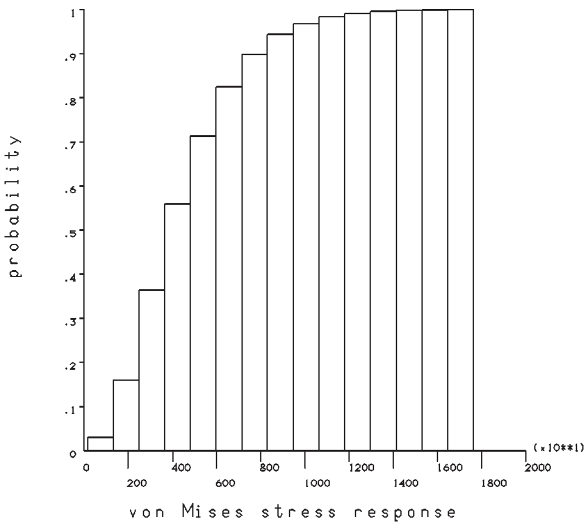
Components of var–covar. matrix	Node 3 of element 3	Node 18 of element 11	Node 15 of element 12	Node 10 of element 8
Var(τ_x)	1386 (2.92 × 10^7)	1369 (2.88 × 10^7)	336 (7.07 × 10^6)	318 (6.68 × 10^6)
Cov(τ_x, τ_y)	729 (1.53 × 10^7)	721 (1.52 × 10^7)	596 (1.25 × 10^7)	553 (1.16 × 10^7)
Cov(τ_x, τ_{xy})	−23.5 (−5.0 × 10^5)	33.6 (7.07 × 10^5)	1.15 (2.42 × 10^4)	49.9 (1.05 × 10^6)
Var(τ_y)	487 (1.02 × 10^7)	480 (1.01 × 10^7)	1274 (2.68 × 10^7)	1221 (2.57 × 10^7)
Cov(τ_y, τ_{xy})	−12.8 (−2.7 × 10^5)	19.6 (4.12 × 10^5)	−8.89 (−1.9 × 10^5)	94.1 (1.98 × 10^6)
Var(τ_{xy})	109 (2.29 × 10^6)	112 (2.35 × 10^6)	92.6 (1.95 × 10^6)	104 (2.19 × 10^6)

Table 2
Statistics of von Mises stress at the critical points

Statistic	Node 3 of element 3	Node 18 of element 11	Node 15 of element 12	Node 10 of element 8
Mean (MPa)	33.86	33.78	31.52	31.68
(psi)	(4910.4)	(4898.6)	(4570.6)	(4593.8)
Standard deviation (MPa)	17.55	17.49	17.06	16.95
(psi)	(2545.2)	(2536.0)	(2474.2)	(2457.8)
Skewness coefficient	0.985	0.981	1.012	1.004
Kurtosis coefficient	4.174	4.173	4.253	4.266



(a)



(b)

Fig. 7. Histograms for (a) probability density function and (b) cumulative distribution function of von Mises stress response at node 3.

non-Gaussian even though the excitations are zero-mean Gaussian processes. For a trial failure probability of $p_f = 0.005$, the proposed technique gives the required design strength as approximately 93.4 MPa (13 540 psi). Using a design strength of 93.4 MPa (13 540 psi) and assuming a coefficient of variation of 10% for this strength due to manufacturing variations, failure analysis yields $p_f = 0.0074$ with a coefficient of variation of 3.7%. Therefore, based on a 95% confidence level, p_f will be less than 0.00786, i.e., the shaker table will have a reliability of 0.992 under random excitations.

6. Summary

In recent years, random vibration analysis has become available in several commercial finite element software for computing the statistics of Cartesian responses, such as displacements, stresses and strains. Reliability-based design of structures subject to random excitations is desirable for a variety of engineering problems, e.g., fatigue, yield and noise control. However, Cartesian responses are often inadequate for expressing realistic structural design criteria. For ductile material, the von Mises (or distortion energy) theory, which is based on a single equivalent stress response, is preferred. Statistics of the von Mises stress response are computed by means of Monte Carlo simulation (MCS) using the 6×6 variance–covariance matrix of Cartesian stresses at each node computed through random vibration analysis. Consequently, the scatter of the von Mises stresses due to random excitations can be quantified. The probability of failure corresponding to a specified design strength can be estimated enabling the reliability-based design of structures subjected to random excitations. In addition, the required design strength corresponding to a target failure probability can also be estimated yielding useful information for manufacturing. A simple example consisting of a shaker table subject to random vibrations is used to demonstrate the advantages of the proposed techniques.

The combination of random vibration analysis and MCS yields a powerful yet simple method for statistical analysis. Since the von Mises stress can be related to Cartesian stresses in closed-form, the MCS is very efficient. It should be noted that the proposed technique is also applicable to other measures used in failure analysis, such as the stress intensity factor in fracture mechanics, the maximum shear stress for soil mechanics problems, etc. In practice, engineering structures are usually designed with reliabilities ranging from 95% to 99.99% and the required number of simulations is feasible using engineering workstations. On the other hand, advanced probabilistic analysis methods, such as efficient MCS, FORM and SORM, need to solve for the most probable failure point using a nonlinear optimization technique which offsets the computational efficiency of these methods.

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