2mu4

Steady-State Decoupling and Design of Linear Multivariable Systems

Research Report

Grant No. NGR 05-017-010

June 1972 - June 1974

G.J. Thaler
Investigator

Jen-Yen Huang
Research Assistant

This report is prepared for Ames Research Center,
NASA, Moffett Field, California 94035
Principal Investigator: D.D. Siljak

(NASA-CR-138815) STEADY-STATE DECOUPLING AND DESIGN OF LINEAR MULTIVARIABLE SYSTEMS Research Report, Jun. 1972 - Jun. 1974 (Santa Clara Univ.) ~~154~~ 155 p HC $10.75 CSCL 09C G3/10

N74-27715

Unclas
43491

The University of Santa Clara • California

STEADY-STATE DECOUPLING AND DESIGN OF LINEAR MULTIVARIABLE SYSTEMS

Research Report
Grant No. NGR 05-017-010
June 1972 - June 1974

G. J. Thaler
Investigator

Jen-Yen Huang
Research Assistant

This report is prepared for Ames Research Center,
NASA, Moffett Field, California 94035
Principal Investigator: D. D. Siljak

ABSTRACT

This report consists of three parts:

Part I : Steady State Decoupling

Part II : Stability and Design

Part III: Application to STOL Aircraft

Part I presents a constructive criterion for decoupling the steady states of a linear time-invariant multivariable system. This criterion consists of a set of inequalities which, when satisfied, will cause the steady states of a system to be decoupled. It turns out that pure integrators in the loops play an important role. Stability analysis and a new design technique for such systems are given in Part II. A new and simple connection between single-loop and multivariable cases is found. This makes possible the application of the existing single-loop methods to multivariable cases. These results are then applied in Part III to the compensation design for NASA STOL C-8A aircraft. Both steady-state decoupling and stability are justified through computer simulations.

NOMENCLATURE

1.	r	nx1 input vector
2.	y	nx1 output vector
3.	$H(s)$	closed-loop transfer function matrix
4.	$G_p(s)$	nxm plant matrix
5.	$G_c(s)$	mxn compensator matrix
6.	T_p	nxm type number matrix of the nxm plant $G_p(s)$
7.	T_c	mxn type number matrix of the mxn compensator $G_c(s)$
8.	P_g	poles of any transfer function g(s)
9.	Z_g	zeros of any transfer function g(s)
10.	$Max\{\cdots\}$	maximum value among all the elements in the brackets
11.	$LCD\{\cdots\}$	least common denominator of the elements in the brackets
12.	$LCM\{\cdots\}$	least common multiplier of the elements in the brackets
13.	$(G_p)_{ij}$	cofactor of the ijth element of G_p
14.	$(I+G_pG_c)_{ij}$	cofactor of the ijth element of $I+G_pG_c$
15.	$det(I+G_pG_c)$	determinant of the matrix $I+G_pG_c$
16.	$G\begin{pmatrix} i_1, \cdots, i_\ell \\ j_1, \cdots, j_\ell \end{pmatrix}$	minor of matrix G formed from rows $i_1, \cdots, i_\ell$ and columns $j_1, \cdots, j_\ell$

TABLE OF CONTENTS

PART I: STEADY-STATE DECOUPLING

PART II: STABILITY AND DESIGN

PART I

STEADY-STATE DECOUPLING

1. INTRODUCTION

Considerable research has been done on the decoupling of linear multivariable systems (e.g. see (1)). Such decoupling, referred to as *total decoupling* in this report, requires the system to be characterized by a nonsingular, diagonal transfer function matrix, and in general, linear state variable feedbacks have been employed.

The advantage of total decoupling is obvious, however, due to the restriction of having a diagonal transfer function matrix, less freedom should be expected when stability of the system is concerned.

This loss of freedom can be recovered to some extent by requiring only the steady states to be decoupled. Loosely speaking, a steady-state decoupled system is one in which changes in each input (i-th) are reflected in a corresponding output and only that output, when steady state is reached. Thus, different from total decoupling described above, mutual interactions are allowed during the transient period (but only during this period).

Necessary and sufficient conditions for decoupling the steady states of a system via linear state variable feedback were obtained by Wolovich (2).

His result, in terms of transfer function matrix representation is as follows:

A system characterized by an $(n \times m)$ proper rational transfer function matrix, $G_p(s)$ having no poles at the origin $(s = 0)$, can be steady-state decoupled (via linear state variable feedback or perhaps some other less ambitious scheme) if and only if

$$\rho(G_p(0)) = n \tag{1.1}$$

where $\rho(G_p(0))$ denotes rank of the matrix $G_p(s)$ as s approaches zero.

However, it is found that if classical cascade feedback compensation other than linear state variable feedback is used, the rank condition (1.1) is no more necessary. Furthermore, the precluded poles at the origin are allowed. Actually, such poles are very helpful for decoupling the steady states of a multivariable system. Therefore, significant advantages over the linear state variable approach can be obtained through classical feedback configuration which then is obviously not "less ambitious".

The constructive criterion for steady-state decoupling will be derived in this part of the report. It will be shown that this criterion consists of $n(n-1)$ inequalities (n is the number of outputs of the given plant), with the type numbers of the compensator transfer functions as unknowns. These unknowns are chosen to satisfy the inequalities and hence achieve a steady-state decoupling scheme. Fundamental mathematical relations are derived in Chapter 2. Two simple applications for 2 x 2 and 3 x 3 cases are given in Chapter 3. Finally, the general case is considered in Chapter 4. Direct comparison of the result to that of the state variable approach is included in Chapter 9, which marks the end of this report.

The research reported herein was included in the Jen-Yen Huang M.S.E.E. thesis at the Department of Electrical Engineering and Computer Science, University of Stanta Clara, Santa Clara, California. The thesis was supervised by G. J. Thaler, U.S. Naval Postgraduate School, Monterey, California.

2. FUNDAMENTAL RESULTS

The system under consideration in this thesis is shown below in Figure 2.1:

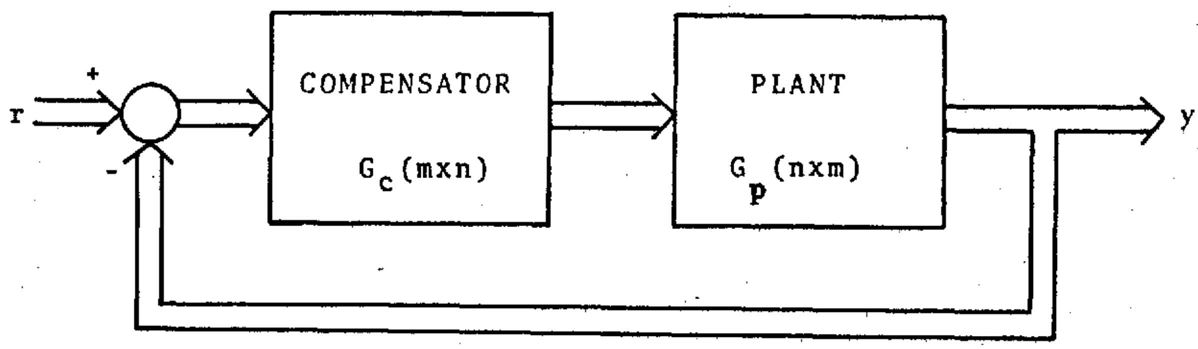

FIGURE 2.1

Where $G_p(nxm)$ characterizes the given m-input, n-output plant, $G_c(mxn)$ is the n-input, m-output compensator to be designed. N unity feedbacks are used and complete controllability and observability are assumed (3), (4), to assure the complete description of the system by transfer function matrices. r, y are the nx1 input and output vectors, respectively.

Let $H(s) = (h_{ij}(s))_{nxn}$ be the closed loop proper transfer function matrix, then by the above assumption, it characterizes the system completely, and we have:

$$y(s) = H(s)\cdot r(s)$$

or

$$y_i(s) = \sum_{j=1}^{n} h_{ij}(s)\cdot r_j(s)$$

$$= h_{ii}(s)r_i(s) + \sum_{\substack{j=1 \\ j\neq i}}^{n} h_{ij}(s)r_j(s) \qquad i=1,\cdots,n \qquad (2.1)$$

By (2.1) and the Final Value Theorem, we have:

$$\lim_{t\to\infty} y_i(t) = \lim_{s\to o} sy_i(s)$$

$$= \lim_{s\to o} s\, h_{ii}(s)\cdot r_i(s) + \lim_{s\to o} s \sum_{\substack{j=1 \\ j\neq i}}^{n} h_{ij}(s)\cdot r_j(s) \qquad (2.2)$$

$$i=1,\cdots,n$$

Then the following formal definition can be given:

DEFINITION:

A system with the transfer function matrix H(s) is steady-state decoupled if and only if it is asymptotically stable[1] and

$$\lim_{s\to o} \sum_{\substack{j=1 \\ j\neq i}}^{n} s\, h_{ij}(s)\cdot r_j(s) = o \quad \text{for all } i=1,\cdots,n \qquad (2.3)$$

[1] i.e., all the poles of the closed loop system lie in the open left half plane (Re(s)<o). This guarantees the application of the final value theorem.

For systems as shown in Figure 2.1, it is well known that the closed loop transfer function matrix $H^{(2)}$ can be expressed as

$$H = (I + G_pG_c)^{-1} G_pG_c \tag{2.4}$$

where I is the nxn identity matrix.

(2.4) can be simplified further as follows:

$$H = (I + G_pG_c)^{-1} G_pG_c$$

$$= (I + G_pG_c)^{-1} (I + G_pG_c) - (I + G_pG_c)^{-1}$$

$$= I - (I + G_pG_c)^{-1} \tag{2.5}$$

(2.5) shows that the elements of H depend in a very simple way on the cofactors of the elements of the matrix $I + G_pG_c$, i.e.,

$$h_{ii} = 1 - \frac{(I+G_pG_c)_{ii}}{\det(I+G_pG_c)} \qquad i = 1,\cdots,n \tag{2.6}$$

$$h_{ij} = - \frac{(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} \qquad \begin{matrix} i, j = 1,\cdots,n \\ i \neq j \end{matrix} \tag{2.7}$$

[2]The argument s will be dropped whenever no confusion exists.

where $\det(I + G_pG_c)$ denotes the determinant of the nxn matrix $I + G_pG_c$ and $(I + G_pG_c)_{ji}$ denotes the cofactor of the jith element of $I + G_pG_c$.

Let the inputs to the system be polynominal inputs with only one term, e.g., step, ramp or parabola inputs, which are of primary importance. The Laplace transform of each input $r_j(t)$. $j = 1, \cdots, n$, is then

$$r_j(s) = L\,(r_j(t)) = \frac{r_j}{s^{k_j}} \tag{2.8}$$

where r_j (without argument) is a constant, and k_j is a positive integer, e.g., if the jth input is a step then $k_j=1$, a ramp then $k_j=2$, etc.

Then, by (2.7), the steady-state decoupling criterion (2.3) becomes:

$$\lim_{s \to o} \sum_{\substack{j=1 \\ j \neq i}}^{n} S \cdot \frac{r_j}{S^{k_j}} \cdot \frac{-(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} = o \quad \text{for all } i=1,\cdots,n$$

Thus, the following fundamental theorem for steady-state decoupling is developed:

THEOREM 2.1:

Assume that the given plant $G_p(nxm)$ is stabilizable through the configuration of Figure 2.1, then the system is steady-state decoupled if and only if:

$$\lim_{s\to o} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{r_j}{s^{(k_j-1)}} \cdot \frac{(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} = o \qquad \text{for all } i=1,\cdots,n \tag{2.9}$$

Theorem 2.1 can be simplified further for systems whose inputs are not fixed. This is desirable in most practical applications, for example, consider an aircraft as our plant G_p, the thrust, flap and elevator inputs must not be fixed in order to perform different functions.

Thus, for inputs with arbitrary constants r_j, we have

THEOREM 2.2:

Assume that the given plant $G_p(nxm)$ is stabilizable through the configuration of Figure 2.1, and that the constants r_j in all the inputs $r_j(t)$, $j=1\cdots,n$ are arbitrary, then the system is steady-state decoupled if and only if

$$\lim_{s\to o} \frac{1}{S^{(k_j-1)}} \cdot \frac{(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} = 0$$

for all i.j=1,···,n and i≠j (2.10)

PROOF:

a) Necessity:

Suppose there exists i′,j′ such that (2.10) is not true, then, by choosing $r_{j'}(t)$ as the only non-zero input, we have

$$\lim_{s\to o} \sum_{\substack{j=1 \\ j\neq i'}}^{n} \frac{r_j}{S^{(k_j-1)}} \frac{(I+G_pG_c)_{ji'}}{\det(I+G_pG_c)} \tag{2.11}$$

$$= \lim_{s\to o} \frac{r_{j'}}{S^{(k_{j'}-1)}} \frac{(I+G_pG_c)_{j'i'}}{\det(I+G_pG_c)} \tag{2.12}$$

By our hypothesis, (2.12) is non-zero, hence (2.11) is non-zero, then by Theorem 2.1, the system is not steady-state decoupled.

b) Sufficiency:

Since (2.9) is simply a sum of (2.10) for different values of i j, if (2.10) is true, (2.9) is obviously true, hence the proof.

Q.E.D.

Note that by adjusting the value of k_j (=1,2,3,$\cdots$) associated with the jth input, both Theorem 2.1 and Theorem 2.2 can be applied to systems whose inputs are either all of the same type (e.g. all inputs are steps) or hybrid (e.g. input 1 is step, input 2 is ramp, input 3 is parabolic, etc.)

Both Theorem 2.1 and 2.2 are in neat mathematical forms. However, they cannot be applied directly, since our objective is to determine specifically what to put in the matrix G_c as the compensator functions in order to decouple the steady states of the system. Therefore, further result than (2.9) and (2.10) is necessary.

Direct approach, which utilizes the expansions of both the determinant and the cofactors of a matrix, is used. A general result will be given in Chapter 4. Before going into the general problem, however, two simple cases are treated first in the following chapter.

3. SIMPLE CASES

In this chapter, 2-input, 2-output and 3-input, 3-output plants, both compensated by diagonal G_c using the feedback configuration Figure 2.1, will be considered.

Details for the 2x2 case are presented in Section 3.1. Then, in Section 3.2, the outline and results for the 3x3 case are given.

3.1 2x2 CASE

For a given 2-input, 2-output plant,

$$G_p = \begin{bmatrix} g_{p11} & g_{p12} \\ g_{p21} & g_{p22} \end{bmatrix}$$

if the diagonal compensator matrix

$$G_c = \begin{bmatrix} g_{c11} & 0 \\ 0 & g_{c22} \end{bmatrix}$$

is used, the system configuration in Figure 2.1 becomes

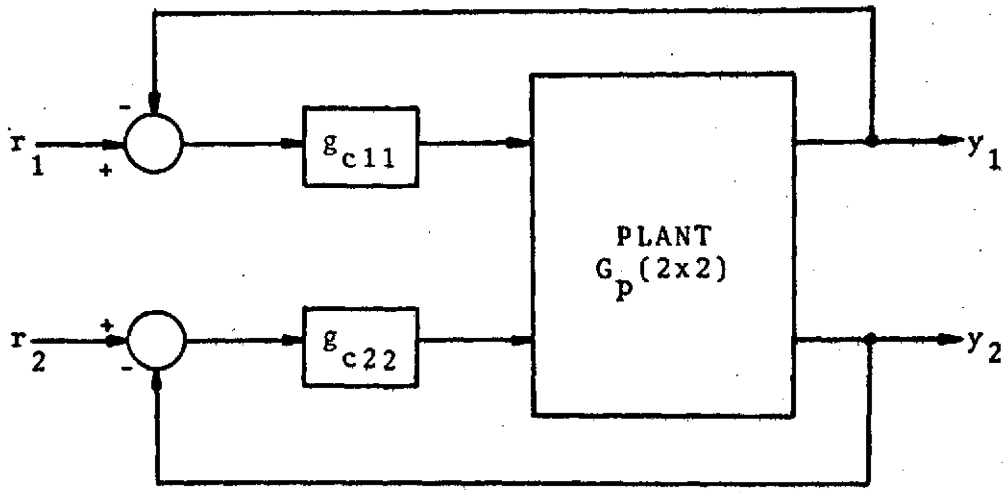

FIGURE 3.1

Since

$$G_pG_c = \begin{bmatrix} g_{p11}g_{c11} & g_{p12}g_{c22} \\ g_{p21}g_{c11} & g_{p22}g_{c22} \end{bmatrix}$$

we have

$$\det(I+G_pG_c) = 1+g_{p11}g_{c11}+g_{p22}g_{c22}+g_{p11}g_{p22}g_{c11}g_{c22} - g_{p12}g_{p21}g_{c11}g_{c22} \tag{3.1}$$

$$(I+G_pG_c)_{12} = -\ g_{p21}g_{c11} \tag{3.2}$$

$$(I+G_pG_c)_{21} = -\ g_{p12}g_{c22} \tag{3.3}$$

By Theorem 2.2, for arbitrary constants in both of the inputs $r_1(t)$ and $r_2(t)$, the system is steady-state decoupled if and only if

$$\lim_{s\to o} \frac{1}{S^{(k_1-1)}} \cdot \frac{(I+G_pG_c)_{12}}{\det(I+G_pG_c)} = 0 \tag{3.4}$$

and

$$\lim_{s\to o} \frac{1}{S^{(k_2-1)}} \cdot \frac{(I+G_pG_c)_{21}}{\det(I+G_pG_c)} = 0 \tag{3.5}$$

where k_1 and k_2 are defined as in (2.8).

Let the inputs $r_1(t)$ and $r_2(t)$ be two step functions with arbitrary amplitudes, then by (2.8), $k_1 = k_2 = 1$ and r_1, r_2 are two arbitrary constants.

Then, by substituting (3.1), (3.2), and (3.3) into (3.4) and (3.5), we have

$$\lim_{s\to o} \frac{g_{p21}g_{c11}}{(1+g_{p11}g_{c11}+g_{p22}g_{c22}+g_{p11}g_{p22}g_{c11}g_{c22} - g_{p12}g_{p21}g_{c11}g_{c22})} = 0 \tag{3.6}$$

$$\text{and} \quad \lim_{s \to o} \frac{g_{p12}g_{c22}}{(1+g_{p11}g_{c11}+g_{p22}g_{c22}+g_{p11}g_{p22}g_{c11}g_{c22} - g_{p12}g_{p21}g_{c11}g_{c22})} = 0 \qquad (3.7)$$

Note that in both (3.6) and (3.7), the s factor from the Final Value Theorem was cancelled by the $\frac{1}{s}$ factor in the input transforms, hence no explicit powers of s appears in (3.6) and (3.7).

Thus, for systems as in Figure 3.1 with arbitrary amplitude step inputs, the necessary and sufficient conditions for steady-state decoupling are (3.6) and (3.7).

For a given plant, all the g_{pij} are known, hence the design for steady-state decoupling is simply the determination of g_{c11} and g_{c22}, such that (3.6) and (3.7) are satisfied.

For example, consider

$$G_p(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+3} & \frac{1}{s+4} \end{bmatrix}$$

By (3.6) and (3.7), if

$$g_{c11}(s) = \frac{1}{s} \cdot g'_{c11}(s)$$
$$g_{c22}(s) = \frac{1}{s} \cdot g'_{c22}(s) \qquad (3.8)$$

where $g'_{c11}(s)$ and $g'_{c22}(s)$ do not contain any pole or zero at the origin, or alternatively, $\lim_{s\to o} g'_{c11}$ and $\lim_{s\to o} g'_{c22}$ are non-zero finite constants, then

$$g_{c11}(s) \cdot g_{c22}(s) = \frac{1}{s^2} g'_{c11}(s) \cdot g'_{c22}(s)$$

Since,

$$\lim_{s\to o} (g_{p11}g_{p22} - g_{p12}g_{p21}) = \frac{1}{12} \neq 0 \qquad (3.9)$$

the following term in the denominator of both (3.6) and (3.7), $(g_{p11}g_{p22}g_{c11}g_{c22} - g_{p12}g_{p21}g_{c11}g_{c22})$ which contains a $1/s^2$ factor, will go to infinity faster than both of the numerators in (3.6) and (3.7) as s approaches zero. Thus, (3.6) and (3.7) are satisfied and the system is steady-state decoupled. It is seen that the pure integrators in g_{c11} and g_{c22} and the constraint (3.9) are important. These constitute the highlights of the analysis that follows.

Let

$$g_{pij} = s^{-t_{pij}} g'_{pij}$$

$$g_{cij} = s^{-t_{cij}} g'_{cij} \qquad (3.10)$$

where t_{pij}, t_{cij} are integers that will be referred to as the type numbers of the corresponding transfer functions, and g'_{pij}, g'_{cij} are such that: $\lim_{s\to o} g'_{pij}$ and $\lim_{s\to o} g'_{cij}$ are non-zero constants (i.e. the numerators and denominators of g'_{pij} and g'_{cij} do not contain powers of s as their factors). Whenever $g_{pij} \equiv 0$ or $g_{cij} \equiv 0$, the corresponding g'_{pij} and g'_{cij} are defined to be identically zero, however t_{pij} and t_{cij} become indefinite in this case, and we will use the symbol x to identify them for reasons that will be clear in Chapter 4.

The matrices $T_p = (t_{pij})_{nxm}$ and $T_c = (t_{cij})_{mxn}$ will be referred to as the TYPE NUMBER MATRICES of the plant and compensator respectively.

For example, given

$$G_p = \begin{bmatrix} 5 & \frac{2}{s(s+3)} \\ \frac{s}{s+2} & 0 \end{bmatrix}$$

the type number matrix is

$$T_p = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$$

By separating the powers of s in each of the transfer functions as in (3.10), (3.6) and (3.7) can be expressed as:

$$\lim_{s\to o} \frac{s^{-(t_{p21}+t_{c11})} \cdot g'_{p21} \cdot g'_{c11}}{\Delta} = 0 \qquad (3.11)$$

$$\lim_{s\to o} \frac{s^{-(t_{p12}+t_{c22})} \cdot g'_{p12} \cdot g'_{c22}}{\Delta} = 0 \qquad (3.12)$$

where

$$\begin{aligned} \Delta \triangleq\ & 1+s^{-(t_{p11}+t_{c11})} g'_{p11} g'_{c11} + s^{-(t_{p22}+t_{c22})} g'_{p22} g'_{c22} \\ & + s^{-(t_{p11}+t_{p22}+t_{c11}+t_{c22})} g'_{p11} g'_{p22} g'_{c11} g'_{c22} \\ & - s^{-(t_{p12}+t_{p21}+t_{c11}+t_{c22})} g'_{p12} g'_{p21} g'_{c11} g'_{c22} \qquad (3.13) \end{aligned}$$

Thus, the necessary and sufficient conditions (3.6) and (3.7) assume different forms in (3.11) and (3.12). Again, once a plant is given, g_{pij}, and hence t_{pij}, g'_{pij} are known. Therefore, only t_{cij} and g'_{cij} in (3.11) and (3.12) are left adjustable. It was shown in a previous example that pure integrators in g_{c11} and g_{c22} (see (3.8)) are important in steady-state decoupling. In terms of the expressions given in (3.10), this is the same as saying that the values of t_{c11} and t_{c22} are the key factors in the attainment of a steady-state decoupling scheme.

In order to find out the constraints on t_{c11} and t_{c22}, such that (3.11) and (3.12) are satisfied, the following theorem is developed:

THEOREM 3.1

Let

a)

$$C_0(s) = \frac{\sum_{i=1}^{k} s^{-n_i} P_i(s)}{1+\sum_{j=1}^{\ell} s^{-m_j} q_j(s)} \tag{3.14}$$

be a rational function in s, where $P_i(s)$, $q_j(s)$ are themselves rational functions such that $\lim_{s\to o} P_i(s)$ and $\lim_{s\to o} q_j(s)$ are non-zero finite constants and n_i, m_j are integers for all $i = 1,\cdots,k$, $j = 1,\cdots,\ell$

b) $N \triangleq \text{Max}\ \{n_1, \cdots, n_k\}$

$M \triangleq \text{Max}\ \{m_1, \cdots, m_\ell\}$

c) $$\lim_{s \to o} \sum_{\substack{i=1 \\ n_i = N}}^{k} P_i(s) \neq 0 \tag{3.15}$$

$$\lim_{s \to o} \sum_{\substack{j=1 \\ m_j = M}}^{\ell} q_j(s) \neq 0 \tag{3.16}$$

$$\lim_{s \to o} \sum_{\substack{j=1 \\ m_j = 0}}^{\ell} q_j(s) \neq -1 \tag{3.17}$$

where $\sum_{\substack{i=1 \\ n_i = N}}^{k} P_i(s)$ means that the summation is only over those P_i in (3.14) that have s^{-N} as their multiplication factor. The other two summations in (3.16) and (3.17) are defined similarly.

Then $\lim_{s \to o} C_o(s) = 0$ if and only if

$$\text{either} \quad M > N \qquad \text{or} \quad M < 0,\ N < 0. \tag{3.18}$$

PROOF:

$C_o(s)$ can be expressed as

$$C_o(s) = \frac{\sum\limits_{\substack{i=1\\n_i>o}}^{k} s^{-n_i}P_i(s) + \sum\limits_{\substack{i=1\\n_i=o}}^{k} s^{-n_i}P_i(s) + \sum\limits_{\substack{i=1\\n_i<o}}^{k} s^{-n_i}P_i(s)}{1+\sum\limits_{\substack{j=1\\m_j>o}}^{\ell} s^{-m_j}q_j(s) + \sum\limits_{\substack{j=1\\m_j=o}}^{\ell} s^{-m_j}q_j(s) + \sum\limits_{\substack{j=1\\m_j<o}}^{\ell} s^{-m_j}q_j(s)}$$

Since

$$\lim_{s\to o} \sum_{\substack{i=1\\n_i<o}}^{k} s^{-n_i}P_i(s) = \lim_{s\to o} \sum_{\substack{j=1\\m_j<o}}^{\ell} s^{-m_j}q_j(s) = 0 \text{ by (a)}$$

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} \sum\limits_{\substack{i=1\\n_i>o}}^{k} s^{-n_i}p_i(s) + \lim\limits_{s\to o} \sum\limits_{\substack{i=1\\n_i=o}}^{k} P_i(s)}{1+\lim\limits_{s\to o} \sum\limits_{\substack{j=1\\m_j>o}}^{\ell} s^{-m_j}q_j(s) + \lim\limits_{s\to o} \sum\limits_{\substack{j=1\\m_j=o}}^{\ell} q_j(s)}$$

The limit value can be determined for each of the following nine possible cases:

1) $N>0,\ M>0$

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} \sum\limits_{\substack{i=1\\ n_i>o}}^{k} s^{M-n_i} P_i(s) + \lim\limits_{s\to o} \sum\limits_{\substack{i=1\\ n_i=o}}^{k} s^M P_i(s)}{\lim\limits_{s\to o} s^M + \lim\limits_{s\to o} \sum\limits_{\substack{j=1\\ m_j>o}}^{\ell} s^{M-m_j} q_j(s) + \lim\limits_{s\to o} \sum\limits_{\substack{j=1\\ m_j=o}}^{\ell} s^M q_j(s)}$$

$$= \frac{\lim\limits_{s\to o} \sum\limits_{\substack{i=1\\ n_i>o}}^{k} s^{M-n_i} P_i(s)}{\lim\limits_{s\to o} \sum\limits_{\substack{j=1\\ m_j=M}}^{\ell} q_j(s)} = 0 \text{ if and only if } M>N, \quad \text{by (3.16) \& (3.15)}$$

2) $N>0,\ M=0$

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} s^{-N} \sum\limits_{\substack{i=1\\ n_i=N}}^{k} P_i(s)}{1 + \sum\limits_{\substack{j=1\\ m_j=o}}^{\ell} q_j(s)} = \pm\infty, \quad \text{by (3.15)}$$

3) $N>0,\ M<0$

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} s^{-N} \sum\limits_{\substack{i=1\\ n_i=N}}^{k} P_i(s)}{1} = \pm\infty, \quad \text{by (3.15)}$$

4) N=0, M>0

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} \sum\limits_{\substack{i=1\\ n_i=o}}^{k} P_i(s)}{\lim\limits_{s\to o} s^{-M} \sum\limits_{\substack{j=1\\ m_j=M}}^{\ell} q_j(s)} = 0, \qquad \text{by (3.16)}$$

5) N=0, M=0

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} \sum\limits_{\substack{i=1\\ n_i=0}}^{k} P_i(s)}{1+\lim\limits_{s\to o} \sum\limits_{\substack{j=1\\ m_j=o}}^{\ell} q_j(s)} \neq 0, \qquad \text{by (3.15)}$$

6) N=0, M<0

$$\lim_{s\to o} C_o(s) = \frac{\lim\limits_{s\to o} \sum\limits_{\substack{i=1\\ n_i=o}}^{k} P_i(s)}{1} \neq 0, \qquad \text{by (3.15)}$$

7) N<0, M>0

$$\lim_{s\to o} C_o(s) = \frac{0}{\lim\limits_{s\to o} s^{-M} \sum\limits_{\substack{j=1\\ m_j=M}}^{\ell} q_j(s)} = 0, \qquad \text{by (3.16)}$$

8) $N<0,\ M=0$

$$\lim_{s\to o} C_o(s) = \frac{0}{1+\lim\limits_{s\to o} \sum\limits_{\substack{j=1\\ m_j=o}}^{\ell} q_j(s)} = 0, \qquad \text{by (3.17)}$$

9) $N<0,\ M<0$

$$\lim_{s\to o} C_o(s) = \frac{0}{1} = 0$$

Thus, $\lim\limits_{s\to o} C_o(s) = 0$ if and only if one of (1), (4), (7), (8), (9) is true. Since the conditions in (1) to (8) are equivalent to $\lim\limits_{s\to o} C_o(s) = 0$ if and only if $M>N$, and (9) gives $M<0$, $N<0$, the theorem is proved.

Q.E.D.

Note that (3.18) contains only the powers of s, neither p_i nor q_j appears in this expression. Also, note that $C_o(s)$ is of exactly the same form as the rational functions in (3.11) and (3.12), therefore, the theorem can be applied directly.

Compare (3.11) and (3.12) with (3.14), and by the definition of M and N in (b) of Theorem 3.1, we have:

$$M=\text{Max}\ \{t_{p11}+t_{c11}, t_{p22}+t_{c22}, t_{p11}+t_{p22}+t_{c11}+t_{c22},$$
$$t_{p12}+t_{p21}+t_{c11}+t_{c22}\}$$

$$N_{12}=\text{Max}\ \{t_{p21}+t_{c11}\} = t_{p21}+t_{c11} \qquad (3.19)$$

$$N_{21}=\text{Max}\ \{t_{p12}+t_{c22}\} = t_{p12}+t_{c22}$$

Where the notation Max {···} denotes the maximum value among all the elements in the brackets and the subscripts on N are used in accordance with (3.2) and (3.3) to distinguish them from each other.

Since t_{pij} are known for a given plant, the only unknowns in (3.19) are t_{c11} and t_{c22}, which can be chosen to satisfy (3.18). Once t_{c11}, t_{c22} are chosen, M, N_{12}, N_{21} are known, and (3.15), (3.16), (3.17) can then be written down explicitly.

In general, these expressions contain both g'_{pij} and g'_{cij}. Since the pole and zero locations and the gains of each g'_{cij} are free parameters, they can hopefully be adjusted to satisfy (3.15), (3.16), and (3.17).

These free parameters should also be designed for stability and transient response of the system, therefore, they cannot be adjusted with complete freedom. However, as was mentioned before, (3.18) does not depend on g'_{cij}, therefore, the design of stability will not destroy the steady-state decoupling as long as (3.15), (3.16) and (3.17) are not violated. Hence, once t_{c11} and t_{c22} are determined, stability can be considered.

After all g'_{cij} are designed, however, (3.15), (3.16) and (3.17) must be checked. If satisfied, the design is completed, if not, slight adjustments of the free parameters, under the allowance of stability, can be made in order to satisfy these constraints and hence guarantees that the steady states are decoupled.

It might happen that in some cases, no adjustment in g'_{cij} is possible to satisfy these constraints, e.g.,

$$\lim_{s \to o} (g'_{p11} g'_{p22} - g'_{p12} g'_{p21})\, g'_{c11} g'_{c22} \neq 0 \text{ is not possible if}$$

$$\lim_{s \to o} (g'_{p11} g'_{p22} - g'_{p12} g'_{p21}) = 0 \text{ happens to be true for the}$$
given plant.

In cases like this, another choice of t_{c11} and t_{c22} is necessary.

Following this procedure, we can, at present, assume that (3.15), (3.16) and (3.17) are satisfied. Then, by Theorem 3.1, if the constraints (3.15), (3.16) and (3.17) for the rational function in (3.11) are satisfied, then (3.11) is true if and only if

$$\begin{aligned} &\text{either} \quad M>N_{12} \\ &\text{or} \quad M<0,\ N_{12}<0 \end{aligned} \tag{3.20}$$

Similarly, if the constraints (3.15), (3.16) and (3.17) for the function in (3.12) are satisfied, then (3.12) is true if and only if

$$\begin{aligned} &\text{either} \quad M>N_{21} \\ &\text{or} \quad M<0,\ N_{21}<0 \end{aligned} \tag{3.21}$$

For steady-state decoupling, both (3.11) and (3.12) must be true, therefore, combining (3.20) and (3.21), each of the following four sets of criteria can be used:

$$\begin{aligned} &M>N_{12} \\ &M>N_{21} \end{aligned} \tag{3.22}$$

$$M > N_{12}, \quad M < 0,\ N_{21} < 0 \tag{3.23}$$

$$M > N_{21}, \quad M < 0,\ N_{12} < 0 \tag{3.24}$$

$$M < 0, \quad N_{12} < 0, \quad N_{21} < 0 \tag{3.25}$$

It should be noticed that (3.23) and (3.24) are redundant since they are contained in (3.25).

The best choice among these four sets will depend on the type number matrix of the given plant.

Consider the following example: Given the 2x2 plant[3]

$$G_p(s) = \begin{bmatrix} \frac{-s+1}{(s+1)^2} & \frac{-s+2}{(s+1)^2} \\ \frac{-3s+1}{3(s+1)^2} & \frac{-s+1}{(s+1)^2} \end{bmatrix} \tag{3.26}$$

[3] The plant is taken from an example in (5).

Find a steady-state decoupling scheme using diagonal G_c and the configuration of Figure 2.1 assuming that the inputs are arbitrary steps.

By inspection, $t_{pij} = 0$ for all i, j=1, 2, then by (3.19),

$$M=Max\ \{t_{c11}, t_{c22}, t_{c11}+t_{c22}, t_{c11}+t_{c22}\}$$

$$N_{12}= t_{c11} \qquad (3.27)$$

$$N_{21}= t_{c22}$$

If (3.22) is used, $t_{c11}=t_{c22}=1$ is the simplest solution (note that the solution is not unique). For this particular choice, $g_{c11}= \frac{1}{s} g'_{c11}$, $g_{c22}=\frac{1}{s} g'_{c22}$, hence the introduction of pure integrators in the loops will cause the steady states to be decoupled.

(3.23), (3.24) and (3.25) can also be used, however, in this case, the solutions for both t_{c11} and t_{c22} will turn out to be negative, which corresponds to the introduction of differentiators in G_c, and is physically undesirable.

Since only one term appears in the numerators of (3.11) and (3.12), (3.15) is satisfied automatically by definition of g'_{pij} and g'_{cij} (see (3.10)).

By inspection of (3.13) and by noting that both of the last two terms contain s^{-M} for the above choice of t_{c11} and t_{c22}, we have for (3.16)

$$\lim_{s \to o} (g'_{p11} g'_{p22} g'_{c11} g'_{c22} - g'_{p12} g'_{p21} g'_{c11} g'_{c22}) \neq 0$$

Since

$$\lim_{s \to o} (g'_{p11} g'_{p22} - g'_{p12} g'_{p21}) = 1/3 \neq 0$$

We have

$$\lim_{s \to o} g'_{c11} g'_{c22} \neq 0$$

which is again satisfied automatically.

Similarly, by inspection of (3.13), (3.17) is also satisfied automatically, since by the above choice of t_{c11} and t_{c22}, none of the terms in (3.13) has 0 as the power of the associated s factor.

Therefore, we are guaranteed to have a steady-state decoupled system by introducing one pure integrator in each of g_{c11} and g_{c22}.

Actually, in this case we don't need (3.15) and (3.17), since $M=2>0$, $N_{12}=N_{21}=1>0$, and by (1) in the proof of Theorem 3.1, only (3.16) is sufficient.

It should also be noticed from the proof of Theorem 3.1 that the constraint (3.15) was used only to make (3.18) also a necessary condition. If (3.15) is not true, the sufficient part of the theorem is still guaranteed by (3.16) and (3.17). Therefore, it is usually only necessary to check (3.16) and (3.17) in practical design.

For inputs other than steps, Theorem 3.1 must be generalized as follows:

THEOREM 3.2

Let

$$C_t(s) = \frac{1}{s^t} C_o(s) \tag{3.28}$$

where $t = 0, 1, 2, \cdots$, $C_o(s)$ is as defined in (3.14) and (b), (c) are the same as in Theorem 3.1.

Then

$$\lim_{s \to o} C_t(s) = 0 \quad \text{if and only if}$$

$$\text{either } M<0,\ N+t<0 \qquad\qquad (3.29)$$

$$\text{or} \quad M>N+t$$

PROOF: By writing:

$$C_t(s) = \frac{\sum_{i=1}^{k} s^{-(n_i + t)} P_i(s)}{1 + \sum_{j=1}^{\ell} s^{-m_j} q_j(s)},$$

the result follows immediately by Theorem 3.1.

Q.E.D.

Now let input 2 be a ramp, while input 1 is still a step (i.e., $k_1=1$, $k_2=2$). Then by (3.4), (3.6) and (3.11) are still the same. However, for input 2, since $k_2=2$ in (3.5), there will be an additional s factor in the denominators of (3.7) and (3.12). Since the only difference is this additional s, (3.19) remains unchanged, and the application of Theorem 3.2 gives $M>N_{21}+1$. Therefore, the conditions corresponding to (3.22) becomes:

$M > N_{12}$

$M > N_{21} + 1$

For the plant (3.26), by (3.27) we have $t_{c11}=2$, $t_{c22}=1$ as the simplest solution.

Thus, we need one more integrator in g_{c11} in order to decouple the steady states, if input 2 is a ramp instead of a step.

3.2 3x3 CASE

For a given 3-input, 3-output plant,

$$G_p = \begin{bmatrix} g_{p11} & g_{p12} & g_{p13} \\ g_{p21} & g_{p22} & g_{p23} \\ g_{p31} & g_{p32} & g_{p33} \end{bmatrix}$$

if the diagonal compensator matrix

$$G_c = \begin{bmatrix} g_{c11} & 0 & 0 \\ 0 & g_{c22} & 0 \\ 0 & 0 & g_{c33} \end{bmatrix}$$

is used, the system configuration is as shown in Figure 3.2.

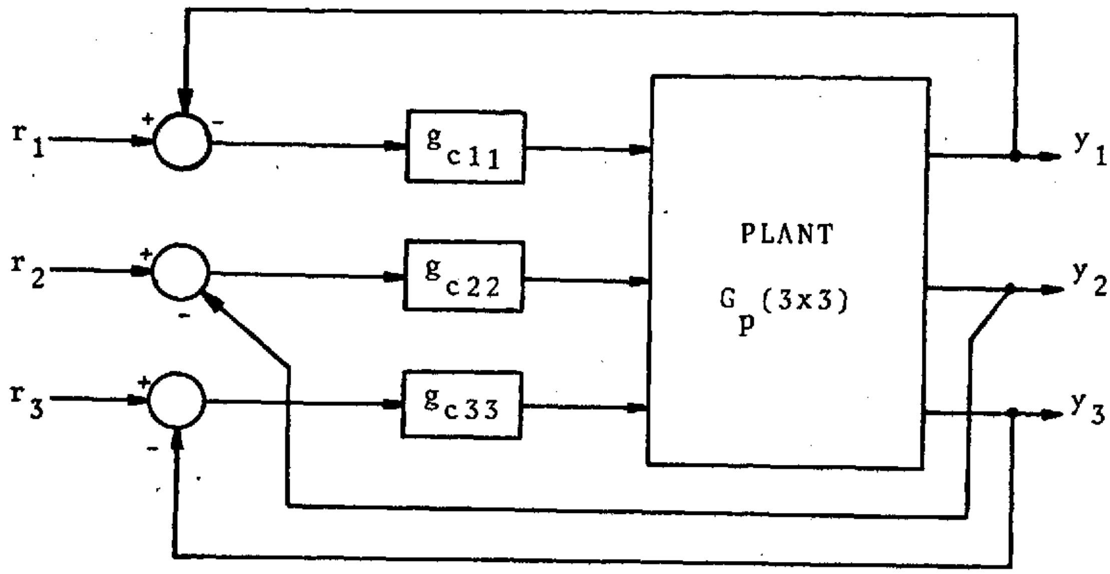

FIGURE 3.2

Expressions for $\det(I+G_pG_c)$ and $(I+G_pG_c)_{ij}$ can be obtained either by direct expansion as was done in Section 3.1 (see (3.1), (3.2) and (3.3)), or by using the formulae (4.5) and (4.6), then by (2.10) of Theorem 2.2 and assuming step inputs (i.e. $k_1=k_2=k_3=1$), a set of 6 limit expressions similar to (3.11) and (3.12) can be obtained. Compare these with (3.14), we have

$$\begin{aligned}M=\text{Max}\,\{&t_{p11}+t_{c11},t_{p22}+t_{c22},t_{p33}+t_{c33},t_{p11}+t_{p22}+t_{c11}+t_{c22},\\&t_{p12}+t_{p21}+t_{c11}+t_{c22},t_{p11}+t_{p33}+t_{c11}+t_{c33},t_{p13}+t_{p31}\\&+t_{c11}+t_{c33},t_{p22}+t_{p33}+t_{c22}+t_{c33},t_{p23}+t_{p32}+t_{c22}+t_{c33},\\&t_{p11}+t_{p22}+t_{p33}+t_{c11}+t_{c22}+t_{c33},t_{p12}+t_{p23}+t_{p31}+t_{c11}\\&+t_{c22}+t_{c33},t_{p13}+t_{p21}+t_{p32}+t_{c11}+t_{c22}+t_{c33},t_{p13}+t_{p22}\\&+t_{p31}+t_{c11}+t_{c22}+t_{c33},t_{p12}+t_{p21}+t_{p33}+t_{c11}+t_{c22}+t_{c33},\\&t_{p11}+t_{p23}+t_{p32}+t_{c11}+t_{c22}+t_{c33}\}\end{aligned} \qquad (3.30)$$

$$\begin{aligned}
N_{12} &= \text{Max } \{t_{p21}+t_{c11}, t_{p21}+t_{p33}+t_{c11}+t_{c33}, \\
&\quad t_{p23}+t_{p31}+t_{c11}+t_{c33}\} \\
N_{13} &= \text{Max } \{t_{p31}+t_{c11}, t_{p31}+t_{p22}+t_{c11}+t_{c22}, \\
&\quad t_{p32}+t_{p21}+t_{c11}+t_{c22}\} \\
N_{21} &= \text{Max } \{t_{p12}+t_{c22}, t_{p12}+t_{p33}+t_{c22}+t_{c33}, \\
&\quad t_{p13}+t_{p32}+t_{c22}+t_{c33}\} \\
N_{23} &= \text{Max } \{t_{p32}+t_{c22}, t_{p32}+t_{p11}+t_{c11}+t_{c22}, \\
&\quad t_{p31}+t_{p12}+t_{c11}+t_{c22}\} \\
N_{31} &= \text{Max } \{t_{p13}+t_{c33}, t_{p13}+t_{p22}+t_{c22}+t_{c33}, \\
&\quad t_{p23}+t_{p12}+t_{c22}+t_{c33}\} \\
N_{32} &= \text{Max } \{t_{p23}+t_{c33}, t_{p23}+t_{p11}+t_{c11}+t_{c33}, \\
&\quad t_{p21}+t_{p13}+t_{c11}+t_{c33}\}
\end{aligned} \tag{3.30}$$

Note that if $g_{pij} = 0$ for some i,j in a given plant, any term in (3.30) that contains the corresponding t_{pij} has to be dropped. In Chapter 4, an analytical scheme will be designed to take care of this.

Similar to (3.22) through (3.25), we have 64 $(=2^{n(n-1)})$ possible sets of criteria here to choose from. However, similar to the previous case, only the two corresponding to (3.22) and (3.25) are not redundant, these are:

$$\begin{aligned} &M > N_{12} \\ &M > N_{13} \\ &M > N_{21} \\ &M > N_{23} \\ &M > N_{31} \\ &M > N_{32} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} &M < 0 \\ &N_{12} < 0 \\ &N_{13} < 0 \\ &N_{21} < 0 \\ &N_{23} < 0 \\ &N_{31} < 0 \\ &N_{32} < 0 \end{aligned} \tag{3.32}$$

Consider the following example:

EXAMPLE:

Given a 3x3 plant[4] with

$$
\begin{aligned}
g_{p11}(s) &= 0.081(s-0.205)(s+0.967+j1.379)\\
&\quad (s+0.967-j1.379)/D\cdot D_{TH}\\
g_{p12}(s) &= -6.12(s+0.837)(s+0.947+j1.144)\\
&\quad (s+0.947-j1.144)/D\cdot D_{F}\\
g_{p13}(s) &= -202(s+1.885)(s-13.037)/D\cdot D_{E}\\
g_{p21}(s) &= -0.00163(s+2.881)(s+0.032+j0.313)\\
&\quad (s+0.032-j0.313)/D\cdot D_{TH}\\
g_{p22}(s) &= -0.153(s+0.824)(s-0.047+j0.205)\\
&\quad (s-0.047-j0.205)/D\cdot D_{F}\\
g_{p23}(s) &= -9.07(s+26.339)(s+0.03+j0.361)\\
&\quad (s+0.03-j0.361)/D\cdot D_{E}\\
g_{p31}(s) &= -0.00209(s-1.049)(s+0.268)/D\cdot D_{TH}\\
g_{p32}(s) &= 0.0995(s-0.12)(s+3.485)/D\cdot D_{F}\\
g_{p33}(s) &= -235.5(s+0.361+j0.076)\\
&\quad (s+0.361-j0.076)/D\cdot D_{E}
\end{aligned}
\tag{3.33}
$$

[4]NASA STOL C-8A aircraft, with thrust, flap angle and elevator angle as the inputs and velocity, angle of attack, pitch angle as the outputs.

where

$$D(s) = (s+0.018+j0.336)\ (s+0.018-j0.336)\ (s+1.103+j1.277)\ (s+1.103-j1.277)$$

$$D_{TH}(s) = (s+0.99+j0.479)\ (s+0.99-j0.479)$$

$$D_E(s) = (s+3.3+j10.49)\ (s+3.3-j10.49)$$

$$D_F(s) = s + 1$$

Find a steady-state decoupling scheme using diagonal G_c, and the configuration of Figure 3.2, assuming that the inputs are arbitrary steps.

By inspection of (3.33), $t_{pij} = 0$ for all $i,j = 1,2,3$. Thus (3.30) assumes the following simple form:

$$\begin{aligned}
M &= \text{Max}\ \{t_{c11}, t_{c22}, t_{c33}, t_{c11}+t_{c22}, t_{c11}+t_{c33}, \\
&\qquad t_{c22}+t_{c33}, t_{c11}+t_{c22}+t_{c33}\} \\
N_{12} &= \text{Max}\ \{t_{c11}, t_{c11}+t_{c33}\} \\
N_{13} &= \text{Max}\ \{t_{c11}, t_{c11}+t_{c22}\} \\
N_{21} &= \text{Max}\ \{t_{c22}, t_{c22}+t_{c33}\} \\
N_{23} &= \text{Max}\ \{t_{c22}, t_{c22}+t_{c11}\} \\
N_{31} &= \text{Max}\ \{t_{c33}, t_{c33}+t_{c22}\} \\
N_{32} &= \text{Max}\ \{t_{c33}, t_{c33}+t_{c11}\}
\end{aligned} \tag{3.34}$$

By (3.31) and (3.34), it is clear that M must be $t_{c11}+t_{c22}+t_{c33}$, and t_{c11}, t_{c22}, t_{c33} must be positive (otherwise $t_{c11}+t_{c22}+t_{c33}$ cannot be a maximum). Hence the simplest solution is $t_{c11} = t_{c22} = t_{c33} = 1$. This means that the introduction of one pure integrator in each of the compensators g_{c11}, g_{c22}, g_{c33} will cause the steady states of the system to be decoupled.

Since $M>0$, $N_{ij}>0$, only (3.16) has to be checked. It can readily be found that this is satisfied, therefore we are guaranteed to have a steady-state decoupled system.

(3.32) can also be used, however, as in the 2x2 example of Section 3.1, the result requires pure differentiators in g_{c11}, g_{c22} and g_{c33}, hence also not desirable for this particular plant.

4. GENERAL nxm CASE

Results of Chapter 3 show that, for the two special cases considered, the steady-state decoupling criterion can be written as a set of n(n-1) inequalities, where n is the number of outputs of the plant (or number of inputs or outputs of the system).

These results will be generalized in this chapter to systems consisting of m-input, n-output plant G_p(nxm), n-input, m-output compensator G_c(mxn), and unity feedbacks are employed as shown in Figure 2.1. Exactly the same approach as in Chapter 3 is presumed and it will be seen that both Theorem 3.1 and 3.2 are applicable.

As was shown in Chapter 3, the first step is to obtain expressions for $\det(I+G_pG_c)$ and $(I+G_pG_c)_{ji}$ as in (3.1), (3.2) and (3.3). For the general case, this can be accomplished by using the following formulae which are proved in Appendix A.

For any nxn square matrix G

$$\det\,(I+G) = 1 + \sum_{\ell=1}^{n} \sum_{1 \le k_1 < \cdots < k_\ell \le n} G\begin{pmatrix} k_1, \cdots, k_\ell \\ k_1, \cdots, k_\ell \end{pmatrix} \tag{4.1}$$

$$(I+G)_{ji} = -\sum_{\ell=0}^{n-2} \sum_{\substack{1 \le k_1 < \cdots < k_\ell \le n \\ k_1, \cdots, k_\ell \ne i,j}} G\begin{pmatrix} i, k_1, \cdots, k_\ell \\ j, k_1, \cdots, k_\ell \end{pmatrix} \tag{4.2}$$

where $G\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$ denotes the minor formed from rows $i_1,\cdots,i_\ell$ and columns $j_1,\cdots,j_\ell$ of the matrix G with $1 \le \ell \le n$.

Let $G = G_pG_c$ where G_p is the nxm plant matrix and G_c is the mxn compensator matrix. Then by Binet-Cauchy formula (6), we have

$$G\begin{pmatrix} k_1,\cdots,k_\ell \\ k_1,\cdots,k_\ell \end{pmatrix} = \begin{cases} \sum\limits_{1\le\sigma_1<\cdots<\sigma_\ell\le m} G_p\begin{pmatrix} k_1,\cdots,k_\ell \\ \sigma_1,\cdots,\sigma_\ell \end{pmatrix} G_c \begin{pmatrix} \sigma_1,\cdots,\sigma_\ell \\ k_1,\cdots,k_\ell \end{pmatrix} & \ell\le m \\ 0 & \ell>m \end{cases} \tag{4.3}$$

and

$$G\begin{pmatrix} i,k_1,\cdots,k_\ell \\ j,k_1,\cdots,k_\ell \end{pmatrix} = \begin{cases} \sum\limits_{1\le\rho_0<\cdots<\rho_\ell\le m} G_p\begin{pmatrix} i,k_1,\cdots,k_\ell \\ \rho_0,\rho_1,\cdots,\rho_\ell \end{pmatrix} G_c\begin{pmatrix} \rho_0,\rho_1,\cdots,\rho_\ell \\ j,k_1,\cdots,k_\ell \end{pmatrix} & \ell\le m-1 \\ 0 & \ell>m-1 \end{cases} \tag{4.4}$$

Combining (4.1) and (4.3), (4.2) and (4.4), respectively, we have

$$\det(I+G_pG_c)=1+\sum_{\ell=1}^{\min(n,m)}\ \sum_{1\le k_1<\cdots<k_\ell\le n}\ \sum_{1\le\sigma_1<\cdots<\sigma_\ell\le m} G_p\begin{pmatrix}k_1,\cdots,k_\ell\\ \sigma_1,\cdots,\sigma_\ell\end{pmatrix} G_c\begin{pmatrix}\sigma_1,\cdots,\sigma_\ell\\ k_1,\cdots,k_\ell\end{pmatrix} \tag{4.5}$$

$$(I+G_pG_c)_{ji}=-\sum_{\ell=0}^{\min(n-2,m-1)}\ \sum_{\substack{1\le k_1<\cdots<k_\ell\le n\\ k_1,\cdots,k_\ell\ne i,j}}\ \sum_{1\le\rho_0<\cdots<\rho_\ell\le m} G_p\begin{pmatrix}i,k_1,\cdots,k_\ell\\ \rho_0,\rho_1,\cdots,\rho_\ell\end{pmatrix} G_c\begin{pmatrix}\rho_0,\rho_1,\cdots,\rho_\ell\\ j,k_1,\cdots,k_\ell\end{pmatrix} \tag{4.6}$$

Then, by expanding the associated minors, (4.5) can be written as

$$\det(I+G_pG_c)=1+\sum_{\ell=1}^{\min(n,m)}\ \sum_{1\le k_1<\cdots<k_\ell\le n}\ \sum_{1\le\sigma_1<\cdots<\sigma_\ell\le m}$$

$$\sum_{\sigma'}\delta^{\sigma_1,\cdots,\sigma_\ell}_{\sigma'_1,\cdots,\sigma'_\ell}\ G_p\begin{pmatrix}k_1\\ \sigma'_1\end{pmatrix}\cdots G_p\begin{pmatrix}k_\ell\\ \sigma'_\ell\end{pmatrix}$$

$$\sum_{\sigma''}\delta^{\sigma_1,\cdots,\sigma_\ell}_{\sigma''_1,\cdots,\sigma''_\ell}\ G_c\begin{pmatrix}\sigma''_1\\ k_1\end{pmatrix}\cdots G_c\begin{pmatrix}\sigma''_\ell\\ k_\ell\end{pmatrix}$$

$$=1+\sum_{\ell=1}^{\min(n,m)} \sum_{1\le k_1<\cdots<k_\ell\le n} \sum_{1\le\sigma_1<\cdots<\sigma_\ell\le m} \sum_{\sigma'}\sum_{\sigma''} \delta^{\sigma'_1,\cdots,\sigma'_\ell}_{\sigma''_1,\cdots,\sigma''_\ell} G_p\begin{pmatrix}k_1\\ \sigma'_1\end{pmatrix}\cdots G_p\begin{pmatrix}k_\ell\\ \sigma'_\ell\end{pmatrix} G_c\begin{pmatrix}\sigma''_1\\ k_1\end{pmatrix}\cdots G_c\begin{pmatrix}\sigma''_\ell\\ k_\ell\end{pmatrix} \quad (4.7)$$

where $\delta^{\sigma'_1 \quad \sigma'_\ell}_{\sigma''_1,\cdots,\sigma''_\ell}$

$$=\begin{cases}1 \text{ if } \sigma'_1,\cdots,\sigma'_\ell \text{ is an even permutation of } \sigma''_1,\cdots,\sigma''_\ell\\ -1 \text{ if } \sigma'_1,\cdots,\sigma'_\ell \text{ is an odd permutation of } \sigma''_1,\cdots,\sigma''_\ell\end{cases}$$

and $\sum_{\sigma'}$, $\sum_{\sigma''}$ represent summations over all possible permutations of $\sigma'_1,\cdots,\sigma'_\ell$, and $\sigma''_1,\cdots,\sigma''_\ell$, respectively.

The identity $\delta^{\sigma'_1,\cdots,\sigma'_\ell}_{\sigma''_1,\cdots,\sigma''_\ell} = \delta^{\sigma_1,\cdots,\sigma_\ell}_{\sigma'_1,\cdots,\sigma'_\ell}\ \delta^{\sigma_1,\cdots,\sigma_\ell}_{\sigma''_1,\cdots,\sigma''_\ell}$ is

used, which can be proved by first rearranging $\sigma'_1,\cdots,\sigma'_\ell$ into $\sigma_1,\cdots,\sigma_\ell$, then $\sigma''_1,\cdots,\sigma''_\ell$ and by using simple reasoning.

It can easily be shown by letting n=m=2 in (4.7) that (3.1) can be obtained through this expression. Similarly, general expressions for the cofactors $(I+G_pG_c)_{ji}$ can be obtained by (4.6). Thus, by using (4.1), (4.2), (4.3), and (4.4), the problem of expressing $\det(I+G_pG_c)$ and $(I+G_pG_c)_{ji}$ in terms of the transfer functions g_{pij} and g_{cij} explicitly is solved.

Then, following the approach of Chapter 3, limit expressions similar to (3.6) and (3.7) can be obtained. In order to evaluate the values of these limits, it was found convenient to express each transfer function as in (3.10). By doing so, (4.7) can be written as:

$$\det(I+G_pG_c)=1+\sum_{\ell=1}^{\min(n,m)}\ \sum_{1\le k_1<\cdots<k_\ell\le n}\ \sum_{1\le\sigma_1<\cdots<\sigma_\ell\le m}\ \sum_{\sigma'}$$

$$\sum_{\sigma''}\delta\begin{matrix}\sigma_1',\cdots,\sigma_\ell'\\ \sigma_1'',\cdots,\sigma_\ell''\end{matrix}\ S^{-\left[T_p\binom{k_1}{\sigma_1'}+\cdots+T_p\binom{k_\ell}{\sigma_\ell'}+T_c\binom{\sigma_1''}{k_1}+\cdots+T_c\binom{\sigma_\ell''}{k_\ell}\right]}$$

$$\cdot\ G_p'\binom{k_1}{\sigma_1'}\cdots G_p'\binom{k_\ell}{\sigma_\ell'}\cdot G_c'\binom{\sigma_1''}{k_1}\cdots G_c'\binom{\sigma_\ell''}{k_\ell} \tag{4.8}$$

Note that there are

$$J \triangleq \sum_{\ell=1}^{\min(n,m)} {}_nC_\ell\cdot{}_mC_\ell\cdot\ell!\cdot\ell! = \sum_{\ell=1}^{\min(n,m)} {}_nP_\ell\cdot{}_mP_\ell \tag{4.9}$$

terms in (4.8). Similarly, (4.6) can be manipulated into the following form:

$$(I+G_pG_c)_{ji} = -\sum_{\ell=0}^{\min(n-2,m-1)} \sum_{\substack{1\le k_1<\cdots<k_\ell\le n \\ k_1,\cdots,k_\ell\ne i,j}} \sum_{1\le\rho_0<\cdots<\rho_\ell\le m}$$

$$\cdot \sum_{\rho'} \sum_{\rho''} \delta^{\rho_0' \quad \rho_\ell'}_{\rho_0'' \quad \rho_\ell''}$$

$$S^{-\left[T_p\binom{i}{\rho_0'}+T_p\binom{k_1}{\rho_1'}+\cdots+T_p\binom{k_\ell}{\rho_\ell'}+T_c\binom{\rho_0''}{j}+T_c\binom{\rho_1''}{k_1}+\cdots+T_c\binom{\rho_\ell''}{k_\ell}\right]}$$

$$\cdot\ G_p'\binom{i}{\rho_0'}\ G_p'\binom{k_1}{\rho_1'}\cdots G_p'\binom{k_\ell}{\rho_\ell'}\ \cdot\ G_c'\binom{\rho_0''}{j}\ G_c'\binom{\rho_1''}{k_1}\cdots G_c'\binom{\rho_\ell''}{k_\ell} \qquad (4.10)$$

Again, note that there are

$$L \triangleq \sum_{\ell=0}^{\min(n-2,m-1)} {}_{n-2}C_\ell \cdot {}_mC_{\ell+1} \cdot (\ell+1)!\ \cdot (\ell+1)!$$
$$= \sum_{\ell=0}^{\min(n-2,m-1)} (\ell+1) \cdot {}_{n-2}P_\ell \cdot {}_mP_{\ell+1} \qquad (4.11)$$

terms in (4.10).

By (4.8), (4.10) and Theorem 2.2 (assuming arbitrary r_j), n(n-1) limit expressions similar to (3.11) and (3.12) can be obtained. These limits are, according to Theorem 2.2, necessary and sufficient conditions for steady-state decoupling. In order to satisfy these conditions, Theorem 3.1 and 3.2 were developed to find the constraints on t_{cij}.

By comparing (4.8) and (4.10) to the denominator and the numerator of (3.14), it can be seen that they are of exactly the same form, only that n_i, m_j, p_i, q_j assume more complicated forms here. Therefore, similar to what was done in Chapter 3, constraints on t_{cij} for steady-state decoupling can be obtained by applying Theorem 3.1 (or Theorem 3.2, if the inputs contain ramps or parabolas besides steps) to each of these n(n-1) limits.

To be more precise, let's go through these step by step as follows:

1. Assume step inputs with arbitrary amplitudes,(i.e., k_j=1, r_j arbitrary for all j=1,···n), and consider the configuration Figure 2.1. By Theorem 2.2, the system is steady-state decoupled if and only if

$$\lim_{s\to o} \frac{(I+G_pG_c)_{ji}}{\det(I+G_pG_c)} = 0 \qquad (4.12)$$

for all i,j=1,···,n and i≠j.

Note that $1/(s^{k_j} - 1) \equiv 1$ since $k_j = 1$ for step inputs.

2. By comparing the denominator and the numerator of (3.14) to those of (4.12) which are given respectively in (4.8) and (4.10), we have J m's, J q's, L n's and L p's (see (4.9) and (4.11)) as follows:

$$m_{j'} = T_p\binom{k_1}{\sigma'_1} + \cdots + T_p\binom{k_\ell}{\sigma'_\ell} + T_c\binom{\sigma''_1}{k_1} + \cdots + T_c\binom{\sigma''_\ell}{k_\ell} \tag{4.13}$$

$$n_{i'} = T_p\binom{i}{\rho'_o} + T_p\binom{k_1}{\rho'_1} + \cdots + T_p\binom{k_\ell}{\rho'_\ell} + T_c\binom{\rho''_o}{j} + T_c\binom{\rho''_1}{k_1} + \cdots + T_c\binom{\rho''_\ell}{k_\ell} \tag{4.14}$$

$$q_{j'} = G'_p\binom{k_1}{\sigma'_1} \cdots G'_p\binom{k_\ell}{\sigma'_\ell} G'_c\binom{\sigma''_1}{k_1} \cdots G'_c\binom{\sigma''_\ell}{k_\ell} \tag{4.15}$$

$$p_{i'} = G'_p\binom{i}{\rho'_o} G'_p\binom{k_1}{\rho'_1} \cdots G'_p\binom{k_\ell}{\rho'_\ell} G'_c\binom{\rho''_o}{j} G'_c\binom{\rho''_1}{k_1} \cdots G'_c\binom{\rho''_\ell}{k_\ell} \tag{4.16}$$

where each possible combination of k's σ's and ρ's under the restrictions in (4.8) and (4.10) contributes to one of the above.

3. Then, for each of the n(n-1) limits (4.12), Theorem 3.1 can be applied and a set of inequalities consisting of

$$\text{Max } \{m_{j'} \mid j'=1,\cdots,J\} \tag{4.17}$$

and $$\text{Max } \{n_{i'} \mid i'=1,\cdots,L\} \tag{4.18}$$

can be obtained as in Chapter 3. Again, since all the T'_ps in (4.13) and (4.14) are known for any given plant, the only unknowns are the T'_cs which can be chosen to satisfy the inequalities and hence achieve a steady-state decoupling scheme.

4. Whenever any transfer function in G_p or G_c is identically zero, those terms in the summations of (4.8) and (4.10) that contain such a factor will also be identically zero, hence the number of non-trivial terms in (4.8) and (4.10) will be less than J and L, respectively. The number of m_j', and n_i', will also be reduced. Thus, those m_j', and n_i', in (4.17) and (4.18) associated with the identical zero term should be dropped, since they don't even appear in (3.14). In order to express this analytically, the identification symbol "x" introduced in Chapter 3 (see the discussion following (3.10)) will be used. Also, the following definition of annihilation sum is needed:

The annihilation sum is defined to be a summation, which will sum up to be an empty set whenever there exists at least one identification symbol x in the summands, otherwise it is the same as algebraic sum. The symbol (+) will be used for such kind of summation, e.g.

$$1(+)2(+)3 = 1+2+3 = 6$$

$$1(+)x(+)3 = \emptyset, \text{ an empty set.}$$

By using these concepts, (4.13), (4.14) become

$$m_{j'} = T_p\begin{pmatrix} k_1 \\ \sigma_1' \end{pmatrix} (+) \cdots (+) T_p\begin{pmatrix} k_\ell \\ \sigma_\ell' \end{pmatrix} (+) T_c\begin{pmatrix} \sigma_1'' \\ k_1 \end{pmatrix} + \cdots (+) T_c\begin{pmatrix} \sigma_\ell'' \\ k_\ell \end{pmatrix} \quad (4.19)$$

$$n_{i'} = T_p\begin{pmatrix} i \\ \rho_o' \end{pmatrix} (+) T_p\begin{pmatrix} k_1 \\ \rho_1' \end{pmatrix} (+) \cdots (+) T_p\begin{pmatrix} k_\ell \\ \rho_\ell' \end{pmatrix} (+) T_c\begin{pmatrix} \rho_o'' \\ j \end{pmatrix} (+) T_c\begin{pmatrix} \rho_1'' \\ k_1 \end{pmatrix}$$

$$(+) \cdots (+) T_c\begin{pmatrix} \rho_\ell'' \\ k_\ell \end{pmatrix} \quad (4.20)$$

Now, starting from the type number matrices T_p and T_c (see Chapter 3), we know immediately from (4.19) and (4.20) which $m_{j'}$ and $n_{i'}$ are to be included and which should be discarded.

5. Define M, N_{ji} to be the 1xJ and 1xL vectors with their elements corresponding to the J and L annihilation sums given in (4.19) and (4.20). Note that some of their elements can be an empty set, whenever "x" is contained in those particular elements. In this way, whether a term should be dropped or not is expressed analytically. Then, by Theorem 2.2 and Theorem 3.2, the following general theorem for steady-state decoupling can be given:

<u>Theorem 4:</u> Let the given nxm plant G_p be compensated by an mxn G_c as in Figure 2.1. T_p (nxm), T_c (mxn) are the type number matrices of the plant and compensator, respectively. M, N_{ji}

are defined to be the maximum among the elements of M and N_{ji}, respectively. Where M and N_{ji} are the 1xJ, and 1xL vectors defined above, then under the following constraints:

(i)
$$\lim_{s\to o} \sum_{\substack{\ell=0 \\ \langle N_{ji}\rangle}}^{\min(n-2,m-1)} \sum_{\substack{1\le k_1<\cdots<k_\ell\le n \\ k_1,\cdots,k_\ell\ne i,j}} \sum_{1<\rho_0<\cdots<\rho_\ell\le m} \sum_{\rho'} \sum_{\rho''} \delta^{\rho'_0,\cdots,\rho'_\ell}_{\rho''_0,\cdots,\rho''_\ell} G'_p\begin{pmatrix} i \\ \rho'_0 \end{pmatrix} G'_p\begin{pmatrix} k_1 \\ \rho'_1 \end{pmatrix} \cdots G'_p\begin{pmatrix} k_\ell \\ \rho'_\ell \end{pmatrix} G'_c\begin{pmatrix} \rho''_0 \\ j \end{pmatrix} G'_c\begin{pmatrix} \rho''_1 \\ k_1 \end{pmatrix} \cdots G'_c\begin{pmatrix} \rho''_\ell \\ k_\ell \end{pmatrix} \ne 0 \qquad (4.21)$$

(ii)
$$\lim_{s\to o} \sum_{\substack{\ell=1 \\ \langle M\rangle}}^{\min(n,m)} \sum_{1\le k_1<\cdots<k_\ell\le n} \sum_{1\le\sigma_1<\cdots<\sigma_\ell\le m} \sum_{\sigma'} \sum_{\sigma''} \delta^{\sigma'_1,\cdots,\sigma'_\ell}_{\sigma''_1,\cdots,\sigma''_\ell} G'_p\begin{pmatrix} k_1 \\ \sigma'_1 \end{pmatrix} \cdots G'_p\begin{pmatrix} k_\ell \\ \sigma'_\ell \end{pmatrix} G'_c\begin{pmatrix} \sigma''_1 \\ k_1 \end{pmatrix} \cdots G'_c\begin{pmatrix} \sigma''_\ell \\ k_\ell \end{pmatrix} \ne 0 \qquad (4.22)$$

(iii)
$$\lim_{s\to o} \sum_{\substack{\ell=1 \\ \langle 0\rangle}}^{\min(n,m)} \sum_{1\le k_1<\cdots<k_\ell\le n} \sum_{1\le\sigma_1<\cdots<\sigma_\ell\le m} \sum_{\sigma'} \sum_{\sigma''} \delta^{\sigma'_1,\cdots,\sigma'_\ell}_{\sigma''_1,\cdots,\sigma''_\ell} G'_p\begin{pmatrix} k_1 \\ \sigma'_1 \end{pmatrix} \cdots G'_p\begin{pmatrix} k_\ell \\ \sigma'_\ell \end{pmatrix} G'_c\begin{pmatrix} \sigma''_1 \\ k_1 \end{pmatrix} \cdots G'_c\begin{pmatrix} \sigma''_\ell \\ k_\ell \end{pmatrix} \ne -1 \qquad (4.23)$$

Where $\langle N_{ji} \rangle$ under the summation sign of (4.21) denotes that the sum is only over all those terms with $s^{-N_{ji}}$ as their multiplication factor in (4.10). The $\langle M \rangle$ and $\langle 0 \rangle$ in (4.22) and (4.23) are defined similarly.

The system is steady-state decoupled if and only if either

$$M > N_{ji} + k_j - 1 \qquad \text{for all } i = j$$

$$\text{or} \quad M < 0 \quad N_{ji} < 1 - k_j \qquad i,j = 1,\cdots,n$$

Theorem 4 looks formidable, however, for systems with less than 4 inputs and 4 outputs, long-hand calculation is still feasible, especially when G_c assumes some simple forms like being diagonal, as were shown in Chapter 3, which is often of practical importance.

Besides, due to its analytical nature, Theorem 4 can be programmed into computers, thus making the design easier.

PART II

STABILITY AND DESIGN

5. INTRODUCTION TO PART II

Part I gives the scheme for decoupling the steady states of a system, however, it should be noticed that:

1. The result does not guarantee stability.

2. The whole discussion is meaningful only when the closed-loop system is stable.

Therefore, stability must be considered after the steady-state decoupling scheme is achieved.

The problem of stability and design of multivariable systems has been widely investigated (e.g. (5), (7), (8), (9), (10), (11)), and a survey of the existing methods was given by Anderson (12). In general, efforts have been made to utilize the beauty of the existing single-loop techniques such as Nyquist-Bode-Nichol's methods and root locus design.

In this part of the thesis, a new connection between single-loop and multivariable systems is seen by properly factorizing the closed-loop characteristic equation. This makes the design of multivariable systems possible by using any suitable classical single-loop method. An extended root locus method is developed and diagonal G_c will be considered primarily.

6. DESIGN OF CLOSED-LOOP SYSTEMS WITH 2x2 PLANT AND DIAGONAL G_c

The characteristic equation for single-loop systems with cascade compensation and unity feedback is

$$1 + g_p(s) \cdot g_c(s) = 0 \tag{6.1}$$

where $g_p(s)$ is the given plant and $g_c(s)$ is the cascade compensator function to be designed.

Two major techniques for the design of $g_c(s)$ are Bode's method and the root locus method (see (14)). However, only one unknown function can be handled in each of these methods. Therefore, they cannot be applied directly to multivariable cases, since in general, there exists $n \cdot m$ unknown compensator functions to be designed.

A simple factorization which shows the connection between single-loop and multivariable cases will be given in this chapter. It can then be seen that the above-mentioned single-loop methods can still be applied for multivariable systems.

The design philosophy will be illustrated through an example in Section 6.4. Before that, however, several important steps must be established. These are given in Section 6.1 to 6.3 as follows:

6.1 CHARACTERISTIC EQUATION

It is proved in (7) and (8) that the stability of a multivariable system as shown in Figure 2.1 is determined by the zeros of $N_1(s)$ and $\hat{N}(s)$, which are defined as follows:

$$\frac{N_1(s)}{D_1(s)} \triangleq \det(I+G_p(s)\cdot G_c(s)) \tag{6.2}$$

$$\hat{N}(s) \triangleq \frac{\Delta_c(s)\cdot\Delta_p(s)}{D_1(s)} \tag{6.3}$$

Where the rational function $N_1(s)/D_1(s)$ is in irreducible form, i.e., no common factor between $N_1(s)$ and $D_1(s)$ is left uncancelled. And $\Delta_c(s)$ $\Delta_p(s)$ represent the characteristic polynomials of the rational transfer function matrices $G_c(s)$ and $G_p(s)$, respectively.

The characteristic polynomial of a proper rational transfer function matrix G(s) is defined to be the least common denominator of all the minors (in irreducible rational form) of G(s) (see e.g. (13)).

Therefore, by (4.5), if all the minors of G_p and G_c are non-zero, and no common pole exists between G_p and G_c, the denominator of $\det(I+G_pG_c)$ is $\Delta_c(s)\ \Delta_p(s)$ before any pole-zero cancellation is performed. Since $N_1(s)/D_1(s)$ is in irreducible form, it is clear by (6.3) that $\hat{N}(s)$ simply consists of all those factors that were cancelled in getting the irreducible $N_1(s)/D_1(s)$ form. Therefore, in this case, if all the common factors in (6.2) are left uncancelled, the zeros of (6.2) alone determine the stability of the system. Unfortunately, the same conclusion is not true in general if zero minor(s) of G_p and (or) G_c exists. Furthermore, if "all" the common factors are left uncancelled, erroneous results can still be obtained as was shown by Chen in (7).

However, it is found (see Appendix B) that if

1. cancellations are selected systematically by using (4.5),

2. poles of G_c are carefully selected,

then (6.2) alone determines the stability of the system.

Since (1) above can always be done and (2) can be taken care of fairly easily in the process of design, the mathematical possibilities in which zeros of (6.3) must be considered can be bypassed.

Thus, $\det(I+G_pG_c) = 0$ (6.4)

will be referred to as the characteristic equation for multi-variable systems as shown in Figure 2.1.

Details are given in Appendix B.

6.2 CONNECTION BETWEEN SINGLE-LOOP AND MULTIVARIABLE CASES

For 2x2 plant G_p, and 2x2 diagonal G_c,

$$\det(I+G_pG_c) = 1+g_{p11}g_{c11}+g_{p22}g_{c22}+(\det G_p)g_{c11}g_{c22} \tag{6.5}$$

which can be factored as

$$\det(I+G_pG_c) = (1+g_{p11}g_{c11}) \left[1+ \frac{g_{p22}+(\det G_p)g_{c11}}{1+g_{p11}g_{c11}} g_{c22} \right] \tag{6.6}$$

$$= (1+g_{p11}g_{c11}) \; (1+G_{eq}g_{c22}) \tag{6.7}$$

where $$G_{eq} \triangleq \frac{g_{p22}+(\det G_p)g_{c11}}{1+g_{p11}g_{c11}} = \frac{g_{p22}\left[1+ \frac{\det G_p}{g_{p22}} g_{c11}\right]}{1+g_{p11}g_{c11}} \tag{6.8}$$

By (6.7), the roots of the characteristic equation are simply the zeros of the rational function $(1+g_{p11}g_{c11})\,(1+G_{eq}g_{c22})$.

It is also clear by (6.6) that for non-trivial cases ($G_{eq}(s) \neq 0$), all the zeros of the first factor will be cancelled exactly by some of the poles of the second factor. Therefore, the roots of

$$1+G_{eq} \cdot g_{c22} = 0 \tag{6.9}$$

will determine the stability of the system.

The similarity of the form of (6.9) to that of (6.1) suggests immediately the possibility of applying the single-loop methods mentioned above to multivariable cases. But, unlike the g_p in (6.1), which is known for a given plant, G_{eq} of (6.9) is not a known function. Hence, neither Bode plot nor root locus for G_{eq} can be drawn at this stage. By inspection of (6.8), the only unknown function contained in G_{eq} is g_{c11}. We can, of course, choose an arbitrary function for g_{c11}, then the only unknown function left to be designed is g_{c22}, and the design is reduced to that of the single-loop case. For example, let g_{c11} be an arbitrary constant, say 3, then for any given plant, G_{eq} can be obtained through (6.8), and the design of g_{c22} can be carried through by using (6.9).

However, in doing so, G_{eq} may turn out to be very unstable, which will make the design of g_{c22} extremely difficult. Therefore, instead of choosing it arbitrarily, a guide in designing g_{c11} is preferable.

By inspection of (6.8), it can be seen that the roots of

$$1 + \frac{\det G_p}{g_{p22}} \cdot g_{c11} = 0 \tag{6.10}$$

and
$$1 + g_{p11}g_{c11} = 0 \tag{6.11}$$

constitute part of the zeros and poles of G_{eq}, respectively.

Again, both (6.10) and (6.11) are of the standard form (6.1). Furthermore, $\det G_p/g_{p22}$ and g_{p11} are now known functions.

Therefore, any single-loop method be used in designing g_{c11}, to place the roots of (6.10) and (6.11) at desirable locations. Since these roots will be part of the poles and zeros of G_{eq}, what is meant by placing them at desirable locations is that g_{c11} should be designed such that these roots, together with the other known poles and zeros (see (6.8)) form a reasonably good pole-zero pattern for G_{eq} (i.e., G_{eq} is not badly unstable). Once g_{c11} is designed, all the poles and zeros of G_{eq} are known, and the problem of designing g_{c22} is, by (6.9), reduced to that of the single-loop case, and can be done by either Bode's or root locus method.

In summary, what has been accomplished so far is that the effect of g_{c11} on the pole-zero pattern of G_{eq} and hence on the design of g_{c22} can be seen through (6.10) and (6.11). Therefore, (6.10) and (6.11) serve as a guide in designing g_{c11} in order to make easy the design of g_{c22}.

Because of the standard forms involved, both Bode's and root locus design techniques can be applied. For better insight of the problem, the root locus method will be considered primarily.

6.3 IDENTIFICATION OF POLES AND ZEROS

Since the design will be concerned with (6.9), the poles and zeros of $G_{eq}(s)$ must be well identified, and the problem of pole-zero cancellation must be considered carefully. This can be done generally by considering the sum and ratio of the following two rational functions $G_1(s)$ and $G_2(s)$:

$$G_1(s) \triangleq \frac{N_1(s)}{D_1(s)} \tag{6.12}$$

$$G_2(s) \triangleq \frac{N_2(s)}{D_2(s)} \tag{6.13}$$

Where $N_i(s)$, $D_i(s)$ are the numerator and the denominator polynomials of $G_i(s)$ $(i=1,2)$, also note that $N_1(s)$, $D_1(s)$ here are different from those in (6.2).

For simplicity, the argument s will be omitted in the following discussion.

Let D_{12} be the greatest common factor between D_1 and D_2, and D_{11}, D_{22} are the remaining factors as shown below:

$$D_1 = D_{11} D_{12}$$
$$D_2 = D_{22} D_{12} \tag{6.14}$$

Then, the sum of G_1 and G_2 is

$$G_1 + G_2 = \frac{N_1}{D_1} + \frac{N_2}{D_2}$$

$$= \frac{N_1 D_{22} + N_2 D_{11}}{D_{11} D_{12} D_{22}} \tag{6.15}$$

Note that $D_{11} D_{12} D_{22}$ is the least common multiplier of D_1 and D_2.

Now, consider the sum $1 + G_2/G_1$, by (6.12), (6.13) and (6.14)

$$1 + \frac{G_2}{G_1} = 1 + \frac{N_2}{D_2} \cdot \frac{D_1}{N_1}$$

$$= 1 + \frac{N_2}{D_{22}\, D_{12}} \cdot \frac{D_{12}\, D_{11}}{N_1} \tag{6.16}$$

If the common factor D_{12} between D_1 and D_2 is cancelled, and no other cancellation is performed, (6.16) becomes

$$1 + \frac{G_2}{G_1} = \frac{D_{22}N_1 + D_{11}N_2}{D_{22}\, N_1} \tag{6.17}$$

The numerator of (6.17) is exactly that of (6.15). Therefore, the zeros of G_1+G_2, before any possible cancellation by the poles, would be the same as those of $1 + \frac{G_2}{G_1}$, if D_{12} (and only D_{12}) is cancelled. No other cancellation in G_2/G_1 is allowed, even if it can be done. Otherwise, some zero of (6.15) would not appear in (6.17).

Therefore, if only D_{12} is cancelled in forming G_2/G_1, the root locus for G_2/G_1 would give all the zeros of G_1+G_2 if no cancellation is done between the numerator and the denominator of (6.15).

Let $P_G = \{\cdots\}$ and $Z_G = \{\cdots\}$ denote the set of poles and zeros of the rational function G, where the multiplicity of each pole or zero is counted, e.g.,

let
$$G(s) = \frac{(s+3)^3(s+5)^2}{s(s+1)^2(s+2)}$$

$$H(s) = \frac{(s+3)^2(s+5)^2}{s^2(s+1)^3(s+2)}$$

then

$$\begin{aligned} P_G &= \{0,-1,-1,-2\} \\ Z_G &= \{-3,-3,-3,-5,-5\} \\ P_H &= \{0,0,-1,-1,-1,-2\} \\ Z_H &= \{-3,-3,-5,-5\} \end{aligned} \qquad (6.18)$$

Also, let $s_1 \cap s_2$ denote the intersection of the two sets s_1 and s_2, defined as in set theory only that multiplicity is taken into account here, e.g., in (6.18)

$$P_G \cap P_H = \{0,-1,-1,-2\}$$
$$Z_G \cap Z_H = \{-3,-3,-5,-5\}$$

Furthermore, s_1+s_2 is defined to be the set consisting of all the elements of s_1 and those of s_2, (counting multiplicity), e.g., in (6.18)

$$P_G + P_H = \{0,-1,-1,-2,0,0,-1,-1,-1,-2\}$$

Similarly, s_1-s_2 is defined to be the set formed by taking away all the elements (counting multiplicity) of s_2 from s_1, and is defined only when s_2 is a subset of s_1, e.g., in (6.18)

$$P_H - P_G = \{0,-1\}$$
$$Z_G - Z_H = \{-3\}$$

By using these notations and by inspection of the denominator of (6.15), we have

$$P_{G_1+G_2} = P_{G_1}+P_{G_2} - (P_{G_1} \cap P_{G_2}) \tag{6.19}$$

Similarly, by comparing the numerators of (6.15) and (6.17), we have

$$Z_{G_1+G_2} = Z_1 + \frac{G_2}{G_1} \tag{6.20}$$

The poles and zeros for the ratio G_2/G_1 can also be obtained in a similar way by inspection of the second term in (6.16)

$$^{P}G_2/G_1 = {}^{Z}G_1 + {}^{P}G_2 - ({}^{P}G_1 \cap {}^{P}G_2) \tag{6.21}$$

$$^{Z}G_2/G_1 = {}^{Z}G_2 + {}^{P}G_1 - ({}^{P}G_1 \cap {}^{P}G_2) \tag{6.22}$$

Note that in (6.19), (6.20), (6.21) and (6.22), D_{12} is cancelled, and no other cancellation is performed.

The application of (6.19), (6.20), (6.21) and (6.22) will be illustrated through a design example in the next section. At present, however, (6.20) will be used to justify one statement pointed out in Section 6.2, i.e., the roots of (6.9) will determine the stability of the system.

Let $\quad G_1 = 1+g_{p11}g_{c11}$

$$G_2 = g_{p22}+(\det G_p)g_{c11}$$

then (6.5) becomes

$$\det(I+G_pG_c) = G_1+G_2g_{c22} \tag{6.23}$$

By (6.20),

$$Z_{G_1+G_2 g_{c22}} = Z_{1+\frac{G_2}{G_1} g_{c22}}$$

if the common factors between the denominators of G_1 and $G_2 g_{c22}$ (and only these common factors) are cancelled.

Since $G_2/G_1 = G_{eq}$ by (6.8),

$$Z_{1+\frac{G_2}{G_1} g_{c22}} = Z_{1+G_{eq} \cdot g_{c22}}$$

Hence, the zeros of (6.23) are exactly the same as those of $1+G_{eq} \cdot g_{c22}$, i.e., the roots of (6.9) are the same as those of the characteristic equation (6.4). Therefore, they do determine the stability of the system (under the restrictions given in Appendix B). No pole would be lost on account of the factorization and the using of (6.19), (6.20), (6.21) and (6.22).

6.4 DESIGN EXAMPLE

Consider the 2x2 plant

$$G_p = \begin{bmatrix} \frac{s+3}{s(s+1)} & \frac{4}{s+1} \\ \frac{3}{s+2} & -\frac{2}{s} \end{bmatrix} \tag{6.24}$$

For diagonal G_c, the characteristic equation (6.4) is, by (6.5)

$$1+g_{p11}g_{c11}+g_{p22}g_{c22}+(\det G_p)g_{c11}g_{c22} = 0 \tag{6.25}$$

By the factorization (6.7) and the discussions in Section 6.2 and Section 6.3, the design philosophy follows:

1. Design g_{c11} according to (6.10) and (6.11), to achieve a reasonably good pole-zero pattern for G_{eq}.

2. Design g_{c22} according to (6.9) to meet system specifications.

where (6.9), (6.10) and (6.11) are repeated below:

$$1+G_{eq}\cdot g_{c22} = 0 \tag{6.9}$$

$$1+\frac{\det G_p}{g_{p22}}\,g_{c11} = 0 \tag{6.10}$$

$$1+g_{p11}g_{c11} = 0 \tag{6.11}$$

By (6.24),

$$g_{p11} = \frac{s+3}{s(s+1)} \tag{6.26}$$

$$g_{p22} = -\frac{2}{s} \tag{6.27}$$

$$\det G_p = -\frac{14s^2+10s+12}{s^2(s+1)(s+2)} \tag{6.28}$$

$$\frac{\det G_p}{g_{p22}} = \left[-\frac{14s^2+10s+12}{s^2(s+1)(s+2)}\right]\cdot\left(-\frac{s}{2}\right)$$

$$= \frac{7s^2+5s+6}{s(s+1)(s+2)} \tag{6.29}$$

Note that the common factor s between the denominators of det G_p and g_{p22} is cancelled. Also note that the use of diagonal G_c is allowed in this example, since multiplicity of each plant pole is not reduced in the minor det G_p (see Appendix B).

Let

$$G_1 = 1+g_{p11}g_{c11} \tag{6.30}$$

$$G_2 = g_{p22}+(\det G_p)g_{c11} \tag{6.31}$$

then, by (6.8),

$$G_{eq} = \frac{G_2}{G_1} \tag{6.32}$$

The poles and zeros of G_{eq} are by (6.21) and (6.22),

$$
\begin{aligned}
{}^{P}G_{eq} &= {}^{Z}G_1 + {}^{P}G_2 - ({}^{P}G_2 \cap {}^{P}G_1) \\
{}^{Z}G_{eq} &= {}^{Z}G_2 + {}^{P}G_1 - ({}^{P}G_2 \cap {}^{P}G_1)
\end{aligned}
\tag{6.33}
$$

The poles and zeros of G_1 and G_2 are by (6.19) and (6.20),

$$
{}^{P}G_1 = {}^{P}g_{p11}g_{c11} = {}^{P}g_{p11} + {}^{P}g_{c11} = \{0,-1\} + {}^{P}g_{c11} \tag{6.34}
$$

$$
{}^{Z}G_1 = {}^{Z}1 + g_{p11}g_{c11} \tag{6.35}
$$

$$
\begin{aligned}
{}^{P}G_2 &= {}^{P}g_{p22} + {}^{P}(\det G_p)g_{c11} - \left[{}^{P}g_{p22} \cap {}^{P}(\det G_p)g_{c11} \right] \\
&= \{0\} + \{0,0,-1,-2\} + {}^{P}g_{c11} - \left[\{0\} \cap (\{0,0,-1,-2\} + {}^{P}g_{c11}) \right] \\
&= \{0,0,0,-1,-2\} + {}^{P}g_{c11} - \{0\} \\
&= \{0,0,-1,-2\} + {}^{P}g_{c11}
\end{aligned}
\tag{6.36}
$$

$$
{}^{Z}G_2 = {}^{Z}1 + \frac{\det G_p}{g_{p22}} . g_{c11} \tag{6.37}
$$

By (6.34) and (6.36),

$$
{}^{P}G_1 \cap {}^{P}G_2 = \{0,-1\} + {}^{P}g_{c11} = {}^{P}G_1 \tag{6.38}
$$

Then by (6.33),

$$P_{G_{eq}} = Z_{1+g_{p11}g_{c11}} + \{0,0,-1,-2\} + P_{g11} - \left[\{0,-1\} + P_{g_{c11}}\right]$$

$$= Z_{1+g_{p11}g_{c11}} + \{0,-2\} \tag{6.39}$$

$$Z_{G_{eq}} = Z_{G_2} = Z_{1+\frac{\det G_p}{g_{p22}} \cdot g_{c11}} \tag{6.40}$$

Thus, G_{eq} has two known poles at s=0 and s=-2, the other poles are the roots of $1+g_{p11}g_{c11}=0$, which is (6.11). The zeros of G_{eq} are simply the roots of $1 + \frac{\det G_p}{g_{p22}} \cdot g_{c11} = 0$, which is (6.10).

Then, the design procedure follows:

1. Prepare the root loci for

$$1+k_1 g_{p11} = 0$$

and $1+k_1 \frac{\det G_p}{g_{p22}} = 0$

where k_1 is a real parameter. The result is shown in Figure 6.1 and 6.2. The loci for positive and negative values of k_1 are represented by solid and dashed curves, respectively.

2. The roots of Figure 6.1 will be poles of G_{eq}, therefore, for negative values of k_1, there will be one pole of G_{eq} in the right half plane (on branch ①). Similarly, by Figure 6.2, a zero of G_{eq} will be in the right half plane(on branch ①) if $k_1<0$. This is certainly undesirable. Thus, negative k_1 will not be considered.

3. Mark down the known poles s=0, and s=-2 of G_{eq} on Figure 6.3 and superimpose the loci of Figure 6.1 and 6.2 corresponding to positive k_1 on top of it. Note that roots on branch ① and ② correspond to poles and zeros of G_{eq}, respectively.

4. Increase the value of k_1 from 0 to ∞, and observe the change of pole-zero pattern. It can be seen that

   (i) $0<k_1<<5$ is not desirable, since the poles will be clustered together near the origin.

   (ii) $k_1>>5$ is also not desirable, since the pole on branch ① will be pushed into the negative real axis, hence, dominant roots of (6.9) will probably be determined by the two known poles s=0, s=-2 and the zeros on branch ②, which are too close to the origin.

5. When $k_1 \tilde{=} 5$, the roots on branch ② which are the zeros of G_{eq} will be close to the two zeros on Figure 6.2 and the pole on branch ① is as shown. Another zero, by Figure 6.2 will be on the negative real axis at about s = -35. This pole-zero configuration looks to be the best, since it is possible to confine the roots of $1+k_2G_{eq} = 0$, corresponding to the two poles s=0 and s=-2 and the two conjugate zeros, to be on the negative real axis. And at the same time, the root of $1+k_2G_{eq} = 0$ on the branch starting from the pole at $k_1=5$ will be someplace to the upper left of the pole, which is a good location for dominant root. Therefore, try $k_1=5$.

6. Once $k_1=5$ is determined, all the poles and zeros of G_{eq} are known. The root locus gain (see Appendix C) of G_{eq}, denoted by k_{eq} is found as follows:

$$G_{eq} = \frac{g_{p22}+(\det G_p)g_{c11}}{1+g_{p11}g_{c11}}$$

$$= \frac{-\frac{2}{s} - \frac{14s^2+10s+12}{s^2(s+1)(s+2)}\cdot 5}{1 + \frac{s+3}{s(s+1)}\cdot 5}$$

$$= \frac{-2s^3+\cdots}{s^4+\cdots}$$

$$= \frac{-2(s-z_1)(s-z_2)(s-z_3)}{(s-p_1)(s-p_2)(s-p_3)(s-p_4)} \qquad (6.41)$$

where z_i, and p_i are the zeros and poles. By inspection of (6.41) and the definition of root locus gain, we have

$$K_{eq} = -2$$

7. The root locus for $1+k_2G_{eq}=0$ is then drawn as shown in Figure 6.4. The choice of k_2 is now strictly that of a single-loop problem. It is easily found that the two small real roots meet each other at about $k_2k_{eq} = 0.36$, i.e. $k_2 = \frac{0.36}{k_{eq}} = -0.18$. For this value of k_2, the smallest real root on branch ① will be at approximately the breakaway point and will hopefully be the best among all the possible locations on this particular branch. The root on branch ② corresponding to $k_2k_{eq} = 0.36$ is also shown in Figure 6.4. It is seen that this is a pretty good dominant root.

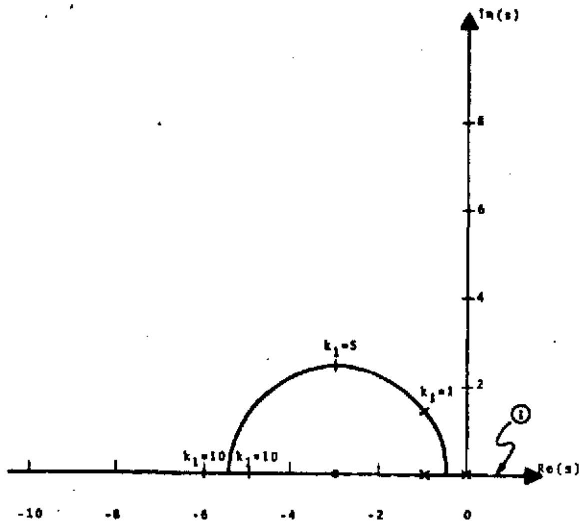

FIGURE 6.1 root locus for $1+k_1 \cdot g_{p11}=0$

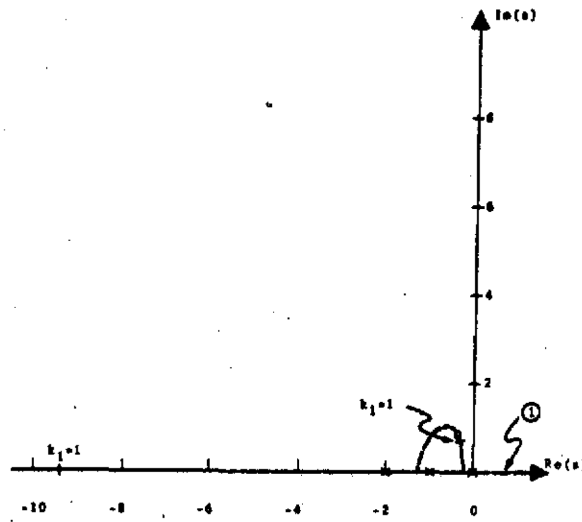

FIGURE 6.2 root locus for $1+k_1 \cdot \frac{\det G_p}{g_{p22}}=0$

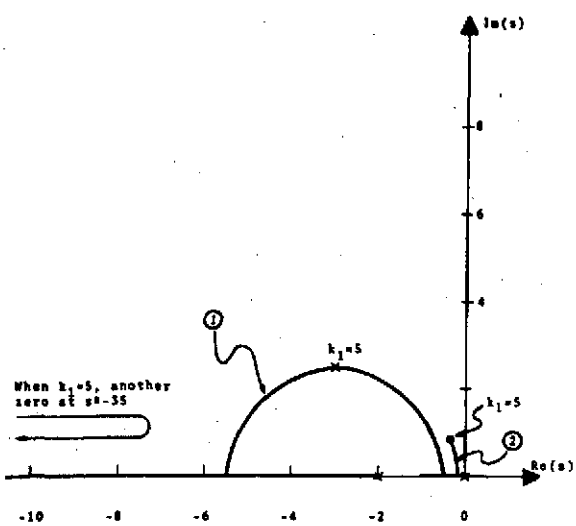

FIGURE 6.3 SUPERPOSITION OF FIGURE 6.1 and FIGURE 6.2 FOR $k_1>0$

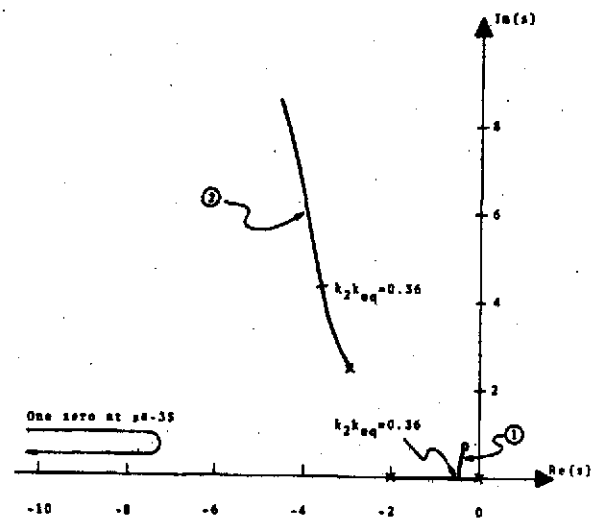

FIGURE 6.4 root locus for $1+k_2 \cdot G_{eq}=0$

The schematic for the designed system is as shown in Figure 6.5.

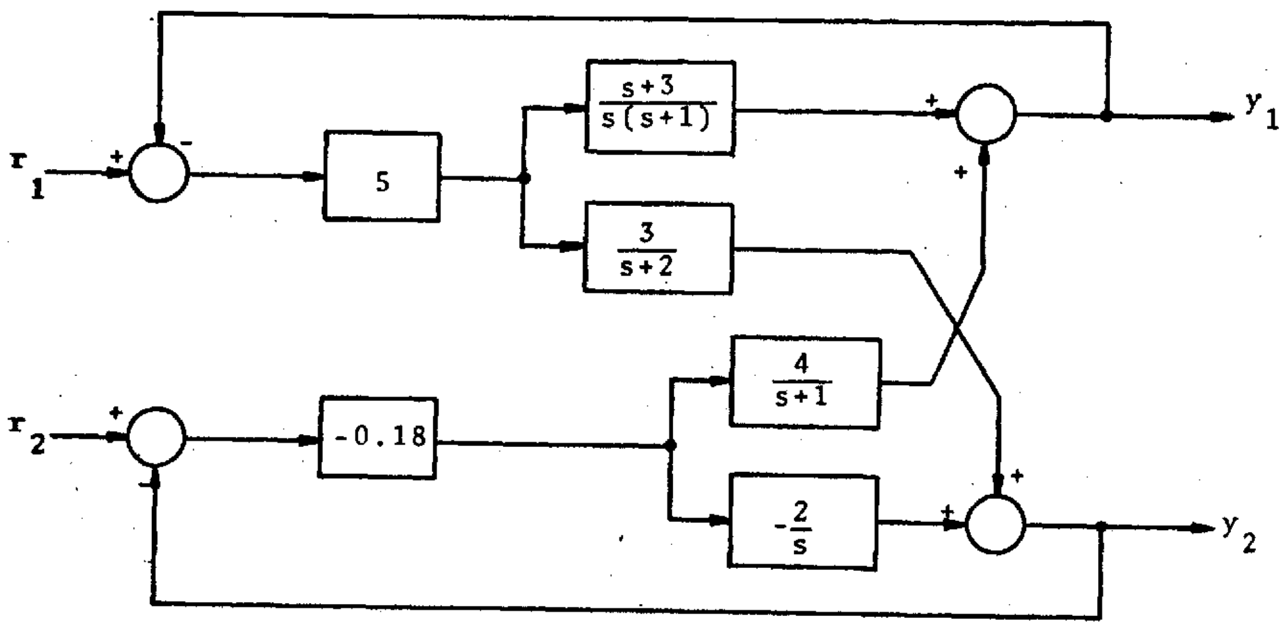

FIGURE 6.5

The simulation result is shown in Figure 6.6, in which $r_1=10$ and $r_2=5$ were used as reference inputs and the time responses for the two outputs y_1 and y_2 are as shown.

It is clear that stability has been achieved. The transient response of y_2 is very good, however, that of y_1 is kind of slow. If this is not allowed by the specification, a redesign is necessary. However, as in the single-loop case, every trial, despite of its failure, provides some guide for the next trial.

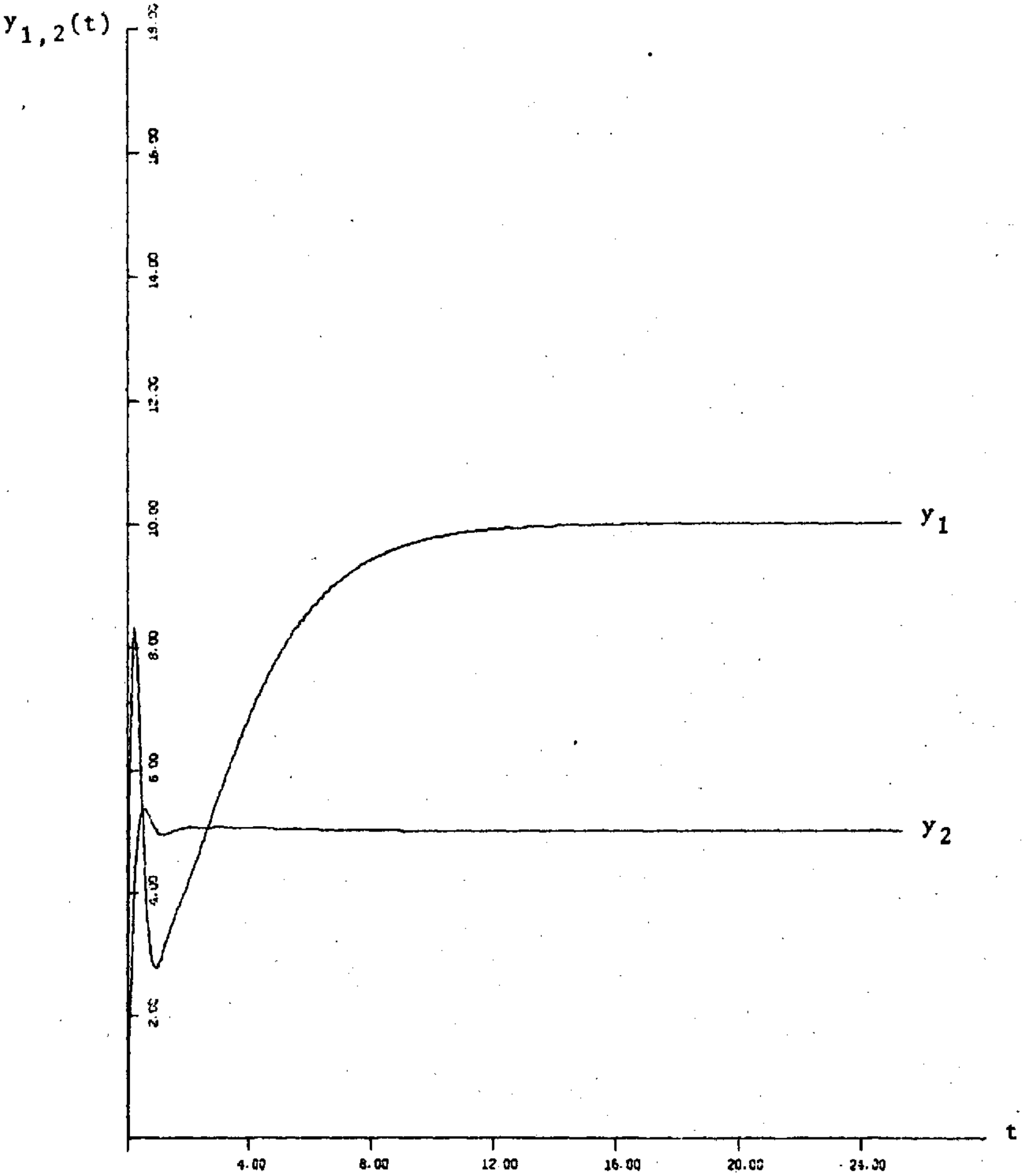

FIGURE 6.6

TIME RESPONSE OF OUTPUTS y_1 & y_2 IN FIGURE 6.5 WITH

$r_1=10\cdot u(t)$

$r_2=5\cdot u(t)$

u(t): UNIT STEP

In this example, it can be seen from Figure 6.4 that the roots on branch 1 are, compared with that on branch 2, very close to the origin. This is probably the cause of the slow response exhibited in y_1. What we can do is put pole(s) and zero(s) in g_{c11} (instead of just using a pure gain) to push the roots of Figure 6.2 farther away from $j\omega$ axis. Then the two conjugate zeros in Figure 6.4 will also be farther away from the $j\omega$ axis and the resultant root locus for G_{eq} will move toward the left, thus improving the transient response.

Incidentally, the steady states of the outputs are decoupled. This is because g_{p11} and g_{p22} are of type 1 and can be proved very easily by (3.19) and (3.22).

7. DESIGN OF CLOSED-LOOP SYSTEMS WITH 3x3 PLANT AND DIAGONAL G_c

7.1 GENERAL

It will be shown in this chapter that the same design philosophy given in Chapter 6 can be carried over to 3-input, 3-output plant. Again, the central idea is in the factorization of the characteristic equation $\det(I+G_pG_c) = 0$.

For 3x3 G_p and diagonal 3x3 G_c, the system configuration is as shown in Figure 3.2 and the characteristic equation is by (4.5),

$$\begin{aligned}\det(I+G_pG_c) &= 1+g_{p11}g_{c11}+g_{p22}g_{c22}+g_{p33}g_{c33}+(G_p)_{33}g_{c11}g_{c22} \\ &\quad + (G_p)_{22}g_{c11}g_{c33}+(G_p)_{11}g_{c22}g_{c33}+(\det G_p)g_{c11}g_{c22}g_{c33} \\ &= 0 \qquad (7.1)\end{aligned}$$

where $(G_p)_{11}$, $(G_p)_{22}$, $(G_p)_{33}$ denote the cofactors of g_{p11}, g_{p22} and g_{p33}, respectively.

(7.1) can be factored as

$$(1+g_{p11}g_{c11})\left[1+\frac{g_{p22}+(G_p)_{33}g_{c11}}{1+g_{p11}g_{c11}}\,g_{c22}+\frac{g_{p33}+(G_p)_{22}g_{c11}}{1+g_{p11}g_{c11}}\,g_{c33}\right.$$

$$\left.+\frac{(G_p)_{11}+(\det G_p)g_{c11}}{1+g_{p11}g_{c11}}\cdot g_{c22}g_{c33}\right]=0 \qquad (7.2)$$

By defining,

$$G_1 \triangleq 1+g_{p11}g_{c11}$$

$$G_2 \triangleq g_{p22}+(G_p)_{33}g_{c11}$$

$$G_3 \triangleq g_{p33}+(G_p)_{22}g_{c11} \qquad (7.3)$$

$$G_4 \triangleq (G_p)_{11}+(\det G_p)g_{c11}$$

(7.1) and (7.2) become

$$G_1+G_2g_{c22}+G_3g_{c33}+G_4g_{c22}g_{c33} = 0 \qquad (7.4)$$

$$G_1\left[1+\frac{G_2}{G_1}g_{c22}+\frac{G_3}{G_1}g_{c33}+\frac{G_4}{G_1}g_{c22}\;g_{c33}\right]=0 \qquad (7.5)$$

Similar to the 2x2 case considered in Chapter 6, the roots of (7.4) will be exactly the same as those of:

$$1+\frac{G_2}{G_1}g_{c22}+\frac{G_3}{G_1}g_{c33}+\frac{G_4}{G_1}g_{c22}g_{c33} = 0 \qquad (7.6)$$

if the common factors between the denominators of G_2 and G_1, G_3 and G_1, G_4 and G_1 (and only these common factors) are cancelled in G_2/G_1, G_3/G_1 and G_4/G_1, respectively.

Thus, if the poles and zeros of G_2/G_1, G_3/G_1 and G_4/G_1 are obtained through (6.21) and (6.22), which were designed to meet the above cancellation restrictions, the roots of (7.6) are the same as those of the characteristic equation (7.1).

By defining

$$F_1 \triangleq \frac{G_2}{G_1} = \frac{g_{p22}+(G_p)_{33}g_{c11}}{1+g_{p11}g_{c11}} \qquad (7.7)$$

$$F_2 \triangleq \frac{G_3}{G_1} = \frac{g_{p33}+(G_p)_{22}g_{c11}}{1+g_{p11}g_{c11}} \qquad (7.8)$$

$$F_3 \triangleq \frac{G_4}{G_1} = \frac{(G_p)_{11}+(\det G_p)g_{c11}}{1+g_{p11}g_{c11}} \qquad (7.9)$$

(7.6) becomes

$$1+F_1 g_{c22}+F_2 g_{c33}+F_3 g_{c22} g_{c33} = 0 \tag{7.10}$$

(7.10) is of the same form as (6.5), therefore, the same factorization can be done on (7.10) to give

$$(1+F_1 g_{c22}) \cdot \left[1 + \frac{F_2+F_3 g_{c22}}{1+F_1 g_{c22}} g_{c33} \right] = 0 \tag{7.11}$$

Again, by (6.20), if the common factor between the denominators of $F_2+F_3 g_{c22}$ and $1+F_1 g_{c22}$ (and only this common factor) is cancelled in

$$G_{eq} \triangleq \frac{F_2+F_3 g_{c22}}{1+F_1 g_{c22}} \tag{7.12}$$

the roots of (7.10) will be the same as those of

$$1 + G_{eq} \cdot g_{c33} = 0 \tag{7.13}$$

Since (7.10) is simply (7.6), and the roots of (7.6) are the same as those of the characteristic equation (7.1), if (6.21) and (6.22) are used in determining the poles and zeros of F_1, F_2 and F_3, the following conclusion can be made:

The roots of (7.13) are the same as those of the characteristic equation (7.1), if (6.19), (6.20), (6.21) and (6.22) are used in determining the poles and zeros at each stage.

Thus, stability design can be considered through (7.13). The form of (7.13) is exactly that of a single-loop characteristic equation (6.1). Therefore, similar to the design of g_{c22} for the 2x2 case in Chapter 6, any single-loop design method can be applied in designing g_{c33}, once all the poles, zeros and the root locus gain of G_{eq} are known.

The ingredients of the poles and zeros of G_{eq} can be found by substituting (7.7), (7.8) and (7.9) into (7.12) to express G_{eq} in terms of the elements of G_p and G_c explicitly. However, due to the fact that some cancellations must be done while some others are not allowed, this approach may sometimes lead to an erroneous result. Therefore, the analytical schemes (6.19), (6.20), (6.21) and (6.22) designed to take care of the pole-zero cancellations, are recommended.

By applying (6.21) and (6.22) on (7.12), we have

$$^{P}G_{eq} = {}^{Z}1{+}F_1g_{c22} + {}^{P}F_2{+}F_3g_{c22} - ({}^{P}F_2{+}F_3g_{c22} \cap {}^{P}1{+}F_1g_{c22}) \quad (7.14)$$

$$^{Z}G_{eq} = {}^{Z}F_2{+}F_3g_{c22} + {}^{P}1{+}F_1g_{c22} - ({}^{P}F_2{+}F_3g_{c22} \cap {}^{P}1{+}F_1g_{c22}) \quad (7.15)$$

(7.14) tells us that the roots of

$$1 + F_1 g_{c22} = 0 \tag{7.16}$$

constitute part of the poles of G_{eq}.

Also, by (6.20),

$$Z_{F_2+F_3 \cdot g_{c22}} = Z_{1+\frac{F_3}{F_2} g_{c22}}$$

Hence, by (7.15), the roots of

$$1 + \frac{F_3}{F_2} g_{c22} = 0 \tag{7.17}$$

constitute part of the zeros of G_{eq}.

These justify what can be seen by inspection of (7.12). Actually, these were done by inspection in Section 6.2 (see (6.8), (6.9), (6.10) and (6.11)), before the development of (6.19), (6.20), (6.21) and (6.22).

Thus, root loci for (7.16) and (7.17) identify part of the poles and zeros of G_{eq}, and similar to the 2x2 case in Chapter 6, these root loci can be used in the design of g_{c22} (note: this corresponds to g_{c11} in Chapter 6, compare (6.10), (6.11) with (7.16) and (7.17)).

However, unlike the 2x2 case, these two root loci cannot be drawn directly, since F_1 and F_3/F_2 depend on g_{c11}, which is also to be designed (hence not known yet!). The dependences of F_1, F_2 and F_3 on g_{c11} are given in (7.7), (7.8) and (7.9).

Again, by repeated application of (6.19), (6.20), (6.21) and (6.22) on (7.7), (7.8) and (7.9), or simply by inspection, it can be seen that the root loci for

$$1 + g_{p11}g_{c11} = 0 \qquad (7.18)$$

and

$$1 + \frac{(G_p)_{33}}{g_{p22}} g_{c11} = 0 \qquad (7.19)$$

give part of the poles and the zeros of F_1, respectively.

And the root loci for

$$1 + \frac{(G_p)_{22}}{g_{p33}} g_{c11} = 0 \tag{7.20}$$

and
$$1 + \frac{\det G_p}{(G_p)_{11}} g_{c11} = 0 \tag{7.21}$$

give part of the poles and the zeros of F_3/F_2, respectively.

Now, let's look back and see what we've got:

1. We concluded that roots of (7.13) are the same as those of the characteristic equation (7.1). Therefore, the design of g_{c33} can be done through (7.13) if G_{eq} is known.

2. Some of the poles and zeros of G_{eq} are adjustable through g_{c22}, and the relationships are given in (7.16) and (7.17). Thus, the effect of g_{c22} on the pole-zero pattern of G_{eq} and, hence, on the design of g_{c33}, can be seen through (7.16) and (7.17). Therefore, (7.16) and (7.17) serve as a guide in designing g_{c22} to make the design of g_{c33} not formidable. This is exactly what was obtained in Section 6.2 for the 2x2 case.

3. Since both (7.16) and (7.17) are of the standard single-loop form (6.1), g_{c22} can be designed if F_1, and F_3/F_2 are known.

4. Again, some poles and zeros of F_1 and F_3/F_2 are adjustable through g_{c11}, and the relationships are given in (7.18), (7.19), (7.20) and (7.21). Thus, the effect of g_{c11} on the pole-zero pattern of F_1 and F_3/F_2, and hence on the design of g_{c22}, can be seen through (7.18), (7.19), (7.20) and (7.21). Therefore, these four equations can be used as a guide in designing g_{c11}.

Thus, it is clear that (7.16), (7.17), (7.18), (7.19), (7.20) and (7.21) are important in stability design. By the standard forms they assume and by their similarity to those in Chapter 6, it can be concluded that root loci for these six equations can help us design g_{c11} and g_{c22} to get a stable enough G_{eq} such that g_{c33} can be designed according to (7.13).

Since g_{p11}, g_{p22}, g_{p33}, $(G_p)_{11}$, $(G_p)_{22}$, $(G_p)_{33}$ and $\det G_p$ are known for a given plant, the four uncompensated root loci (i.e., $g_{c11}=k$, a free parameter) for (7.18), (7.19), (7.20) and (7.21) can be constructed right after the plant is given. However, the other two root loci for (7.16) and (7.17) cannot be drawn until after g_{c11} is designed, since both F_1 and F_3/F_2 depend on g_{c11}. As mentioned before, F_1 and F_3/F_2 have some poles and zeros other than those given by (7.18), (7.19), (7.20) and (7.21). Therefore, the identification of these poles and zeros is necessary, both for constructing root loci for (7.16) and (7.17) and for guiding the design of g_{c11}. This constitutes the topic of the following section.

7.2 IDENTIFICATION OF POLES AND ZEROS

As mentioned in the previous section, direct algebraic manipulation may lead to an erroneous result, so let's apply the analytical schemes (6.19), (6.20), (6.21) and (6.22) to identify all the poles and zeros we are interested in.

The expressions for all the poles and zeros of G_{eq} have already been given in (7.14) and (7.15). The poles and zeros in the sets $Z_{1+F_1g_{c22}}$ and $Z_{F_2+F_3g_{c22}}$ are the roots of (7.16) and (7.17), respectively, and can be taken care of by the corresponding root loci. The other poles and zeros left to be identified are respectively the elements of the following two sets:

$$P_{F_2+F_3g_{c22}} - (P_{F_2+F_3g_{c22}} \cap P_{1+F_1g_{c22}})$$
$$P_{1+F_1g_{c22}} - (P_{F_2+F_3g_{c22}} \cap P_{1+F_1g_{c22}}) \qquad (7.22)$$

In order to express these explicitly in terms of the poles and zeros in the plant G_p and the compensator G_c, repeated applications of (6.19), (6.20), (6.21), and (6.22) on $1+F_1g_{c22}$, $F_2+F_3g_{c22}$, F_1, F_2, F_3/F_2 are necessary. The results can be written down by inspection as follows:

$$P_{F_2+F_3g_{c22}} = P_{F_2} + P_{F_3} + P_{g_{c22}} - (P_{F_2} \cap P_{F_3g_{c22}})$$
$$P_{1+F_1g_{c22}} = P_{F_1} + P_{g_{c22}} \tag{7.23}$$

$$P_{F_1} = Z_{1+g_{p11}g_{c11}} + P_{g_{p22}+(G_p)_{33}g_{c11}} - \{P_{1+g_{p11}g_{c11}} \cap P_{g_{p22}+(G_p)_{33}g_{c11}}\}$$
$$Z_{F_1} = Z_{g_{p22}+(G_p)_{33}g_{c11}} + P_{1+g_{p11}g_{c11}} - \{P_{1+g_{p11}g_{c11}} \cap P_{g_{p22}+(G_p)_{33}g_{c11}}\} \tag{7.24}$$
$$P_{F_2} = Z_{1+g_{p11}g_{c11}} + P_{g_{p33}+(G_p)_{22}g_{c11}} - \{P_{1+g_{p11}g_{c11}} \cap P_{g_{p33}+(G_p)_{22}g_{c11}}\}$$

$$
\begin{aligned}
{}^{Z}F_2 &= {}^{Z}g_{p33}+(G_p)_{22}g_{c11} + {}^{P}1+g_{p11}g_{c11} \\
&\quad - \{{}^{P}1+g_{p11}g_{c11} \cap {}^{P}g_{p33}+(G_p)_{22}g_{c11}\} \\
{}^{P}F_3 &= {}^{Z}1+g_{p11}g_{c11} + {}^{P}(G_p)_{11}+(\det G_p)g_{c11} \\
&\quad - \{{}^{P}1+g_{p11}g_{c11} \cap {}^{P}(G_p)_{11}+(\det G_p)g_{c11}\} \\
{}^{Z}F_3 &= {}^{Z}(G_p)_{11}+(\det G_p)g_{c11} + {}^{P}1+g_{p11}g_{c11} \\
&\quad - \{{}^{P}1+g_{p11}g_{c11} \cap {}^{P}(G_p)_{11}+(\det G_p)g_{c11}\}
\end{aligned}
\tag{7.24}
$$

where

$$
\begin{aligned}
{}^{P}g_{p22}+(G_p)_{33}g_{c11} &= {}^{P}g_{p22} + {}^{P}(G_p)_{33} + {}^{P}g_{c11} \\
&\quad - \{{}^{P}g_{p22} \cap {}^{P}(G_p)_{33}g_{c11}\} \\
{}^{P}g_{p33}+(G_p)_{22}g_{c11} &= {}^{P}g_{p33} + {}^{P}(G_p)_{22} + {}^{P}g_{c11} \\
&\quad - \{{}^{P}g_{p33} \cap {}^{P}(G_p)_{22}g_{c11}\} \\
{}^{P}(G_p)_{11}+(\det G_p)g_{c11} &= {}^{P}(G_p)_{11} + {}^{P}(\det G_p) + {}^{P}g_{c11} \\
&\quad - \{{}^{P}(G_p)_{11} \cap {}^{P}(\det G_p)g_{c11}\}
\end{aligned}
\tag{7.25}
$$

$$P_{1+g_{p11}g_{c11}} = P_{g_{p11}} + P_{g_{c11}} \tag{7.25}$$

Also

$$\begin{aligned} P_{F_3/F_2} &= Z_{F_2} + P_{F_3} - (P_{F_2} \cap P_{F_3}) \\ Z_{F_3/F_2} &= Z_{F_3} + P_{F_2} - (P_{F_2} \cap P_{F_3}) \end{aligned} \tag{7.26}$$

By proper substitutions of (7.25) into (7.24), then (7.26), (7.23) and (7.22), the poles and zeros of interest can be identified.

This will be illustrated through a numerical example in the following section.

At present, however, some simplifications on the above general expressions can be made. It is observed that if,

$$\begin{aligned} &P_{g_{p22}} \subset P_{(G_p)_{33}} \\ &P_{g_{p33}} \subset P_{(G_p)_{22}} \\ &P_{(G_p)_{11}} \subset P_{\det G_p} \end{aligned} \tag{7.27}$$

where $A \subset B$ denotes that set A is a subset of set B, again multiplicities of the elements in each set are counted. Then, we have

$$
\begin{aligned}
&P_{g_{p22}} \cap P_{(G_p)_{33} g_{c11}} = P_{g_{p22}} \\
&P_{(G_p)_{11}} \cap P_{(\det G_p) g_{c11}} = P_{(G_p)_{11}} \\
&P_{g_{p33}} \cap P_{(G_p)_{22} g_{c11}} = P_{g_{p33}}
\end{aligned}
\tag{7.28}
$$

then (7.25) becomes

$$
\begin{aligned}
&P_{g_{p22}+(G_p)_{33} g_{c11}} = P_{(G_p)_{33}} + P_{g_{c11}} \\
&P_{g_{p33}+(G_p)_{22} g_{c11}} = P_{(G_p)_{22}} + P_{g_{c11}} \\
&P_{(G_p)_{11}+(\det G_p) g_{c11}} = P_{(\det G_p)} + P_{g_{c11}} \\
&P_{1+g_{p11} g_{c11}} = P_{g_{p11}} + P_{g_{c11}}
\end{aligned}
\tag{7.29}
$$

and

$$
\begin{aligned}
{}^{P}1+g_{p11}g_{c11} \cap {}^{P}g_{p22}+(G_p)_{33}g_{c11} &= {}^{P}g_{c11}+\{{}^{P}g_{p11} \cap {}^{P}(G_p)_{33}\} \\
{}^{P}1+g_{p11}g_{c11} \cap {}^{P}g_{p33}+(G_p)_{22}g_{c11} &= {}^{P}g_{c11}+\{{}^{P}g_{p11} \cap {}^{P}(G_p)_{22}\} \qquad (7.30) \\
{}^{P}1+g_{p11}g_{c11} \cap {}^{P}(G_p)_{11}+(\det G_p)g_{c11} &= {}^{P}g_{c11}+\{{}^{P}g_{p11} \cap {}^{P}(\det G_p)\}
\end{aligned}
$$

By (7.29) and (7.30), (7.24), (7.26) become

$$
\begin{aligned}
{}^{P}F_1 &= {}^{Z}1+g_{p11}g_{c11}+{}^{P}(G_p)_{33}-\{{}^{P}g_{p11} \cap {}^{P}(G_p)_{33}\} \\
{}^{Z}F_1 &= {}^{Z}g_{p22}+(G_p)_{33}g_{c11}+{}^{P}g_{p11}-\{{}^{P}g_{p11} \cap {}^{P}(G_p)_{33}\} \\
{}^{P}F_2 &= {}^{Z}1+g_{p11}g_{c11}+{}^{P}(G_p)_{22}-\{{}^{P}g_{p11} \cap {}^{P}(G_p)_{22}\} \qquad (7.31) \\
{}^{Z}F_2 &= {}^{Z}g_{p33}+(G_p)_{22}g_{c11}+{}^{P}g_{p11}-\{{}^{P}g_{p11} \cap {}^{P}(G_p)_{22}\} \\
{}^{P}F_3 &= {}^{Z}1+g_{p11}g_{c11}+{}^{P}(\det G_p)-\{{}^{P}g_{p11} \cap {}^{P}(\det G_p)\} \\
{}^{Z}F_3 &= {}^{Z}(G_p)_{11}+(\det G_p)g_{c11}+{}^{P}g_{p11}-\{{}^{P}g_{p11} \cap {}^{P}(\det G_p)\}
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{P}F_3/F_2 = {}^{Z}g_{p33} + (G_p)_{22}g_{c11} + {}^{P}g_{p11} - \{{}^{P}g_{p11} \cap {}^{P}(G_p)_{22}\} + {}^{P}(\det G_p) \\
- \{{}^{P}g_{p11} \cap {}^{P}(\det G_p)\} - \{{}^{P}(\det G_p) - ({}^{P}g_{p11} \cap {}^{P}(\det G_p))\} \cap \\
\{{}^{P}(G_p)_{22} - ({}^{P}g_{p11} \cap {}^{P}(G_p)_{22})\}
\end{aligned}
\tag{7.32}
$$

$$
\begin{aligned}
{}^{Z}F_3/F_2 = {}^{Z}(G_p)_{11} + (\det G_p)g_{c11} + {}^{P}g_{p11} - \{{}^{P}g_{p11} \cap {}^{P}(\det G_p)\} + {}^{P}(G_p)_{22} \\
- \{{}^{P}g_{p11} \cap {}^{P}(G_p)_{22}\} - \{{}^{P}(\det G_p) - ({}^{P}g_{p11} \cap {}^{P}(\det G_p))\} \cap \\
\{{}^{P}(G_p)_{22} - ({}^{P}g_{p11} \cap {}^{P}(G_p)_{22})\}
\end{aligned}
$$

Once a plant is given, ${}^{P}(G_p)_{33} - \{{}^{P}g_{p11} \cap {}^{P}(G_p)_{33}\}$ is a known set. Therefore, by ${}^{P}F_1$ of (7.31), all the poles of F_1 are well identified. They consist of all the roots of (7.18) which are adjustable through g_{c11}, and some other fixed poles given by the above set which is known.

Similarly, by inspection of (7.31) and (7.32), all the zeros of F_1, all the poles and zeros of F_3/F_2 are well identified. Some of them are fixed, the others, which are the roots of (7.19), (7.20) and (7.21) respectively, are adjustable through g_{c11}. Therefore, g_{c11} can be designed according to the four root loci for (7.18), (7.19), (7.20) and (7.21), to realize a presumably good pole-zero pattern for F_1 and F_3/F_2, such that the design of

g_{c22} according to (7.16) and (7.17) will not be formidable.

As anyone who is familiar with root locus design knows, there is a certain amount of trial-and-error involved. This is more so in the multivariable case because of the successive dependence of the root loci described so far. However, a little experience can always lead to good judgements that would reduce the amount of the trial-and-error. For example, in the design of g_{c11} described above, if the root loci for (7.18), (7.19), (7.20) and (7.21) extend well into the right-half-plane, more sections for g_{c11} is, in general, recommended. Otherwise, it is very probable that the resulting F_1 or F_3/F_2 (or both) contain poles and zeros well in the right-half-plane, thus making g_{c22} difficult to design. This is, of course, a trade-off between g_{c11} and g_{c22}. If more sections of compensation are used in g_{c11}, F_1 and F_3/F_2 can be made more stable, hence, less sections are required in g_{c22}. Conversely, if g_{c11} is chosen to be too simple, more sections should be needed in g_{c22}. Judicious choice can be made by investigating the four root loci (7.18), (7.19), (7.20) and (7.21), and g_{c11} can be designed accordingly.

The poles and zeros of G_{eq} can also be identified in a similar way. This will be clearer after the consideration of the numerical example in the following section.

7.3 DESIGN EXAMPLE

As was found in Section 7.1 and 7.2, g_{c11} can be designed according to the four root loci

$$1 + g_{p11}g_{c11} = 0 \tag{7.18}$$

$$1 + \frac{(G_p)_{33}}{g_{p22}}g_{c11} = 0 \tag{7.19}$$

$$1 + \frac{(G_p)_{22}}{g_{p33}}g_{c11} = 0 \tag{7.20}$$

$$1 + \frac{\det G_p}{(G_p)_{11}}g_{c11} = 0 \tag{7.21}$$

And g_{c22} can be designed by the other two

$$1 + F_1 g_{c22} = 0 \tag{7.16}$$

$$1 + \frac{F_3}{F_2}g_{c22} = 0 \tag{7.17}$$

Finally, g_{c33} is designed according to

$$1 + G_{eq} \cdot g_{c33} = 0 \tag{7.13}$$

where F_1, F_2, F_3 and G_{eq} are given in (7.7), (7.8), (7.9), and (7.12), respectively.

The general design procedure then follows:

1. Identify all the poles and zeros of F_1, F_3/F_2 and G_{eq}.

2. Prepare root loci for (7.18), (7.19), (7.20) and (7.21) with $g_{c11} = k_1$, a free parameter.

3. By varying the value of k_1 from $-\infty$ to ∞, observe the accompanying changes in root locations.

4. Choose the value of k_1 that corresponds to the best pole-zero pattern for F_1 and F_3/F_2 in the sense that g_{c22} can be designed most easily to give good pole-zero pattern for G_{eq}.

5. If no value of k_1 gives satisfactory F_1 and F_3/F_2, use pole-zero pair as necessary in g_{c11} to pull the loci toward the left and determine the gain value for best pole-zero pattern for F_1 and F_3/F_2.

6. Construct the root loci for (7.16) and (7.17), using $g_{c22}=k_2$, a free parameter.

7. Adjust k_2 as in 3 to find best pole-zero pattern for G_{eq}. Put in poles and zeros as necessary, as in 5.

8. Construct root locus for (7.13).

9. Design g_{c33} to meet specifications.

Consider the 3-input, 3-output plant (3.33) repeated below:

$$
\begin{aligned}
g_{p11}(s) &= 0.081(s-0.205)(s+0.967+j1.379) \\
&\quad (s+0.967-j1.379)/D\cdot D_{TH} \\
g_{p12}(s) &= -6.12(s+0.837)(s+0.947+j1.144) \\
&\quad (s+0.947-j1.144)/D\cdot D_{F} \\
g_{p13}(s) &= -202(s+1.885)(s-13.037)/D\cdot D_{E} \\
g_{p21}(s) &= -0.00163(s+2.881)(s+0.032+j0.313) \\
&\quad (s+0.032-j0.313)/D\cdot D_{TH} \\
g_{p22}(s) &= -0.153(s+0.824)(s-0.047+j0.205) \\
&\quad (s-0.047-j0.205)/D\cdot D_{F} \\
g_{p23}(s) &= -9.07(s+26.339)(s+0.03+j0.361) \\
&\quad (s+0.03-j0.361)/D\cdot D_{E} \\
g_{p31}(s) &= -0.00209(s-1.049)(s+0.268)/D\cdot D_{TH} \\
g_{p32}(s) &= 0.0995(s-0.12)(s+3.485)/D\cdot D_{F} \\
g_{p33}(s) &= -235.5(s+0.361+j0.076) \\
&\quad (s+0.361-j0.076)/D\cdot D_{E}
\end{aligned}
\qquad (7.33)
$$

where

$$D(s)=(s+0.018+j0.336)(s+0.018-j0.336)$$
$$(s+1.103+j1.277)(s+1.103-j1.277)$$
$$D_{TH}(s)=(s+0.99+j0.479)(s+0.99-j0.479) \quad (7.34)$$
$$D_E(s)=(s+3.3+j10.49)(s+3.3-j10.49)$$
$$D_F(s)=s+1$$

By manipulation,

$$(G_p)_{11}=g_{p22}g_{p33}-g_{p23}g_{p32}=\frac{36.85(s-0.096)}{D\cdot D_F\cdot D_E}$$

$$(G_p)_{22}=g_{p11}g_{p33}-g_{p13}g_{p31}=\frac{-19.08(s+0.229)}{D\cdot D_E\cdot D_{TH}} \quad (7.35)$$

$$(G_p)_{33}=g_{p11}g_{p22}-g_{p12}g_{p21}=\frac{-0.0224(s^2+1.656s+0.694)}{D\cdot D_F\cdot D_{TH}}$$

$$\det G_p=\frac{5.232}{D\cdot D_E\cdot D_F\cdot D_{TH}}$$

By inspection of (7.33) and (7.35), it is clear that (7.27) is true. Therefore, by (7.31),

$$P_{F_1} = Z_{1+g_{p11}g_{c11}}+Z_{D_F}$$
$$(7.36)$$
$$Z_{F_1} = Z_{g_{p22}+(G_p)_{33}g_{c11}}$$

$$ {}^{P}F_{2} = {}^{Z}1{+}g_{p11}g_{c11} + {}^{Z}D_{E} $$
(7.37)
$$ {}^{Z}F_{2} = {}^{Z}g_{p33}{+}(G_{p})_{22}g_{c11} $$

$$ {}^{P}F_{3} = {}^{Z}1{+}g_{p11}g_{c11} + {}^{Z}D_{E} + {}^{Z}D_{F} $$
(7.38)
$$ {}^{Z}F_{3} = {}^{Z}(G_{p})_{11}{+}(\det G_{p})g_{c11} $$

where ${}^{Z}D = \{(-0.018\pm j0.336),(-1.103\pm j1.277)\}$

${}^{Z}D_{TH} = \{-0.99\pm j0.479\}$

${}^{Z}D_{E} = \{-3.3\pm j10.49\}$

${}^{Z}D_{F} = \{-1\}$

are the sets of the zeros of $D(s)$, $D_{TH}(s)$, $D_{E}(s)$, $D_{F}(s)$ in (7.34), respectively,

and note that ${}^{P}(G_{p})_{33} = {}^{Z}D + {}^{Z}D_{F} + {}^{Z}D_{TH}$, etc.,

and by (7.32) or by (7.26), (7.37) and (7.38)

$$ {}^{P}F_{3}/F_{2} = {}^{Z}g_{p33}{+}(G_{p})_{22}g_{c11} + {}^{Z}D_{F} $$
(7.39)
$$ {}^{Z}F_{3}/F_{2} = {}^{Z}(G_{p})_{11}{+}(\det G_{p})g_{c11} $$

Then, by (7.23)

$$P_{F_2+F_3g_{c22}} = Z_{1+g_{p11}g_{c11}} + Z_{D_E} + Z_{D_F} + P_{g_{c22}}$$

$$P_{1+F_1g_{c22}} = Z_{1+g_{p11}g_{c11}} + Z_{D_F} + P_{g_{c22}} \qquad (7.40)$$

Finally, by (7.14), (7.15) and (7.40), we have

$$P_{G_{eq}} = Z_{1+F_1g_{c22}} + Z_{D_E}$$

$$Z_{G_{eq}} = Z_{F_2+F_3g_{c22}} \qquad (7.41)$$

The outline of the design then follows:

1. Prepare the four root loci for (7.18), (7.19), (7.20) and (7.21), using $g_{c11}=k_1$. These are shown in Figure 7.1 to Figure 7.4 (note that dashed curves are the loci corresponding to negative k_1).

   By inspection of these loci, the following can be observed:

   (i) $k_1<0$ is undesirable, since for negative k_1, there will be one branch in each of these four plots that extends along positive real axis to $+\infty$ and this will tend to produce a pole-zero pair on positive real axis for G_{eq}, which is certainly undesirable.

(ii) When $k_1 \cong 50$, there will be a pole-zero pair of G_{eq} close to the point (0.5, 1.5). The reason is that the roots corresponding to $k_1 \cong 50$ on branch ① of Figure 7.1 and Figure 7.2 are close to each other. By (7.36), these two roots will be one pole and one zero of F_1 respectively. Since they are close to each other, the root for $1+F_1 g_{c22}=0$ corresponding to this pole-zero pair is very difficult to push far away from this region. Thus, by (7.41), a pole of G_{eq} will be around (0.5, 1.5). Similarly, by Figure 7.3 and Figure 7.4, there will be a zero of G_{eq} close to the same point. Thus, a pole-zero pair of G_{eq} exists near the point (0.5, 1.5) in the right-half-plane. This will most probably cause the corresponding root of (7.13) close to the same point, hence, undesirable.

(iii) For $k_1>50$, the situation is obviously worse. Therefore, the range of k_1 that remains to be investigated is $0<k_1<50$.

(iv) It can be seen that when k_1 is too close to 0, the root on branch ① of each plot will all be close to the origin, hence also not desirable.

2. When $k_1 \approx 10$, the pole-zero locations seem to be the best. Thus, try $k_1 = 10$.

3. Using $g_{c11} = k_1 = 10$, construct the root loci for (7.16) and (7.17) with $g_{c22} = k_2$, a free parameter. The results are shown in Figure 7.5 and Figure 7.6, respectively. By inspection of these two plots, the following can be observed.

   (i) $k_2 > 0$ is undesirable for the same reason as that in 1(i) above.

   (ii) $0 > k_2 >> -10$ looks better than the other range, since the root on branch ① in both Figure 7.5 and Figure 7.6 will be farther away from the $j\omega$ axis, hence, more stable G_{eq} can be expected (note that roots in Figure 7.5 and Figure 7.6 give poles and zeros of G_{eq} respectively, see (7.41)).

   (iii) Although the root on branch ② of Figure 7.5 will be close to the origin for $0 > k_2 >> -10$, it can still be tolerated because the same thing does not happen in Figure 7.6. Thus, the root of (7.13) corresponding to this pole-zero pair of G_{eq} can still be adjusted to be not too close to the origin.

Thus, try $g_{c22} = k_2 = -1$.

4. The roots corresponding to $g_{c22} = -1$ can be read off from Figure 7.5 and Figure 7.6. By (7.14), these constitute part of the poles and zeros of G_{eq}. Together with the other known poles (roots of $D_E(s) = 0$, by (7.41)), the root locus for (7.13) can be constructed as shown in Figure 7.7, with $g_{c33} = k_3$, a free parameter. By inspection, $g_{c33} = k_3 = -2$ is a good value.

Thus, we have determined all three compensator functions with only gain adjustments. The resulting system schematic is as shown in Figure 7.8. For the three step inputs $r_1 = 126.7$ ft./sec., $r_2 = -0.25$ radian and $r_3 = -0.5$ radian, the simulation results for the three outputs $y_1(t)$, $y_2(t)$ and $y_3(t)$ are given in Figure 7.9, Figure 7.10, and Figure 7.11, respectively. It is clear that the resulting system is stable.

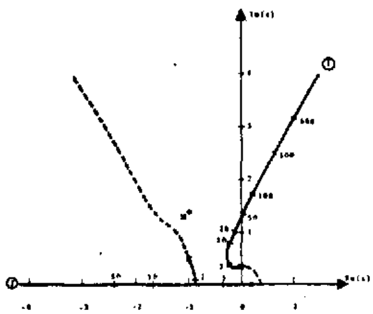

FIGURE 7.1 root locus for $1+g_{p11}g_{c11}=0$

$g_{c11}=k_1$

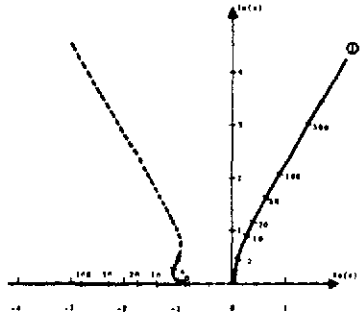

FIGURE 7.2 root locus for $1+\frac{(G_p)_{11}}{g_{p22}}\cdot g_{c11}=0$

$g_{c11}=k_1$

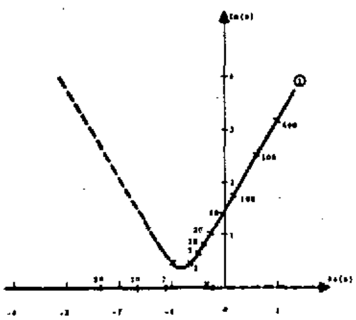

FIGURE 7.3 root locus for $1+\frac{(G_p)_{22}}{g_{p11}}\cdot g_{c11}=0$

$g_{c11}=k_1$

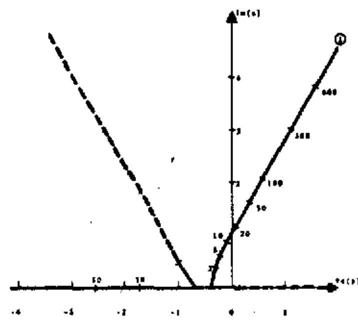

FIGURE 7.4 root locus for $1+\frac{\det G_p}{(G_p)_{11}}\cdot g_{c11}=0$

$g_{c11}=k_1$

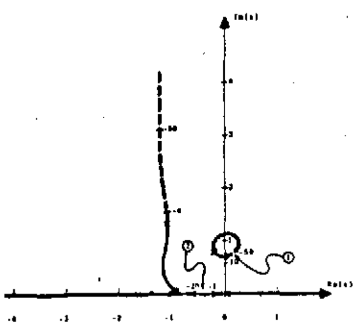

FIGURE 7.5 root locus for $1+P_1' g_{c22}=0$

$g_{c22}=k_2$, $k_1=10$

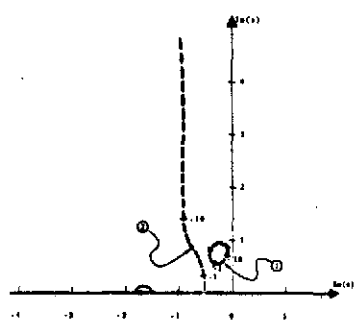

FIGURE 7.6 root locus for $1+\frac{P_1}{P_2}\cdot g_{c22}=0$

$g_{c22}=k_2$, $k_1=10$

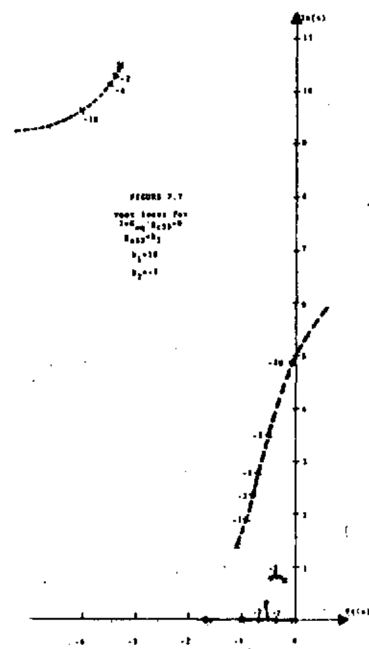

FIGURE 7.7

root locus for

$1+G_{eq}' g_{c33}=0$

$g_{c33}=k_3$

$k_1=10$

$k_2=-1$

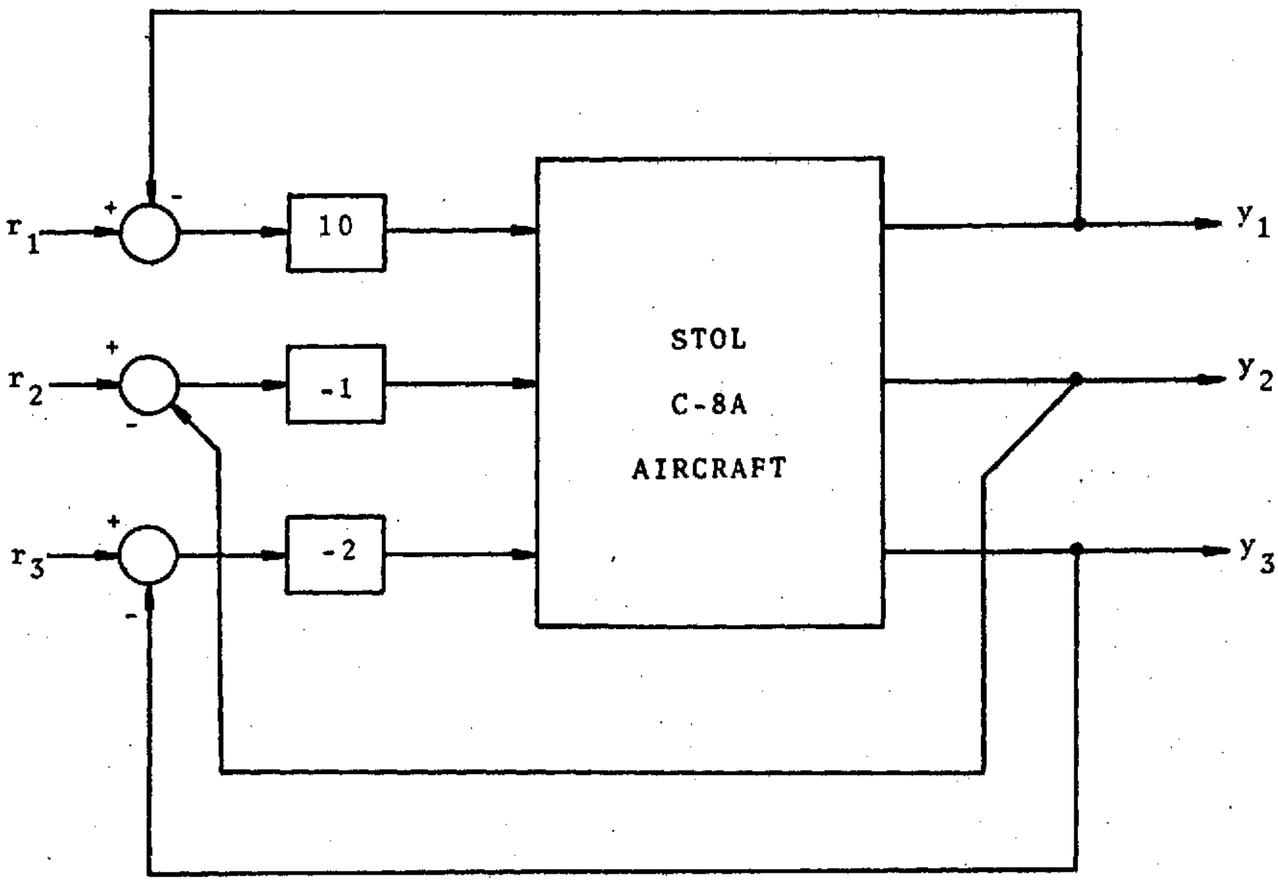

FIGURE 7.8

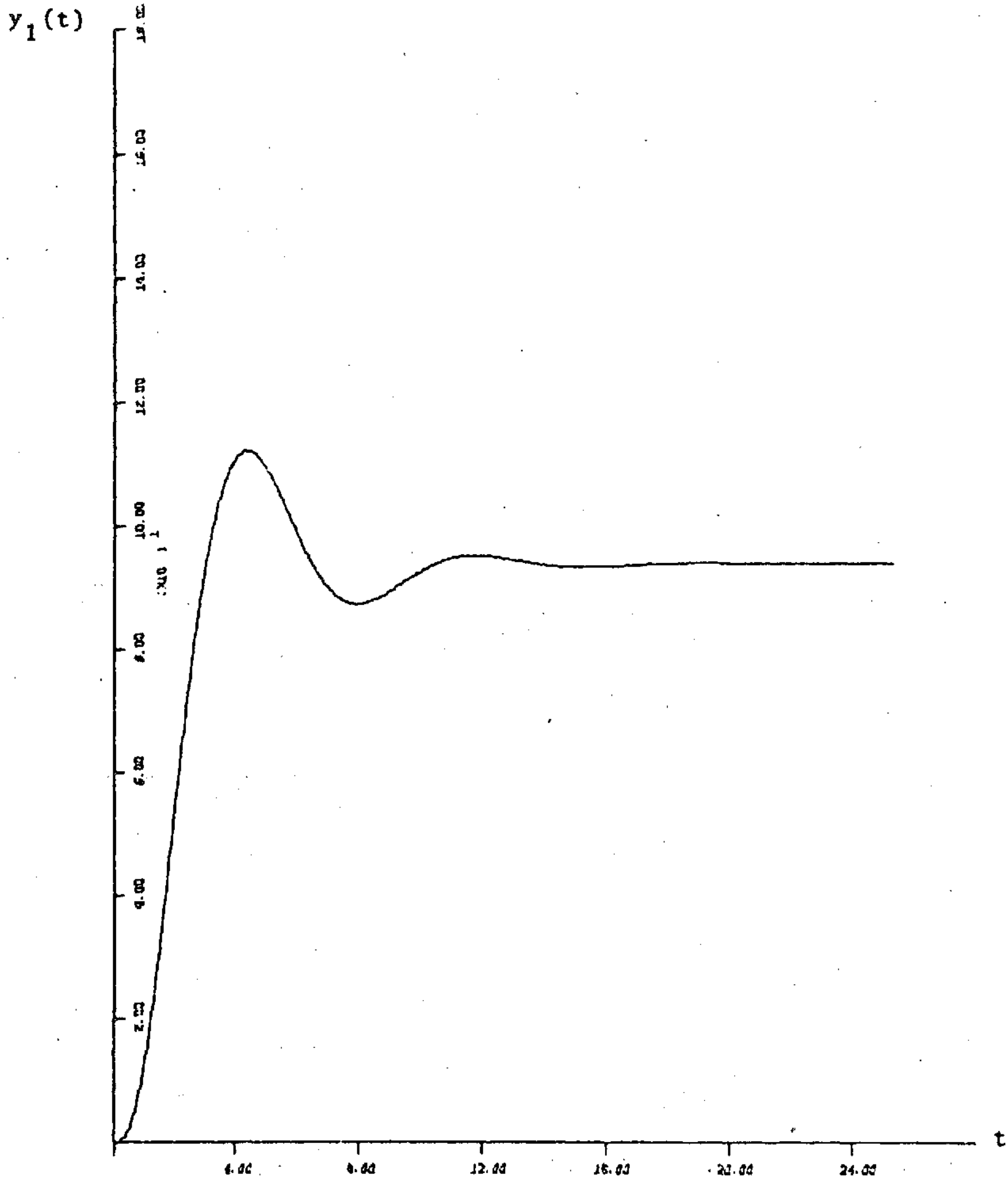

FIGURE 7.9

TIME RESPONSE OF OUTPUT y_1 IN FIGURE 7.8 WITH

$r_1=126.7\cdot u(t)$

$r_2=-0.25\cdot u(t)$

$r_3=-0.5\cdot u(t)$

u(t): UNIT STEP

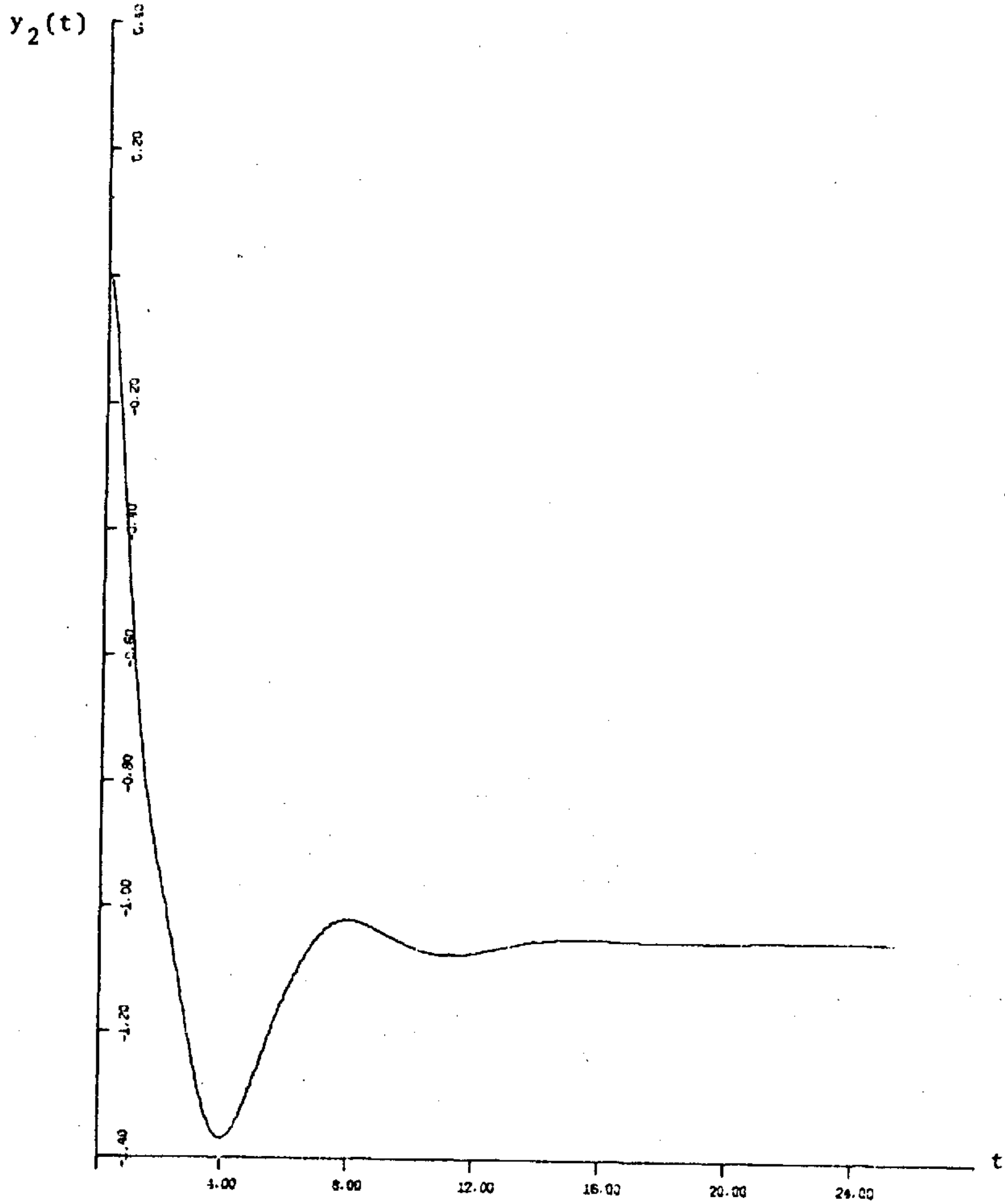

FIGURE 7.10

TIME RESPONSE OF OUTPUT y_2 IN FIGURE 7.8 WITH

$r_1=126.7\cdot u(t)$

$r_2=-0.25\cdot u(t)$

$r_3=-0.5\cdot u(t)$

u(t): UNIT STEP

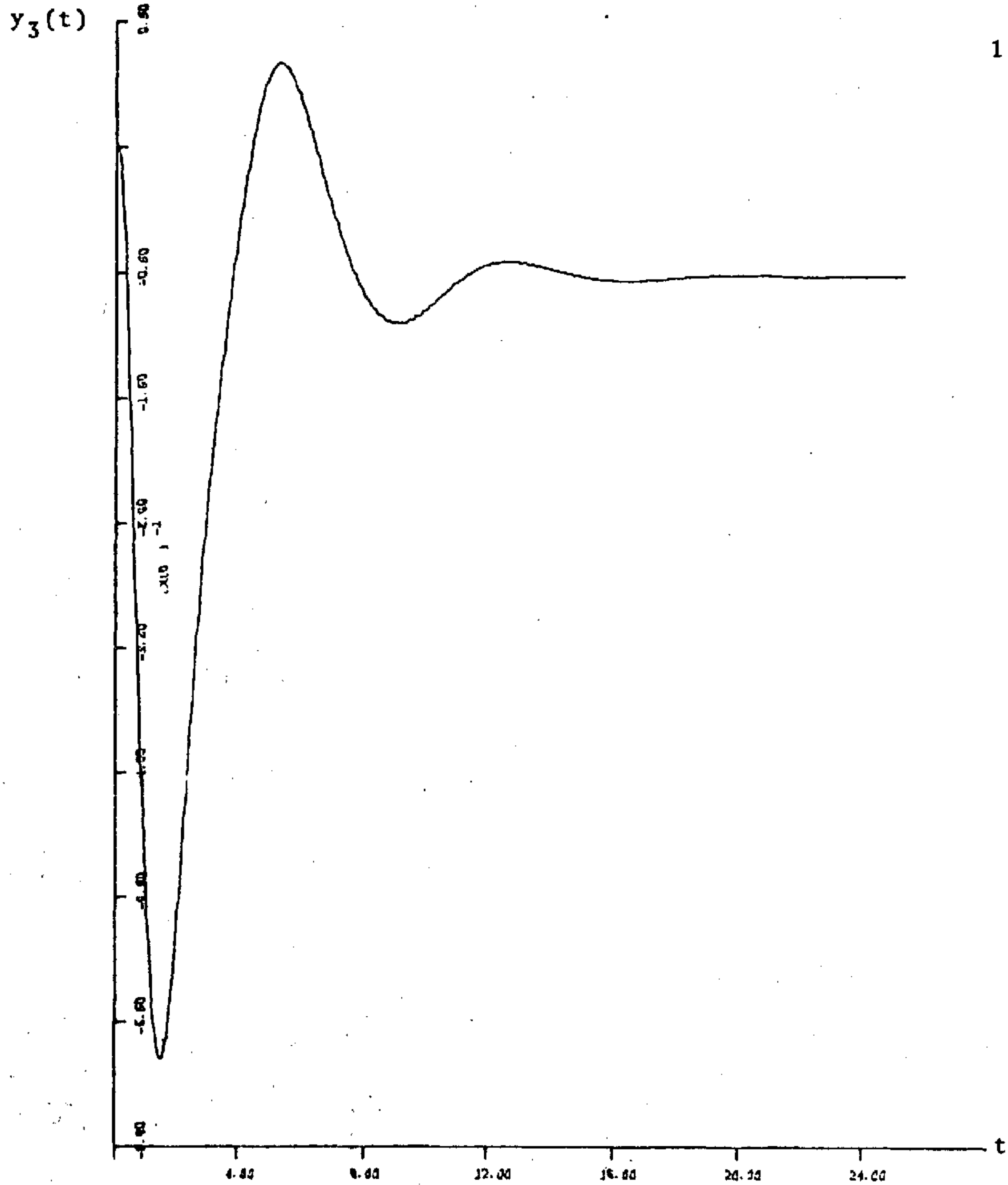

FIGURE 7.11

TIME RESPONSE OF OUTPUT y_3 IN FIGURE 7.8 WITH

$r_1 = 126.7 \cdot u(t)$

$r_2 = -0.25 \cdot u(t)$

$r_3 = -0.5 \cdot u(t)$

u(t): UNIT STEP

PART III

APPLICATION TO STOL AIRCRAFT

8. COMPENSATION DESIGN FOR STOL C-8A AIRCRAFT WITH STEADY-STATE DECOUPLING

8.1 GENERAL

The simulation results, Figure 7.9, Figure 7.10 and Figure 7.11, for the system in Figure 7.8, justify that the root locus technique developed in Chapter 6 and Chapter 7, can be used for designing both the stability and the transient response of a multivariable system. Stability and transient response are certainly the most important factors to consider when designing a system, however, some other factors are also important. Among them (e.g., steady-state accuracy, integrity, sensitivity, etc.), steady-state accuracy is usually the most important. In single-loop theory, the restriction on steady-state accuracy usually makes it impossible to adjust some parameters with complete freedom. In root locus terminology, root relocation zones (14) exist, which limits some of our abilities to relocate those roots of interest.

For the multivariable case, due to the existence of mutual coupling, the problem of steady-state accuracy is more complex. This can be seen by comparing the steady-state values in Figure 7.9, Figure 7.10 and Figure 7.11 to the input commands. For example, input r_1 is a step of magnitude 126.7 ft./sec., while the velocity output y_1 is only 93.8 ft./sec. when steady-state is reached.

One way to reduce or eliminate the steady-state errors in multivariable systems is decoupling the steady states of the system. The concept of steady-state decoupling was developed in Part I of this thesis, and the schemes obtained can be used directly to determine what types of functions should be used in the compensator. Then, results of Part II are applied to stabilize the system.

8.2 DESIGN PROCEDURE

The plant under consideration is the longitudinal mode of the NASA STOL (Short Take Off and Landing) C-8A Buffalo Aircraft which is a 3-input, 3-output plant as shown below in block diagram form.

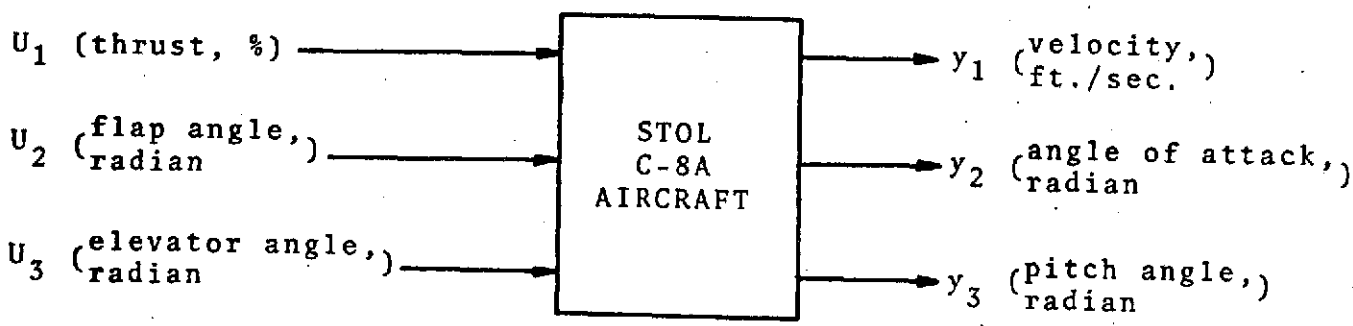

FIGURE 8.1

Thus, the plant can be represented by a 3x3 matrix G_p. The transfer functions $g_{p_{ij}}$ are given in (3.33) and (7.33).

By using the feedback configuration as that in Figure 3.2, a steady-state decoupling scheme was obtained in Section 3.2 to be $t_{c11}=t_{c22}=t_{c33}=1$, i.e.,

$$g_{c11}(s) = \frac{1}{s}\, g'_{c11}(s)$$

$$g_{c22}(s) = \frac{1}{s}\, g'_{c22}(s) \qquad (8.1)$$

$$g_{c33}(s) = \frac{1}{s}\, g'_{c33}(s)$$

where, as defined in (3.10), numerators and denominators of $g'_{c11}(s)$, $g'_{c22}(s)$ and $g'_{c33}(s)$ do not contain power(s) of s as their factors.

(8.1) tells us that the introduction of one pure integrator in each of the three compensator transfer functions will cause the steady states of the system to be decoupled. Thus, one pole (at the origin) is required in each of the three unknown functions $g_{c11}(s)$, $g_{c22}(s)$ and $g_{c33}(s)$. The other poles and zeros and the gain values are unknown and have to be determined for stability and transient response. The design of these unknowns can be done in exactly the same manner as that in Chapter 7, except that the existence of the extra pole (s=0) in each of $g_{c11}(s)$, $g_{c22}(s)$, and $g_{c33}(s)$ has to be taken into account. Also note that the pole-zero expressions (7.41) (7.36) (7.39) are still valid, since the same plant is considered. Then the design procedure follows:

1) Prepare the 4 root loci for (7.18), (7.19), (7.20) and (7.21) with $g^{5}_{c11}=\frac{k_1}{s}$, where k_1 is a free parameter.

These are shown in Figure 8.2 to Figure 8.5 (note that these loci can be drawn by adding the additional pole at the origin in Figure 7.1 to Figure 7.4).

By inspection, it is observed that

(i) $k_1<0$ is not desirable, for the same reason as 1 (i) of the design in Section 7:3.

(ii) For $k_1>0$, branch ① on all four plots are quite unstable, which is the effect of the introduced pole at the origin. By the same argument as that in 1 (ii) of the design in Section 7.3, it can be concluded that no range of k_1 is desirable.

Therefore, only one pole (at the origin) and no zero in g_{c11} is most probably not enough to give satisfactory system performance, some other pole(s) and (or) zero(s) are recommended.

[5]The argument s will be dropped whenever no confusion exists.

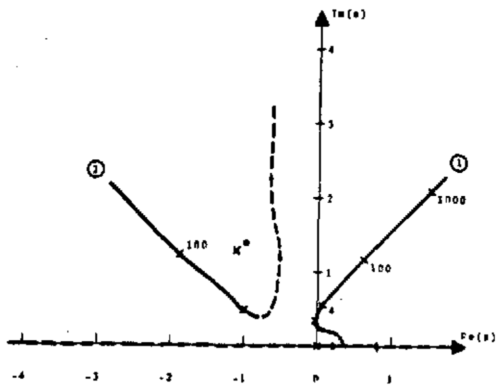

FIGURE 6.2 root locus for $1+g_{p11}\cdot g_{c11}=0$

$g_{c11}=k_1/s$

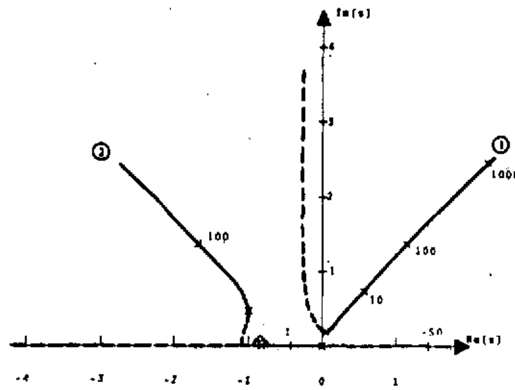

FIGURE 6.3 root locus for $1+\frac{(G_p)_{33}}{g_{p22}}\cdot g_{c11}=0$

$g_{c11}=k_1/s$

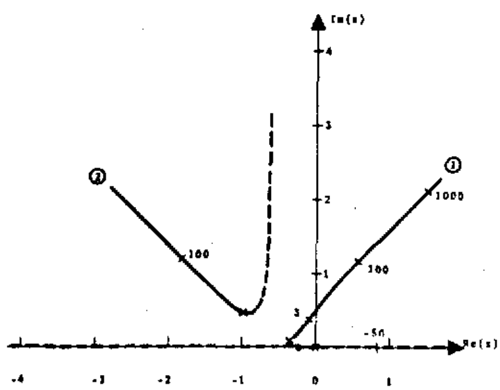

FIGURE 6.4 root locus for $1+\frac{(G_p)_{22}}{g_{p33}}\cdot K_{c11}=0$

$g_{c11}=k_1/s$

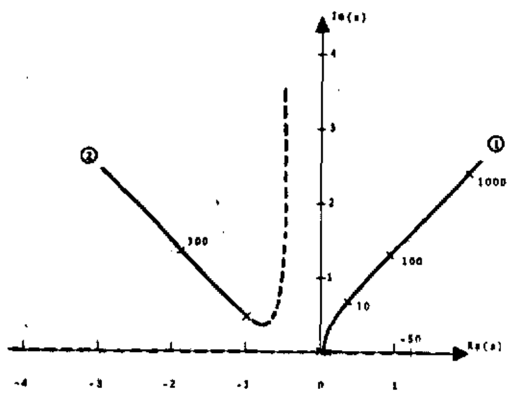

FIGURE 6.5 root locus for $1+\frac{\det G_p}{(G_p)_{11}}\cdot K_{c11}=0$

$K_{c11}=k_1/s$

2) By spirule check, or by noting that any introduced zero (that goes with the decoupling pole at the origin to form one section of filter) on the negative real axis cannot overcome the effect of the pole at the origin, it can be seen that if only one section of filter (pole at origin, zero on negative real axis) is used, the situations will always be worse than those in Figure 7.1 to Figure 7.4. Since that design was only marginally successful and the situations now are worse, several sections of filter are recommended.

3) By spirule, or by inspection (if experienced enough) it can be seen that two sections of filters are enough to pull branch ① in each of these four plots into the left-half-plane. Then no poles and zeros of F_1 and F_3/F_2 are in the right-half-plane, (Note that by (7.36), roots in Figure 8.3 are the zeros of F_1. Those in Figure 8.2, together with the known pole at s=-1, which is the root of $D_F(s)=0$, give all the poles of F_1. Similarly, by (7.39), roots in Figure 8.4, Figure 8.5 and the known pole at s=-1 give all the poles and zeros of F_3/F_2), and the design of g_{c22} and g_{c33} can be continued in the same manner as that in Section 7.3. However, by noting that each new root locus that has to be constructed out of these four has also an additional pole at the origin (comes from g_{c22} and g_{c33}, see (8.1)). This will tend to destabilize the results and make the design of g_{c22} and g_{c33} more difficult. Thus, try a three section lead compensator.

4) By spirule (or by inspection), three zeros close to the origin and two poles far away from the origin can stabilize Figure 8.2 to Figure 8.5 to a great extent. For one specific choice,

$$g_{c11}(s)=\frac{k_1(s+1)(s+0.5)^2}{s(s+4)(s+10)} \quad \text{where } k_1 \text{ is a free parameter,}$$

the results are shown in Figure 8.6 to Figure 8.9. Note that the loci for $k_1<0$ are not shown, since $k_1<0$ is not desirable for the same reason as in 1 (i).

5) By inspection of these root loci and by considering the effect of k_1 on the pole-zero pattern of F_1 and F_3/F_2 (see (7.36) and (7.39)), $k_1 = 1000$ is chosen for the same reason as in the previous design of Section 7.3. For this value of k_1, the root loci for F_1 and F_3/F_2 can be drawn. Then, g_{c22} can be designed according to these two root loci and their relationship with the poles and zeros of G_{eq} (see (7.41)). By trial-and-error (or by inspection), $g_{c22}=\frac{k_2(s+0.5)}{s}$ (k_2 is a free parameter) was found to be good. For this g_{c22}, the root loci for (7.16) and (7.17) are shown in Figure 8.10 and Figure 8.11. Loci for $k_2>0$ are not shown since for $k_2>0$, one branch in each plot extends along positive real axis to $+\infty$, hence undesirable.

6) By inspection of Figure 8.10 and Figure 8.11 and by (7.41), $k_2 = -400$ is a good value.

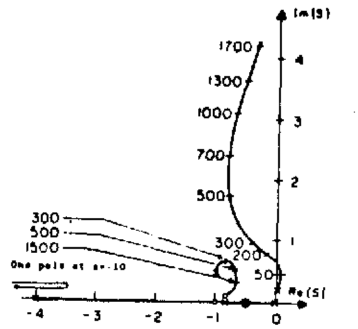

FIGURE 8.7

root locus for $1+\frac{(G_p)_{33}}{g_{p22}}\cdot g_{c11}=0$

$g_{c11}=k_1\cdot\frac{(s+1)(s+0.5)^2}{s(s+4)(s+10)}$

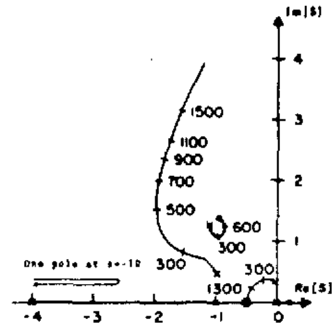

FIGURE 8.6

root locus for $1+g_{p11}\cdot g_{c11}=0$

$g_{c11}=k_1\cdot\frac{(s+1)(s+0.5)^2}{s(s+4)(s+10)}$

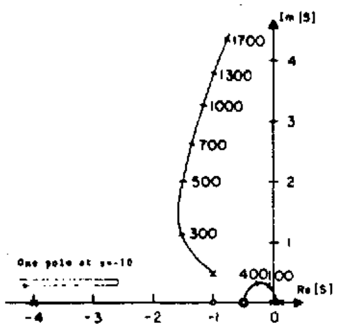

FIGURE 8.9

root locus for $1+\frac{\det G_p}{(G_p)_{11}}\cdot g_{c11}=0$

$g_{c11}=k_1\cdot\frac{(s+1)(s+0.5)^2}{s(s+4)(s+10)}$

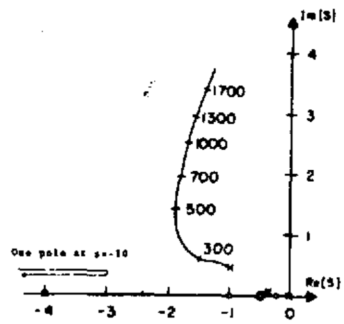

FIGURE 8.8

root locus for $1+\frac{(G_p)_{22}}{g_{p33}}\cdot g_{c11}=0$

$g_{c11}=k_1\cdot\frac{(s+1)(s+0.5)^2}{s(s+4)(s+10)}$

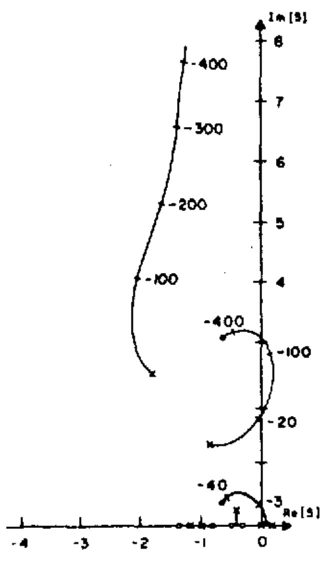

FIGURE 8.10

root locus for $1+F_1\cdot g_{c22}=0$

$g_{c22}=k_2\cdot\frac{s+0.5}{s}$, $k_1=1000$

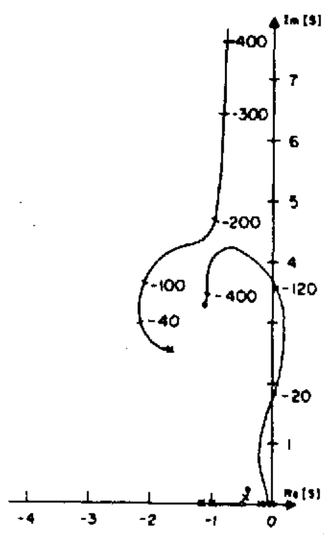

FIGURE 8.11

root locus for $1+\frac{F_3}{F_2}\cdot g_{c22}=0$

$g_{c22}=k_2\cdot\frac{s+0.5}{s}$, $k_1=1000$

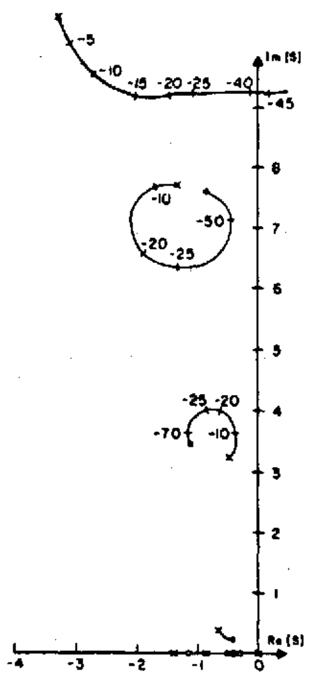

FIGURE 8.12

root locus for $1+G_{eq}\cdot g_{c33}=0$

$g_{c33}=k_3\cdot\frac{(s+0.5)(s+0.4)}{s(s+10)}$, $k_1=1080$, $k_2=-400$

7) Using $k_2 = -400$, the root locus for (7.13) with $g_{c33} = \frac{k_3(s+0.3)(s+0.4)}{s(s+10)}$ can be drawn (see (7.41)), and g_{c33} can be designed accordingly. The result is shown in Figure 8.12, from which the best value for k_3 can be seen by inspection to be $k_3 \cong -20$.

   Thus, a design with

$$g_{c11}(s) = 1000(s+1)(s+0.5)^2/s(s+4)(s+10)$$

$$g_{c22}(s) = -400(s+0.5)/s \qquad (8.2)$$

$$g_{c33}(s) = -20(s+0.3)(s+0.4)/s(s+10)$$

   is completed.

   The schematic diagram for the designed system is shown in Figure 8.13.

For step inputs of magnitudes 126.7 ft./sec., -0.25 radian, -0.5 radian in r_1, r_2, and r_3, the simulation results for y_1, y_2, and y_3 are shown in Figure 8.14, Figure 8.15 and Figure 8.16, respectively. It is seen that both stability and steady-state decoupling have been achieved. Furthermore, due to the introduced pole at the origin in each of g_{c11}, g_{c22}, and g_{c33}, the steady-state error in each of the outputs is zero, which is the most desirable situation. Thus, by decoupling the steady states, steady-state accuracy has also been achieved as a byproduct.

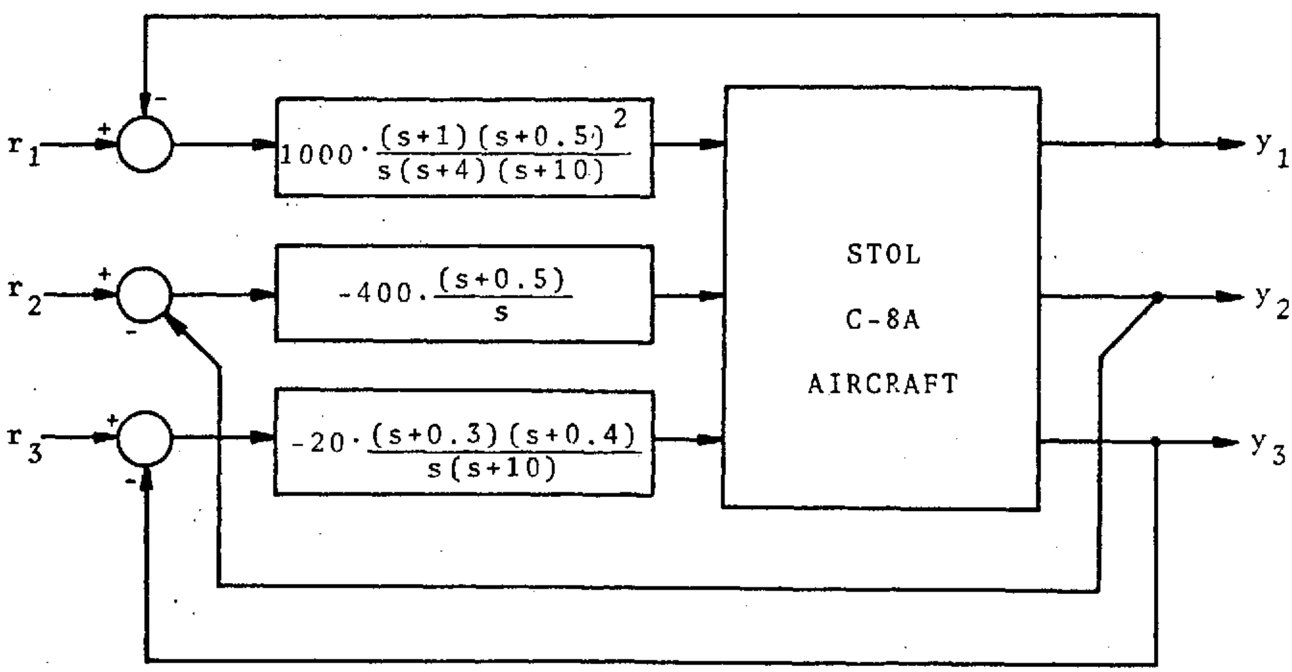

FIGURE 8.13

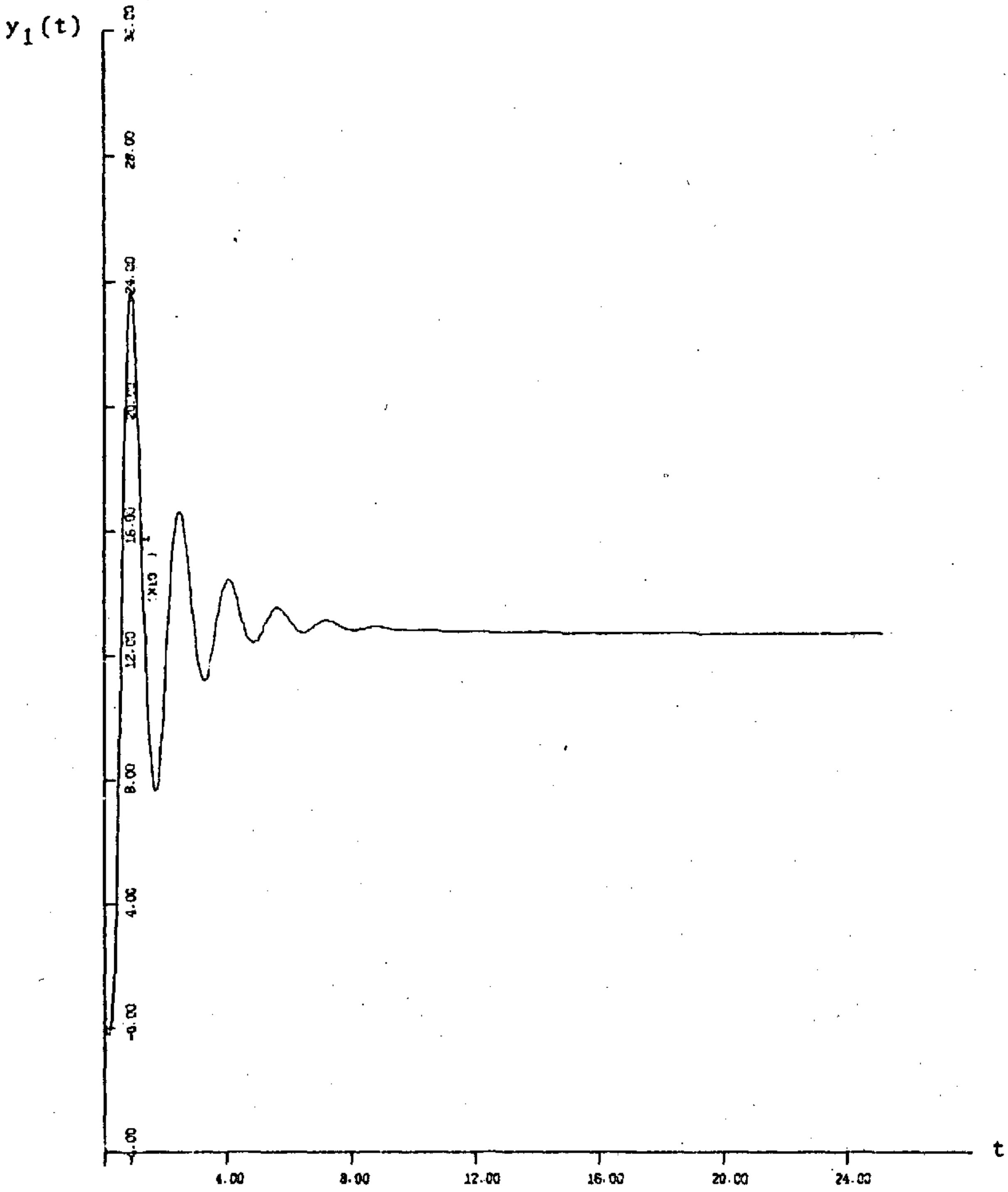

FIGURE 8.14

TIME RESPONSE OF OUTPUT y_1 IN FIGURE 8.13 WITH

$r_1=126.7\cdot u(t)$

$r_2=-0.25\cdot u(t)$

$r_3=-0.5\cdot u(t)$

u(t): UNIT STEP

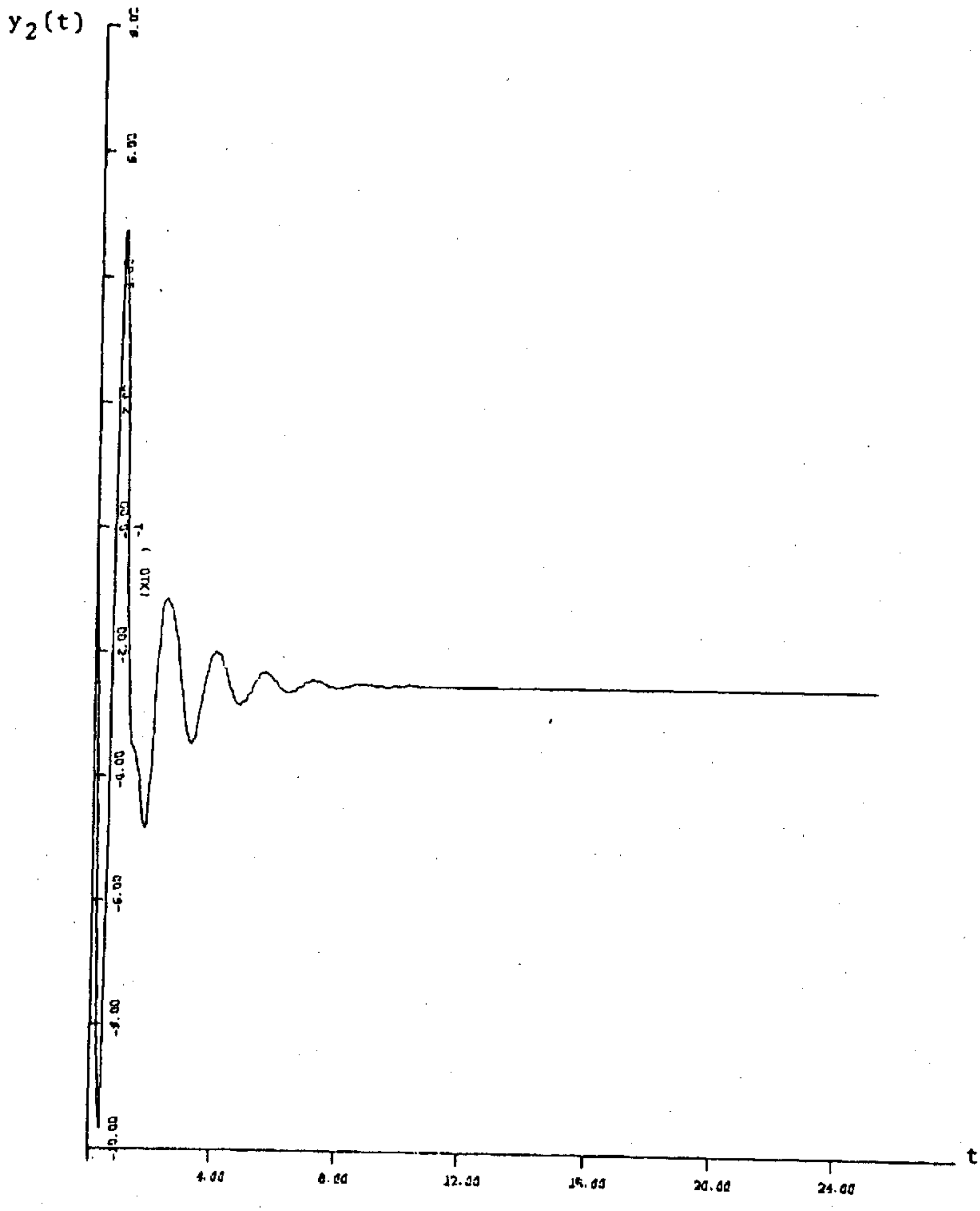

FIGURE 8.15

TIME RESPONSE OF OUTPUT y_2 IN FIGURE 8.13 WITH

$r_1=126.7 \cdot u(t)$

$r_2=-0.25 \cdot u(t)$

$r_3=-0.5 \cdot u(t)$

u(t): UNIT STEP

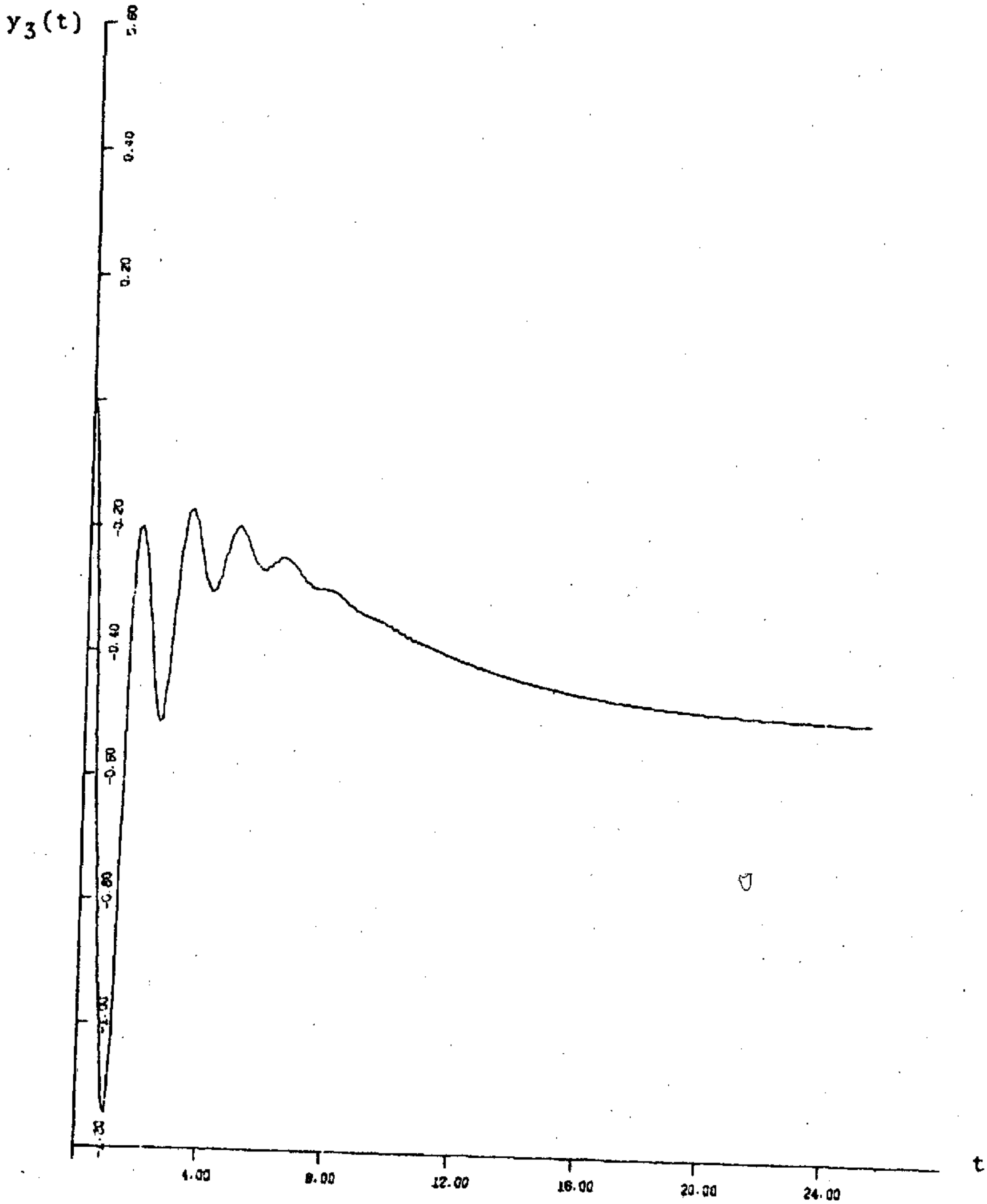

FIGURE 8.16

TIME RESPONSE OF OUTPUT y_3 IN FIGURE 8.13 WITH

$r_1 = 126.7 \cdot u(t)$

$r_2 = -0.25 \cdot u(t)$

$r_3 = -0.5 \cdot u(t)$

u(t): UNIT STEP

9. CONCLUSIONS

9.1 COMPARISON OF THE RESULT IN PART I TO THAT OF STATE VARIABLE FEEDBACK APPROACH

In part I, a constructive criterion for decoupling the steady states of a linear time-invariant multivariable system was developed. Transfer function matrix representation, unity feedbacks and cascade compensation were used as shown in Figure 2.1.

Another approach using linear state variable feedbacks was investigated by Wolovich (2). The result in terms of transfer function matrix representation is given in Chapter 1, and repeated here as follows:

> A system characterized by an (nxm) proper rational transfer function matrix, $G_p(s)$, having no poles at the origin (s=0) can be steady-state decoupled (via linear state variable feedback or perhaps some other less ambitious scheme) if and only if
>
> $$\rho(G_p(0)) = n \qquad (9.1)$$
>
> where $\rho(G_p(0))$ denotes the rank of the matrix $G_p(s)$ when s approaches zero.

Several advantages of the result given in Part I over that described above are in order:

1. Cascade compensators and output feedbacks are much easier to implement than measuring the states.

2. The rank condition (9.1) is not necessary, e.g., given the 2x2 plant (n=m=2):

$$G_p(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} \\ \frac{s}{s+3} & \frac{1}{s+4} \end{bmatrix} \qquad (9.2)$$

   By (3.19), (3.10) and (3.18), steady states can be decoupled by introducing two and one pure integrators in g_{c11} and g_{c22} in Figure 3.1, respectively. However, by (9.1), this cannot be done through linear state variable feedbacks.

3. Poles at the origin in the given plant are allowed. Actually, such poles are very helpful for steady-state decoupling as was shown in Figure 6.6 for the plant (6.24).

9.2 POSSIBLE GENERALIZATION OF THE SYSTEM CONFIGURATION

The discussion of this thesis has been restricted to the configuration given in Figure 2.1. For the general feedback configuration in which the unity feedbacks are replaced by a transfer function matrix $G_f(s)$ (nxn), the simple relation (2.5) is no longer valid. However, if $G_f(s)$ is diagonal and nonsingular (in the field of rational functions of s), which are of practical importance, similar result to (2.5) can still be obtained as follows:

$$H = (I+G_pG_cG_f)^{-1}G_pG_c$$

$$= (I+G_pG_cG_f)^{-1}G_pG_cG_fG_f^{-1}$$

$$= (I-(I+G_pG_cG_f)^{-1})\ G_f^{-1}$$

Since G_f is diagonal, G_f^{-1} is also diagonal. Therefore, simple expressions for the off-diagonal elements of H(s) can still be obtained easily. Then, with a slight modification, the results in Part I can still be applied.

9.3 STABILITY AND DESIGN

In Parts II and III, stability of a linear time-invariant multivariable system was considered. A design technique, using an extended root locus method was also developed and applied successfully to 2x2 and 3x3 cases. The major achievement is the revelation of the simple connection between single-loop and multivariable cases. Such connections made the application of single-loop design methods to multivariable systems possible, as was seen through the design examples in Chapter 6, Chapter 7 and Chapter 8. Some other advantages of the design techniques are:

1. Consideration of integrity problems is possible in the process of the design by forming pertinent root loci. This means that the system can be designed such that possible failure of any loop (or combination of loops) do not cause the system to be unstable (e.g., see (10)).

2. The problem of input output permutation, like "which output should be fed back to a particular input?" can be solved to some extent by inspection.

3. More insight to the problem is achieved through the root locus approach.

4. The problem of meeting system specifications can be done in the same manner as in any single-loop design method.

A few disadvantages, however, do exist. For example, the successive dependence of each root locus on the previous ones causes more design difficulty as the number of inputs and outputs of the plant increases. Also, like the single-loop frequency domain methods, trial-and-error is inherent in this technique.

However, with the help of computers, these problems can be minimized and the design can be done within a reasonable amount of time. Besides, with some experience in handling the root locus, the effort can be further reduced.

REFERENCES

(1) P. L. Falb and W. A. Wolovich, "Decoupling In The Design and Synthesis of Multivariable Control Systems" IEEE Trans. Auto. Control Vol. AC-12(6), December, 1967, pp. 651-659.

(2) W. A. Wolovich, "Static Decoupling" Joint Auto. Control Conference, 1973.

(3) E. G. Gilbert, "Controllability and Observability in Multivariable Control Systems" SIAM J. Control Ser. A Vol.2, pp. 128-151, 1963.

(4) C. T. Chen, "Representations of Linear Time-Invariant Composite Systems" IEEE Trans. Auto. Control, Vol. AC-13(3), June, 1968.

(5) H. H. Rosenbrock, "Design of Multivariable Control Systems Using the Inverse Nyquist Array" Proc. IEE, Vol. 116, pp. 1929-1936, 1969.

(6) F. R. Gantmacher, "The Theory of Matrices" Vol. 1, pp. 8-12, Chelsea Publishing Co., 1960.

(7) C. T. Chen, "Stability of Linear Multivariable Feedback Systems" Proc. IEEE, Vol. 56, No. 5, May, 1968.

(8) C. H. Hsu and C. T. Chen, "A Proof of The Stability of Multivariable Feedback Systems" Proc. IEEE, Nov., 1968.

(9) C. T. Chen and C. H. Hsu, "Design of Dynamic Compensators for Multivariable Systems" Proc. 1971 JACC, pp. 893-900.

(10) J. J. Belletrutti and A. G. J. MacFarlane, "Characteristic Loci Techniques in Multivariable-Control-System Design" Proc. IEE, Vol. 118, pp. 1291-1297, 1971.

(11) A. G. J. MacFarlane, "Return-Difference and Return-Ratio Matrices and Their Use in Analysis and Design of Multivariable Feedback Control Systems" Proc. IEE, Vol. 117, pp. 2037-2049, 1970.

(12) B. D. O. Anderson, "Linear Multivariable Control Systems - A Survey" Proc. IFAC 1972.

(13) C. T. Chen, "Introduction to Linear System Theory" Holt, Rinehart and Winston, New York, 1970.

(14) G. J. Thaler, "Design of Feedback Systems" Dowden, Hutchinson & Ross, Inc., 1973.

(15) D. Hazony, "Elements of Network Synthesis" Reinhold, New York, 1963.

APPENDIX A: FORMULAE (4.1) AND (4.2)

The proof of (4.1) can be done by applying Lemma A.1 in Appendix A of (15) directly to the two nxn square matrices I and G. For (4.2) however, direct application of this lemman has a little difficulty.

For better analyticity, an independent proof using mathematical induction has been developed. An outline of the proof is given below:

1. (4.1), (4.2) are satisfied for n=2 and n=3 by direct expansion.

2. Suppose (4.1) is true for n=N, (4.2) is true for n=N+1, then (4.1) is true for n=N+1.

3. Suppose (4.1) is true for both n=N-1, and n=N, then (4.2) is true for n=N+1.

Thus, starting from N=2, it can be induced that (4.1) and (4.2) are true for any positive integer.

Details of the proof in 2 and 3 above are omitted.

APPENDIX B: CHARACTERISTIC EQUATION

It was pointed out in Section 6.1 that the stability of a multivariable system as shown in Figure 2.1 is determined by the zeros of both (6.2) and (6.3). It will be shown in this appendix that if pole-zero cancellations are done deliberately, and if the compensator poles are chosen carefully, zeros of (6.2) alone can determine the stability. The constraints under which this is true are very practical and can be fulfilled in a systematic manner. Thus, the mathematical possibilities in which zeros of (6.3) must be considered are bypassed.

By the definition of characteristic polynomials given in Section 6.1, $\Delta_c(s)$ and $\Delta_p(s)$ can be expressed analytically as:

$$\Delta_c(s)=\mathrm{LCD}\left\{G_c\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix} \middle| \ \ell=1,\cdots,\min(n,m),\ 1\le i_1<\cdots<i_\ell\le m,\ 1\le j_1<\cdots<j_\ell\le n\right\} \quad \text{(B.1)}$$

$$\Delta_p(s)=\mathrm{LCD}\left\{G_p\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix} \middle| \ \ell=1,\cdots,\min(n,m),\ 1\le i_1<\cdots<i_\ell\le n,\ 1\le j_1<\cdots<j_\ell\le m\right\} \quad \text{(B.2)}$$

Where, as in Chapter 4, $G_c\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$, $G_p\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$ denote the

ℓth-order minors of G_c and G_p formed from rows $i_1,\cdots,i_\ell$ and columns $j_i,\cdots,j_\ell$ of each matrix, respectively.

LCD $\{\cdots\}$ denotes the Least Common Denominator of all the rational functions described in the brackets and all the minors are assumed to be in irreducible rational forms.

Let $N_p\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$, $N_c\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$, $D_p\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$ and

$D_c\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$ denote the numerators and denominators of the irreducible minors, $G_p\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$ and $G_c\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix}$ respectively. Then,

$$\Delta_c(s)=\text{LCM}\{D_c\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix} \mid \ell=1,\cdots,\min(n,m)\ \ 1\leq i_1<\cdots<i_\ell\leq m,\ 1\leq j_1<\cdots<j_\ell\leq n\} \tag{B.3}$$

$$\Delta_p(s)=\text{LCM}\{D_p\begin{pmatrix} i_1,\cdots,i_\ell \\ j_1,\cdots,j_\ell \end{pmatrix} \mid \ell=1,\cdots,\min(n,m)\ \ 1\leq i_1<\cdots<i_\ell\leq n,\ 1\leq j_1<\cdots<j_\ell\leq m\} \tag{B.4}$$

Where LCM $\{\cdots\}$ denotes the Least Common Multiplier of all the polynomials described in the brackets.

These analytical expressions for $\Delta_c(s)$ and $\Delta_p(c)$ will be useful later in this appendix.

For single-loop systems,

$$G_p(s) = \left[g_p(s)\right]_{1x1}$$

$$G_c(s) = \left[g_c(s)\right]_{1x1}$$

Let $N_p(s)$, $N_c(s)$, $D_p(s)$, $D_c(s)$ denote the numerators and the denominators of $g_p(s)$ and $g_c(s)$ (both in irreducible rational forms) respectively, we have for (6.2)

$$\frac{N_1(s)}{D_1(s)} = 1 + \frac{N_p(s)}{D_p(s)} \cdot \frac{N_c(s)}{D_c(s)} \tag{B.5}$$

(Note that $N_1(s)/D_1(s)$ is in irreducible form.)

Also, by definition of the characteristic polynomial,

$$\Delta_p(s) = D_p(s)$$

$$\Delta_c(s) = D_c(s)$$

Hence, (6.3) becomes

$$\hat{N}(s) = \frac{D_c(s)\ D_p(s)}{D_1(s)} \tag{B.6}$$

The right hand side of (B.5) can be written as

$$\frac{D_p(s)\ D_c(s) + N_p(s)\ N_c(s)}{D_p(s)\ D_c(s)}$$

Let $D_o(s) \triangleq D_p(s)\ D_c(s)$

$N_o(s) \triangleq D_p(s)\ D_c(s) + N_p(s)\ N_c(s)$

which are the denominator and numerator of (6.2) before cancellation (if any). Also, let C(s) denote the greatest common factor between $D_o(s)$ and $N_o(s)$ ($C(s) \equiv 1$, if no common factor exists), then

$$\begin{aligned} D_o(s) &= C(s) \cdot D_1(s) \\ N_o(s) &= C(s) \cdot N_1(s) \end{aligned} \tag{B.7}$$

By (B.6) and (B.7), we have:

$$\begin{aligned} N_1(s) &= \frac{N_o(s)}{C(s)} \\ \hat{N}(s) &= \frac{D_o(s)}{D_1(s)} = C(s) \end{aligned} \tag{B.8}$$

(B.8) tells us that the zeros of $N_1(s)$, together with those of $\hat{N}(s)$ are the zeros of $N_o(s)$ alone.

Therefore, the following conclusion, which is well-known in single-loop theory, can be made.

The stability for systems shown in Figure B.1 is determined by the roots of

$$1 + g_p(s)g_c(s) = 0 \tag{B.9}$$

if and only if no pole-zero cancellation is allowed in (B.9), even if a common factor exists.

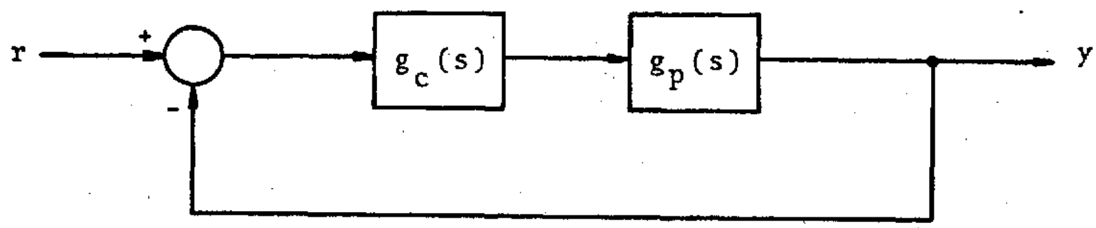

FIGURE B.1

As an example, consider the system shown in Figure B.1 with

$$g_p(s) = \frac{s-1}{s+1}$$

$$g_c(s) = \frac{1}{s-1}$$

By (6.1) and (6.3),

$$\frac{N_1(s)}{D_1(s)} = \frac{s+2}{s+1}$$

$$\hat{N}(s) = \frac{(s-1)(s+1)}{s+1} = s-1$$

Since the zero of $\hat{N}(s)$ is in the right-half plane, the system is unstable.

By (B.9),

$$1+\frac{1}{s-1}\cdot\frac{s-1}{s+1} = 0$$

If the common factor (s-1) is cancelled, we have $\frac{s+2}{s+1}=0$. Therefore, only the s=-2 pole is retained and erroneous conclusion that the system is stable is reached.

However, if (s-1) is not cancelled, we have $\frac{(s-1)(s+2)}{(s-1)(s+1)}=0$. Hence, both the zero of $N_1(s)$ and $\hat{N}(s)$ are retained.

Therefore, for single-loop systems, if no cancellation is allowed, (6.2) alone gives all the zeros of $N_1(s)$ and $\hat{N}(s)$, hence determines the stability.

The following question then arises naturally: "Can we use (6.2) alone in determining the stability of a multivariable system by the same requirement that no cancellation is allowed?" The answer is "no" as was shown by Chen (7) through the following 2x2 example:

Consider

$$G_p(s) = \begin{bmatrix} \frac{-s^2+s+1}{(s+1)\ (s-1)} & \frac{1}{s-1} \\ \frac{1}{(s+1)\ (s-1)} & \frac{1}{s-1} \end{bmatrix} \tag{B.10}$$

$$G_c(s) = I \tag{B.11}$$

where I denotes the 2x2 unity matrix.

Then,

$$\begin{aligned} \det(I+G_pG_c) &= \frac{s^2}{(s+1)(s-1)^2} - \frac{1}{(s+1)(s-1)^2} \\ &= \frac{(s+1)(s-1)}{(s-1)^2(s+1)} \end{aligned} \tag{B.12}$$

There exists a right-half-plane zero in (B.12) at s=1.

However, by (6.2) and (6.3)

$$\begin{aligned} N_1(s) &= 1 \\ \hat{N}(s) &= s+1 \end{aligned} \tag{B.13}$$

(Note: $\Delta_c(s) = 1$, $\Delta_p(s) = (s+1)(s-1)$ and $D_1(s) = s-1$) the system is clearly stable. Thus, leaving all the common factors uncancelled does not work.

However, if the general formula (4.5) is used and let all the minors of G_p be in irreducible forms, we have

$$\begin{aligned}\det(I+G_pG_c) &= \det(I+G_p) \\ &= 1+g_{p11}+g_{p22}+\det G_p \\ &= 1+\frac{-s^2+s+1}{(s+1)(s-1)} + \frac{1}{s-1} + \frac{-s}{(s+1)(s-1)} \\ &= \frac{s+1}{(s+1)(s-1)} \qquad \text{(B.14)}\end{aligned}$$

It is seen that if the common factor (s+1) is not cancelled, the zero of (B.14) is exactly the same as those given by $N_1(s)$ and $\hat{N}(s)$ as shown in (B.13).

Note that the misleading factor (s-1) in (B.12) does not appear in (B.14). The reason is that we started out with irreducible minors and in forming the 2nd order irreducible minor $\det G_p$, the (s-1) factor was cancelled.

Therefore, by selecting cancellations, stability of a multivariable system can still be determined by (6.2) alone. All common factors in the minors must be cancelled to get the irreducible forms, while the others are not allowed. This strict rule together with the application of formula (4.5) make the whole procedure completely systematic, no confusion will arise.

As another example, consider

$$G_p(s) = \begin{bmatrix} \frac{-s}{s-1} & \frac{s}{s+1} \\ 1 & \frac{-2}{s+1} \end{bmatrix},$$

$$G_c(s) = I$$

which is also an example in (7).

By direct manipulation, it can be found that det $(I+G_pG_c) = -1$. However, by selecting cancellations as described above, we have

$$\begin{aligned} \det(I+G_pG_c) &= 1+g_{p11}+g_{p22}+\det G_p \\ &= 1+\frac{-s}{s-1} + \frac{-2}{s+1} + \frac{-s^2+3s}{(s+1)(s-1)} \\ &= -\frac{(s+1)(s-1)}{(s+1)(s-1)} \end{aligned} \quad \text{(B.15)}$$

Once again, if the common factor (s+1)(s-1) is not cancelled, the two zeros of (B.15) are exactly what one would obtain as zeros for $N_1(s)$ and $\hat{N}(s)$ by (6.2) and (6.3). Thus, these two uncancelled zeros of $\det(I+G_pG_c)$ determine the stability of the system. Since one of them is in the right-half-plane, the feedback system is unstable.

These two examples suggest that (6.2) alone can be used as characteristic equation for a multivariable system if we select cancellations as described above. But, is this true in general? To answer this question, consider the general expression (4.5) for feedback system as shown in Figure 2.1. For simplicity, consider 2x2 G_p and G_c first.

By (4.5),

$$\begin{aligned}\det(I+G_pG_c) &= 1+G_p\binom{1}{1}G_c\binom{1}{1}+G_p\binom{1}{2}G_c\binom{2}{1}+G_p\binom{2}{1}G_c\binom{1}{2}+G_p\binom{2}{2}G_c\binom{2}{2}\\ &\quad +G_p\binom{1,2}{1,2}\;G_c\binom{1,2}{1,2}\\ &= 1+\frac{N_p\binom{1}{1}}{D_p\binom{1}{1}}\cdot\frac{N_c\binom{1}{1}}{D_c\binom{1}{1}}+\frac{N_p\binom{1}{2}}{D_p\binom{1}{2}}\cdot\frac{N_c\binom{2}{1}}{D_c\binom{2}{1}}+\frac{N_p\binom{2}{1}}{D_p\binom{2}{1}}\cdot\frac{N_c\binom{1}{2}}{D_c\binom{1}{2}}\\ &\quad +\frac{N_p\binom{2}{2}}{D_p\binom{2}{2}}\cdot\frac{N_c\binom{2}{2}}{D_c\binom{2}{2}}+\frac{N_p\binom{1,2}{1,2}}{D_p\binom{1,2}{1,2}}\cdot\frac{N_c\binom{1,2}{1,2}}{D_c\binom{1,2}{1,2}} \qquad \text{(B.16)}\end{aligned}$$

Where the notations for numerator and denominator of each irreducible minor used in (B.3) and (B.4) are employed, e.g., $N_p\binom{1}{1}$, $D_p\binom{1}{1}$

denote the numerator and denominator of the first order minor $G_p\binom{1}{1}$ ($=g_{p11}$), etc. Note that no common factor exists between $N_p\binom{1}{1}$ and $D_p\binom{1}{1}$, $N_c\binom{1}{1}$ and $D_c\binom{1}{1}$, etc.

Let $N_o(s)$, $D_o(s)$ be the numerator and denominator of (B.16), after collecting all the terms at the right-hand-side under the restriction that no pole-zero cancellation is allowed (even if a common factor exists). Then,

$$D_o(s) = \mathrm{LCM}\{D_p\binom{1}{1}D_c\binom{1}{1}, D_p\binom{1}{2}D_c\binom{2}{1}, D_p\binom{2}{1}D_c\binom{1}{2}, D_p\binom{2}{2}D_c\binom{2}{2}, D_p\binom{1,2}{1,2}D_c\binom{1,2}{1,2}\} \tag{B.17}$$

Also, by (B.3) and (B.4),

$$\Delta_c(s)=\mathrm{LCM}\{D_c\binom{1}{1}, D_c\binom{2}{1}, D_c\binom{1}{2}, D_c\binom{2}{2}, D_c\binom{1,2}{1,2}\} \tag{B.18}$$

$$\Delta_p(s)=\mathrm{LCM}\{D_p\binom{1}{1}, D_p\binom{1}{2}, D_p\binom{2}{1}, D_p\binom{2}{2}, D_p\binom{1,2}{1,2}\} \tag{B.19}$$

If no common factor exists between $\Delta_c(s)$ and $\Delta_p(s)$, which can be realized by not using any plant pole[6] as a pole in the compensator,

---

[6] $G(s)$ has a pole at $s=\lambda$, whenever at least one element of $G(\lambda)$ is ∞.

we have

$$D_o(s) = \Delta_c(s) \cdot \Delta_p(s) \tag{B.20}$$

This can be proved by the following arguments:

1. If no common factor exists between $\Delta_c(s)$ and $\Delta_p(s)$, then no common factor can exist between any element in (B.18) and any of those in (B.19) Otherwise, a common factor will exist between $\Delta_c(s)$ and $\Delta_p(s)$.

2. Any factor of $D_o(s)$ must be a factor of either $\Delta_c(s)$ or $\Delta_p(s)$ and with the same multiplicity. The reason is that any factor of $D_o(s)$ must exist in at least one of the five elements in (B.17). And the multiplicity of this factor must be the same as that of the element that has the maximum multiplicity of the same factor. By 1, this factor can be either in the D_p's or in the D_c's of (B.17), but not both. Therefore, by (B.19) or (B.18), the factor with the same multiplicity must appear in either $\Delta_p(s)$ or $\Delta_c(s)$.

3. Any factors of $\Delta_c(s)$ $\Delta_p(s)$ must also be a factor of $D_o(s)$ with the same multiplicity. This can be seen by similar arguments as that in 2 above, but starting from (B.18) and (B.19) instead of (B.17).

Now, similar to what was done in the 1x1 case, let C(s) be the common factor between $N_o(s)$ and $D_o(s)$. Then,

$$D_o(s) = C(s)\cdot D_1(s)$$

$$N_o(s) = C(s)\cdot N_1(s)$$

By (6.3) and (B.20),

$$\hat{N}(s) = \frac{\Delta_c(s)\cdot\Delta_p(s)}{D_1(s)}$$

$$= \frac{D_o(s)}{D_1(s)}$$

$$= C(s)$$

Therefore, the zeros of $N_o(s)$ are exactly those of $N_1(s)$ together with those of $\hat{N}(s)$. Thus, stability of a 2x2 multivariable system can be considered by (6.2) alone; If (1) irreducible minors are used in (4.5), (2) all the other cancellations are not allowed, (3) no plant pole is used in the compensator, and (4) $g_{cij}(s) \not\equiv 0$ for all $i=1,\cdots,m$ $j=1,\cdots,n$.

The condition $g_{cij}(s) \not\equiv 0$ in (4) was added because any zero g_{cij} will cause one corresponding element of (B.17) to be missing which will impair the equality of (B.20) if the associated D_p term happens to be the only one among all the D_p's that contain the highest multiplicity of any factor. This is shown by the following example:

$$\text{Let} \quad G_p(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s-5)} \\ \frac{s-5}{(s+3)(s+1)} & \frac{1}{s+2} \end{bmatrix} \tag{B.21}$$

$$G_c(s) = \begin{bmatrix} g_{c11}(s) & g_{c12}(s) \\ g_{c21}(s) & g_{c22}(s) \end{bmatrix}$$

Then

$$g_{p11}(s) = \frac{1}{s+1}$$

$$g_{p12}(s) = \frac{1}{s-5}$$

$$g_{p21}(s) = \frac{s-5}{(s+3)(s+1)}$$

$$g_{p22}(s) = \frac{1}{s+2}$$

$$\det G_p(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

By (B.19), we have

$$\Delta_p(s)=\text{LCM}\{(s+1),\ (s-5),\ (s+3)(s+1),\ (s+2),\ (s+1)(s+2)(s+3)\}$$
$$= (s+1)(s+2)(s+3)(s-5)$$

Note that the highest multiplicity for the factor (s-5) comes from the second element in the bracket, which is $D_p\binom{1}{2}$, denominator of $g_{p12}(s)$ (or $G_p\binom{1}{2}$). Now if $g_{c21}(s) = 0$ is used (e.g., diagonal G_c), the third term in (B.16) is zero, hence the element $D_p\binom{1}{2}\ D_c\binom{2}{1}$ will not appear in (B.17). Therefore, the factor (s-5) will not appear in $D_o(s)$. This makes (B.20) to be not true. The (s-5) factor will show up in $\hat{N}(s)$ of (6.3) which makes the system always unstable as long as $g_{c21}(s) \equiv 0$. Therefore, it is impossible to stabilize the system with diagonal compensator matrix $G_c(s)$.

Therefore, whenever diagonal $G_c(s)$ is employed, care must be taken to see if situation like this happens. This can be checked very easily by forming all the pertinent irreducible minors of $G_p(s)$ and then check to see if the multiplicity of each plant pole is retained in these minors. If yes, zero element is allowed in $G_c(s)$ and (6.2) alone can determine the stability. If not, (6.3) must also be considered as was shown in the above example. Incidentally, this is a trivial case in single-loop systems. Since there is only one element in $G_c(s)_{1x1}$, which obviously cannot be identically zero.

The above analysis for 2x2 $G_p(s)$ can be carried over in exactly the same manner to the nxm case.

Therefore, for systems as shown in Figure 2.1, if

1. The general formula (4.5) for $\det(I+G_pG_c)$ is used.

2. All the common factors in the minors of G_p are cancelled to get irreducible forms.

3. All other cancellations are not allowed.

4. Plant poles are not used as poles of G_c.

5. Appearance of zero elements in G_c is carefully checked.

then the zeros of $\det(I+G_pG_c)$ alone determine the stability of the system. If all of them are in the open left-half-plane, the system is stable, otherwise it is not.

Therefore, as in the single-loop theory,

$$\det(I+G_pG_c)=0 \tag{B.22}$$

is referred to as the characteristic equation for multivariable systems and stability can be considered through this equation.

APPENDIX C: ROOT LOCUS GAIN

When root locus approach is used, it is convenient to express transfer functions in terms of root locus forms (14) as shown below:

$$G(s) = \frac{k\prod^{j}(s+z_j)}{s^N\prod^{k}(s+p_k)} \qquad \text{(C.1)}$$

where j, k are positive integers and $s=-z_j$, $s=-p_k$ are the zeros and poles of G(s) respectively.

The gain constant k in (C.1) is referred to as the "root locus gain" of the function G(s) (see 14).

In step 6 of Section 6.4, some algebraic manipulation was performed to find the root locus gain k_{eq} of the function $G_{eq}(s)$. In most cases, this step can be bypassed as shown below:

Let k_1, k_2 be the root locus gains of the two transfer functions $G_1(s)$ and $G_2(s)$ respectively. Let ($G_s(s)$, $G_r(s)$ be the sum and the ratio (G_2/G_1) of $G_1(s)$ and $G_2(s)$, in root locus forms and k_s, k_r denote the corresponding root locus gains. Also, let the order of the numerator and denominator of any rational function H(s) be denoted by ON(H) and OD(H) respectively.

Then, we have

THEOREM C.1

The root locus gain k_s of $G(s)=G_1(s)+G_2(s)$ is

(i) $k_s = k_1$ if and only if $ON(G_1)+OD(G_2)>ON(G_2)+OD(G_1)$

(ii) $k_s = k_2$ if and only if $ON(G_1)+OD(G_2)<ON(G_2)+OD(G_1)$

(iii) $k_s = k_1+k_2$ if and only if $ON(G_1)+OD(G_2)=ON(G_2)+OD(G_1)$ and $k_1 \neq -k_2$

THEOREM C.2

The root locus gain k_r of $G(s) = G_2(s)/G_1(s)$ is $k_r=k_2/k_1$.

Proofs for both Theorem C.1 and C.2 are straightforward, hence, omitted.

Both of these theorems are very simple in nature, however, they are very useful tools in evaluating the root locus gains, as illustrated below for the determination of k_{eq} in Section 6.4 (see (6.41)).

By (6.8),

$$G_2 = g_{p22}+(\det G_p)g_{c11}$$

$$G_1 = 1+g_{p11}g_{c11}$$

and $$G_{eq}=\frac{G_2}{G_1}$$

For G_2,

$$ON(g_{p22})+OD(\det G_p \cdot g_{c11}) = 0+4 = 4$$

$$ON(\det G_p \cdot g_{c11})+OD(g_{p22}) = 2+1 = 3$$

Since 4>3, we have by Theorem C.1 (i)

$$k_{G_2} = k_{g_{p22}} = -2$$

where k_{G_2}, $k_{g_{p22}}$ denote the root locus gains of $G_2(s)$ and $g_{p22}(s)$ respectively.

Similarly, for G_1

$$ON(1)+OD(g_{p11}g_{c11}) = 0+2 = 2$$

$$ON(g_{p11}g_{c11})+OD(1) = 1+0 = 1$$

Again, since 2>1, by Theorem C.1 (i), we have

$$k_{G_1} = 1$$

Then, by Theorem C.2,

$$k_{eq} = \frac{k_{G_2}}{k_{G_1}} = -2$$

which agrees with what was obtained in Section 6.4 through algebraic manipulation.

Remark: Whenever $k_1+k_2=0$ in case (iii) of Theorem C.1, no conclusion can be obtained through the theorem, since no obvious analytic expression exists for the coefficient of the second higher order terms. However, direct algebraic manipulation can always be used in such cases.