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STEADY THERMAL CONVECTION FROM A CONCENTRATED SOURCE IN A POROUS MEDIUM*

C. E. Hickox
Division 5512
Fluid Mechanics and Heat Transfer

and

H. A. Watts
Division 2623
Applied Mathematics
Sandia Laboratories†
Albuqueruque, New Mexico 87185

ABSTRACT

Solutions for the axisymmetric velocity and temperature fields associated with a point source of thermal energy in a fluid-saturated porous medium are obtained numerically through use of similarity transformations. The two cases considered are those of a point source located on the lower boundary of a semi-infinite region and a point source embedded in an infinite region. Tabulated results are presented from which complete descriptions of the velocity and temperature fields can be constructed for Rayleigh numbers of 0.1, 1, 10, and 100.

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C. E. Hickox and H. A. Watts

I. Introduction

In a previous paper, Wooding[1] demonstrated the utility of simple models for the description of geothermal flow processes. Of particular interest in the present context, Wooding considered the case of a point source of thermal energy located on the lower boundary of a semi-infinite, fluid-saturated, porous medium. When vertical gradients of velocity and thermal conduction are negligible compared with horizontal gradients, and the Boussinesq approximation invoked, Wooding was able to show that similarity solutions for planar or axisymmetric flow can be obtained by direct analogy with the solutions given by Schlichting[2] for planar or axisymmetric, incompressible, laminar jets. The approximations imposed are, generally speaking, valid only for sufficiently large values of the Rayleigh number.

Since point source solutions can be of considerable practical utility, a more complete analysis, which does not depend on approximations like those mentioned above, is developed in the present paper. Two cases, both concerned with axisymmetric flow and valid for any value of the Rayleigh number, are considered. In the first instance, the

physical arrangement is identical to that considerd by Wooding. The second case treats the problem of axisymmetric flow induced by a point source imbedded in an infinite, fluid-saturated, porous medium. Both cases are analyzed through the introduction of similarity transformations resulting in sets of nonlinear ordinary differential equations which are then solved numerically in order to provide descriptions of the velocity and temperature fields.

II. General Theory

In this section, mathematical models are developed for the description of axisymmetric free convection in a fluid saturated porous medium. The medium is assumed to be rigid, homogeneous, and isotropic and the fluid incompressible, with density changes occurring only as a result of changes in the temperature according to

$$\rho = \rho_\infty \, [1 - \beta(T - T_\infty)], \tag{1}$$

where ρ is the density, T is the temperature, β is the coefficient of thermal expansion, and the subscripts refer to reference conditions. It is also assumed that the fluid motion obeys Darcy's law.

The steady-state equations of continuity, motion, and thermal transport are then

$$\text{div}\, \underset{\sim}{v} = 0, \tag{2}$$

$$\frac{\mu}{K} \underset{\sim}{v} = -\text{grad}\, (p + \rho g h), \tag{3}$$

$$\underset{\sim}{v} \cdot \text{grad } T = \alpha \text{ div}(\text{grad } T), \tag{4}$$

where $\underset{\sim}{v}$, k, μ, α, p, and g are, respectively, the velocity vector, permeability, dynamic viscosity, effective thermal diffusivity, pressure, and acceleration due to gravity. The elevation h is measured vertically upward and g is oppositely directed.

In accordance with the usual Boussinesq approximation, density changes are accounted for only in the buoyancy term in the equations of motion. It is furthermore assumed that permeability, viscosity, and thermal diffusivity are constants and that dispersion effects are negligible.

A. Point Source Below Semi-Infinite Region

Here we wish to consider the axisymmetric flow induced by a point source of strength Q (energy generated per unit time) situated on the lower, insulated, edge of a semi-infinite region. Cylindrical polar coordinates (r,z) with associated velocity components (u,w) are used in the subsequent analysis. The origin of the coordinate system is coincident with the point source, and the z-axis is directed vertically upward.

The basic formulation now proceeds in a straightforward manner from Equations (1) through (4). Equation (1) admits the introduction of a stream function ψ defined by

$$ru = -\frac{\partial\psi}{\partial z}, \quad rw = \frac{\partial\psi}{\partial r} \quad . \tag{5}$$

Introduction of the similarity transformations

$$\eta = r/z, \ \psi = \alpha z f(\eta), \ T - T_\infty = \left(\frac{\mu\alpha}{\rho_\infty g k \beta}\right) \frac{\theta(\eta)}{z} \quad , \tag{6}$$

as suggested by Yih[3], allows Equations (1) through (4) to be reduced to the set of nonlinear ordinary differential equations

$$(\eta^3 + \eta) f'' - f' = \eta^2 \theta, \tag{7}$$

$$-(f\theta)' = (\theta'\eta)' + (\eta^2\theta)' + (\eta^3\theta')', \tag{8}$$

where primes denote differentiation with respect to η. An additional requirement obtained upon integration of Equation (4) over a plane taken normal to the z-axis is

$$\int_0^\infty (f'\theta + \eta\theta + \eta^2\theta')\, d\eta = R_A, \tag{9}$$

where R_A is the Rayleigh number

$$R_A = Q g k \beta / 2\pi\alpha^2 \mu c, \tag{10}$$

and c is the specific heat of the fluid. For future reference, the relationships between velocities and similarity parameters are

$$u = \frac{\alpha}{z}\, \eta \left(\frac{f}{\eta}\right)', \quad w = \frac{\alpha}{z} \left(\frac{f'}{\eta}\right) . \tag{11}$$

From physical arguments, it is expected that u, w, and $(T - T_\infty)$ should approach zero for large r. In addition, u as well as the radial gradients of w and T should be zero along the axis of symmetry. These requirements yield the boundary conditions

$$f(0) = 0 = \theta'(0), \tag{12a}$$

$$f'(\infty) = 0 = \theta(\infty). \tag{12b}$$

Solution of Equations (7) and (8) subject to the boundary conditions (12) and integral relation (9) will provide a complete description of the thermal and flow fields.

B. Point Source in an Infinite Medium

In order to analyze the axisymmetric flow field induced by a point source in an infinite medium, it is appropriate to adopt spherical polar coordinates (R,Φ) with associated velocity components (v_R, v_Φ). The angle Φ is measured from the vertical axis (z-axis as introduced in Section II.A) to the radial position vector $\underset{\sim}{R}$. The velocity component v_R is in the direction of $\underset{\sim}{R}$. The other velocity component v_Φ is normal to $\underset{\sim}{R}$, lies in the plane of R and z, and is positive in the direction of increasing Φ. The origin of coordinates is again taken coincident with the point source.

The formulation for this case, in essence, parallels that presented in the first part of this section. A stream function ψ is defined such that

$$v_R = \frac{1}{R^2 \sin\Phi} \frac{\partial\psi}{\partial\Phi}, \quad v_\Phi = \frac{-1}{R\sin\Phi} \frac{\partial\psi}{\partial R}. \tag{13}$$

Based on certain apparent similarities between the present system of equations and those derived by Squire[4] in his study of the round laminar jet, we introduce the following substitutions

$$\eta = \cos\Phi, \ \psi = \alpha R f(\eta), \ T - T_\infty = \frac{\mu\alpha}{\rho_\infty g k \beta} \frac{\theta(\eta)}{R} . \tag{14}$$

The original system of partial differential equations can now be reduced to the system of ordinary differential equations

$$f'' = -(\eta\theta)' , \tag{15}$$

$$(f\theta)' = \theta'' - (\eta^2\theta')' , \tag{16}$$

where primes denote differentiation with respect to η. Integration of Equation (4) over a sphere centered about the origin provides the relation

$$\int_{-1}^{1} (1 - f')\theta d\eta = R_A, \tag{17}$$

where R_A is the Rayleigh number as defined in (10). Velocity components are now given by

$$v_R = -\frac{\alpha}{R} f'(\eta), \ v_\Phi = -\frac{\alpha}{R} (1 - \eta^2)^{-1/2} f(\eta) . \tag{18}$$

As before, it is expected that v_R, v_Φ, and $(T - T_\infty)$ should approach zero for large R and that the solution should exhibit symmetry about a vertical axis. These physical requirements translate into the conditions

$$f(1) = 0 = f(-1) , \tag{19a}$$

$$\theta, \theta' \text{ bounded for } \eta = \pm 1. \tag{19b}$$

Although, in some instances, the notation used in this section is in conflict with that used in Section II.A, no confusion should exist so long as the two cases are considered separately.

III. Approximate Analytical Solutions

When the Rayleigh number is sufficiently small, approximate analytical results can be obtained for both cases under consideration. For large Rayleigh numbers, Wooding[1] obtained an approximate solution for the semi-infinite region by analogy with the results of Schlichting[2] for incompressible laminar jets. Presently, we know of no corresponding solution for the point source in an infinite region. The various approximate solutions are presented in the remainder of this section.

A. Point Source Below Semi-Infinite Region

For small Rayleigh number, solutions to Equations (7), (8), and (9) can be obtained by straightforward expansion of the dependent variables in terms of the Rayleigh number. The leading terms resulting from this process provide the approximations

$$f = -\frac{1}{2} R_A \left[(\eta^2 + 1)^{-1/2} - 1\right] , \tag{20}$$

$$\theta = R_A (\eta^2 + 1)^{-1/2} , \tag{21}$$

where it may be noted that (21) produces the steady-state conduction solution associated with a point source.

For a sufficiently large Rayleigh number, Wooding's[1] analysis provides the results

$$f = \frac{3}{8} R_A \eta^2 \left(1 + \frac{3}{32} R_A \eta^2\right)^{-1} , \tag{22}$$

$$\theta = \frac{3}{4} R_A \left(1 + \frac{3}{32} R_A \eta^2\right)^{-2} . \tag{23}$$

B. Point Source in an Infinite Medium

For small Rayleigh number, the leading terms of an expansion in terms of the Rayleigh number provide the approximate results[5]

$$f = \frac{1}{4} R_A (1 - \eta^2) + \frac{1}{24} R_A^2 \eta (1 - \eta^2) , \tag{24}$$

$$\theta = \frac{1}{2} R_A + \frac{1}{8} \eta R_A^2 , \tag{25}$$

where it may be noted that substitution of the first term of (25) into (14) produces the steady-state conduction solution associated with a point source.

The results of this section are useful in that they provide bounds for the more general numerical results which will be presented subsequently.

IV. Computational Approach

Although the use of similarity transformations results in simplified formulations for the cases considered, the resulting equations are still of sufficient complexity

to warrant the use of numerical techniques in order to obtain general solutions. Efficient, state-of-the-art computer codes, currently under development at Sandia Laboratories and designed specifically for the solution of two-point boundary value problems, were utilized for the numerical solutions. In the remainder of this section numerical results are presented along with brief descriptions of the techniques employed in the analysis.

A. Point Source Below Semi-Infinite Region[6]

Equations (7) and (8) can be integrated once to provide

$$(\eta + \frac{1}{\eta})f' - f = \theta + C_1, \tag{26}$$

$$-f\theta = (\eta^3 + \eta)\theta' + \eta^2\theta \ , \tag{27}$$

where C_1 is a constant of integration, and the constant of integration associated with (27) is equal to zero by virtue of (12a). Further integration of the differential equations by analytical methods does not appear feasible.

In order to obtain numerical solutions, equations (26) and (27) could be solved subject to the boundary conditions (12) for a specified value of the constant C_1. Then C_1 could be varied until the constraint condition (9) is satisfied. However, it is desirable to automate the entire process within the framework of solving only differential equations. Thus, a differential equation for C_1 is added and (9) is modified to provide

$$S(\eta) = \int_0^{\eta} (f'\theta + \zeta\theta + \zeta^2\theta')d\zeta \ , \tag{28}$$

where the integration variable is now ζ. The original boundary value problem can now be expressed by the system of first order differential equations

$$\theta'(\eta) = -(\eta + \eta^3)^{-1} (f + \eta^3)\theta, \ \eta > 0 \ , \tag{29a}$$

$$f'(\eta) = \eta(1 + \eta^2)^{-1} (\theta + f + C_1) \ , \tag{29b}$$

$$C_1'(\eta) = 0 \ , \tag{29c}$$

$$S'(\eta) = \eta(1 + \eta^2)^{-1} (\theta + C_1 + 1)\theta \ , \tag{29d}$$

subject to the boundary conditions

$$f(0) = 0 = S(0), \tag{30a}$$

$$f(\infty) = -C_1, \ S(\infty) = R_A \ . \tag{30b}$$

The condition $\theta'(0) = 0$ is used directly to give proper definition to (29a) at the origin. Equation (29d) is obtained upon differentiation of (28) and substitution from (29a) and (29b). The boundary conditions on S follow directly from consideration of (9) and (28). The condition $f'(\infty) = 0$ is automatically satisfied by the differential equation (29b) under the assumption of boundedness for $f(\eta)$ and $\theta(\eta)$. An even stronger condition emerges under our physical constraints; namely $\lim_{\eta\to\infty} \eta f'(\eta) = 0$. Using this requirement along

with the condition $\theta(\infty) = 0$ in (29b) leads to the new boundary condition on f, given by $f(\infty) + C_1 = 0$. The motivation for using this condition stems from difficulties encountered in attempting to obtain satisfactory numerical solutions when the condition $\theta(\infty) = 0$ was imposed. It was observed that θ was driven to zero for large η regardless of the acceptability of the other variables. With the alternate boundary condition, the solution algorithm was less sensitive to poor initial guesses for $\theta(0)$ and C_1.

Numerical solutions of (29) subject to the boundary conditions (30) were obtained for values of the Rayleigh number in the range 0.1 to 100 and are tabulated in Tables 1 through 3. The computer code SHOOT2[7] was employed for the solution of the boundary value problem. This code is based on a shooting procedure which uses current state-of-the-art variable step size integration methods. The integration was carried out to a value of 1000 for η in all cases. This interval was found to be of sufficient magnitude to insure the accuracy of solutions in regions of physical interest. Tabulated results appear to be accurate to four significant figures.

B. Point Source in an Infinite Medium

The numerical solution procedure utilized for this problem closely parallels that of the preceding case. The basic differential equations (15) and (16) can be integrated once to provide

$$f' = -\eta\theta + C_2, \tag{31}$$

$$f\theta = (1 - \eta^2)\theta' , \tag{32}$$

where C_2 is a constant of integration, and the constant of integration associated with (32) is zero by virtue of the boundary conditions (19). As before, the integral constraint (17) is rewritten as

$$S(\eta) = \int_{-1}^{\eta} (1 - f')\theta d\zeta . \tag{33}$$

Now the original boundary value problem can be expressed by the system

$$\theta'(\eta) = (1 - \eta^2)^{-1} f\theta , \tag{34a}$$

$$f'(\eta) = -\eta\theta + C_2 , \tag{34b}$$

$$C_2'(\eta) = 0 , \tag{34c}$$

$$S'(\eta) = (1 + \eta\theta - C_2)\theta , \tag{34d}$$

subject to the boundary conditions

$$f(-1) = 0 = f(1), \tag{35a}$$

$$S(-1) = 0, \; S(1) = R_A . \tag{35b}$$

A complication arises due to the singular nature of (34a) at the boundary points. This difficulty is, however, easily removed since a straightforward analysis using (34a) and

(34b) shows that $\theta'(\pm 1) = \mp (1/2)\,\theta(\pm 1) f'(\pm 1)$. The solution of the differential system (34) subject to the boundary conditions (35) was obtained through use of the computer code SUPOR Q,[8] a code utilizing quasilinearization techniques coupled with superposition and an orthonormalization process. Results, apparently accurate to four significant figures, are tabulated in Tables 4 through 6.

V. Discussion

Accurate numerical results have been obtained which provide for the complete description of the axisymmetric flow and temperature fields associated with a point source situated on the lower edge of a semi-infinite region or embedded in an infinite porous medium. Representative sketches of the isotherms and streamlines for each case considered are presented in Figures 1 and 2 for a Rayleigh number of 10. A careful examination of the tabulated results indicates that for a Rayleigh number of 0.1 the temperature distribution can be closely approximated by the steady-state conduction solution. Isothermal surfaces are approximately hemispherical or spherical surfaces centered about the point source. The approximate results given by (20), (21), (24), and (25) provide reasonably accurate estimates to the true solutions. When the Rayleigh number is 100 the solutions exhibit a plume-like behavior more in line with the boundary layer model proposed by Wooding[1]. For the semi-infinite

region, the approximate results given by (22) and (23) are found to be reasonably accurate, even for a relatively modest value of 100 for the Rayleigh number.

It is readily observed that the vertical velocity along the axis of symmetry above the point source is proportional to $1/z$ for both cases considered, where z is the distance above the source. An upper bound to the vertical transport of a fluid particle can, thus, readily be found by integration. Estimates of this type can be useful in the analysis of induced fluid motion due to the emplacement of thermally active canisters of nuclear waste material in a fluid-saturated porous medium.

Another interesting observation is that, for the semi-infinite region, the constant C_1 can be obtained from the condition $C_1 = -f(\infty)$. Similarly, it is noted that, for the infinite region, $C_2 = f'(0)$. Hence, values for the constants of integration can be obtained from the tabulated results. This information is useful in that it can be used to provide estimates for the constants of integration for cases which are not tabulated. Integration of the appropriate differential equations can then be carried out in a straightforward manner for any Rayleigh number. It is perhaps also worth noting that the limiting values for C_1 and C_2 are, respectively, for large Rayleigh number: -4 and 2, and for small Rayleigh number: $-R_A/2$ and $R_A^2/24$.

Nomenclature

f	-nondimensional function related to streamfunction
g	-acceleration of gravity
h	-elevation
k	-intrinsic permeability
p	-pressure
Q	-strength of point source
$\underset{\sim}{R}$	-position vector in spherical polar coordinates
R_A	-Rayleigh number
(r,z)	-cylindrical polar coordinates
(R,ϕ)	-spherical polar coordinates
T	-temperature
(u,w)	-velocity components in cylindrical polar coordinates
$(v_R,v\phi)$	-velocity components in spherical polar coordinates
$\underset{\sim}{v}$	-velocity vector

Greek

α	-thermal diffusivity
β	-coefficient of volumetric thermal expansion
η	-similarity variable
θ	-nondimensional function related to temperature
μ	-dynamic viscosity
ρ	-density
ψ	-stream function
ζ	-integration variable

Subscript

∞	-refers to conditions far from point source

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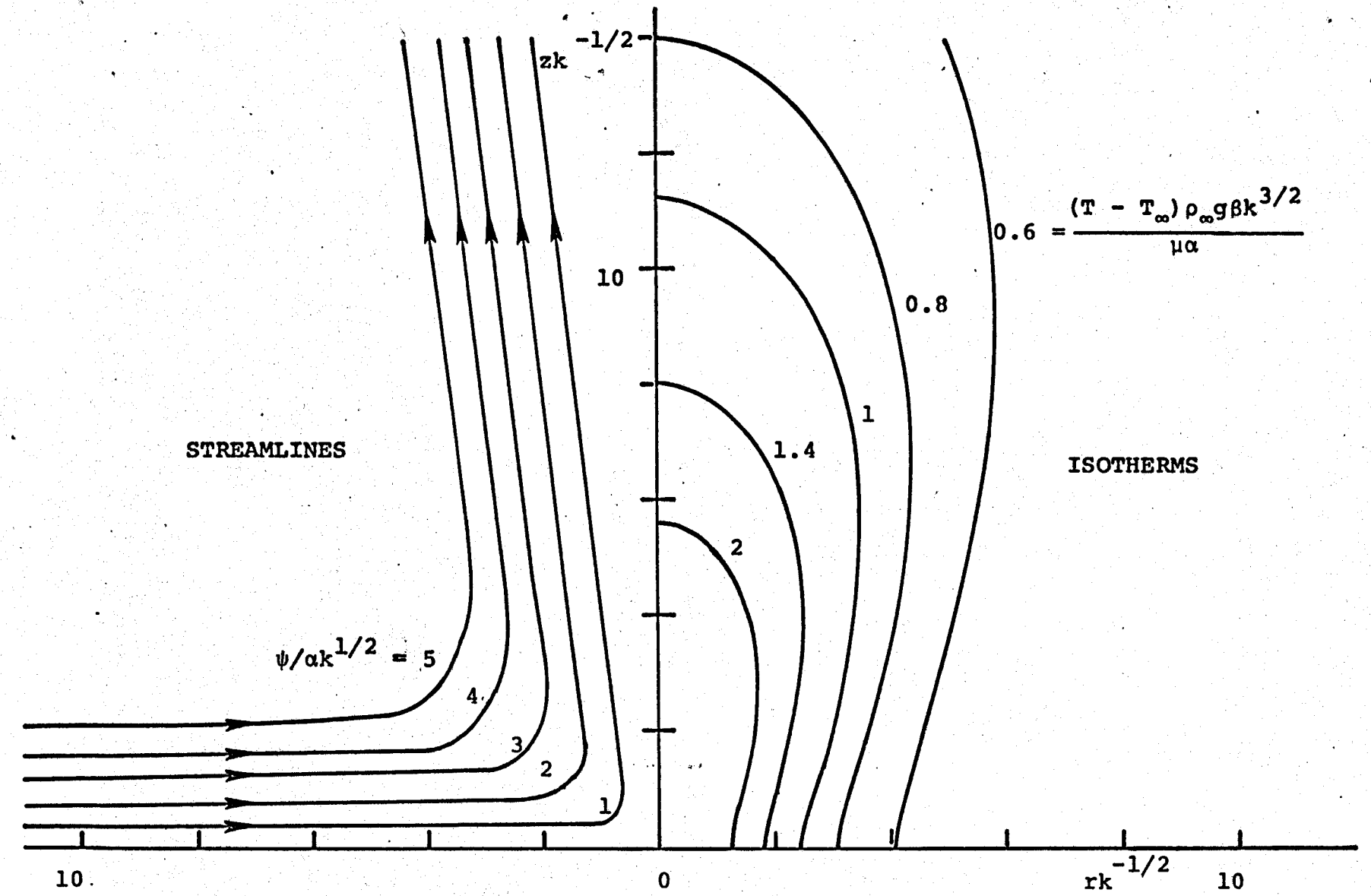

Figure 1. Isotherms and Streamlines for a Point Source Below a Semi-Infinite Region, $R_A = 10$.

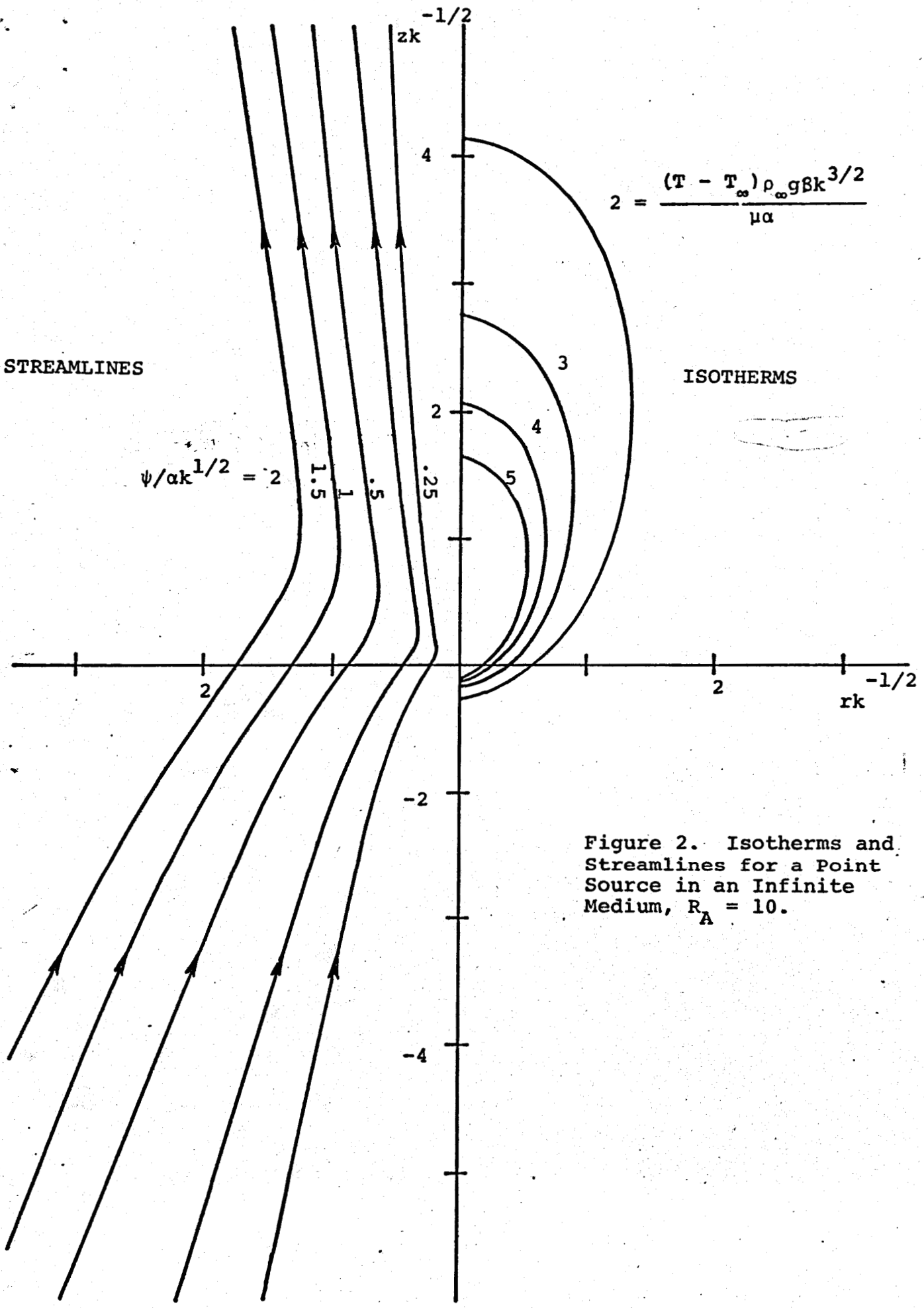

Figure 2. Isotherms and Streamlines for a Point Source in an Infinite Medium, $R_A = 10$.

η	Rayleigh Number 0.1	1	10	100
0.0000	1.0106E-01	1.0959E+00	1.1213E+01	8.0541E+01
.0100	1.0106E-01	1.0958E+00	1.1210E+01	8.0383E+01
.0200	1.0104E-01	1.0956E+00	1.1201E+01	7.9911E+01
.0300	1.0102E-01	1.0953E+00	1.1186E+01	7.9134E+01
.0400	1.0099E-01	1.0949E+00	1.1166E+01	7.8066E+01
.0500	1.0093E-01	1.0941E+00	1.1140E+01	7.6726E+01
.0600	1.0083E-01	1.0934E+00	1.1107E+01	7.5136E+01
.0700	1.0081E-01	1.0924E+00	1.1070E+01	7.3324E+01
.0800	1.0073E-01	1.0914E+00	1.1027E+01	7.1317E+01
.0900	1.0065E-01	1.0902E+00	1.0978E+01	6.9145E+01
.1000	1.0055E-01	1.0889E+00	1.0925E+01	6.6840E+01
.2000	9.9053E-02	1.0685E+00	1.0143E+01	4.2135E+01
.3000	9.6699E-02	1.0369E+00	9.0635E+00	2.4005E+01
.4000	9.3669E-02	9.9683E-01	7.8948E+00	1.3837E+01
.5000	9.0159E-02	9.5129E-01	5.7849E+00	8.4378E+00
.6000	8.6360E-02	9.0301E-01	5.8085E+00	5.4945E+00
.7000	8.2434E-02	8.5417E-01	4.9859E+00	3.8047E+00
.8000	7.8505E-02	8.0634E-01	4.3082E+00	2.7784E+00
.9000	7.4666E-02	7.6056E-01	3.7549E+00	2.1214E+00
1.0000	7.0977E-02	7.1740E-01	3.3036E+00	1.6806E+00
2.0000	4.4675E-02	4.3085E-01	1.3879E+00	4.7022E-01
3.0000	3.1533E-02	2.9888E-01	5.5469E-01	2.6843E-01
4.0000	2.4165E-02	2.2732E-01	6.2983E-01	1.8959E-01
5.0000	1.9532E-02	1.8302E-01	4.9642E-01	1.4733E-01
6.0000	1.5369E-02	1.5303E-01	4.1018E-01	1.2081E-01
7.0000	1.4079E-02	1.3144E-01	3.4972E-01	1.0254E-01
8.0000	1.2347E-02	1.1516E-01	3.0492E-01	8.9144E-02
9.0000	1.0992E-02	1.0245E-01	2.7038E-01	7.8888E-02
10.0000	9.9037E-03	9.2267E-02	2.4291E-01	7.0772E-02
20.0000	4.9695E-03	4.6225E-02	1.2074E-01	3.5023E-02
30.0000	3.3152E-03	3.0827E-02	8.0407E-02	2.3304E-02
40.0000	2.4869E-03	2.3123E-02	6.0281E-02	1.7466E-02
50.0000	1.9898E-03	1.8500E-02	4.8216E-02	1.3969E-02
60.0000	1.6582E-03	1.5417E-02	4.0176E-02	1.1639E-02
70.0000	1.4214E-03	1.3215E-02	3.4434E-02	9.9749E-03
80.0000	1.2437E-03	1.1563E-02	3.0129E-02	8.7274E-03
90.0000	1.1056E-03	1.0278E-02	2.6781E-02	7.7573E-03
100.0000	9.9503E-04	9.2506E-03	2.4102E-02	6.9814E-03
200.0000	4.9753E-04	4.6254E-03	1.2050E-02	3.4903E-03
300.0000	3.3169E-04	3.0836E-03	8.0334E-03	2.3268E-03
400.0000	2.4877E-04	2.3127E-03	6.0250E-03	1.7451E-03
500.0000	1.9901E-04	1.8502E-03	4.8200E-03	1.3961E-03
600.0000	1.6585E-04	1.5418E-03	4.0167E-03	1.1634E-03
700.0000	1.4215E-04	1.3215E-03	3.4429E-03	9.9720E-04
800.0000	1.2438E-04	1.1563E-03	3.0125E-03	8.7255E-04
900.0000	1.1056E-04	1.0279E-03	2.6778E-03	7.7560E-04
1000.0	9.9507E-05	9.2508E-04	2.4100E-03	6.9804E-04

Table 1. The function θ(η) for the semi-infinite region

η	Rayleigh Number 0.1	1	10	100
0.0000	0.	0.	0.	0.
.0100	2.5443E-06	2.9455E-05	4.2736E-04	3.8334E-03
.0200	1.0175E-05	1.1779E-04	1.7084E-03	1.5285E-02
.0300	2.2885E-05	2.6492E-04	3.8402E-03	3.4211E-02
.0400	4.0662E-05	4.7069E-04	6.8176E-03	6.0377E-02
.0500	6.3491E-05	7.3490E-04	1.0634E-02	9.3464E-02
.0600	9.1351E-05	1.0573E-03	1.5279E-02	1.3309E-01
.0700	1.2422E-04	1.4375E-03	2.0744E-02	1.7877E-01
.0800	1.6206E-04	1.8751E-03	2.7014E-02	2.3002E-01
.0900	2.0494E-04	2.3698E-03	3.4077E-02	2.8630E-01
.1000	2.5253E-04	2.9210E-03	4.1915E-02	3.4703E-01
.2000	9.8789E-04	1.1396E-02	1.5846E-01	1.0787E+00
.3000	2.1446E-03	2.4637E-02	3.2673E-01	1.7698E+00
.4000	3.6351E-03	4.1539E-02	5.2028E-01	2.2821E+00
.5000	5.3624E-03	6.0908E-02	7.1735E-01	2.6361E+00
.6000	7.2337E-03	8.1636E-02	9.0399E-01	2.8795E+00
.7000	9.1693E-03	1.0281E-01	1.0732E+00	3.0505E+00
.8000	1.1109E-02	1.2375E-01	1.2228E+00	3.1740E+00
.9000	1.3005E-02	1.4399E-01	1.3534E+00	3.2655E+00
1.0000	1.4830E-02	1.6324E-01	1.4666E+00	3.3352E+00
2.0000	2.7879E-02	2.9545E-01	2.0509E+00	3.6004E+00
3.0000	3.4424E-02	3.5893E-01	2.2577E+00	3.6704E+00
4.0000	3.8100E-02	3.9379E-01	2.3605E+00	3.7025E+00
5.0000	4.0413E-02	4.1561E-01	2.4218E+00	3.7211E+00
6.0000	4.1993E-02	4.3045E-01	2.4524E+00	3.7333E+00
7.0000	4.3137E-02	4.4116E-01	2.4914E+00	3.7418E+00
8.0000	4.4003E-02	4.4925E-01	2.5131E+00	3.7482E+00
9.0000	4.4680E-02	4.5558E-01	2.5299E+00	3.7531E+00
10.0000	4.5224E-02	4.6065E-01	2.5433E+00	3.7571E+00
20.0000	4.7591E-02	4.8363E-01	2.6038E+00	3.7747E+00
30.0000	4.8518E-02	4.9133E-01	2.6239E+00	3.7805E+00
40.0000	4.8933E-02	4.9519E-01	2.6339E+00	3.7834E+00
50.0000	4.9192E-02	4.9750E-01	2.6400E+00	3.7852E+00
60.0000	4.9348E-02	4.9905E-01	2.6440E+00	3.7863E+00
70.0000	4.9467E-02	5.0015E-01	2.6469E+00	3.7872E+00
80.0000	4.9557E-02	5.0098E-01	2.6490E+00	3.7878E+00
90.0000	4.9626E-02	5.0163E-01	2.6507E+00	3.7883E+00
100.0000	4.9682E-02	5.0215E-01	2.6521E+00	3.7887E+00
200.0000	4.9936E-02	5.0451E-01	2.6582E+00	3.7905E+00
300.0000	5.0024E-02	5.0532E-01	2.6604E+00	3.7911E+00
400.0000	5.0070E-02	5.0575E-01	2.6615E+00	3.7914E+00
500.0000	5.0100E-02	5.0603E-01	2.6622E+00	3.7916E+00
600.0000	5.0121E-02	5.0623E-01	2.6627E+00	3.7918E+00
700.0000	5.0138E-02	5.0639E-01	2.6631E+00	3.7919E+00
800.0000	5.0152E-02	5.0652E-01	2.6635E+00	3.7920E+00
900.0000	5.0164E-02	5.0663E-01	2.6638E+00	3.7921E+00
1000.0	5.0174E-02	5.0673E-01	2.6640E+00	3.7921E+00

Table 2. The function f(η) for the semi-infinite region.

η	Rayleigh Number 0.1	1	10	100
0.0000	0.	0.	0.	0.
.0100	5.0882E-04	5.8307E-03	4.5455E-02	7.6587E-01
.0200	1.0172E-03	1.1775E-02	1.7071E-01	1.5221E+00
.0300	1.5246E-03	1.7548E-02	2.5555E-01	2.2592E+00
.0400	2.0305E-03	2.3508E-02	3.3981E-01	2.9585E+00
.0500	2.5348E-03	2.9333E-02	4.2325E-01	3.6422E+00
.0600	3.0368E-03	3.5134E-02	5.0571E-01	4.2732E+00
.0700	3.5359E-03	4.0903E-02	5.8599E-01	4.8559E+00
.0800	4.0319E-03	4.6524E-02	6.6592E-01	5.3959E+00
.0900	4.5243E-03	5.2302E-02	7.4533E-01	5.8601E+00
.1000	5.0125E-03	5.7927E-02	8.2205E-01	6.2767E+00
.2000	9.5395E-03	1.1023E-01	1.4587E+00	7.5310E+00
.3000	1.3395E-02	1.5270E-01	1.8512E+00	6.8507E+00
.4000	1.6251E-02	1.8333E-01	1.9931E+00	4.2503E+00
.5000	1.8139E-02	2.0219E-01	1.9353E+00	2.9127E+00
.6000	1.9153E-02	2.1085E-01	1.7961E+00	2.0215E+00
.7000	1.9463E-02	2.1153E-01	1.5950E+00	1.4391E+00
.8000	1.9239E-02	2.0552E-01	1.3985E+00	1.0533E+00
.9000	1.8645E-02	1.9781E-01	1.2154E+00	7.9295E-01
1.0000	1.7815E-02	1.8595E-01	1.0531E+00	6.1154E-01
2.0000	8.9517E-03	8.7323E-02	3.0399E-01	1.1140E-01
3.0000	4.7347E-03	4.5295E-02	1.3752E-01	4.4005E-02
4.0000	2.8447E-03	2.6913E-02	7.6781E-02	2.3529E-02
5.0000	1.8739E-03	1.7673E-02	4.8580E-02	1.4573E-02
6.0000	1.3277E-03	1.2447E-02	3.3625E-02	1.0344E-02
7.0000	9.8583E-04	9.2223E-03	2.4792E-02	7.3128E-03
8.0000	7.5998E-04	7.0993E-03	1.8950E-02	5.5553E-03
9.0000	6.0335E-04	5.6310E-03	1.4955E-02	4.3792E-03
10.0000	4.9339E-04	4.5735E-03	1.2103E-02	3.5368E-03
20.0000	1.2393E-04	1.1535E-03	3.0172E-03	8.7381E-04
30.0000	5.5242E-05	5.1375E-04	1.3403E-03	3.8873E-04
40.0000	3.1117E-05	2.8935E-04	7.5455E-04	2.1867E-04
50.0000	1.9941E-05	1.8539E-04	4.8330E-04	1.4003E-04
60.0000	1.3865E-05	1.2591E-04	3.3597E-04	9.7335E-05
70.0000	1.0201E-05	9.4837E-05	2.4715E-04	7.1597E-05
80.0000	7.8223E-06	7.2722E-05	1.8950E-04	5.4895E-05
90.0000	6.1911E-06	5.7553E-05	1.4998E-04	4.3445E-05
100.0000	5.0244E-06	4.6711E-05	1.2171E-04	3.5255E-05
200.0000	1.2935E-06	1.2025E-05	3.1331E-05	9.0749E-06
300.0000	6.0255E-07	5.6013E-06	1.4594E-05	4.2271E-06
400.0000	3.6071E-07	3.3534E-06	8.7363E-06	2.5304E-06
500.0000	2.4377E-07	2.3127E-06	6.0251E-06	1.7451E-06
600.0000	1.8795E-07	1.7474E-06	4.5522E-06	1.3185E-06
700.0000	1.5123E-07	1.4065E-06	3.6642E-06	1.0513E-06
800.0000	1.2743E-07	1.1853E-06	3.0873E-06	8.9437E-07
900.0000	1.1115E-07	1.0335E-06	2.6927E-06	7.7991E-07
1000.0	9.9503E-08	9.2508E-07	2.4103E-06	6.9504E-07

Table 3. The function f'(η) for the semi-infinite region.

Φ (deg)	Rayleigh Number 0.1	1	10	100
0	5.1246E-2	6.1577E-1	8.2570E-0	7.6984E+1
10	5.1226	6.1308	7.8103	4.6587
20	5.1167	6.0522	6.6875	1.7113
30	5.1072	5.9272	5.3290	6.4265E-0
40	5.0943	5.7643	4.0811	2.8029
50	5.0784	5.5735	3.0850	1.4112
60	5.0602	5.3652	2.3437	7.9816E-1
70	5.0401	5.1493	1.8093	4.9562
80	5.0188	4.9347	1.4276	3.3193
90	4.9971	4.7285	1.1544	2.3668
100	4.9754	4.5364	9.5715E-1	1.7792
110	4.9546	4.3621	8.1350	1.3998
120	4.9353	4.2085	7.0827	1.1467
130	4.9178	4.0773	6.3112	9.7406E-2
140	4.9029	3.9695	5.7509	8.5545
150	4.8908	3.8854	5.3541	7.7486
160	4.8820	3.8254	5.0898	7.2276
170	4.8765	3.7894	4.9386	6.9352
180	4.8746	3.7774	4.8893	6.8408

Table 4. The function $\theta(\Phi)$ for the infinite region.

Φ (deg)	Rayleigh Number 0.1	1	10	100
0	0.0000	0.0000	0.0000	0.0000
10	7.6654E-4	8.6744E-3	1.0858E-1	8.7685E-1
20	2.9696E-3	3.3349E-2	3.8509	2.0273E-0
30	6.3404	7.0327	7.2188	2.6011
40	1.0461E-2	1.1413E-1	1.0206E-0	2.7770
50	1.4824	1.5853	1.2278	2.7381
60	1.8901	1.9760	1.3303	2.5787
70	2.2195	2.2644	1.3377	2.3480
80	2.4309	2.4179	1.2692	2.0748
90	2.4992	2.4234	1.1457	1.7792
100	2.4169	2.2862	9.8693E-1	1.4761
110	2.1944	2.0277	8.0981	1.1782
120	1.8589	1.6813	6.2897	8.9666E-1
130	1.4510	1.2877	4.5654	6.4118
140	1.0197	8.9062E-2	3.0252	4.2028
150	6.1601E-3	5.3116	1.7463	2.4076
160	2.8780	2.4586	7.9001E-2	1.0836
170	7.4179E-4	6.3010E-3	1.9974	2.7318E-2
180	0.0000	0.0000	0.0000	0.0000

Table 5. The function f(Φ) for the infinite region.

Φ (deg)	Rayleigh Number 0.1	1	10	100
0	-5.0829E-2	-5.7670E-1	-7.4306E-0	-7.5255E+1
10	-5.0031	-5.6469	-6.8651	-4.4150
20	-4.7666	-5.2965	-5.4578	-1.4352
30	-4.3812	-4.7422	-3.7884	-3.8365E-0
40	-3.8606	-4.0247	-2.2997	-4.1815E-1
50	-3.2228	-3.1919	-1.1566	8.2177
60	-2.4884	-2.2918	-3.4541E-1	1.3298E-0
70	-1.6821	-1.3703	2.0765	1.5594
80	-8.2963E-3	-4.6592E-2	5.7860	1.6712
90	4.1638E-4	3.9074	8.2643	1.7289
100	9.0538E-3	1.1783E-1	9.9259	1.7597
110	1.7361E-2	1.8826	1.1046E-0	1.7767
120	2.5093	2.4950	1.1806	1.7862
130	3.2028	3.0116	1.2321	1.7915
140	3.7973	3.4314	1.2670	1.7944
150	4.2771	3.7555	1.2901	1.7960
160	4.6292	3.9855	1.3047	1.7968
170	4.8441	4.1226	1.3128	1.7972
180	4.9163	4.1682	1.3154	1.7973

Table 6. The function f'(Φ) for the infinite region.