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STEADY PERIODIC WAVES BIFURCATING FOR FIXED-DEPTH ROTATIONAL FLOWS

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Abstract. We consider steady periodic water waves for rotational flows with a specified fixed depth over a flat bed. We construct a modified height function, which explicitly introduces the mean depth into the rotational water wave problem, and use the Crandall-Rabinowitz local bifurcation theorem to establish the existence of solutions of the resulting problem.

1. Introduction. In the following, we prove the existence of steady periodic water waves for rotational flows with a specified fixed depth over a flat bed. Until recently, most of the rigorous analytical work concerning existence for water waves focused on the irrotational case; see [51] for a survey of much of this work. While irrotational flows may be regarded as being suitable for modelling waves which enter a body of still water [37, 42], more physically complicated and realistic flows generally possess vorticity, for example flows which model wave-current interactions [38, 47] or flows generated by wind-shear [39] (see [8] for a comprehensive discussion on this topic).

In 1802, Gerstner [28] found an explicit solution to the full water wave equations, which determined a periodic travelling wave, and where the resulting underlying flow was rotational with a very specific vorticity distribution (see [5, 33] for a modern treatment of Gerstner's wave, and [6] for an adapted flow which describes edge-waves). In 1934, Dubreil-Jacotin [23] used power series to demonstrate the existence of waves of small amplitude; however, a rigorous proof of the existence of large amplitude waves proved elusive until the breakthrough paper in 2004 by Constantin and Strauss [14] (a noteworthy first approach to addressing this question, using numerical simulations, is given in [22]). Constantin and Strauss used local and global bifurcation methods to establish the existence of a global continuum of solutions to the water wave problem for periodic steady flows with general vorticity. This breakthrough was followed by a wide body of work on flows with vorticity, establishing such properties as stability of solutions

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[16], the symmetry of solutions [10, 11, 12], the analyticity of the surface profile and the streamlines for waves with vorticity [13, 35, 34, 44, 45] and extending the proof of the existence of solutions to more general rotational flows, such as flows which experience surface tension, have critical layers or stagnation points, or flows with discontinuous vorticity [4, 15, 18, 19, 43, 49, 50].

In this paper we wish to determine the local existence of solutions for the water wave problem with a general vorticity distribution over a fixed mean depth d. This differs markedly from the approach in [14], where the mass-flux p_0 was fixed and the existence of small and large amplitude waves was proven for the resulting water wave problem. In order to fix the depth, we use a novel reformulation of the water wave problem for rotational flows whereby we introduce what we call a "modified height function". Fixing the depth rather than the mass-flux is the more quintessentially physical approach, since for any given body of water it is naturally easier to measure the mean depth of the flow than the mass-flux. It is important to note that fixing the mass-flux p_0 does not fix the depth d; in fact recently it was noticed in [40, 41] that, for any fixed p_0, the depth d varies along the bifurcation curve of solutions established in [14]. The mean depth of a mass of water over a flat bed is an inherent property which we can directly determine, whereas the mass-flux is a more opaque and variable characteristic. These considerations are the motivating factors behind the current paper.

2. Governing equations. We consider two-dimensional steady periodic travelling surface waves propagating over water of a fixed depth $d > 0$, where d is fixed. We allow for rotational flow, and the dominant external restoration force is gravity. The presence of vorticity severely complicates the mathematical problem for water waves, but it is a physically far more relevant scenario [8, 14, 37, 42], since rotational flows model wave-current interaction [38, 47], among other complicated phenomena, whereas irrotational flows generally occur only for waves propagating on the surface of a body of water which was initially undisturbed [37, 42]. The two-dimensional nature of water waves is commonly observed for many flows, such as swell propagating on the surface of the ocean, and for such flows we can restrict our analysis to a cross section of the flow which is perpendicular to the wave crest line [37, 42]. Indeed, it was recently shown in [8] that for waves with nonzero constant vorticity the flow must be two-dimensional. We present three equivalent formulations of the gravity water wave problem: the standard physical form of the governing equations, which describes the flow in terms of the velocity field and pressure distribution function, which we then formulate in terms of the stream function, and by applying the Dubreil-Jacotin [23] semi-hodograph transformation to the variables, and introducing a modified height function in the new variables, we finally get the modified height form of the governing equations. It is this form of the equations to which we apply local bifurcation methods to show the existence of waves.

We finally note that the local bifurcation approach bestows an additional benefit, in that following the linearisation it provides detailed information about the entire flow, and can thus be used to study the particle paths in waves of small amplitude, as was done in [20, 25, 32, 48]. Results of this type are sometimes even indicative of the pattern in waves of large amplitude, as was proved for irrotational flows in [7, 17, 31].

2.1. *Standard governing equations.* We use the Cartesian (x, y)-coordinates to formulate the standard governing equations in a frame moving alongside the wave. If the undisturbed mass of water has a depth $d > 0$ and we take $y = 0$ to represent the location of the undisturbed water surface, then the flat bed is located at $y = -d$. Suppose the wave has period $2L$. Then basic physical considerations coupled with periodicity imply that for any fixed time t_0,

$$\int_{-L}^{L} \eta(x, t_0) dx = 0,$$

where $\eta(x, t)$ is the wave surface profile. In the following we take $L = \pi$ for convenience, and in doing so we lose no generality since our analysis is equally applicable to waves of any set period following scaling arguments. If we denote the constant speed of the travelling waves by $c > 0$, then the velocity field takes the form $(u(x - ct, y), v(x - ct, y))$ and the wave surface profile is given by $\eta(x - ct)$. The wave profile η is a free surface since it is a priori undetermined and thus represents an unknown in the problem. Dealing with steady travelling waves enables us simplify matters by transforming to a new reference frame moving alongside the wave, with constant speed $c > 0$, by using the change of coordinates $(x - ct, y) \mapsto (x, y)$. In this frame the flow is steady and we are now working with a time-independent problem.

We denote the closure of the fluid domain by $\overline{D_\eta} = \{(x, y) \in \mathbb{R}^2 : -d \leq y \leq \eta(x)\}$. The Eulerian governing equations of motion then take the following form in the moving frame coordinates. The assumption of incompressibility of the fluid leads to the equation of mass conservation,

$$u_x + v_y = 0, \tag{2.1}$$

and the equations of motion for inviscid fluids experiencing the external restorational force of gravity are given by Euler's equations,

$$\begin{cases} (u - c)u_x + vu_y = -P_x, \\ (u - c)v_x + vv_y = -P_y - g, \end{cases} \tag{2.2}$$

which hold throughout the domain D_η. Here $P = P(x, y)$ is the pressure distribution function and g is the gravitational constant. Since the free boundary of the fluid domain always consists of the same particles we have the kinematic surface condition

$$v = (u - c)\eta_x \qquad \text{on } y = \eta(x). \tag{2.3}$$

At the surface, the dynamic boundary condition which decouples the motion of the air from that of the free surface particles is given by

$$P = P_{atm} \qquad \text{on } y = \eta(x), \tag{2.4}$$

where P_{atm} is the constant atmospheric pressure. On the flat bed we have the kinematic boundary condition

$$v = 0 \qquad \text{on } y = -d, \tag{2.5}$$

which tells us that the rigid bed is impenetrable. The Eulerian governing equations for the gravity water wave problem are embodied by the nonlinear free boundary problem (2.1)–(2.5) [37, 42] along with the equation which describes vorticity

$$\omega = u_y - v_x. \tag{2.6}$$

We make the additional assumption that

$$u < c, \tag{2.7}$$

throughout the fluid. Physically, it is known that the assumption (2.7) is valid for flows which are not near breaking, a claim which is supported by field evidence [37, 42]. These flows do not contain any stagnation points, and the individual fluid particles move with a horizontal velocity which is less than the speed with which the surface wave propagates. Mathematically this assumption is of fundamental importance in permitting us to perform the Dubreil-Jacotin transformation in Section 2.3 below. In the following we consider solutions (u, v, P, η) of (2.1)-(2.6) in the class $C^{2+\alpha}_{per}(D_\eta) \times C^{2+\alpha}_{per}(D_\eta) \times C^{2+\alpha}_{per}(D_\eta) \times C^{3+\alpha}_{per}(\mathbb{R})$ of Hölder continuously differentiable functions, with Hölder exponent $\alpha \in (0,1)$, and where the *per* subscript indicates that our solutions are $2\pi-$periodic. Furthermore, our solutions will have a single crest located at $x = 0$ and troughs located at $x = \pm\pi$, and the condition (2.7) on u and c will hold.

2.2. *The stream function formulation.* We define the stream function ψ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{2.8}$$

and we fix the constant by setting $\psi = 0$ on $y = \eta(x)$. Relations (2.3) and (2.5) tell us that ψ is constant on both boundaries of D_η, and so it follows from integrating (2.8) and using (2.7) that $\psi = -p_0$ on $y = -d$, where

$$p_0 = \int_{-d}^{\eta(x)} (u(x,y) - c)dy < 0.$$

The above expression is usually referred to as the relative mass flux. From writing

$$\psi(x,y) = -p_0 + \int_{-d}^{y} (u(x,s) - c)ds,$$

we can see that ψ is also periodic, with period $2L$. We can deduce from (2.8) that the level sets of the stream function $\psi(x,y)$ are the streamlines of the fluid motion. Mathematically, the assumption (2.7) is vital for a number of reasons, primarily in as far as we are concerned because it enables us to apply the semi-hodograph change of variables which we introduce in Section 2.3 below. We can see by direct calculation, using (2.6) and (2.8), that

$$\Delta\psi = \omega.$$

Setting

$$\tilde{\Gamma}(p) = \int_0^p p_0 \gamma(s) ds,$$

where γ is the vorticity function, which we will define below using relation (2.14), then upon integrating (2.2) and using various other manipulations we derive Bernoulli's law, which states that the expression

$$E := \frac{(u-c)^2 + v^2}{2} + g(y+d) + P - \tilde{\Gamma}(\frac{\psi}{p_0})$$

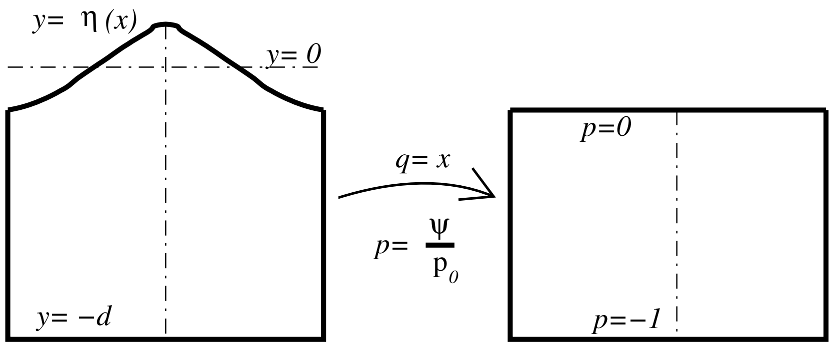
is constant throughout the fluid domain $\overline{D_\eta}$. We can reformulate the governing equations in the moving frame in terms of the stream function as follows:

$$\Delta\psi = \omega \quad \text{in} \quad -d < y < \eta(x), \tag{2.9a}$$
$$|\nabla\psi|^2 + 2g(y+d) = Q \quad \text{on} \quad y = \eta(x), \tag{2.9b}$$
$$\psi = 0 \quad \text{on} \quad y = \eta(x), \tag{2.9c}$$
$$\psi = -p_0 \quad \text{on} \quad y = -d. \tag{2.9d}$$

2.3. *The modified height function formulation.* The next step is to introduce the semi-hodograph transformation of Dubreil-Jacotin [23] given by

$$\begin{cases} q = x, \\ p = \frac{\psi(x,y)}{p_0}. \end{cases} \tag{2.10}$$

It is now obvious that the assumption (2.7) of there being no stagnation points is vital, in order to ensure that the change of variables represents an isomorphism. The semi-hodograph transformation has the advantage of transforming the fluid domain D_η, with the a priori unknown free boundary η, into the fixed semi-infinite rectangular strip $\overline{R} = \mathbb{R} \times [-1, 0]$.



Next we define the modified height function in the (q,p)-variables,

$$h(q,p) = \frac{y}{d} - p. \tag{2.11}$$

Here $y = y(q,p)$ is regarded as a function of the new variables. The nomenclature “modified height function” expresses the fundamental difference between the approach here and the approach taken by Constantin and Strauss [14]. In [14] the mass-flux p_0 was fixed and the existence of small and large amplitude waves was proven for the resulting water wave problem. In this paper we wish to determine the local existence of solutions for the water wave problem over a fixed depth d. This is the more quintessentially physical approach, since for any given body of water it is naturally easier to measure the mean depth of the flow than the mass-flux. It is important to note that fixing the mass-flux p_0 does not fix the depth d; indeed it was observed in [41] that, for any fixed p_0, the depth d varies along the bifurcation curve of solutions. Additionally, for a wave with a given mass-flux p_0, it is not trivial to directly determine the resulting mean depth d of the flow, as we will see from the relations we establish in Section 7 for flows with constant vorticity. We can directly determine the mean depth of a mass of water over a flat bed whereas the mass-flux is a more variable characteristic for any given flow. Therefore we aim to recast the water wave problem in such a fashion as to allow us to fix the mean depth. This requires a marked difference in approaches from that of [14] where the choice of the height function $h = y$ eliminates the parameter d from the problem. Here we have chosen the particular form of the modified height function (2.11) in order to introduce the depth d into the water wave problem while retaining two very important characteristics, namely

$$\int_{-\pi}^{\pi} h(q,0)dq = 0 \tag{2.12}$$

and

$$h(q,-1) = 0.$$

We note the following relations:

$$\partial_x = \frac{\psi_x}{p_0}\partial_p + \partial_q, \quad \partial_y = \frac{\psi_y}{p_0}\partial_p, \tag{2.13}$$
$$\partial_p = \frac{p_0}{u-c}\partial_y = \frac{p_0}{\psi_y}\partial_y, \quad \partial_q = \partial_x + \frac{v}{u-c}\partial_y = \partial_x - \frac{\psi_x}{\psi_y}\partial_y,$$

and we see from taking the curl of the Euler equations (2.2) that

$$\partial_q \omega = \omega_x + \frac{v}{u-c}\omega_y = 0. \tag{2.14}$$

It follows that the vorticity is a function of p alone; hence $\omega = \gamma(p)$, where γ will be referred to as the vorticity function. We use the above relations to reformulate Bernouilli's condition (2.9b) on the surface in terms of h as

$$\frac{1}{d^2} + h_q^2 + \frac{(h_p+1)^2}{p_0^2}[2gd(h+1) - Q] = 0 \quad \text{on} \quad p = 0.$$

Composing further x and y derivatives in terms of the new variables we can reexpress (2.9a) as

$$\left(\frac{1}{d^2} + h_q^2\right) h_{pp} - 2h_q h_{pq}(h_p+1) + h_{qq}(h_p+1)^2 + \frac{\gamma(p)}{p_0}(h_p+1)^3 = 0.$$

Furthermore, we can see that the condition which excludes stagnation points (2.7) is equivalent to

$$h_p + 1 > 0. \tag{2.15}$$

To summarise, the semi-hodograph change of variables (2.10) transforms the stream function system of equations (2.9a)-(2.9d) on an unknown domain into the following modified height function system of equations. We are seeking a solution $h(q,p) \in C^{3+\alpha}(\overline{R})$ of the equations

$$\left(\frac{1}{d^2}+h_q^2\right)h_{pp}-2h_q(h_p+1)h_{pq}+(h_p+1)^2h_{qq}+\frac{\gamma(p)}{p_0}(h_p+1)^3=0,$$
$$\text{in } -1<p<0, \qquad (2.16a)$$
$$\frac{1}{d^2}+h_q^2+\frac{(h_p+1)^2}{p_0^2}[2gd(h+1)-Q]=0, \quad p=0, \qquad (2.16b)$$
$$h=0, \quad p=-1, \qquad (2.16c)$$

where h is even and $2\pi-$periodic in q, and conditions (2.15)–(2.12) hold.

2.4. *Equivalency of the systems.* It is known that the standard system of governing equations (2.1)–(2.5) is equivalent to the stream function form of the governing equations (2.9); cf. [14] for a detailed proof. What we now propose to show is that the modified height system (2.16) is also equivalent to the stream function system (2.9), and we will achieve this by showing that if we are given a solution $h(q,p)$ of (2.16), for fixed $d>0, p_0<0$, then we can recover a solution $\tilde{\psi}(x,y)$ for the system (2.9).

We see immediately that the free surface is given by $\eta(x)=dh(x,0)$, since $q=x$. For a fixed x in the fluid domain, given $p\in[-1,0]$ we wish to recover $y\in[-d,\eta(x)]$ by solving

$$y=d[h(x,p)+p] \qquad (2.17)$$

for $p\in[-1,0]$. When $p=0$ we get $y=\eta(x)$, and $y=-d$ when $p=-1$. Differentiating, we see that

$$\partial_p(d[h(x,p)+p])=d[h_p+1]>0$$

and so (2.17) defines a homeomorphism from $[-1,0]$ to $[-d,\eta(x)]$. In particular, we can regard p as being a function of the variables x and y. This allows us to define a new function in x and y as

$$\tilde{\psi}(x,y)=p_0p(x,y) \qquad \text{for} \quad x\in\mathbb{R}, -d\le y\le\eta(x). \qquad (2.18)$$

We will show that the function (2.18) solves the system (2.9). It follows directly from (2.11) that

$$\tilde{\psi}(x,-d)=p_0(-1)=-p_0$$

and

$$\tilde{\psi}(x,\eta(x))=p_0(0)=0,$$

and so $\tilde{\psi}$ satisfies the boundary data (2.9c)-(2.9d). Differentiating (2.17) we get

$$y_x=0=d[h_q+h_pp_x+p_x]\Rightarrow p_x=-\frac{h_q}{1+h_p}, \qquad (2.19)$$
$$y_y=1=d[h_pp_y+p_y]\Rightarrow p_y=\frac{1}{d(1+h_p)}, \qquad (2.20)$$

and so from (2.16b), (2.18), (2.19) and (2.20) we get

$$\tilde{\psi}_x = -\frac{p_0 h_q}{h_p+1}, \quad \tilde{\psi}_y = \frac{p_0}{d(h_p+1)},$$

$$\begin{aligned}|\nabla\tilde{\psi}|^2 + 2g(y+d) &= \frac{p_0^2 h_q^2}{(h_p+1)^2} + \frac{p_0^2}{d^2(h_p+1)^2} + 2gd(1+\frac{y}{d}) \\ &= \left(\frac{1}{d^2}+h_q^2\right)\frac{p_0^2}{(h_p+1)^2} + 2gd(1+h+p) = Q \quad \text{ on } y=\eta(x).\end{aligned}$$

This shows that $\tilde{\psi}$ satisfies (2.9b). Finally, by direct calculation we have

$$\Delta\tilde{\psi} = \left(\frac{h_q}{h_p+1}\partial_p - \partial_q\right)\frac{p_0 h_q}{h_p+1} + \left(\frac{1}{d(h_p+1)}\partial_p\right)\frac{p_0}{d(h_p+1)} = \gamma(p).$$

3. The bifurcation setting. The Crandall-Rabinowitz [21] local bifurcation theorem will be used to prove the existence of nontrivial solutions to (2.16):

THEOREM 3.1 (Crandall-Rabinowitz). Let X, Y be Banach spaces and let $\mathcal{F} \in C^k(X \times \mathbb{R}, Y)$ with $k \geq 2$ satisfy:

(1) $\mathcal{F}(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$.
(2) The Fréchet derivative $\mathcal{F}_x(0,\lambda^*)$ is a Fredholm operator of index zero with a one-dimensional kernel:

$$\ker(\mathcal{F}_x(0,\lambda^*)) = \{sx_0 : s \in \mathbb{R}, 0 \neq x_0 \in X\}.$$

(3) The tranversality condition holds:

$$\mathcal{F}_{\lambda x}(0,\lambda^*)[(x_0,1)] \notin \operatorname{range}(\mathcal{F}_x(0,\lambda^*)).$$

Then λ^* is a bifurcation point in the sense that there exists $\epsilon_0 > 0$ and a branch of solutions

$$\{(x,\lambda) = \{(s\chi(s), \Lambda(s)) : s \in \mathbb{R}, |s| < \epsilon_0\} \subset X \times \mathbb{R}\},$$

with $\mathcal{F}(x,\lambda) = 0$, $\Lambda(0) = 0, \chi(0) = x_0$, and the maps

$$s \mapsto \Lambda(s) \in \mathbb{R}, \qquad s \mapsto s\chi(s) \in X,$$

are of class C^{k-1} on $(-\epsilon_0, \epsilon_0)$. Furthermore there exists an open set $U_0 \in X \times \mathbb{R}$ with $(0,\lambda_0) \in U_0$ and

$$\{(x,\lambda) \in U_0 : \mathcal{F}(x,\lambda) = 0, x \neq 0\} = \{(s\chi(s), \Lambda(s)) : 0 < |s| < \epsilon_0\}.$$

We remark that the Crandall-Rabinowitz theorem as stated above does not actually require the X-component to be zero; it applies equally to the point $(\lambda^*, \tilde{x})$: if all three conditions, appropriately adapted, in Theorem 3.1 hold at the point $(\lambda^*, \tilde{x})$, then local bifurcation occurs at this point. We refer to [2] for a detailed discussion of local and global bifurcation theory, including a proof of Theorem 3.1.

The plan for implementing the Crandall-Rabinowitz theorem in the context of the water wave problem (2.16) goes as follows. Firstly, we regard the system (2.16) as an operator $\mathcal{F}(h,\lambda) : X \times \mathbb{R} \to Y$, where the exact form of the Banach spaces X, Y is a delicate matter which we will deal with later on in Section 6. In Section 4 we find the laminar flow solutions $H(p) = H(p,\lambda)$ of the system (2.16), and it turns out that a suitable bifurcation parameter λ is suggested naturally by the structure of the laminar flow solutions. What then remains is to check, for the water wave problem (2.16), whether there exists a value λ^* for which each of the three conditions in Theorem 3.1 hold at the point $(H(p), \lambda^*)$. The local bifurcation curve at this point will then consist of non-laminar water waves, thus proving local existence of water waves. Whether such a value λ^* exists will depend on the form of the vorticity function, as we will see.

4. Laminar flow solutions of (2.16)**.** We wish to find laminar flow solutions of the modified height system (2.16), that is, solutions $H(p)$ which have no $q-$dependence and where the streamlines of the resulting flow are horizontal. Therefore such an $H(p)$ solves

$$\begin{aligned} \frac{H_{pp}}{(H_p+1)^3} &= -\frac{d^2\gamma(p)}{p_0} \text{ in } -1<p<0, \\ (H_p(0)+1)^2 &= \frac{p_0^2}{d^2}[Q-2gd(H(0)+1)]^{-1}, \quad p=0, \\ & \qquad\qquad H=0, \quad p=-1. \end{aligned}$$

If

$$\Gamma(p) = 2\int_0^p \frac{d^2\gamma(s)}{p_0}ds, \quad -1\le p\le 0, \qquad \lambda = \left.\frac{1}{(1+H_p)^2}\right|_{p=0},$$

with

$$\Gamma_{\min} = \min_{p\in[-1,0]} \Gamma(p) \le 0, \tag{4.1}$$

then for $\lambda > -\Gamma_{\min}$ we solve to get

$$\begin{aligned} H(p) &= \int_0^p \frac{ds}{\sqrt{\lambda+\Gamma(s)}} + \frac{1}{2gd}\left[Q - \frac{p_0^2}{d^2}\lambda\right] - (p+1), \quad -1<p\le 0, \\ &= \int_{-1}^p \frac{ds}{\sqrt{\lambda+\Gamma(s)}} - (p+1), \qquad -1<p\le 0, \\ H(-1) = 0 &= \int_0^{-1} \frac{ds}{\sqrt{\lambda+\Gamma(s)}} + \frac{1}{2gd}\left[Q-\frac{p_0^2}{d^2}\lambda\right]. \end{aligned}$$

From

$$Q(\lambda) = 2gd\int_{-1}^0 \frac{ds}{\sqrt{\lambda+\Gamma(s)}} + \frac{p_0^2}{d^2}\lambda > 0,$$

we can determine that Q is a positive, convex function of λ, with minimum occurring at the unique value $\lambda_0 > 0$, where

$$\frac{p_0^2}{gd^3} = \int_{-1}^0 \frac{ds}{(\lambda_0+\Gamma(s))^{\frac{3}{2}}}. \tag{4.2}$$

It follows that $Q(\lambda)$ is monotonically decreasing for $-\Gamma_{\min} < \lambda < \lambda_0$ and monotonically increasing for $\lambda > \lambda_0$. With this in mind we choose λ as our bifurcation parameter. For all $\lambda > -\Gamma_{\min}$ we have $\mathcal{F}(H(p), \lambda) = 0$, where $\mathcal{F}$ is the operator associated to (2.16), and so the first condition in Theorem 3.1 is satisfied. We now need to find whether a value λ^* exists such that the second and third conditions of Theorem 3.1 hold. If so the curve of laminar solutions bifurcates in the sense of Crandall and Rabinowitz, giving us a local curve of nonlaminar solutions.

We will actually identify this bifurcation value λ^* by first carefully examining what it would mean for the final two conditions of Theorem 3.1 to hold for the water wave problem (2.16a)–(2.16c).

5. The Fréchet derivative $\mathcal{F}_x(h,\lambda)$. Checking the second condition of Theorem 3.1 involves calculating the Fréchet derivative $\mathcal{F}_x(H(p),\lambda)$, where $\mathcal{F}$ is the operator defining the water wave problem (2.16). We must check, for some value λ^*, that $\mathcal{F}_x(H(p),\lambda)$ has a one-dimensional kernel and a range with codimension one.

In order to calculate the Fréchet derivative we look at the linearisation of the problem. We look for solutions of (2.16) of the form $h(q,p) = H(p;\lambda) + \epsilon m(q,p)$, where $m \in C^{3+\alpha}_{per}(\overline{R})$ is even in q, and study the equations with ϵ at the first order. The resulting equations look like

$$\frac{1}{d^2} m_{pp} + (H_p+1)^2 m_{qq} + \frac{\gamma(p)}{p_0} 3(H_p+1)^2 m_p = 0, \qquad (q,p) \in D, \tag{5.1a}$$

$$2\frac{(H_p+1)}{p_0^2} m_p[2gd(H+1) - Q] + \frac{(H_p+1)^2}{p_0^2} 2gdm = 0, \qquad p = 0, \tag{5.1b}$$

$$m = 0 \qquad p = -1. \tag{5.1c}$$

Setting $a(p;\lambda) = \frac{1}{H_p+1} = \sqrt{\lambda + \Gamma(p)}$, for $\lambda > -\Gamma_{\min}$, we have $a_p = \frac{d^2}{p_0}\gamma(p)a^{-1}$, and so we rewrite (5.1) as

$$(a^3 m_p)_p + d^2 a m_{qq} = 0, \qquad (q,p) \in D, \tag{5.2a}$$

$$a^3 m_p = \frac{gd^3}{p_0^2} m, \qquad p = 0, \tag{5.2b}$$

$$m = 0 \qquad p = -1. \tag{5.2c}$$

From standard Fourier analysis [9, 24] we can assume that the even function m has the Fourier series representation

$$m(q,p) = \sum_{k=0}^{\infty} m_k(p)\cos(kq) \in C^2_{per}(\overline{R}), \tag{5.3}$$

with $C^{3+\alpha}[-1,0]$ coefficients

$$m_0(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} m(q,p)dq, \quad m_k(p) = \frac{1}{\pi}\int_{-\pi}^{\pi} m(q,p)\cos(kq)dq, \ k \geq 1.$$

and so m is a solution to (5.2) if and only if for each value k the function $m_k(p)$ solves the following Sturm-Liouville problem:

$$(a^3 m_p)_p = k^2 d^2 a m, \quad -1 < p < 0, \tag{5.4a}$$

$$a^3 m_p = \frac{gd^3}{p_0^2} m, \qquad p = 0, \tag{5.4b}$$

$$m = 0 \qquad p = -1, \tag{5.4c}$$

where the $q-$dependency of m is assured if $m_k \neq 0$ for some $k \geq 1$. Since we wish to find solutions $m(p) \in C^{3+\alpha}[-1,0]$ of (5.4) which have period 2π, we deal only with the case $k = 1$ in (5.4). Such a solution m would represent a $2\pi-$periodic solution of the linearisation of the water wave problem (2.16) with $\mathcal{F}_x(H(p),\lambda)m = 0$.

5.1. *The variational approach to solving* (5.4). We prove the existence of solutions to the Sturm-Liouville problem (5.4) by recasting it in the variational setting. Supposing we have a solution m of (5.4), if we multiply equation (5.4a) by m and integrate we get

$$\int_{-1}^{0}(a^3 m_p)_p m dp = \left[a^3 m_p m\right]_{-1}^{0} - \int_{-1}^{0} a^3 m_p^2 dp = \int_{-1}^{0} d^2 a m^2 dp,$$

$$\left[\frac{gd^3}{p_0^2}m^2(0)\right] - \int_{-1}^{0} a^3 m_p^2 dp = \int_{-1}^{0} d^2 a m^2 dp,$$

$$\frac{-gd^3 m^2(0) + p_0^2 \int_{-1}^{0} a^3 m_p^2 dp}{p_0^2 d^2 \int_{-1}^{0} a m^2 dp} = -1.$$

We now associate to (5.4) the minimisation problem

$$\mu(\lambda) = \inf_{\phi\in H^1(-1,0),\ \phi(-1)=0,\ \phi\not\equiv 0} \mathbb{F}(\phi,\lambda), \tag{5.5}$$

$$\text{with}\quad \mathbb{F}(\phi,\lambda) = \frac{-gd^3\phi^2(0) + p_0^2\int_{-1}^{0} a^3\phi_p^2 dp}{p_0^2 d^2 \int_{-1}^{0} a\phi^2 dp}.$$

Here the Hilbert space $H^1(-1,0)$ is the standard Sobolev space of square summable functions on $[-1,0]$ whose first derivative is also square summable [26]. We wish to show that $\mu(\lambda^*) = -1$ for some value λ^* and that the value $\mu(\lambda^*) = \mathbb{F}(m,\lambda^*) = -1$ is attained by some $m \in C^{3+\alpha}[-1,0]$. It then follows [14] that m is also a solution of the corresponding Sturm-Liouville problem (5.4).

We begin by showing that the variational problem is well-posed, that is, $\mu(\lambda) > -\infty$. If we denote $\epsilon(\lambda) = \inf_{p\in[-1,0]} a(p,\lambda) > 0$, then we have

$$\begin{aligned}2gd^3\phi^2(0) &= 2gd^3\int_{-1}^{0}(\phi^2)_p dp = 4gd^3\int_{-1}^{0}\phi\phi_p dp\\ &\leq p_0^2\int_{-1}^{0}\epsilon^3\phi_p^2 dp + \int_{-1}^{0} 4\frac{g^2d^6}{\epsilon^3 p_0^2}\phi^2 dp \leq p_0^2\int_{-1}^{0} a^3\phi_p^2 dp + 4\frac{g^2d^6}{\epsilon^4 p_0^2}\int_{-1}^{0} a\phi^2 dp,\end{aligned}$$

and hence we have the strict inequality

$$\mu(\lambda) = \frac{-gd^3\phi^2(0) + p_0^2\int_{-1}^{0} a^3\phi_p^2 dp}{p_0^2 d^2\int_{-1}^{0} a\phi^2 dp} > -C, \quad \text{for } C = (2g)^2\left(\frac{d}{\epsilon p_0}\right)^4 > 0.$$

5.1.1. *The limit* $\mu(\lambda)$ *is attained by* $\phi = M$. We now show that the limit in (5.5) is always attained, that is, $\mu(\lambda) = \mathbb{F}(M,\lambda)$ for some $M \in H^1(-1,0)$. Let ϕ_n be a minimising sequence satisfying $\lim_{n\to\infty}\mathbb{F}(\phi_n) \to \mu(\lambda)$. Since we can see from the definition of (5.5) that $\mathbb{F}(t\phi,\lambda) = \mathbb{F}(\phi,\lambda)$ for any $t \neq 0$, we can normalise the sequence $\{\phi_n\}$ by setting

$p_0^2 d^2 \int_{-1}^0 a\phi_n^2 dp = 1$ for each n. We infer that

$$\mathbb{F}(\phi_n) = -gd^3\phi_n^2(0) + p_0^2 \int_{-1}^0 a^3(\partial_p \phi_n)^2 dp$$
$$\geq \frac{p_0^2}{2}\int_{-1}^0 a^3(\partial_p\phi_n)^2 dp - \frac{C}{2} \geq \frac{\epsilon(\lambda)^3}{2}\int_{-1}^0 (\partial_p\phi_n)^2 dp - \frac{C}{2},$$

and since $\mathbb{F}(\phi_n,\lambda) \to \mu(\lambda)$ it follows that the sequence $\left\{\int_{-1}^0 (\partial_p\phi_n)^2 dp\right\}_{n\geq 1}$ is bounded. Furthermore since

$$\int_{-1}^0 \phi_n^2 dp \leq \frac{1}{\epsilon(\lambda)}\int_{-1}^0 a\phi_n^2 dp = \frac{1}{\epsilon(p_0 d)^2},$$

it follows that $\{\phi_n\}_{n\geq 1}$ is bounded in $H^1(-1,0)$. We know [26] that $\{\phi_n\}_{n\geq 1}$ must have a weakly convergent subsequence $\{\phi_{n_k}\}$ with limit $M \in H^1(p_0,0)$. Since $\partial_p\phi_{n_k} \rightharpoonup M_p$ in $L^2(-1,0)$ and $\phi_{n_k}(-1)=0$ it follows that

$$\phi_{n_k}(p) = \int_{-1}^p \partial_p\phi_{n_k}(s)ds \to \int_{-1}^p M_p(s)ds = M(p) \quad \text{for each } \ p\in[-1,0].$$

Showing the strong pointwise convergence of the $p-$derivatives of the sequence is more tricky. It follows from

$$\int_{-1}^0 a^3(\partial_p\phi_{n_k})^2 dp - \int_{-1}^0 a^3 M_p^2 dp$$
$$= \int_{-1}^0 a^3(\partial_p\phi_{n_k} - M_p)^2 dp + 2\int_{-1}^0 a^3(\partial_p\phi_{n_k})M_p dp - 2\int_{-1}^0 a^3 M_p^2 dp, \tag{5.6}$$

where the first term in (5.6) is positive, and the last two converge to zero by weak convergence, that

$$-gM^2(0) + \int_{-1}^0 a^3 M_p^2 dp \leq \liminf_{k\to\infty}\{-g\phi_{n_k}^2(0) + \int_{-1}^0 a^3(\partial_p\phi_{n_k})^2 dp\}.$$

Since the sequence $\{\phi_{n_k}\}$ is minimising for $\mathbb{F}(\cdot,\lambda)$, it follows that the infimum $\mu(\lambda)$ is in fact a minimum which is attained by $M\in H^1(-1,0)$.

5.1.2. *M is in $C^{3,\alpha}[-1,0]$.* We now show that M is in $C^{3,\alpha}[-1,0]$. Firstly, as a minimum value, M satisfies the Euler-Lagrange equation [29], where

$$\left.\frac{d}{d\epsilon}\mathbb{F}(M+\epsilon\phi,\lambda)\right|_{\epsilon=0} = 0 \tag{5.7}$$

for every $\phi\in H^1(-1,0)$ with $\phi(-1)=0$. We know that $\mathbb{F}(M,\lambda)=\mu(\lambda)$, and furthermore $\int_{-1}^0 aM^2 dp = 1$ follows from the renormalisation procedure on ϕ_{n_k} together with the dominated convergence theorem. This enables us to express relation (5.7) in the form

$$-gd^3\phi(0)M(0) + p_0^2\int_{-1}^0 a^3\phi_p(p)M_p(p)dp = \mu(\lambda)d^2p_0^2\int_{-1}^0 a\phi(p)M(p)dp. \tag{5.8}$$

Choosing ϕ to be smooth and with compact support, this implies that

$$(a^3M_p)_p = -\mu d^2 aM \quad \text{ in } H^{-1}(-1,0). \tag{5.9}$$

Since $a\in C^{2,\alpha}[-1,0]\subset H^2(-1,0)$ it follows that $aM\in H^1(-1,0)$ on the right-hand side of (5.9). Therefore $a^3M_p \in H^2(-1,0)$ on the left-hand side of (5.9) and consequently

$M_p \in H^2(-1,0)$ implies that $M \in H^3(-1,0) \subset C^2[-1,0]$. We deduce that (5.9) holds classically, and the fact that $a \in C^{2,\alpha}[-1,0]$ allows us to differentiate both sides of (5.9) once more, and we see that $M \in C^{3,\alpha}[-1,0]$. If we multiply (5.9) by any $\phi \in H^1(-1,0)$ with $\phi(-1) = 0$ and integrate we get

$$-a^3(0)M_p(0)\phi_p(0) + \int_{-1}^{0} a^3(s)M_p(s)\phi_p(s)ds = \mu d^2 \int_{-1}^{0} aM,$$

and choosing $\phi(p) = p + 1$ above and in (5.8) we get

$$a^3 M_p = gM \qquad \text{on } p = 0.$$

Putting this all together, the minimiser $M \in C^{3,\alpha}[-1,0]$ of the variational problem (5.5) is a classical solution of the weighted Sturm-Liouville problem

$$(a^3 M_p)_p = -\mu d^2 aM \qquad \text{for } s \in (-1,0), \tag{5.10a}$$

$$a^3 M_p = \frac{gd^3}{p_0^2} M \qquad \text{on } p = 0, \tag{5.10b}$$

$$M = 0 \qquad \text{on } p = -1. \tag{5.10c}$$

5.2. *The groundstate μ dependence on λ.* We will now prove the real-analytic dependence of the groundstate $\mu(\lambda)$ on the parameter λ, and the monotonicity of the groundstate when $\mu(\lambda) < 0$. Therefore, any value λ for which $\mu(\lambda) = -1$ is unique. We then show that for any vorticity function γ we can find λ such that $\mu(\lambda) > -1$. It follows that a sufficient condition for bifurcation to occur is to prove that $\mu(\lambda) < -1$ for some value of λ. We finish this section with a condition on the vorticity function γ which ensures that $\mu(\lambda) < -1$ for some λ, thereby ensuring that the linearised system (5.4) has a solution, and also showing that, for sufficiently negative constant vorticity, no such solution exists.

5.2.1. *Analyticity of $\lambda \mapsto \mu(\lambda)$.* We transform the weighted Sturm-Liouville problem (5.10) into an equivalent standard Sturm-Liouville problem as follows. We introduce the new variable s by the $C^{3,\alpha}-$map

$$p \mapsto s := g(p) = \frac{1}{\mathcal{I}} \int_{-1}^{p} \frac{d\tau}{a(\tau,\lambda)}, \tag{5.11}$$

where $\mathcal{I}(\lambda) = \int_{-1}^{0} \frac{d\tau}{a(\tau,\lambda)}$ is a real-analytic function in λ. The function $s = g(p)$ is a diffeomorphism from $[-1,0]$ to $[-1,0]$, and we denote the inverse diffeomorphism by $f(s)$. The $C^{3,\alpha}$ map $f(s)$ satisfies

$$\begin{aligned} f'(s) &= \mathcal{I}(\lambda)a(p), \quad f''(s) = \mathcal{I}^2(\lambda)a_p(p)a(p), \\ f'''(s) &= \mathcal{I}^3(\lambda)\left(a_{pp}(p)a^2(p) + a_p^2(p)a(p)\right). \end{aligned}$$

If we define the $C^{2,\alpha}[-1,0]$ function $\theta(s)$ by

$$\theta(s) = f'(s)M(f(s)), \tag{5.12}$$

then we can calculate

$$\theta'(s) = f''(s)M(f(s)) + f'(s)^2 M'(f(s)),$$
$$\theta''(s) = f'''(s)M(f(s)) + 3f'(s)f''(s)M'(f(s)) + f'(s)^3 M''(f(s)),$$

and we can transform the weighted Sturm-Liouville problem (5.10) into the equivalent problem

$$\theta''(s) - A(s,\lambda)\theta = -d^2\mu(\lambda)\mathcal{I}^2(\lambda)\theta(s) \qquad \text{for} \quad s \in (-1,0), \tag{5.13a}$$

$$\theta'(s) = \beta(\lambda)\theta(s) \qquad \text{on} \quad s = 0, \tag{5.13b}$$

$$\theta = 0 \qquad \text{on} \quad s = -1. \tag{5.13c}$$

Here we have

$$A(s,\lambda) = \frac{f'''(s)}{f'(s)} = [a_{pp}(p,\lambda)a(p,\lambda) + a_p^2(p,\lambda)]\mathcal{I}^2(\lambda),$$

$$\beta(\lambda) = \frac{f''(0)}{f'(0)} + g\frac{d^3\mathcal{I}^3(\lambda)}{p_0^2 f'(0)^2} = \mathcal{I}(\lambda)\left[-\frac{d^2\gamma(0)}{\sqrt{\lambda}p_0} + \frac{gd^3}{p_0^2\lambda}\right].$$

Analogous to (5.5) we associate to (5.13) the minimisation problem

$$\inf_{\theta \in H^1(-1,0), \theta(-1)=0, \theta\not\equiv 0} \left\{ \frac{-\beta\theta^2(0) + \int_{-1}^0 (\theta')^2 ds + \int_{-1}^0 A(s)\theta^2 ds}{\int_{-1}^0 \theta^2 ds} \right\},$$

and we can easily see from (5.12) that the above expression equals $d^2\mathcal{I}^2(\lambda)\mathbb{F}(M,\lambda)$. Since $\beta(\lambda)$ could be zero for values of λ if $\gamma(0) < 0$, we perform a further change of variables

$$w(s) = \theta(s)e^{-\beta s}, \qquad s \in [-1,0],$$

which transforms (5.13) into the equivalent problem

$$-w''(s) - 2\beta(\lambda)w'(s) + [A(s,\lambda) - \beta^2(s)]w(s) = d^2\mu(\lambda)\mathcal{I}^2(\lambda)w(s) \quad \text{for } s \in (-1,0), \tag{5.14a}$$

$$w'(s) = 0 \quad \text{on } s = 0, \tag{5.14b}$$

$$w(s) = 0 \quad \text{on } s = -1. \tag{5.14c}$$

The system (5.14) is a standard Sturm-Liouville problem with spectral parameter $d^2\mu(\lambda)\mathcal{I}^2(\lambda)$. For $\lambda > 0$ we associate to (5.14) the operator

$$L_\lambda = -\partial_s^2 - 2\beta(\lambda)\partial_s + [A(s,\lambda) - \beta^2(s)], \tag{5.15a}$$

$$D(L_\lambda) = \{w \in H^2(-1,0) : w(-1) = 0, w'(0) = 0\}, \tag{5.15b}$$

whose domain is independent of λ, while to (5.13) we associate the selfadjoint operator

$$S_\lambda = -\partial_s^2 + A(s,\lambda), \tag{5.16a}$$

$$D(S_\lambda) = \{w \in H^2(-1,0) : w(-1) = 0, w'(0) = \beta(\lambda)w(0)\}, \tag{5.16b}$$

whose domain is dependent on λ due to the presence of β. The spectral theory of self-adjoint operators is well known [46], and we will exploit the best features of the operators L_λ and S_λ, namely independence of domain and selfadjointness respectively, by using the conjugacy relation

$$L_\lambda = M_\beta S_\lambda M_{-\beta}, \tag{5.17}$$

where M_s is the linear isomorphism of $L^2[-1,0]$ given by

$$(M_s\theta)(s) = \theta(s)e^{-ms}, \quad s \in [-1,0],$$

with inverse $M_s^{-1} = M_{-s}$. The relation (5.17) implies that L_λ and S_λ have the same eigenvalues. Establishing the following relations,

$$|w(s)| = \left| \int_{-1}^{s} w'(\tau) d\tau \right| \leq \|w'\|_{L^\infty[-1,0]}, \quad s \in [-1, 0],$$

and

$$\int_{-1}^{0} (w')^2 d\tau = -\int_{-1}^{0} w w'' d\tau \leq \frac{\epsilon^2}{2} \int_{-1}^{0} (w'')^2 d\tau + \frac{1}{2\epsilon^2} \int_{-1}^{0} w^2 d\tau,$$

with

$$\begin{aligned} [w'(s)]^2 = -2 \int_{s}^{0} w' w'' d\tau \leq 2 \int_{-1}^{0} w' w'' d\tau &\leq \epsilon \int_{-1}^{0} (w'')^2 d\tau + \frac{1}{\epsilon^2} \int_{-1}^{0} (w')^2 d\tau \\ &\leq \frac{3}{2} \epsilon \|w''\|^2_{L^2[-1,0]} + \frac{1}{2\epsilon^3} \|w\|^2_{L^2[-1,0]}, \end{aligned}$$

we have

$$\|w\|^2_{L^\infty[-1,0]} \leq \|w'\|^2_{L^\infty[-1,0]} \leq \frac{3}{2} \epsilon \|w''\|^2_{L^2[-1,0]} + \frac{1}{2\epsilon^3} \|w\|^2_{L^2[-1,0]}.$$

Defining the operator $E_\lambda : D(L_\lambda) \to L^2[-1, 0]$ by

$$E_\lambda = L_\lambda + e(\lambda),$$

where $e(\lambda)$ is a constant, then the above inequalities ensure that

$$\|E_\lambda w\|_{L^2[-1,0]} \geq \|w\|_{H^2(-1,0)}, \quad w \in D(L_\lambda), \tag{5.18}$$

for a sufficiently large constant $e(\lambda) > 0$. This implies that E_λ is injective as an operator from the closed Banach subspace $D(L_\lambda)$ of $H^2(-1, 0)$ into $L^2[-1, 0]$. Since $E_\lambda = M_\beta(S_\lambda + e(\lambda))M_{-\beta}$, and the range of the selfadjoint operator $S_\lambda + e(\lambda)$ is dense, it follows that the range of E_λ is dense in $L^2[-1, 0]$. Therefore we will have proven the bijectivity of the operator E_λ if we show that its range is closed. In order to do this let us suppose that the sequence $x_n \in D(L_\lambda)$ is such that $E_\lambda x_n \to y$ in $L^2[-1, 0]$. Then (5.18) ensures that the x_n form a Cauchy sequence, with limit x say. Then by the completeness of the Banach spaces we have $E_\lambda x_n \to E_\lambda x = y$. This shows that the range of E_λ is closed and therefore the operator $E_\lambda = L_\lambda + e(\lambda)$ is invertible. Now E_λ^{-1} is an operator from $L^2[-1, 0] \to H^2(-1, 0)$ and the compactness of the embedding $H^2(-1, 0) \subset L^2[-1, 0]$ ensures that E_λ^{-1} is a compact operator from $L^2[-1, 0] \to L^2[-1, 0]$. Since $E_\lambda^{-1} = M_\beta(S_\lambda + e(\lambda))^{-1}M_{-\beta}$ we have that $(S_\lambda + e(\lambda))^{-1}$ is a compact selfadjoint operator.

The spectral theory of selfadjoint compact operators is well known [46]: there is a discrete collection of positive eigenvalues whose only accumulation point is zero, and the eigenfunctions form an orthonormal set in $L^2[-1, 0]$. Now, the conjugacy relation (5.17) implies that $(S_\lambda + e(\lambda))^{-1}$ and E_λ^{-1} must have the same spectrum (although the associated eigenfunctions may be different). Therefore the spectrum of E_λ consists precisely of the inverses of these eigenvalues, and furthermore the spectrum of L_λ is obtained by subtracting $e(\lambda)$ from the spectrum of E_λ. Therefore the spectrum of L_λ consists of isolated eigenvalues with the lowest one being (by (5.14a))

$$\nu(\lambda) = d^2 \mu(\lambda) \mathcal{I}^2(\lambda).$$

We notice that since the differential equation in (5.14a) is linear and $w(-1) = 0$, the eigenfunctions are therefore uniquely determined by the value of $w'(-1)$: the eigenspace of each eigenvalue is one-dimensional.

Since $\mathcal{I}(\lambda)$ is a nonzero real-analytic function of λ, we will prove that the mapping $\lambda \mapsto \mu(\lambda)$ is real-analytic for $\lambda > -\Gamma_{\min}$ by showing that $\nu(\lambda)$ is a real-analytic function in λ. Fixing $\lambda_0 > -\Gamma_{\min} \geq 0$, we let $\rho(L_\lambda) \subset \mathbb{C}$ denote the resolvent set of the operator L_λ on $L^2[-1,0]$.

Lemma 5.1. Let $O \subset \mathbb{C}$ be a bounded open set such that $\overline{O} \subset \rho(L_{\lambda_0})$, that is, $(z - L_{\lambda_0})^{-1}$ exists for $z \in O$. Then for $\lambda \in \mathbb{R}$ with $|\lambda - \lambda_0|$ sufficiently small we have $O \subset \rho(L_\lambda)$, i.e. $(z - L_\lambda)^{-1}$ also exists.

Proof. For $z \in O$ we have

$$z - L_\lambda = z - L_{\lambda_0} + L_{\lambda_0} - L_\lambda = \left(1 - (L_\lambda - L_{\lambda_0})(z - L_{\lambda_0})^{-1}\right)(z - L_{\lambda_0})$$

and if $\epsilon > 0$ is small enough so that

$$\|(L_\lambda - L_{\lambda_0})(z - L_{\lambda_0})^{-1}\|_{\mathcal{B}(L^2[-1,0])} \leq C < 1$$

for $|\lambda_0 - \lambda| < \epsilon$, then $(z - L_\lambda)^{-1}$ exists and is given by the formula

$$(z - L_\lambda)^{-1} = (z - L_{\lambda_0})^{-1} \sum_{k=0}^{\infty} \left((L_{\lambda_0} - L_\lambda)(z - L_{\lambda_0})^{-1}\right)^k,$$

where the right-hand side converges absolutely in $\mathcal{B}(L^2[-1,0])$. □

Corollary 5.2. For $z \in O \subset \rho(L_{\lambda_0})$ the map $\lambda \mapsto (z - L_\lambda)^{-1}$ is real-analytic with values in $\mathcal{B}(L^2[-1,0])$.

We now use the fact that $\nu(\lambda_0)$ is an isolated eigenvalue of L_{λ_0}, together with the previous lemma, to choose $r > 0$ such that $\nu(\lambda_0)$ is the only eigenvalue of L_{λ_0} enclosed by the contour

$$C_r = \{z \in \mathbb{C} : |z - \nu(\lambda_0)| = r\},$$

and $\epsilon > 0$ such that

$$C_r \subset \rho(L_\lambda) \quad \text{for all } |\lambda - \lambda_0| < \epsilon \text{ with } \lambda \in \mathbb{R}.$$

We use the selfadjointness of S_λ to define the Riesz projections [36]

$$P_\lambda = \frac{1}{2\pi} \oint_{C_r} (z - S_\lambda)^{-1} dz, \quad \lambda \in \mathbb{R},\ |\lambda - \lambda_0| < \epsilon.$$

It is well known that each P_λ is the orthogonal projection onto the direct sum of the orthogonal spaces $\ker(S_\lambda - \zeta_k)$, where the ζ_k are the finite number of eigenvalues of S_λ (and hence also L_λ) contained inside C_r. Since $\|P_\lambda - P_{\lambda_0}\|_{\mathcal{B}(L^2[-1,0])} \to 0$ as $\lambda \to \lambda_0$, and we know that the range of P_{λ_0} is, by the choice of C_r, the one-dimensional eigenspace $\ker(S_{\lambda_0} - \nu(\lambda_0))$, it follows that the range of P_λ is also one-dimensional for $|\lambda - \lambda_0| < \epsilon^*$, say. It follows that the operator S_λ has exactly one eigenvalue $\zeta(\lambda)$ in C_r for $|\lambda - \lambda_0| < \epsilon^*$, and P_λ is the orthogonal projection onto $\ker(S_\lambda - \zeta(\lambda))$.

We now choose $w \in D(L_\lambda)$ with $\|M_{-\beta(\lambda_0)}w\|_{L^2[-1,0]} = 1$ and $L_{\lambda_0}w = \nu(\lambda_0)w$. Then $\theta_\lambda = M_{-\beta(\lambda)}w$ is real-analytic in $\lambda > 0$, and $S_{\lambda_0}\theta_{\lambda_0} = \nu(\lambda_0)\theta_{\lambda_0}$. Since P_λ is the orthogonal projection onto the one-dimensional space $\ker(S_\lambda - \zeta(\lambda))$ we have

$$S_\lambda P_\lambda \theta_\lambda = \zeta(\lambda) P_\lambda \theta_\lambda \quad \text{for } |\lambda - \lambda_0| < \epsilon^*,\ \lambda \in \mathbb{R}.$$

Since the set $A_r = \{z : 0 < |z - \zeta(\lambda)| < r\} \subset \rho(L_\lambda) = \rho(S_\lambda)$ we have

$$(z - S_\lambda)^{-1} P_\lambda \theta_\lambda = \frac{1}{z - \zeta(\lambda)} P_\lambda \theta_\lambda, \quad z \in A_r,$$

from which it follows that

$$\int_{-1}^{0} \theta_\lambda \left((z - S_\lambda)^{-1} P_\lambda \theta_\lambda\right) ds = \frac{1}{z - \zeta(\lambda)} \int_{-1}^{0} \theta_\lambda \left(P_\lambda \theta_\lambda\right) ds.$$

Now for $|\lambda - \lambda_0| < \epsilon^*$ the terms θ_λ, $(z - S_\lambda)^{-1}$ and P_λ are each in turn analytic in λ. Therefore both integral terms are analytic in λ and since the second integral term is nonzero (because $\int_{-1}^{0} \theta_\lambda (P_\lambda \theta_\lambda)\, ds \to 1$ as $\lambda \to \lambda_0$, since $P_\lambda \theta_\lambda \to P_{\lambda_0}\theta_{\lambda_0} = \theta_{\lambda_0}$ and $\nu(\lambda_0) = \zeta(\lambda_0)$) it follows that $\lambda \mapsto \zeta(\lambda)$ is analytic near λ_0, with $\nu(\lambda_0) = \zeta(\lambda_0)$. The previous argument can be applied to each eigenvalue of L_{λ_0}, with the eigenvalues ordered at a fixed λ with $\nu(\lambda)$ the lowest; therefore we have $\nu(\lambda) = \zeta(\lambda)$ for each λ. This proves that $\lambda \mapsto \mu(\lambda)$ is real-analytic for $\lambda > -\Gamma_{\min}$.

5.2.2. *Monotonicity of the ground state.* Since we have shown that the mapping $\lambda \mapsto \mu(\lambda)$ is real-analytic, it follows from the smooth dependence of solutions on parameters that the mapping $\lambda \mapsto M(\cdot, \lambda)$ is smooth, since $M(p, \lambda) = \phi(p, \lambda, \mu(\lambda))$ is the unique solution of the linear differential equation

$$(a^3 \phi_p)_p = -\mu a \phi \qquad \text{in } (-1, 0),$$

with initial data $\phi(0) = 1, \phi'(0) = \frac{g}{a^3(0)}$. If $\dot{a}$ is the derivative of a with respect to λ, then we have the relations

$$\dot{a} = \frac{\partial a}{\partial \lambda} = \frac{1}{2a}, \quad \dot{a}_p = -\frac{a_p}{2a^2} = -\frac{d^2\gamma(p)}{2p_0 a^3}.$$

Differentiating equations (5.10) we get

$$(a^3 \dot{M}_p)_p + \frac{3}{2}(a M_p)_p = -\dot{\mu} d^2 a M - \mu \frac{d^2 M}{2a} - \mu d^2 a \dot{M}, \qquad p \in (-1, 0), \quad (5.19a)$$

$$\frac{3}{2} a M_p + a^3 \dot{M}_p = \frac{g d^3}{p_0^2} \dot{M}, \qquad p = 0, \quad (5.19b)$$

$$\dot{M} = 0, \qquad p = -1. \quad (5.19c)$$

Multiplying the above equation by M and (5.10a)–(5.10b) by $\dot{M}$, integrating both equations on $(-1, 0)$ and subtracting the outcomes we obtain

$$\dot{\mu} \int_{-1}^{0} a M^2 dp = -\mu \int_{-1}^{0} \frac{M^2}{2a} dp + \frac{3}{2d^2} \int_{-1}^{0} a M_p^2 dp.$$

Proposition 5.3. The map $\lambda \mapsto \mu(\lambda)$ is increasing on any interval where it is negative, and therefore the solution λ^* to $\mu(\lambda) = -1$, if it exists, is unique.

5.2.3. *Given any γ,$\mu(\lambda) > -1$ for λ sufficiently large.* For $\lambda + \Gamma_{\min} > g\frac{d^2}{p_0^2}$ we have $a > \sqrt{g}\frac{d}{|p_0|}$, and also

$$M^2(0)gd^3 \leq \int_{-1}^{0} \frac{g^2d^4}{p_0^4a^4}(d^2p_0^2)aM^2(s)ds + \int_{-1}^{0} p_0^2a^3M_p(s)ds,$$
$$\mathbb{F}(M,\lambda) = \mu(\lambda) > -C, \text{ where } 0 < C = \frac{g^2d^4}{p_0^4a^4} < 1. \tag{5.20}$$

Therefore for such a λ we have $\mu(\lambda) > -1$. It follows from Proposition 5.3 that

Corollary 5.4. A solution λ^* to $\mu(\lambda) = -1$ exists if and only if $\lim_{\lambda\downarrow\Gamma_{\min}} \mu(\lambda) < -1$.

5.2.4. *Remark: for some $\gamma < 0$ we always have $\mu(\lambda) > -1$.* We now give an example of a flow where $\mu(\lambda) > -1$ for all λ, and so bifurcation does not occur. Let the constant negative vorticity $\gamma < 0$ be such that

$$|\gamma| > \frac{g^{\frac{2}{3}}}{2|p_0|^{\frac{1}{3}}} + \frac{25g^2}{8|p_0|^3},$$

where p_0 is the mass-flux of the resulting flow. We have the following expressions:

$$\Gamma(p) = \frac{2d^2\gamma}{p_0}p, \lambda > -\Gamma_{\min} = \frac{2d^2\gamma}{p_0}, a = \sqrt{\lambda + \Gamma(p)} > \left(\frac{2d^2\gamma}{p_0}\right)^{\frac{1}{2}}(1+p)^{\frac{1}{2}},$$

and we have

$$\begin{aligned}
&\int_{-1}^{0} a^3\phi_p^2dp + d^2\int_{-1}^{0} a\phi^2dp \\
&\quad > \frac{gd^3}{p_0^2}\left(\frac{p_0}{2d^2\gamma}\right)^{\frac{3}{2}}\int_{-1}^{0} a^3\phi_p^2dp + \frac{5gd^3}{2p_0^2}\left(\frac{p_0}{2d^2\gamma}\right)^{\frac{1}{2}}\int_{-1}^{0} a\phi^2dp \\
&\quad > \frac{gd^3}{p_0^2}\int_{-1}^{0}(1+p)^{\frac{3}{2}}\phi_p^2dp + \frac{5gd^3}{2p_0^2}\int_{-1}^{0}(1+p)^{\frac{1}{2}}\phi^2dp \\
&\quad > \frac{gd^3}{p_0^2}\left\{\int_{-1}^{0}(1+p)^{\frac{3}{2}}(\phi_p^2+\phi^2)dp + \frac{3}{2}\int_{-1}^{0}(1+p)^{\frac{1}{2}}\phi^2dp\right\} \\
&\quad > \frac{gd^3}{p_0^2}\left\{\int_{-1}^{0}(1+p)^{\frac{3}{2}}2\phi\phi_pdp + \frac{3}{2}\int_{-1}^{0}(1+p)^{\frac{1}{2}}\phi^2dp\right\} = \frac{gd^3}{p_0^2}\phi^2(0).
\end{aligned}$$

Therefore $\mu(\lambda) > -1$ for all λ and bifurcation cannot occur for these flows. We will discuss in more detail the existence of bifurcation curves for various constant vorticities within the context of dispersion relations in Section 7.

5.3. *Existence of nonlinear waves of small amplitude.* For vorticity functions γ which satisfy condition (5.21), the following proposition, together with Corollary 5.4, proves the existence of values λ^* such that $\mu(\lambda^*) = -1$, thereby proving the existence of solutions to the linearisation (5.4) of the water wave problem (2.16).

Proposition 5.5. Suppose that

$$\frac{\sqrt{2}}{3}\gamma_\infty^{\frac{3}{2}}|p_0|^{\frac{1}{2}}|p_1|^{\frac{1}{2}} + \frac{2\sqrt{2}}{5}\gamma_\infty^{\frac{1}{2}}|p_0|^{\frac{3}{2}}|p_1|^{\frac{3}{2}} < g, \tag{5.21}$$

where $\gamma_\infty = \|\gamma\|_{C[-1,0]}$ and $p_1 = \min\{p \in [-1,0] : \Gamma(p) = \Gamma_{\min}\}$, where $\Gamma_{\min}$ is defined in (4.1). Then there exist nontrivial solutions to the linearised problem (5.10).

REMARK 5.6. We note that $p_1 = 0$ for $\gamma \geq 0$ and so in this case we easily see that (5.21) holds.

Proof. Define for $k > \frac{1}{2}$ and $n \geq 2$ the function

$$\phi_n(p) = \begin{cases} 0, & -1 \leq p \leq p_n, \\ (p-p_n)^k, & p_n \leq p \leq 0, \end{cases}$$

where $p_n = (1-\frac{1}{n})p_1 - \frac{1}{n} < 0$. We see that $\phi_n(0) = |p_n|^k$, $\phi_n(p) \to 0$ as $k \to \infty$, and $\phi_n(p) \searrow 0$ as $n \to \infty$. Then

$$\begin{aligned} a(-\Gamma_{\min}, p) &= \sqrt{-\Gamma_{\min} + \Gamma(p)} = \sqrt{-\Gamma(p_1) + \Gamma(p)} \\ &= \sqrt{\tfrac{2d^2}{p_0}\int_{p_1}^{p} \gamma(s)ds} \leq \sqrt{\tfrac{2d^2}{|p_0|}(p-p_1)\gamma_\infty}, \\ & a(-\Gamma_{\min}, p) \leq d\sqrt{\tfrac{2\gamma_\infty}{|p_0|}} \cdot |p-p_1!^{\frac{1}{2}}. \end{aligned}$$

Then if we let $c_1 = d^3(2\gamma_\infty)^{\frac{3}{2}}|p_0|^{\frac{1}{2}}$ and $c_2 = d^3(2\gamma_\infty)^{\frac{1}{2}}|p_0|^{\frac{3}{2}}$ we have

$$\begin{aligned} & p_0^2 \int_{-1}^{0} a^3(\partial_p\phi_n)^2 dp + p_0^2 d^2 \int_{-1}^{0} a(\phi_n)^2 dp \\ &= c_1 k^2 \int_{p_n}^{0} |p-p_1|^{\frac{3}{2}}(p-p_n)^{2k-2}dp + c_2 \int_{p_n}^{0} |p-p_1|^{\frac{1}{2}}(p-p_n)^{2k}dp \\ &= c_1 k^2 \int_{p_n}^{p_1} |p-p_1|^{\frac{3}{2}}(p-p_n)^{2k-2}dp + c_1 k^2 \int_{p_1}^{0} |p-p_1|^{\frac{3}{2}}(p-p_n)^{2k-2}dp \\ &\quad + c_2 \int_{p_n}^{p_1} |p-p_1|^{\frac{1}{2}}(p-p_n)^{2k}dp + c_2 \int_{p_1}^{0} |p-p_1|^{\frac{1}{2}}(p-p_n)^{2k}dp \\ &\leq \frac{3c_1k^2(p_1-p_n)^{2k+\frac{1}{2}}}{(2k-1)(4k+1)} + \frac{c_1k^2|p_n|^{2k+\frac{1}{2}}}{2k+\frac{1}{2}} + \frac{3c_2(p_1-p_n)^{2k+\frac{3}{2}}}{(2k+1)(4k+3)} + \frac{c_2|p_n|^{2k+\frac{3}{2}}}{2k+\frac{3}{2}} \\ &= \phi_n^2(0)\left\{\frac{c_1k^2|p_n|^{\frac{1}{2}}}{2k+\frac{1}{2}} + \frac{c_2|p_n|^{\frac{3}{2}}}{2k+\frac{3}{2}}\right\} \qquad (5.22) \\ &\quad + \phi_n^2(0)\left\{\frac{3c_1k^2(p_1-p_n)^{2k+\frac{1}{2}}}{|p_n|^{2k}(2k-1)(4k+1)} + \frac{3c_2(p_1-p_n)^{2k+\frac{3}{2}}}{|p_n|^{2k}(2k+1)(4k+3)}\right\}. \qquad (5.23) \end{aligned}$$

We can see that $|p_n| \to |p_1|$, $2K + \frac{1}{2} > \frac{3}{2}$, $2k + \frac{3}{2} > \frac{5}{2}$, $\frac{(p_1-p_n)^{2k}}{|p_n|^{2k}} \leq 1$, and so, for some $\epsilon > 0$, we can choose n large enough and k close enough to $\frac{1}{2}$, and using condition (5.21) we have that

$$\left\{\frac{d^3(2\gamma_\infty)^{\frac{3}{2}}|p_0|^{\frac{1}{2}}k^2|p_n|^{\frac{1}{2}}}{2k+\frac{1}{2}} + \frac{d^3(2\gamma_\infty)^{\frac{1}{2}}|p_0|^{\frac{3}{2}}|p_n|^{\frac{3}{2}}}{2k+\frac{3}{2}}\right\} < d^3(g-\epsilon).$$

We can find n large enough that the large bracket in (5.23) is less than ϵ, which implies that the sum of quantities in (5.22) and (5.23) has value less than $\phi_n^2(0)d^3g$. Therefore, for large enough $n \in \mathbb{N}$, we have $\mathbb{F}(-\Gamma_{\min}) < -1$, and so by continuity in λ we have $\mathbb{F}(\phi_n, \lambda) < -1$ for some $\lambda > -\Gamma_{\min}$, and at this λ we have $\mu(\lambda) < -1$. $\square$

6. Local bifurcation setting. We represent the top and the bottom of the closed rectangle $\overline{R}$ by

$$T = \{(q,p) : q \in [-\pi,\pi], p = 0\}, \quad B = \{(q,p) : q \in [-\pi,\pi], p = p_0\},$$

and we define the Banach spaces

$$X = \{w \in C^{3,\alpha}_{per}(\overline{R}) : w = 0 \text{ on } B\}, \quad Y = C^{1,\alpha}_{per}(\overline{R}) \times C^{2,\alpha}_{per}(T),$$

where the subscript *per* represents periodicity and evenness in the $q-$variable. If $H(p,\lambda)$ are the laminar flows, set

$$h(q,p) = H(p,\lambda) + w(q,p) \quad \text{ with } w \in X,$$

and for $\lambda > -\Gamma_{\min}$ the system (2.16) can be expressed in operator form

$$\mathcal{F}(w,\lambda) = 0 \quad \text{ with } w \in X,$$

where $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : X \times (-\Gamma_{\min}, \infty) \to Y$ is given by

$$\begin{aligned}\mathcal{F}_1(w,\lambda) &= \left(\frac{1}{d^2} + w_q^2\right)(H_{pp} + w_{pp}) - 2w_q w_{qp}(H_p + w_p + 1)\\ &\qquad + w_{qq}(H_p + w_p + 1)^2 + \frac{\gamma(p)}{p_0}(H_p + w_p + 1)^3,\\ \mathcal{F}_2(w,\lambda) &= \frac{1}{d^2} + w_q^2 + \frac{(H_p + w_p + 1)^2}{p_0^2}[2gd(H + w + 1) - Q].\end{aligned}$$

We have $\mathcal{F}(0,\lambda) = 0$ for $\lambda > -\Gamma_{\min}$ since H satisfies the equation for laminar flow. The linearised operator $\mathcal{F}_w = (\mathcal{F}_{1w}, \mathcal{F}_{2w})$, formed by taking the Fréchet derivative of $\mathcal{F}$ with respect to w, is given at $w = 0$ by

$$\mathcal{F}_{1w}(0,\lambda) = \frac{1}{d^2}\partial_{pp} + (H_p + 1)^2\partial_q^2 + \frac{3\gamma(p)}{p_0}(H_p + 1)^2\partial_p \quad \text{ in } R, \tag{6.1}$$

$$\mathcal{F}_{2w}(0,\lambda) = 2\left(\frac{gd}{p_0^2}\lambda^{-1} - \frac{\lambda^{\frac{1}{2}}}{d^2}\partial_p\right)\Bigg|_T. \tag{6.2}$$

We see from (5.1) that a solution m to the linear eigenvalue problem (5.10) belongs to the nullspace of $\mathcal{F}_w(0,\lambda)$.

6.1. *The null space of* $\mathcal{F}_w$. We have shown in Section 5 that once (5.21) holds, there is a unique $\lambda^* > -\Gamma_{\min}$ with $\mu(\lambda^*) = -1$. It follows that the null space of $\mathcal{F}_w(0,\lambda^*)$ contains at least one element $m^*(q,p) = M(p)\cos(q)$, where $M \in C^{3,\alpha}[-1,0]$ is the unique eigenfunction of (5.10) corresponding to the eigenvalue $\mu(\lambda^*) = -1$. We now show that the null space is one-dimensional. Suppose $m \in C^{3,\alpha}_{per}(\overline{R})$ belongs to the null space. Then its Fourier coefficients m_k satisfy (5.4), and so $m_1(p)$ is a constant multiple of $M(p)$ while $m_k \equiv 0$ for all $k \geq 2$, for if not we would have

$$\mathbb{F}(m_k,\lambda) = \frac{-gd^3 m_k^2(0) + p_0^2\int_{-1}^0 a^3(\partial_p m_k)^2 dp}{p_0^2 d^2 \int_{-1}^0 a m_k^2 dp} = -k^2 < -1,$$

which contradicts the minimising value of $\mu(\lambda^*) = -1$.

For m_0, using (5.4) with $k = 0$ we get

$$m_0(p) = \frac{gd^3}{p_0^2} m_0(0) \int_{-1}^{p} a^{-3}(\lambda, s) ds,$$

and setting $p = 0$ we get

$$(\dagger)\ m_0(0) = \frac{gd^3}{p_0^2} m_0(0) \int_{-1}^{0} a^{-3}(\lambda, s) ds \Rightarrow \int_{-1}^{0} a^{-3}(\lambda, s) ds = \frac{p_0^2}{gd^3},$$

but this relation holds only for the unique value $\lambda = \lambda_0$ where the function $Q(\lambda)$ attains its minimum; see (4.2). We use the monotonicity of the function $\lambda \mapsto \mu(\lambda)$ to prove that $\lambda^* < \lambda_0$. This in turn proves that the null space is one-dimensional since it follows that $(\dagger)$ cannot hold.

First we note that $\mu(\lambda^*) = -1$. Now, if we choose $\psi(p) = \int_{-1}^{p} a^{-3}(s, \lambda_0) ds$, then $\psi_p = a^{-3}(p, \lambda_0)$ and $\psi(0) = \frac{p_0^2}{gd^3} = \int_{-1}^{0} a^{-3}(s, \lambda) ds$. It follows that

$$\mathbb{F}(\psi, \lambda) = \frac{-\frac{p_0^4}{gd^3} + p_0^2 \int_{-1}^{0} a^{-3}(\lambda_0, s) ds}{p_0^2 d^2 \int_{-1}^{0} a\phi(p)^2 dp} = 0. \tag{6.3}$$

Therefore $\mu(\lambda_0) \leq 0$ by the minimising property of $\mu(\lambda)$. However, for any $\phi \in H^1(-1, 0)$ such that $\phi(-1) = 0$, we get

$$\begin{aligned} gd^3 \phi^2(0) &= gd^3 \left(\int_{-1}^{0} \phi_p(s) ds \right)^2 \\ &= gd^3 \left(\int_{-1}^{0} a^{\frac{3}{2}}(s, \lambda_0) \phi_p(s) a^{-\frac{3}{2}}(s, \lambda_0) ds \right)^2 \\ &\leq \left(\frac{gd^3}{p_0^2} \int_{-1}^{0} a^{-3}(s, \lambda_0) ds \right) \left(p_0^2 \int_{-1}^{0} a^3(s, \lambda_0) \phi_p^2(s) ds \right) \\ &= \left(p_0^2 \int_{-1}^{0} a^3(s, \lambda_0) \phi_p^2(s) ds \right), \end{aligned}$$

from which it follows that $\mathbb{F}(\phi, \lambda_0) \geq 0$ and thus $\mu(\lambda_0) \geq 0$. Therefore $\mu(\lambda_0) = 0$.

6.2. *The range.* This section is dedicated to proving the following elegant characterisation for the range of the operator $\mathcal{F}_w(0, \lambda^*) : X \to Y$:

Proposition 6.1. The pair $(\mathcal{A}, \mathcal{B}) \in Y$ belong to the range of $\mathcal{F}_w(0, \lambda^*)$ if and only if they satisfy the orthogonality condition

$$\iint_R \mathcal{A}(q, p) a^3(p, \lambda^*) \varphi^*(q, p) dq dp + \frac{1}{2} \int_T \mathcal{B}(q) a^2(0, \lambda^*) \varphi^*(q, 0) dq = 0, \tag{6.4}$$

where

$$\varphi^*(q, p) = M(p) \cos(q) \in X$$

generates $\ker\{\mathcal{F}_w(0, \lambda^*)\}$.

A consequence of the above proposition is that the range $\mathcal{R}(\mathcal{F}_w(0, \lambda^*))$ is clearly closed in Y. Also, if we use the fact that $a(\cdot, \lambda^*) > 0$ implies that $M(0) \neq 0$, since $a^2(0, \lambda^*) = \lambda^*$ and otherwise we could not have $\mathcal{F}(M, \lambda^*) = -1$, it follows that $(0, \cos q)$

does not satisfy the condition (6.4); therefore $(0, \cos q) \notin \mathcal{R}(\mathcal{F}_w(0, \lambda^*))$. Let us suppose $(\mathcal{A}_1, \mathcal{B}_1), (\mathcal{A}_2, \mathcal{B}_2) \in Y \backslash \mathcal{R}(\mathcal{F}_w(0, \lambda^*))$. Then we have

$$(\mathcal{A}_1, \mathcal{B}_1) - c(\mathcal{A}_2, \mathcal{B}_2) \in \mathcal{R}(\mathcal{F}_w(0, \lambda^*))$$

for the constant

$$c = \frac{\iint_R \mathcal{A}_1 a^3 \varphi^* dq dp + \frac{1}{2} \int_T \mathcal{B}_1 a^2 \varphi^* dq}{\iint_R \mathcal{A}_2 a^3 \varphi^* dq dp + \frac{1}{2} \int_T \mathcal{B}_2 a^2 \varphi^* dq}.$$

We have shown that $\mathcal{R}(\mathcal{F}_w(0, \lambda^*))$ has codimension one.

Proof. Let $\mathcal{F}_w(0, \lambda^*)\phi = (\mathcal{A}, \mathcal{B})$. Then multiplying

$$\mathcal{A} = \mathcal{F}_{1w}(0, \lambda^*)\phi = \frac{1}{d^2}\phi_{pp} + (H_p + 1)^2 \phi_{qq} + \frac{3\gamma(p)}{p_0}(H_p + 1)^2 \phi_p$$

by $a^3\varphi^*$ and integrating over R, using integration by parts and the fact that

$$\mathcal{B} = \mathcal{F}_{2w}(0, \lambda^*)\phi = 2\left(\frac{gd}{\lambda^* p_0^2}\phi(q, 0) - \frac{\lambda^{*\frac{1}{2}}}{d^2}\phi_p(q, 0)\right)$$

we find, from the relations $H_p + 1 = \frac{1}{a}$, $a_p = \frac{d^2}{p_0}\gamma(p)a^{-1}$, $a(0) = \sqrt{\lambda^*}$, that condition (6.4) holds.

The proof of the sufficiency is more technically complicated. We define the closed subspaces

$$X_0 = \{\phi \in X : \int_{-\pi}^{\pi} \phi(q, p) dq = 0 \ \text{ for all } \ p \in [-1, 0]\} \subset X,$$

$$Y_0 = \{(\mathcal{A}, \mathcal{B}) \in Y : \int_{-\pi}^{\pi} \mathcal{A}(q, p) dq = 0 \text{ for all } p \in [-1, 0], \ \int_T \mathcal{B} dq = 0\} \subset Y.$$

We note that it is now necessary to split X into X_0 and its topological complement, which corresponds to the zero Fourier mode, to ensure that a coercivity condition holds on X_0 further on in this proof. Given a pair $(\mathcal{A}, \mathcal{B}) \in Y$ such that (6.4) holds we can see upon calculation that $(\mathcal{A}, \mathcal{B}) \in \mathcal{R}(\mathcal{F}_w(0, \lambda^*))$, that is, $(\mathcal{A}, \mathcal{B}) = \mathcal{F}_w(0, \lambda^*)\phi$ for some $\phi \in X$ if and only if

$$(a^3 \phi_0')' = d^2 a^3 \mathcal{A}_0 \qquad \text{in} \quad R, \tag{6.5a}$$

$$\frac{gd}{p_0^2}\phi_0 - \frac{a^3}{d^2}\phi_0' = \frac{1}{2}a^2 \mathcal{B}_0 \qquad \text{for} \quad p = 0, \tag{6.5b}$$

$$\phi_0 = 0 \qquad \text{for} \quad p = -1, \tag{6.5c}$$

for

$$\mathcal{B}_0 = \frac{1}{2\pi}\int_T \mathcal{B} dq, \quad \mathcal{A}_0(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{A}(q, p) dq, \quad \phi_0(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(q, p) dq,$$

where $\mathcal{B}_0 \in \mathbb{R}$, $\mathcal{A}_0(p) \in C^{1,\alpha}[-1, 0]$, $\phi_0(p) \in C^{3,\alpha}[-1, 0]$, and

$$(a^3 \varphi_p)_p + a d^2 \varphi_{qq} = d^2 a^3 (\mathcal{A} - \mathcal{A}_0) \qquad \text{in} \quad R, \tag{6.6a}$$

$$\frac{gd}{p_0^2}\varphi - \frac{a^3}{d^2}\varphi_p = \frac{1}{2}a^2(\mathcal{B} - \mathcal{B}_0) \qquad \text{on} \quad T, \tag{6.6b}$$

$$\varphi = 0 \qquad \text{on} \quad B, \tag{6.6c}$$

for $\varphi = \phi - \phi_0 \in X_0$ and $(\mathcal{A} - \mathcal{A}_0, \mathcal{B} - \mathcal{B}_0) \in Y_0$.

LEMMA 6.2. For any $(\mathcal{A}_0, \mathcal{B}_0) \in C^{1,\alpha}[-1,0] \times \mathbb{R}$ the problem (6.5) has a unique solution $\phi_0 \in C^{3,\alpha}[-1,0]$.

Proof. We see from equation (6.5a) that, for all $p \in [-1,0]$,

$$a^3(p,\lambda^*)\phi_0'(p) = C + \int_{-1}^{p} d^2 a^3(s,\lambda^*)\mathcal{A}_0(s)ds$$

for some constant C, and then

$$\phi_0(p) = C\int_{-1}^{p} a^{-3}(r,\lambda^*)dr + \int_{-1}^{p} a^{-3}(r,\lambda^*)\left(\int_{-1}^{r} d^2a^3(s,\lambda^*)\mathcal{A}_0(s)ds\right)dr.$$

Evaluating the boundary condition at $p=0$ we get

$$\begin{aligned}
&\frac{gd}{p_0^2}\phi_0(0) - \frac{a^3}{d^2}\phi_0'(0) = \frac{1}{2}a^2\mathcal{B}_0\\
&= \frac{gd}{p_0^2}\left(C\int_{-1}^{0} a^{-3}(r,\lambda^*)dr + \int_{-1}^{0} a^{-3}(r,\lambda^*)\left(\int_{-1}^{r} d^2a^3(s,\lambda^*)\mathcal{A}_0(s)ds\right)dr\right)\\
&-\frac{C}{d^2} - \int_{-1}^{0} a^3(s,\lambda^*)\mathcal{A}_0(s)ds\\
&\Rightarrow C\left(\int_{-1}^{0} a^{-3}(r,\lambda^*)dr - \frac{p_0^2}{gd^3}\right)\\
&= \frac{p_0^2}{2gd}a^2(0,\lambda^*)\mathcal{B}_0 + \frac{p_0^2}{gd}\int_{-1}^{0} a^3(0,\lambda^*)(s,\lambda^*)\mathcal{A}_0(s)ds\\
&-\int_{-1}^{0} a^{-3}(r,\lambda^*)\left(\int_{-1}^{r} d^2a^3(s,\lambda^*)\mathcal{A}_0(s)ds\right)dr.
\end{aligned}$$

We saw in the analysis of the null space (6.3) that

$$\frac{p_0^2}{gd^3} = \int_{-1}^{0} a^{-3}(r,\lambda_0)dr < \int_{-1}^{0} a^{-3}(r,\lambda^*)dr$$

and so the constant C is always uniquely determined; therefore a solution of (6.5) always exists. □

It follows from our discussion prior to Lemma 6.2 that sufficiency in the statement (6.4) will be proven if we show that (6.4) implies the existence of a solution $\varphi \in X_0$ to (6.6). It suffices to check this just for $(\mathcal{A}, \mathcal{B}) \in Y_0 \subset Y$ since $(\mathcal{A}_0, \mathcal{B}_0)$ automatically satisfies the orthogonality condition (6.4). Our proof will be complete following the next two lemmas.

LEMMA 6.3. For $(\mathcal{A}, \mathcal{B}) \in Y_0$ and any $\epsilon \in (0,1)$ there exists a unique solution $v^{(\epsilon)} \in X_0$ to the approximate problem

$$\begin{cases}
-\epsilon a^3 v^{(\epsilon)} + (1+\epsilon)\left(a^3 v_p^{(\epsilon)}\right)_p + (1+\epsilon)d^2 a v_{qq}^{(\epsilon)} = d^2a^3\mathcal{A} \quad \text{in } R,\\
\frac{gd}{p_0^2}v^{(\epsilon)} - (1+\epsilon)\frac{a^3}{d^2}v_p^{(\epsilon)} = \frac{1}{2}a^2\mathcal{B} \quad \text{on } T,\\
v^{(\epsilon)} = 0 \quad \text{on } B,
\end{cases} \tag{6.7}$$

where $a = a(p,\lambda^*)$.

Proof. We introduce the space

$$\mathcal{H} = \Big\{\varphi \in H^1_{per}(R) : \varphi \text{ even in } q, \int_{-\pi}^{\pi} \varphi(q,p)dq = 0 \text{ a.e. in } [-1,0], \\ \phi = 0 \text{ a.e. on } B\Big\}.$$

We can see that $\mathcal{H}$ is a Hilbert space in its own right, since it is a closed subspace of the Hilbert space $H^1_{per}(R)$. A function φ is a weak solution of (6.7) if

$$(1+\epsilon)\iint_R a^3\varphi_p\phi_p dpdq + (1+\epsilon)d^2\iint_R a\varphi_q\phi_q dpdq + \epsilon\iint_R a^3\varphi\phi dpdq \\ -\frac{gd^3}{p_0^2}\int_T \varphi\phi dq = -\frac{d^2}{2}\int_T a^2\mathcal{B}\phi dq - d^2\iint_R a^3\mathcal{A}\phi dpdq \tag{6.8}$$

for all $\phi \in \mathcal{H}$. For $\varphi \in \mathcal{H}\cap C^3_{per}(\overline{R})$ we have

$$\varphi(q,p) = \sum_{k=1}^{\infty}\varphi_k(p)\cos(kq) \quad \text{in} \quad C^3_{per}(\overline{R}),$$

where $\varphi_k \in C^3[-1,0]$ is given by

$$\varphi_k(p) = \frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(q,p)\cos(kq)dq, \quad p\in[-1,0],\ k\ge 1.$$

We have $\varphi_k(-1) = 0$ for all $k \ge 1$, and

$$\iint_R a^3(\partial_p\varphi)^2 dqdp = \pi\sum_{k=1}^{\infty}\int_{-1}^{0} a^3(\partial_p\varphi_k)^2 dp,$$

$$\iint_R a(\partial_q\varphi)^2 dqdp = \pi\sum_{k=1}^{\infty}k^2\int_{-1}^{0} a\varphi_k^2 dp, \quad \int_T \varphi^2 dq = \pi\sum_{k=1}^{\infty}\varphi_k^2(0).$$

We infer from the minimisation problem that

$$\iint_R a^3\varphi_p^2 dpdq + d^2\iint_R a\varphi_q^2 dpdq \ge \pi\sum_{k=1}^{\infty}\int_{-1}^{0}[a^3(\partial_p\varphi_k)^2 + d^2a\varphi_k^2]dp \\ \ge \frac{gd^3}{p_0^2}\pi\sum_{k=1}^{\infty}\varphi_k^2(0) = \frac{gd^3}{p_0^2}\int_T\varphi^2 dq,$$

and therefore, since $\mathcal{H}\cap C^3_{per}(\overline{R})$ is dense in $\mathcal{H}$ and $\inf_{p\in[-1,0]}\{a(p,\lambda^*)\} > 0$, we see that the left-hand side of (6.8) defines a bounded and coercive bilinear form on $\mathcal{H}$. Furthermore, the right-hand side defines a bounded linear functional on $\mathcal{H}$, and therefore an application of the Lax-Milgram theorem [30] gives the existence and uniqueness of a weak solution $v^{(\epsilon)} \in \mathcal{H}$ to (6.7). From standard elliptic regularity theory we have $v^{(\epsilon)} \in X_0$ and we also note the Schauder estimates

$$\|v^{(\epsilon)}\|_{C^{1,\alpha}_{per}(\overline{R})} \le C\left(\|a\|_{C^{1,\alpha}_{per}(\overline{R})} + \|\mathcal{A}\|_{C^{1,\alpha}_{per}(\overline{R})} + \|\mathcal{B}\|_{C^{1,\alpha}_{per}(T)} + \|v^{(\epsilon)}\|_{L^\infty(R)}\right), \tag{6.9}$$

where the constant C depends only on $\|a\|_{C^{1,\alpha}_{per}(\overline{R})}$; see [30]. □

LEMMA 6.4. If $(\mathcal{A},\mathcal{B}) \in Y_0$ satisfy (6.4), then for any sequence $\epsilon_k \downarrow 0$ the sequence $\{v^{(\epsilon_k)}\}_{k\geq 1}$ is bounded in $C^{1,\alpha}_{per}(\overline{R})$.

Proof. We prove the claim by contradiction. Suppose that for the sequence $\epsilon_k \downarrow 0$ we have $\|v^{(\epsilon_k)}\|_{C^{1,\alpha}_{per}}(\overline{R}) \to \infty$. Then by (6.9) it follows that $\|v^{(\epsilon_k)}\|_{L^\infty(R)} \to \infty$ and the sequence $v_k = v^{(\epsilon_k)}/\|v^{(\epsilon_k)}\|_{L^\infty(R)}$ is bounded in $C^{1,\alpha}_{per}(\overline{R})$. Since we have the compact embedding $C^{1,\alpha}_{per}(\overline{R}) \subset C^1_{per}(\overline{R})$ we can find a subsequence $\{v_{n_k}\}$ which converges in $C^1_{per}(\overline{R})$ to some v with $\|v\|_{L^\infty(R)} = 1$. If we consider (6.8) for $\epsilon = \epsilon_{n_k}$ and $\varphi = v^{(\epsilon_{n_k})}$, then dividing by $\|v^{(\epsilon_{n_k})}\|_{L^\infty(R)}$ and passing to the limit $n_k \to \infty$ we get

$$\iint_R a^3 v_p \phi_p dpdq + d^2 \iint_R a v_q \phi_q dpdq = \frac{gd^3}{p_0^2}\int_T \varphi\phi dq, \quad \phi \in \mathcal{H}. \tag{6.10}$$

Therefore v is a weak solution in $\mathcal{H}$ of the problem

$$(a^3 v_p)_p + d^2 a v_{qq} = 0, \qquad (q,p) \in R, \tag{6.11a}$$

$$a^3 v_p = \frac{gd^3}{p_0^2} v, \qquad p = 0, \tag{6.11b}$$

$$v = 0, \qquad p = -1, \tag{6.11c}$$

and using standard elliptic regularity theory [30] we have $v \in X_0$. It follows from (5.2) that $v \in \ker\{\mathcal{F}_w(0,\lambda^*)\}$, which we know to be one-dimensional and therefore $v = C\varphi^*$, where $C \in \mathbb{R}$ is a constant and $\varphi^*(q,p) = M(p)\cos(q)$, where $M \in C^{3,\alpha}[-1,0]$ is the eigenfunction of (5.10) for $\mu(\lambda^*) = -1$. If we now set $\epsilon = \epsilon_{n_k}, \varphi = v^{(\epsilon_{n_k})}, \phi = \varphi^*$, then evaluating (6.8) using the relations (6.4) and (6.10) we get

$$\iint_R a^3 v^{(\epsilon_{n_k})}\varphi^* dpdq + \frac{gd^3}{p_0^2}\int_T v^{(\epsilon_{n_k})}\varphi^* dq = 0.$$

If we follow the limiting procedure of dividing by $\|v^{(\epsilon_{n_k})}\|_{L^\infty(R)}$ and passing to the limit $n_k \to \infty$ we get

$$\iint_R a^3 {\varphi^*}^2 dpdq + \frac{gd^3}{p_0^2}\int_T {\varphi^*}^2 dq = 0,$$

which gives us a contradiction since $\varphi^* \not\equiv 0$. □

Since $C^{1,\alpha}_{per}(\overline{R})$ is compactly embedded in $C^1_{per}(\overline{R})$ it follows from Lemma 6.4 that there exists a subsequence $\{v^{(\epsilon_{n_k})}\}$ which converges to some limit $v \in C^1_{per}(\overline{R})$. Taking the limit in (6.8) we see that v is a weak solution of

$$\begin{aligned} (a^3 v_p)_p + a d^2 v_{qq} &= d^2 a^3 \mathcal{A} && \text{in} \quad R, \\ \frac{gd}{p_0^2} v - \frac{a^3}{d^2} v_p &= \frac{1}{2} a^2 \mathcal{B} && \text{on} \quad T, \\ v &= 0 && \text{on} \quad B, \end{aligned}$$

which is exactly (6.6). By standard elliptic regularity theory [30] we can further say that $v \in X_0$ and so it is in fact a classical solution. □

6.3. *Transversality condition.* To apply the Crandall-Rabinowitz theorem we show that

$$\mathcal{F}_{\lambda w}(0,\lambda^*)[(\varphi^*,1)] \notin \mathcal{R}(\mathcal{F}_w(0,\lambda^*)),$$

where

$$\begin{aligned}&\mathcal{F}_{\lambda w}(0,\lambda^*)[(1,\varphi^*)]\\ &\quad= -\left(a^{-4}\varphi^*_{qq} + \frac{3}{d^2}a_p a^{-3}\varphi^*_p, 2\frac{gd}{(p_0\lambda^*)^2}\varphi^*(q,0) + \frac{1}{d^2\sqrt{\lambda^*}}\varphi^*_p(q,0)\right) \neq 0.\end{aligned}$$

In order to do this we show that $\mathcal{F}_{\lambda w}(0,\lambda^*)[(\varphi^*,1)]$ doesn't satisfy the orthogonality condition (6.4). Using the relations $a(0,\lambda)=\sqrt{\lambda}$ and $3a_p\varphi^*_p = d^2a^{-1}\varphi^* - a\varphi^*_{pp}$ (which come from (5.10a)) we get

$$\iint_R a_p\varphi^*\varphi^*_p dqdp = \frac{1}{2}\iint_R \{a(\varphi^*_p)^2 + d^2a^{-1}(\varphi^*)^2\}dqdp - \frac{1}{2}\int_T a\varphi^*\varphi^*_p dq,$$

and we use this relation together with $\varphi^*_{qq} = -\varphi^*$ to compute

$$\begin{aligned}
&\iint_R \{a^{-1}\varphi^*\varphi^*_{qq} + \frac{3}{d^2}a_p\varphi^*\varphi^*_p\}dqdp + \int_T\left\{\frac{gd}{p_0^2\lambda^*}\varphi^*\varphi^* + \frac{\sqrt{\lambda^*}}{2d^2}\varphi^*_p\varphi^*\right\}dq\\
&\qquad= \iint_R \{-a^{-1}(\varphi^*)^2 + \frac{3}{2d^2}a(\varphi^*_p)^2 + \frac{3}{2}a^{-1}(\varphi^*)^2\}dqdp\\
&\qquad\quad -\frac{3}{2d^2}\int_T a\varphi^*\varphi^*_p dq + \int_T\left\{\frac{gd}{p_0^2\lambda^*}\varphi^*\varphi^* + \frac{\sqrt{\lambda^*}}{2d^2}\varphi^*_p\varphi^*\right\}dq\\
&\qquad= \iint_R \{-a^{-1}(\varphi^*)^2 + \frac{3}{2}a(\varphi^*_p)^2 + \frac{3}{2}d^2a^{-1}(\varphi^*)^2\}dqdp\\
&\qquad\quad -\frac{3}{2}\int_T \frac{gd}{\lambda^* p_0^2}(\varphi^*)^2 dq + \int_T\left\{\frac{gd}{p_0^2\lambda^*}\varphi^*\varphi^* + \frac{\sqrt{\lambda^*}}{2d^2}\varphi^*_p\varphi^*\right\}dq\\
&= \iint_R\left\{\frac{1}{2}d^2a^{-1}(\varphi^*)^2 + \frac{3}{2}a(\varphi^*_p)^2\right\}dqdp + \int_T\left\{\frac{\sqrt{\lambda^*}}{2d^2}\varphi^*_p\varphi^* - \frac{gd}{2p_0^2\lambda^*}\varphi^*\varphi^*\right\}dq\\
&\qquad\overset{(5.10b)}{=} \iint_R\left\{\frac{1}{2}d^2a^{-1}(\varphi^*)^2 + \frac{3}{2}a(\varphi^*_p)^2\right\}dqdp > 0.
\end{aligned}$$

Therefore $\mathcal{F}_{\lambda w}(0,\lambda^*)[(1,\varphi^*)]$ does not satisfy the orthogonality condition, and so we have proven the transversality condition required for the Crandall-Rabinowitz theorem.

It follows that if condition (5.21) holds, then the conditions of Theorem 3.1 are satisfied for the unique value λ^* such that $\mu(\lambda^*) = -1$. Therefore for this critical value λ^* local bifurcation occurs at the point $(H(p,\lambda^*),\lambda^*)$ and the points on the resulting bifurcation curve represent small-amplitude solutions of the water wave problem (2.16).

7. Dispersion relations. Since for laminar flows we have (by (2.13))

$$\sqrt{\lambda} = \frac{1}{H_p+1}\bigg|_{p=0} = \frac{d(u-c)}{p_0}\bigg|_{\text{on the flat surface}}, \tag{7.1}$$

we see that small-amplitude waves occur when the speed of the laminar flows reaches the critical value

$$u^* - c = \frac{p_0\sqrt{\lambda^*}}{d}, \tag{7.2}$$

where $c - u^*$ denotes the speed of the laminar flow at the surface. For some simple forms of the vorticity function, namely constant vorticity, we can calculate λ^* in an explicit form which depends on the mean depth d, and this allows us to express the critical value $c - u^*$ in terms of d.

7.1. *Irrotational flows.* Irrotational flow occurs when $\gamma \equiv 0$, and so to compute the value of λ^* explicity in this case, since $a(p;\lambda) \equiv \sqrt{\lambda}$ in (5.4) we look for nontrivial solutions of

$$\begin{aligned} M_{pp} &= d^2\lambda^{-1}M, \quad -1 < p < 0, \\ M_p &= g\frac{d^3}{p_0^2}\lambda^{-3/2}M, \qquad p = 0, \\ M &= 0, \qquad\qquad p = -1. \end{aligned}$$

A general solution for the differential equation which solves the condition on $p = -1$ is given by

$$M(p) = \delta \sinh\left(\frac{d}{\sqrt{\lambda}}(p+1)\right), \qquad -1 < p < 0,$$

and for $\delta \neq 0$ the condition at $p = 0$ implies that

$$\tanh\left(\frac{d}{\sqrt{\lambda}}\right) - \frac{p_0^2}{gd^2}\lambda = 0.$$

This equation has a unique solution, which must be λ^*. Integrating equation (7.2) with respect to $-d \le y \le 0$, we get $\lambda^* = 1$, and therefore from (7.2) it follows that

$$c - u^* = \frac{p_0}{d} = \sqrt{g\tanh(d)}. \tag{7.3}$$

This is the standard dispersion relation for irrotational flow [37]. We see in the shallow water limit, $d \to 0$, that

$$c - u^* = \lim_{d\to 0}\sqrt{gd\frac{\tanh(d)}{d}} = \sqrt{gd},$$

and so in shallow irrotational water all waves travel with the same speed. The term "dispersion relation" comes from the fact that if we were dealing with water waves of wavelength L in the governing equations (2.1)–(2.5), then after performing the following scaling of variables

$$(x, y, t, g, \omega, \eta, u, v, P, c) \mapsto (\kappa x, \kappa y, \kappa t, \kappa^{-1}g, \kappa^{-1}\omega, \kappa\eta, u, v, P, c),$$

where $\kappa = \frac{2\pi}{L}$ is the wavenumber, we end up with a $2\pi-$periodic system in the new variables identical to (2.1)–(2.5) except that g, ω are replaced by $\kappa^{-1}g, \kappa^{-1}\omega$. The dispersion relation (7.3) becomes

$$c - u^* = \sqrt{\frac{g}{\kappa}\tanh(\kappa d)},$$

and we see that waves of different lengths travel at different speeds. This is the dispersive effect.

In the deep-water limit, $\lim_{d\to\infty}\tanh(\kappa d) = 1$, and therefore the dispersion relation in the deep-water limit becomes

$$c - u^* = \sqrt{\frac{gL}{2\pi}}.$$

So, in the deep-ocean, waves with long wavelengths travel faster than those with shorter wavelengths, a situation which is seen in the case of the rapid propagation speeds of such exceptionally long waves as tsunamis.

7.2. *Constant vorticity flows.* For flows with constant vorticity, $\gamma(p) = \gamma \neq 0$ is a constant and $\Gamma(p) = \alpha p$, where $\alpha = \frac{2d^2}{p_0}\gamma$. Therefore $a(p;\lambda) = \frac{1}{H_p+1} = \sqrt{\lambda + \alpha p}$ and $a_p = -\alpha(2a)^{-1}$ in (5.4). It follows upon substituting

$$M(p) = a^{-1} M_0\left(\frac{a}{c}\right),$$

for $c = \frac{d}{p_0}\gamma = \frac{\alpha}{2d}$ into the differential equations (5.4) that we get $M_0'' = M_0$, with $M(-1) = 0$. The general solution of such an equation is

$$M(p) = \frac{1}{\sqrt{\lambda + \alpha p}} \sinh\left(\frac{p_0\left(\sqrt{\lambda+\alpha p} - \sqrt{\lambda - \alpha}\right)}{d\gamma}\right),$$

with $a = \sqrt{\lambda}$ on $p = 0$. The boundary condition at $p = 0$ holds if

$$\tanh\left(\frac{p_0\left(\sqrt{\lambda} - \sqrt{\lambda - \frac{2d^2}{p_0}\gamma}\right)}{d\gamma}\right) = \frac{p_0^2\lambda}{gd^2 + dp_0\gamma\lambda^{\frac{1}{2}}} \tag{7.4}$$

for a unique critical value λ^*, and if (7.4) holds, then $(H(p,\lambda^*),\lambda^*)$ is a bifurcation point, where the bifurcating laminar flow is given by

$$H(p,\lambda^*) = 2\frac{\sqrt{\lambda^* + \alpha p} - \sqrt{\lambda^* - \alpha}}{\alpha} - p - 1, \qquad -1 < p < 0.$$

In the physical coordinates, for laminar flow with constant vorticity, we have $v = 0$ and $u_y = \gamma$, and

$$\int_y^0 u_y dy = -\gamma y \Rightarrow u(y) - c = \frac{p_0\sqrt{\lambda}}{d} + \gamma y, \tag{7.5}$$

$$p_0 = \left(\frac{\gamma d^2}{2(\sqrt{\lambda} - 1)}\right), \tag{7.6}$$

$$\sqrt{\lambda} = 1 + \frac{\gamma d^2}{2p_0}, \tag{7.7}$$

$$\lambda - 2\gamma\frac{d^2}{p_0} = \lambda - 4\sqrt{\lambda} + 4 = (\sqrt{\lambda} - 2)^2. \tag{7.8}$$

It appears from (7.8) that we can formulate the left-hand side of (7.4) as

$$\tanh\left(\frac{p_0(\sqrt{\lambda}-\sqrt{\lambda-2\frac{d^2}{p_0}\gamma})}{d\gamma}\right)=\tanh\left(\frac{p_0(\sqrt{\lambda}\pm(\sqrt{\lambda}-2)}{d\gamma}\right) \tag{7.9}$$

$$=\begin{cases}\tanh\left(\frac{p_0(2\sqrt{\lambda}-2)}{d\gamma}\right)=\tanh(d),\\ \tanh\left(\frac{2p_0}{d\gamma}\right).\end{cases} \tag{7.10}$$

Now the first term in (7.9) implies that $\lambda-2\frac{d^2}{p_0}\gamma\geq 0$. This relation holds universally for $\gamma>0$. In this case, we have

$$0<\tanh\left(\frac{p_0(\sqrt{\lambda}-\sqrt{\lambda-2\frac{d^2}{p_0}\gamma})}{d\gamma}\right)\not\equiv\tanh\left(\frac{2p_0}{d\gamma}\right)<0,$$

and therefore the second possibility in (7.10) does not exist for $\gamma>0$.

For $\gamma<0$ we evaluate (7.5) at $y=-d$, giving us the relation $\frac{p_0\sqrt{\lambda}}{d}<\gamma d$. Together with (7.7) this implies that

$$1+\frac{\gamma d^2}{2p_0}=\sqrt{\lambda}>\frac{d^2\gamma}{p_0}, \tag{*}$$

which means that $\sqrt{\lambda}<2$. However, if the second alternative in relation (7.10) were to hold, we would get

$$\frac{p_0(\sqrt{\lambda}-\sqrt{\lambda-2\frac{d^2}{p_0}\gamma})}{d\gamma}=\frac{2p_0}{d\gamma},$$

which implies that $\sqrt{\lambda}\geq 2$, giving us a contradiction. Therefore (7.4) can be reformulated as

$$\frac{p_0^2\lambda^*}{gd^2+dp_0\gamma\lambda^{*\frac{1}{2}}}=\tanh(d)$$

or

$$p_0^2\lambda^*-dp_0\gamma\lambda^{*\frac{1}{2}}\tanh(d)-gd^2\tanh(d)=0. \tag{7.11}$$

We can solve (7.11) to get

$$\sqrt{\lambda^*}=\frac{dp_0\gamma\tanh(d)\pm\sqrt{d^2p_0^2\gamma^2\tanh^2(d)+4p_0^2gd^2\tanh(d)}}{2p_0^2},$$

and so we obtain the dispersion relation

$$c-u^*=-\frac{\gamma}{2}\tanh(d)+\frac{1}{2}\sqrt{\gamma^2\tanh^2(d)+4g\tanh(d)},$$

where we make the choice of sign above guided by the criteria $u^*-c<0$. Similar to the irrotational case (Section 7.1) the dispersion relation for waves with wavelength L takes the form

$$c-u^*=-\frac{\gamma}{2\kappa}\tanh(\kappa d)+\frac{1}{2\kappa}\sqrt{\gamma^2\tanh^2(\kappa d)+4g\tanh(\kappa d)}. \tag{7.12}$$

Note that setting $\gamma = 0$ here gives us the appropriate irrotational dispersion relation (7.3). The right-hand side of (7.12) is a strictly decreasing function of γ. Therefore the wave speed $c - u^* > 0$ is larger for $\gamma < 0$ than for $\gamma > 0$. We see from (7.5) that

$$u - c|_{y=0} = \frac{p_0}{d}\sqrt{\lambda^*}, \quad u - c|_{y=-d} = \frac{p_0}{d}\sqrt{\lambda^*} - \gamma d = \frac{p_0}{d}\sqrt{\lambda^* - 2\frac{d^2\gamma}{p_0}},$$

and so $u - c$ is smaller on the flat bed than on the surface for $\gamma < 0$, while it is larger on the bottom than on the top for $\gamma > 0$. The case $\gamma < 0$ therefore corresponds to an adverse current while $\gamma > 0$ is a favourable current.

7.2.1. *Existence of small-amplitude solutions for constant vorticity.* To establish necessary and sufficient conditions for the existence of nonlinear bifurcating solutions for constant vorticity we examine the roots of the function

$$f(\lambda) = \tanh(d) - \frac{\gamma^2 d^2 \lambda}{4g(\sqrt{\lambda}-1)^2 + 2d\gamma^2\sqrt{\lambda}(\sqrt{\lambda}-1)} = \tanh(d) - \frac{\gamma^2 d^2}{r(\frac{1}{\sqrt{\lambda}})},$$

where $r(s) = 4gs^2 - (8g + 2d\gamma^2)s + (4g + 2d\gamma^2)$, and for a given γ the point of bifurcation λ^* is located at the roots of $f(\lambda)$. We note that the sign of γ has a bearing on the function f due to the fact that $\lambda < 1$ for positive vorticty, while $\lambda > 1$ for negative vorticity, and it follows that we will learn much about the existence of roots for $f(\lambda)$ by studying the behaviour of $r(\sqrt{\lambda})$.

In order to do this let us solve the quadratic equation

$$4gs^2 - (8g + 2d\gamma^2)s + (4g + 2d\gamma^2) = 0,$$

and find that the roots of $r(\frac{1}{\sqrt{\lambda}})$ are given by $\frac{1}{\sqrt{\lambda}} = 1, 1 + \frac{d\gamma^2}{2g}$ ($\sqrt{\lambda} = \frac{2g}{2g+d\gamma^2}, 1$). We further calculate

$$r'(s) = 8gs - (8g + 2d\gamma^2),$$

and see that $r(\frac{1}{\sqrt{\lambda}})$ has its minimum at $\frac{1}{\sqrt{\lambda}_{\min}} = 1 + \frac{d\gamma^2}{4g}$, or $\sqrt{\lambda}_{\min} = \frac{4g}{4g+d\gamma^2}$, giving us

$$r_{\min} = -\frac{d^2\gamma^4}{4g}.$$

Now we can get an insight into what is happening with $f(\lambda)$ from the two pictures in Figure 1.

For $\gamma > 0$ the bifurcation parameter $\lambda < 1$, by (7.7), and therefore $\frac{1}{\lambda} > 1$. We are dealing with the region to the right of the heavy-dotted line in the second graph, and we can easily see that for every fixed d and $\gamma > 0$ there exists a unique $\sqrt{\lambda} < \frac{2g}{2g+d\gamma^2}$ such that $f(\lambda) = 0$. This follows immediately from the limiting values $\lim_{\sqrt{\lambda}\nearrow \frac{2g}{2g+d\gamma^2}} f(\lambda) = -\infty$ and $\lim_{\lambda\searrow 0} f(\lambda) = \tanh(d)$. Therefore, bifurcation occurs for each fixed d and $\gamma > 0$.

For $\gamma < 0$ we see that $f(\lambda) = 0$ if and only if the curve $r(\frac{1}{\sqrt{\lambda}})$ to the left of the heavy-dotted line in the second picture attains the value $\frac{\tanh(d)}{\gamma^2 d^2}$ for some λ. It follows from relation (*) that $\sqrt{\lambda} < 2$, which implies that $1/\sqrt{\lambda} > 1/2$. This fact, combined with the monotone nature of the curve, implies that bifurcation occurs if and only if $\tanh(d) > \gamma^2 d^2 / r(1/2)$, that is,

$$\gamma^2 d^2 < (g + \gamma^2 d)\tanh(d). \tag{7.13}$$

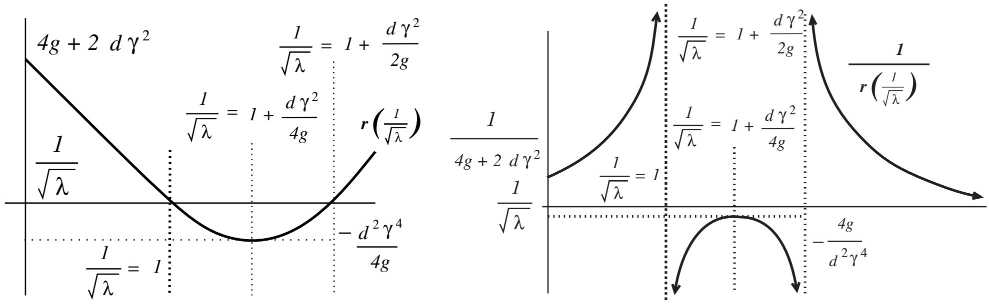


FIG. 1. The first graph plots $r(1/\sqrt{\lambda})$ while the second graph plots $1/r(1/\sqrt{\lambda})$. Small-amplitude waves exist when γ and d are such that $\tanh(d)/\gamma^2 d^2$ equals $1/r(1/\sqrt{\lambda})$.

7.2.2. *Nonconstant vorticity.* For general nonconstant vorticity there are very few qualitative results available, but among these results are the Burns condition for shallow water waves [1, 3, 27], where $\sqrt{\lambda^*} = c - u^*$ must solve

$$\int_{-d}^{0} \frac{dy}{(U(y) - \sqrt{\lambda^*})^2} = \frac{1}{g}, \tag{7.14}$$

where $(U(y), 0)$ is the unique current which generates the given vorticity with $U(0) = 0$, and also $U(y) < \sqrt{\lambda^*}$ by (2.7). If we find $U(y)$, the bifurcating laminar flow is then given by $(U(y) - \sqrt{\lambda^*}, 0)$. We now check how the Burns condition approach compares to that of Section 7.2 when γ is a constant. If γ is constant, then $U(y) = \gamma y$ and the Burns condition (7.14) becomes

$$\frac{1}{\gamma\sqrt{\lambda^*}} - \frac{1}{\gamma(\gamma d + \sqrt{\lambda^*})} = \frac{1}{g},$$

which we regard as a quadratic equation in $\sqrt{\lambda^*}$ and solve to get

$$\sqrt{\lambda^*} = \frac{-\gamma d + \sqrt{\gamma^2 d^2 + 4gd}}{2},$$

where we take the absence of stagnation points into account. But, since $\lim_{t\to 0} \frac{\sinh(t)}{t} = 1$, this is precisely the shallow water limit ($d \to 0$) of the dispersion relation (7.12).

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