Interfaces
Vol. 37, No. 4, July–August 2007, pp. 342–352
ISSN 0092-2102 | EISSN 1526-551X | 07 | 3704 | 0342

DOI 10.1287/inte.1070.0286

Steaming on Convex Hulls

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This is a sea story about using a simple classroom example to save a great deal of money, as well as to convince beginning Postgraduate Naval School operations research students—experienced, skeptical military officers—that mathematical analysis can yield immediate results. The application is planning a ship's transit from one point to another in a fixed amount of time, given that the ship can operate with one or more of its propulsion plants idled to save fuel. Simple analysis yields nonintuitive results that US Navy shipboard energy-conservation guides overlook. One of the authors (Kline) solved this homework problem as a student and subsequently applied this example when he took command of USS AQUILA, a patrol hydrofoil missile ship. AQUILA achieved results so striking in comparison to her sister ships that the squadron material officer inspected her engineering plant to ensure that no safety settings were being overridden to achieve this record. Kline's spreadsheet decision-support tool was provided to other hydrofoil commanders. A more general version has been conveyed to the US Navy. Considering that our navy spends about a billion dollars per year on fuel for surface-combatant ships alone, this development promises substantial, long-term returns.

> "But thou, contracted to thine own bright eyes,
> Feed'st thy light'st flame with self-substantial fuel."
> Shakespeare, Sonnet I

Key words: naval operations; optimization; linear programming; energy conservation; fuel conservation; sea-going vessels.
History: This paper was refereed.

Naval surface-combatant ships consume large quantities of fuel. For example, four LM2500 gas-turbine engines power a DDG 51 Arleigh Burke-class, guided-missile destroyer (Figure 1). The LM2500 is a derivative of the engines in the Boeing 747 and other commercial aircraft. Depending on propulsion-plant configuration and speed, these four turbines can collectively consume from 600 to 7,000 gallons per hour (GPH) of distillate marine fuel (DFM) (United States Navy 2005a). That means that the destroyer's fuel consumption while at sea is between 100,000 and 1,000,000 gallons per week. For surface combatants alone, marine fuel costs our navy about one billion dollars per year (United States Navy 2005b).

The US Navy is keenly aware of its fuel consumption, seeks innovations to conserve fuel, and prescribes how its ships should monitor, control, and minimize fuel use. For instance, since 2000, its Naval Sea Systems Command has administered a program to provide energy-conservation incentives and has saved 10 percent of the total fuel allotment for participating ships. For each ship class, the program conducts sea trials to determine efficient fuel-consumption standards for every propulsion-plant configuration and speed, and then publishes these engineering standards in software that can be used to plan transits and ship plant operating modes (United States Navy 2005b).

First principles of naval architecture (Manning 1956) characterize propulsion energy requirements as a *cubic* function of speed; therefore, speed is the key determinant of fuel consumption. Ronen (1982) suggests an optimization of civilian cargo-ship employment that trades off fuel savings that result from slow steaming with the resulting decrease in revenue to offset fixed operating costs. This provides insight; however, the mission of navy ships often prohibits moving slowly.

Our analysis pertains to a transit along a great circle between points A and B, with no winds, currents, or obstructions in between. The basic idea is to minimize the amount of fuel required to get from A to B in a specified time. We discuss generalizations in the *Other Route-Planning Considerations* section and in the appendix.

Fuel Use as a Function of Speed and Mode

Some ships can operate in any of a number of propulsion-plant operating modes. The number of engines that are powering each shaft often distinguishes the mode. When a ship's captain orders maximum speed, the chief engineer brings all plants online to power all shafts. This is the least fuel-efficient plant-operating mode in terms of gallons per nautical mile. Routine operations call for lower speeds that allow the ship to align its plants in different modes to use less fuel.

The twin-screwed DDG 51-class destroyer in Figure 1 has two LM2500 engines for each shaft and screw. Other than *cold-plant* mode in which the ship is anchored or tied up, there are three common plant-operation modes. (1) In *trail-shaft* mode, only one of the four turbines is online. It drives one shaft, while the idled shaft is said to be "trailing." (2) In *split-plant* mode, two engines are online, each driving one shaft. (3) Under *full-power* mode, all four turbines are online, with two turbines driving each shaft. Engineering restrictions and ship handling limit the use of three engines online.

Figure 1: The O'KANE (DDG 77) is an Arleigh Burke (DDG 51)–class guided-missile destroyer, 505 feet long, with a 59-foot beam, and a displacement of 8,400 tons. Powered by four LM2500 gas-turbine engines that can generate more than 100,000 shaft horsepower, her top speed exceeds 30 knots.

Source. Department of Defense.

Figure 2 illustrates the relationship between speed and fuel consumption for each DDG 51 plant-operation mode, as recorded in sea trials. Note the dramatic increase in fuel consumption as a function of speed. (Figure 2 and similar fuel-consumptions graphs below are exact reproductions of actual navy charts used at sea. Note that the information and abbreviations in the upper portion are not relevant to this paper.)

Figure 3 repeats the functions in Figure 2, adding dotted-line segments to complete the lower convex envelope of the fuel-consumption curves. This envelope is the boundary of the convex hull of the speed curves, regarded as sets of points. It includes the origin, which represents a ship with all engines shut down (i.e., cold-plant mode). The lower envelope plays a central role in our analysis; we will refer to it simply as *the convex hull*. In general, it has the shape of a taut rubber band that supports all the fuel-consumption curves from below.

The left-most dotted segment in Figure 3 connects the origin to the point of greatest fuel efficiency in the sense of gallons per nautical mile. (A nautical mile is 1,852 meters or a bit farther than the 1,609 meters of a statute mile.) For DDG 51, the most fuel-efficient speed is 12 knots in trail-shaft mode—at 800 GPH this is about 66.6 gallons per nautical mile or 0.02 statute miles per gallon. Unfortunately, this point of greatest fuel efficiency is seldom fast enough to satisfy tactical requirements.

The Advantage of Mixed-Mode Operation

A (convex) combination of plant modes and speeds can sometimes permit the ship to be more fuel efficient than using one mode and a constant speed. For example, to make 28 knots on average, the O'KANE

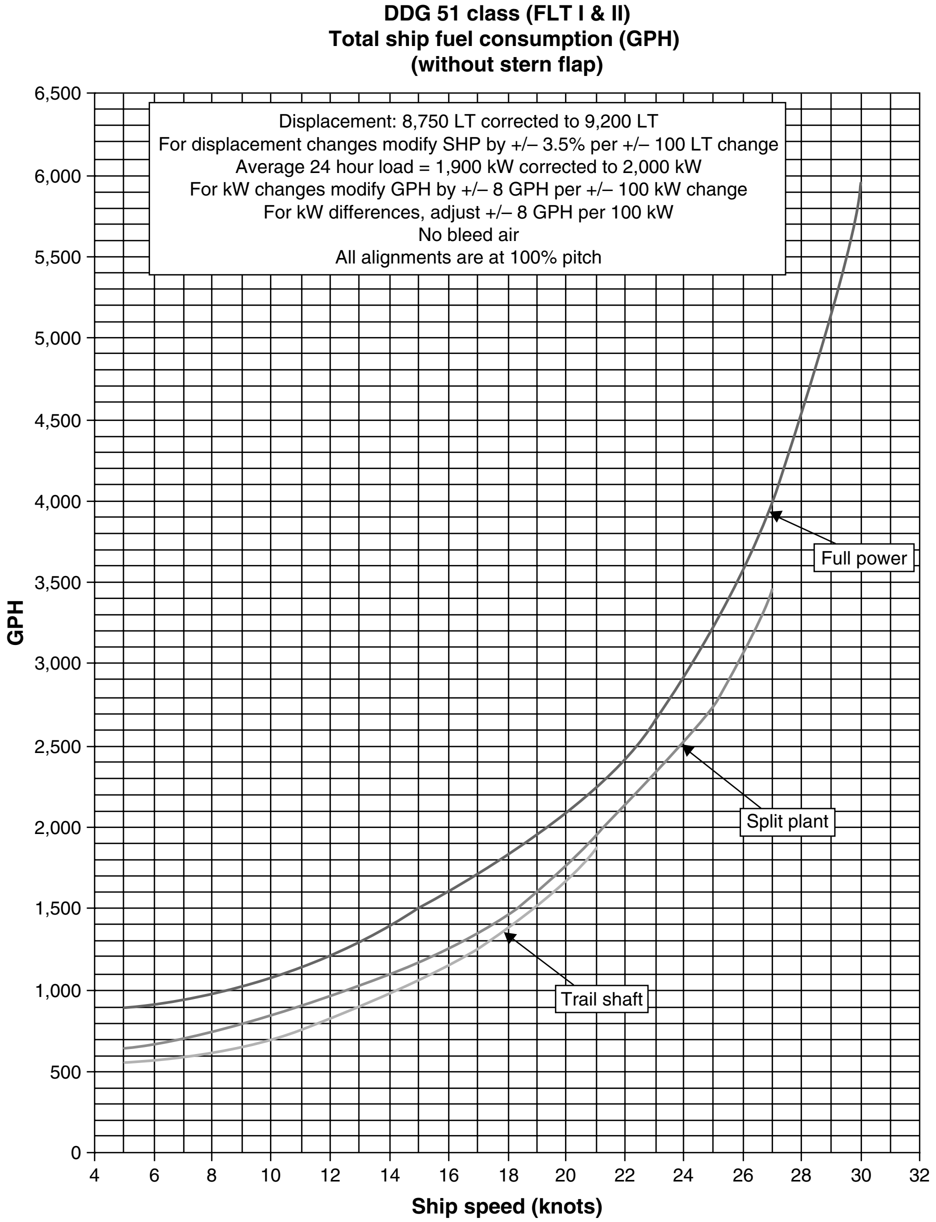


Figure 2: We show DDG 51 Arleigh Burke–class fuel consumption as a function of power-plant mode and speed. In trail-shaft, split-plant, or full-power modes, there are one, two, or four engines online, respectively. No one mode can achieve all speeds. The data in these graphs were obtained empirically in sea trials conducted for a Naval Sea Systems Command fuel-conservation program (United States Navy 2005a).

could operate at a constant speed in full-power mode, consuming 4,500 GPH. It can achieve the same average speed by operating half of the time at 29 knots in full-power mode, and by operating the other half of the time at 27 knots in split-plant mode. The average fuel-consumption rate in that mixed mode is the average of 5,100 GPH and 3,500 GPH, or 4,300 GPH. The mixed mode thus provides a fuel savings of 200 GPH (4.65 percent). A fuel savings is possible for any average speed, *avg_speed*, between 27 and 29 knots. For

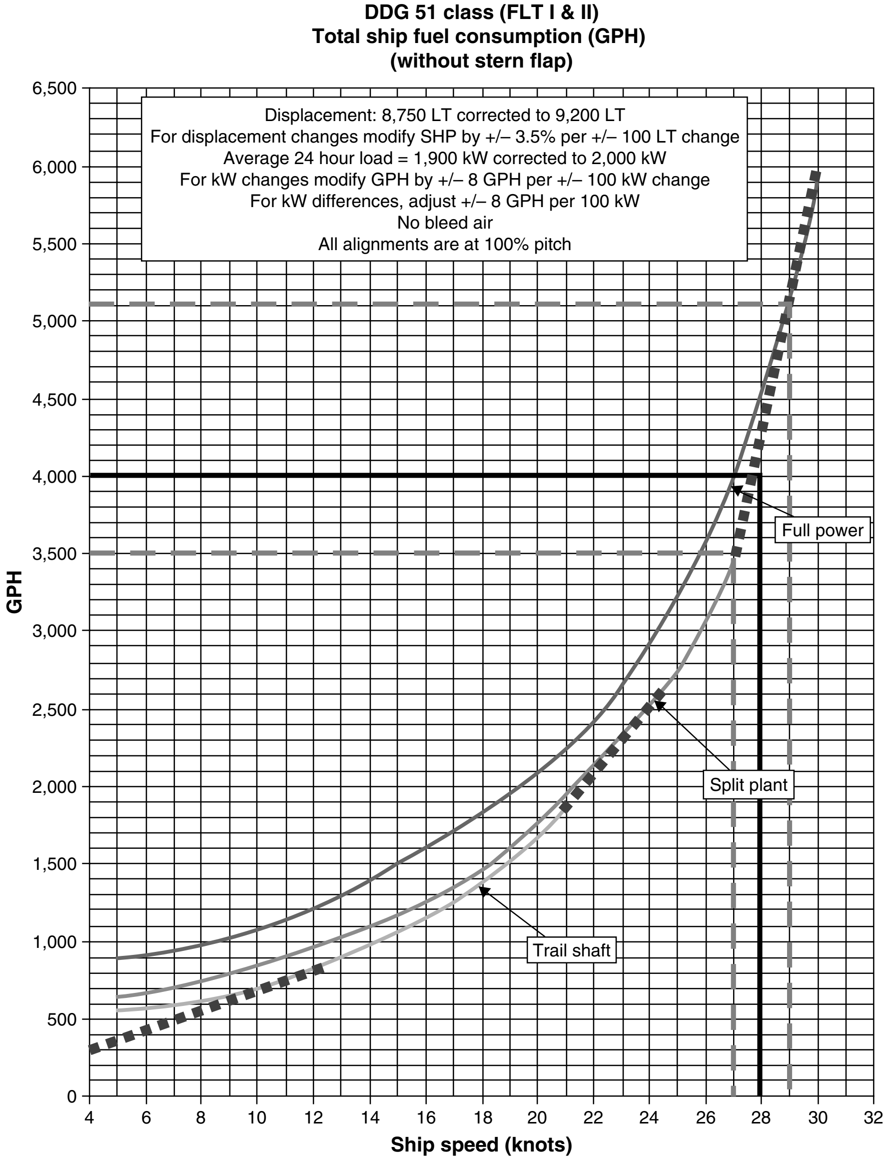


Figure 3: This figure is the same as Figure 2, with additions to illustrate opportunities to improve efficiency by operating in mixed modes. The dotted-line segments complete the lower envelope of the convex hull of these fuel-consumption curves. Any speed corresponding to one of the dotted segments offers some fuel saving if we mix the modes that the line segment connects. For example, at 28 knots, we can save 200 gallons per hour (4.65%) by equally mixing full-power mode (at 29 knots) and split-plant mode (at 27 knots), rather than steaming at a constant 28 knots.

speeds in that range, the average fuel consumption in GPH is given by

$$3{,}500\left(\frac{29-avg\_speed}{29-27}\right)+5{,}100\left(\frac{avg\_speed-27}{29-27}\right)$$
$$=3{,}500+800(avg\_speed-27),$$

the line that succeeds in touching both the full-power and split-plant curves, while never being higher than either of them. The fraction of time spent at full power is $(avg\_speed-27)/(29-27)$, which is between 0 and 1 (therefore feasible) as long as *avg_speed* is in the stated range. Mixed-mode transit yields its largest fuel savings at just over 27 knots, where the savings is 500 GPH (12.5 percent). Split-plant mode alone cannot quite achieve that speed, whereas full-power mode alone at 27 knots requires more fuel. No fuel savings is possible at 26 knots because the convex hull is the same as the split-plant curve at that speed. In general, the fuel savings per hour at any given speed is the difference between the convex hull and the most efficient single operating mode for that speed.

In this DDG 51 example, the mixed mode involves operating at two distinct speeds. It is never necessary to use more than two speeds because any point on the convex hull is a weighted average of at most two points lying on fuel-consumption curves. This is geometrically clear from the taut rubber-band analogy and follows formally from Carathéodory's theorem (Stassinopoulos and Vintner 1977).

There are no significant considerations when switching power modes on a naval ship; perhaps this is in contrast to turning turbines on and off in a civilian power plant. This is because naval engineering crews drill constantly and are extremely adept at changing modes very quickly. Their missions often require frequent speed changes. The incremental fuel usage is negligible. Our analysis determines the optimal combination of modes, not the order. When mixing two modes is necessary, the ship's captain has the flexibility to decide on the ordering of modes for tactical reasons, without sacrificing optimal fuel economy. A prudent captain might go fast initially because this provides some insurance against unanticipated delays.

The results we show for the DDG 51-class destroyer are representative, not exceptional. Figures 4–9 show three additional ship types and their fuel-consumption rates as a function of plant operating mode and speed. The dotted-line portions of the convex hulls show that mixed-mode operations provide considerable opportunities to save fuel. The engines on these ships include diesel engines, steam plants, and gas turbines. All of these power-plant types operate on different engineering principles, but they consume the same DFM fuel.

Figure 4: PC 1-class Patrol Coastal ship HURRICANE displaces 331 tons, is 170 feet long, with a beam of 25 feet, and is powered by four diesel engines, each with its own shaft and screw.
Source. Department of Defense.

SEA FIGHTER, the last of the four ship types we analyze, is an experimental ship. It is one of a number of prototypes designed to play an important role in close-to-shore, or *littoral* operations. Large numbers of these vessels will be commissioned in the coming decade because the US Navy anticipates that littoral operations will grow in importance. The fuel-consumption curves that Figure 9 shows are estimates we derived from Nigel Gee (BMT Nigel Gee and Associates 2004). Empirical data on SEA FIGHTER's fuel consumption has not been collected yet.

From manufacturer's specifications and hydrodynamic tests, we understand how fuel consumption behaves. The (now decommissioned) PEGASUS patrol hydrofoil missile (PHM)-class ships provide a

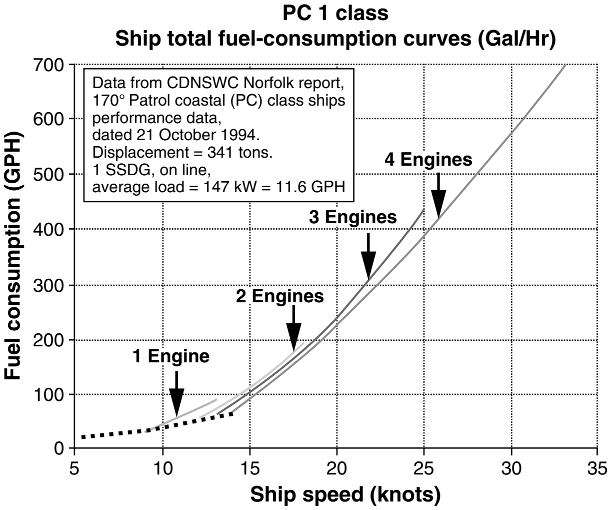


Figure 5: The graph shows PC 1 fuel consumption as a function of power-plant mode and speed (United States Navy 2005a). The convex hull of these fuel curves reveals that it is inefficient to operate on two or three engines.

Figure 6: The Amphibious Assault Ship USS WASP (LHD 1) is 844 feet long, with a beam of 106 feet. Two steam boilers and two shafts that generate 70,000-shaft horsepower power this 40,650-ton ship.
Source. Department of Defense.

Figure 8: The US Navy experimental ship SEA FIGHTER (FSF1) is 262 feet long, with a beam of 72 feet. Four water jets that are powered by two LM2500 gas-turbine engines or two diesels can drive its 950-ton weight over 50 knots.
Source. Department of Defense.

good analogy to SEA FIGHTER for fuel-consumption behavior. We anticipate a slower, relatively efficient operating mode using the diesel engines, and a faster mode using gas turbines. We cannot determine precisely how fuel consumption varies with mode, speed, and vessel weight without extensive sea tests. However, we are confident that such experiments will result in minor reconciliation, not substantive revision, of our estimates. In particular, the form of the fuel-consumption curve will not change. SEA FIGHTER and other newer small ships will feature mixed-propulsion modes, one for slow station keep-

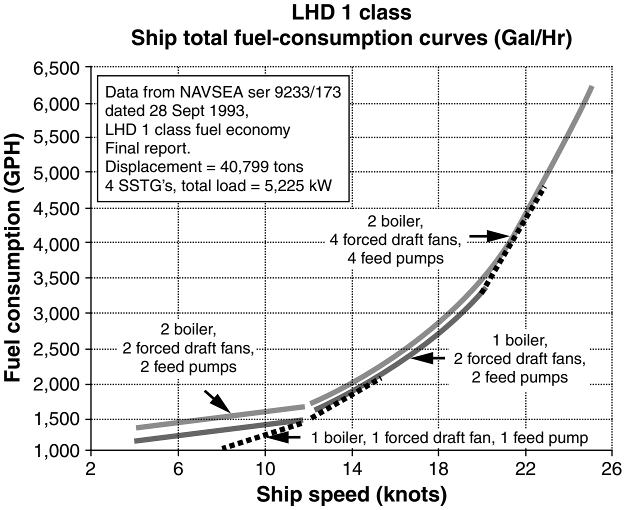


Figure 7: The graph shows LHD 1 WASP fuel consumption as a function of power-plant mode and speed (United States Navy 2005a). The dotted lines in the convex hull show that this ship may have many opportunities to save fuel through mixed-mode plant operation. The largest savings opportunity is at four knots. Rather than transit a long distance at four knots (1,100 GPH), it would be more fuel efficient to stay in port for 2/3 of the period and then transit at 12 knots (1,500 GPH). The net savings would be 600 GPH (54 percent). Of course, tactical reasons may intervene in making such dramatic savings practicable. Nevertheless, mixed-mode operations offer positive savings for many average speeds likely to be used in practice.

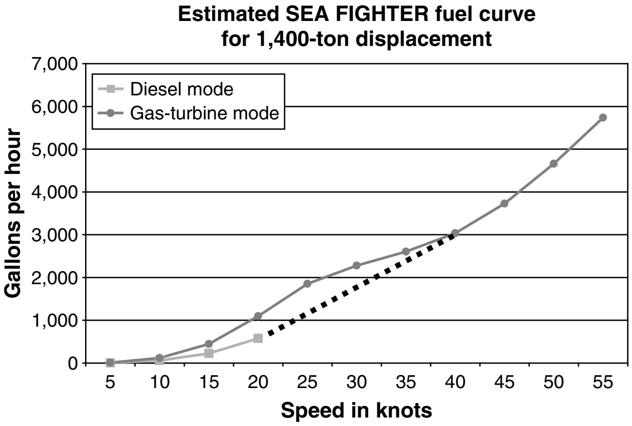


Figure 9: The graph shows estimated fuel curves for SEA FIGHTER at 1,400-ton displacement as a function of plant mode. This new, fast vessel can make 20 knots with its diesels, consuming about 700 gallons per hour, or with its LM2500 gas turbines, using about 1,100 gallons per hour. We estimate the data in this graph because the US Navy has not yet conducted fuel-consumption tests at sea for SEA FIGHTER.

Class	Average speed	Single-mode GPH	Mixed-mode GPH	Savings (GPH)	Savings (%)
DDG 51	27	4,000	3,500	500	12.5
PC 1	12	80	60	20	25
LHD 1	4	1,100	500	600	54.5
LSD 41	18	850	837	13	1.5
FFG 7	26	2,287	2,233	54	2.4
CG 47	27	3,500	3,404	96	2.7

Table 1: We show potential fuel savings in gallons per hour (GPH) for six ship classes. In each case, we show the average speed that maximizes fuel savings in GPH.

ing, and one for high-speed transits. These ships all exhibit the striking mode-change fuel consumption of SEA FIGHTER and offer large advantages for mixed-mode operation.

Table 1 summarizes the potential fuel savings for the DDG 51, PC 1, and LHD 1 classes. In addition, it includes three classes we have not discussed. In each case, we deliberately chose the average speed that we show to maximize the potential fuel savings. There are no savings at some other speeds, and for some ship classes the largest achievable savings is a small percentage of total fuel consumption. Nonetheless, even a small percentage of the US Navy's fuel costs is worth pursuing.

A Spreadsheet Sea Story

Using available US Navy graphs of fuel consumption as a function of speed for each propulsion-plant mode and a straight edge, we can find the convex hull that reveals efficient transit plans. Using calipers to interpolate optimal (convex) combinations of plant configurations is instructive, but it is also instructive to solve such problems digitally. In fact, constructing an optimal solution using linear programming has been a regular classroom exercise at the Naval Postgraduate School for many years.

Author Kline earned his master's of science degree in operations research from the Naval Postgraduate School in 1992. He later applied his "steaming on convex hulls" class exercise when he assumed command of patrol hydrofoil missile boat USS AQUILA (PHM 4). He devised an Excel spreadsheet to recommend minimum-fuel transits for his ship and used it with his navigator and chief engineer for six months. AQUILA's fuel savings were so striking in comparison to her sister ships that the squadron material officer inspected her engineering plant to ensure that no safety settings were being overridden to achieve this record. The secret turned out to be the mathematics, and Kline's spreadsheet became a standard planning tool for the squadron.

Twelve years later, we have developed a similar planning tool for SEA FIGHTER that its crew is currently beta-testing. Figure 10 shows a screen shot of our SEA FIGHTER transit planner, which calls the Excel Solver in a macro.

We also report retaining an onboard fuel reserve of at least 25,000 gallons. Standard US Navy practice is to maintain an onboard reserve as a margin of safety in case of poor weather or unexpected refueling delay.

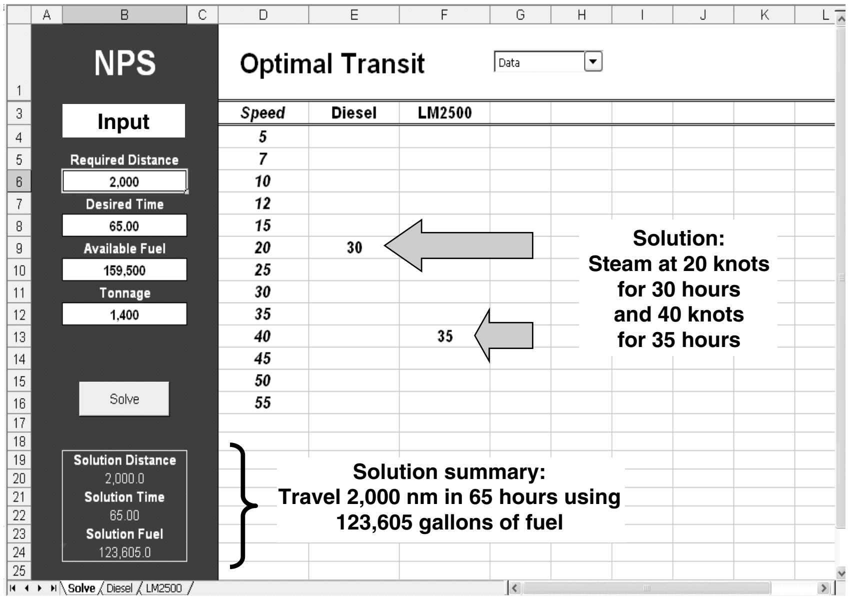
The appendix provides a linear-programming formulation of the mode-mixing problem. The objective of the optimization model is to minimize the total fuel required for a transit over a specified distance in a specified time. The variables are the number of hours spent operating at each of a discrete set of possible speeds (generally but not necessarily multiples of one knot) up to the top speed of the ship. For the fuel consumption at each of these speeds, we use input data from US Navy charts such as the graphs we show in the figures above. We enforce the transit time requirement in a constraint.

The result of the constrained optimization is "soft" guidance in the sense that the optimized variables provide the number of hours at each speed, but not the temporal arrangement of those hours within the transit. In the example in Figure 10, the planner can do 20 knots in the beginning and 40 knots at the end, or vice versa, or mix the two speeds up in some other way. Because fuel consumption is not affected, the temporal arrangement can depend on other tactical needs.

The appendix includes a version of the linear program that is "elasticized" with respect to violations of the reserve-fuel constraint, and also includes a version that minimizes fuel consumption for a group of ships that must transit in company.

Other Route-Planning Considerations

When ocean currents are present, one must distinguish between speed relative to the ocean, the primary



NPS

Input

Required Distance
2,000
Desired Time
65.00
Available Fuel
159,500
Tonnage
1,400

Solve

Solution Distance
2,000.0
Solution Time
65.00
Solution Fuel
123,605.0

Optimal Transit

Speed	Diesel	LM2500
5		
7		
10		
12		
15		
20	30	
25		
30		
35		
40		35
45		
50		
55		

Solution:
Steam at 20 knots
for 30 hours
and 40 knots
for 35 hours

Solution summary:
Travel 2,000 nm in 65 hours using
123,605 gallons of fuel

Figure 10: This Excel spreadsheet suggests SEA FIGHTER propulsion-plant modes to complete a 2,000-nautical-mile transit in 65 hours with minimal fuel consumption. There are two alternate propulsion modes: operating using diesels or gas turbines. The optimal transit plan, which we recommend, operates on diesels for 30 hours at 20 knots, and on gas turbines for 35 hours at 40 knots. This "convex-hull" transit plan consumes 123,605 gallons. Completing this same transit in a single mode (two turbines at a constant speed of about 31 knots) would require 148,200 gallons—20% more fuel. This spreadsheet accommodates fuel-consumption tables for any ship and any mixture of propulsion-plant modes and is easy to modify to reflect secondary effects, e.g., gross displacement.

determinant of fuel consumption, and speed relative to land, which is relevant to meeting timeliness constraints. The magnitude and direction of both waves and wind, as well as the fluctuating displacement of the ship, and requirements for electrical power, also influence fuel consumption. Ships transiting shallow water *squat*, the common name for a Bernoulli suction effect caused by the interaction between the sea bottom and the ship's hull that effectively increases displacement. There are still other subtleties that one can incorporate in the analysis (United States Navy 1998). If not evaded, weather can be severe enough to threaten damage or even loss of the ship involved.

Many naval missions are not simple transits, and even transits are sometimes accompanied by drills or emergencies that require track deviations. Aircraft carriers frequently follow a zigzag path because of the need to launch and recover aircraft, and their ship escorts that are supposed to maintain a fixed geometric relation may find themselves sprinting as the carrier whipsaws them around the ocean. However, we can estimate the overall impacts of these digressions. Rosenthal and Walsh (1996) describe the geometry of such a transit and present an optimization model to plan air launch-recovery cycles to minimize the time required for course diversions.

Considerations such as these complicate the naval-route-planning problem, even for simple transits. Nonetheless, there is some literature pertinent to naval-route optimization. Fagerholt et al. (2000) develop shortest routes that avoid obstacles. When a ship files its movement request (transit plan), the navy provides an optimum track ship routing (Department of the Navy, 2000) that advises how to avoid adverse weather and sea states. Chen et al. (1998) describe how the vessel optimization and safety system (VOSS) uses dynamic programming to route ships optimally in the face of such complications. The navy's ship tracking and routing system (STARS), which has functions similar to VOSS, is now coming into use (United States Navy 2006).

In an unusual application, Philpott and Leyland (2006) describe using optimization to a two-man, ocean-rowing endurance race. Based on weather and sea-state data and the solution of stochastic shortest paths, they prepare a map for the oarsmen that says: if you are *here* and you observe *this* weather, then go in *that* direction. Great circles are not always optimal because of anticipated wind and current. The authors' New Zealand countrymen won the 2003 Transatlantic Challenge, a 3,000-mile race from the Canary Islands to Barbados, giving much credit to their operations research inspired map.

Montes (2005) uses a constrained shortest-path model to optimally plan ship transit routes in bad weather. His inputs are US Navy meteorological forecasts that extend several days into the future with four-hour time resolution on a 30-nautical-mile grid worldwide. His model minimizes fuel consumption while assuring that individual ship limits on sea state by aspect are not exceeded. He takes his examples from weather histories and actual ship transits planned around severe Pacific typhoons. Montes' input fuel-consumption curve acknowledges only a single mode.

The simplest way to include the effect of multiple modes in Montes' work, or in any other system initially designed for a single-propulsion mode, would be to use the convex hull (or hulls, if the curves are indexed by wind and wave) of the several modes as the single fuel-consumption input.

Conclusion

Almost every US Navy surface-warfare officer arriving at our classes already wears the gold "boats-and-cross-sword" insignia of a fully qualified surface-warfare officer. Not one has ever claimed prior knowledge of the material we present here, and we know for sure that at least one (Kline) has applied it after graduation. This is an example of why the US Navy sends its officers to postgraduate education, and of why the operations research profession needs to publicize our capabilities to a wider audience.

Would a ship actually go to these lengths to plan a transit? Yes. The actual execution might not be as ideal as the plan. However, if we precompute that we should spend, for example, one-third of our time in one plant configuration, and two-thirds in another, then simple engineering logs can keep a rough, running tally during transit, suggesting which plant mode is more desirable in meeting our transit goal.

Given the ubiquity and reliability of the standard Excel Solver within the US Navy, we can carry out computations such as those in Figure 10 immediately with no additional cost in software. The flexibility of linear programming permits us to adapt to new constraints or other problem features that may emerge in the future. We have conveyed this convex-hull insight and our spreadsheet decision-support tool to Naval Sea Systems Command and other interested parties. We believe that the idea will help continue to improve our navy's fuel efficiency.

Appendix

Optimizing Transit Plans by Linear Programming

This section presents the mathematical formulations of our basic optimization model and some extensions. We assume the availability of input data on fuel consumption versus speed, which are the observations that sea trials produce. For each knot of speed, we tabulate the minimum fuel consumption in the best (single) propulsion-plant mode available.

In our notation for optimization formulations, lowercase denotes exogenous constants; UPPERCASE denotes decision variables.

The first formulation below is the basis of the spreadsheet we illustrate above. The vessel subscript is not necessary in this formulation because there is only one vessel; however, it permits us to use the same notation in discussing alternate formulations later.

Index use (*∼cardinality*)

$v \in V$	vessel [∼10].
$s \in S$	speed index [∼50].

Given data (*units*)

$distance$	required transit distance (nautical miles).
$speed_s$	speed s (knots $\equiv$ nautical miles/hour).
$frate_{v,s}$	fuel consumption rate for vessel v operating with its most efficient mode at speed s (gallons/hour).

$\overline{fuel}_v$	fuel capacity (gallons).
$\underline{reserve\_fuel}_v$	fuel reserve (gallons).
$\overline{hours}$	maximum allowed transit time (hours).

Decision variables (*units*)

$HOURS_s$	time spent underway at speed s (hours).
$FUEL_v$	fuel consumed by vessel v in transit, when minimizing fuel is the objective (gallons).
$TIME_v$	time required by vessel v for transit, when minimizing transit time is the objective (hours).

To minimize fuel consumption for a given transit, we can use the following linear program.

Formulation (*MINFUEL*) *for vessel* v:

$$\underset{FUEL,\,HOURS}{\text{MIN}} \quad FUEL_v \tag{F0}$$

$$\text{s.t.} \quad \sum_{s\in S} speed_s HOURS_s \geq distance, \tag{F1}$$

$$\sum_{s\in S} HOURS_s \leq \overline{hours}, \tag{F2}$$

$$FUEL_v \geq \sum_{s\in S} frate_{v,s} HOURS_s, \tag{F3}$$

$$FUEL_v \leq \overline{fuel}_v - \underline{reserve\_fuel}_v, \tag{F4}$$

$$HOURS_s \geq 0 \quad \forall\, s \in S. \tag{F5}$$

The objective function (F0) accounts for total fuel consumption in transit. We are required to complete the transit of the required distance (F1), within the required time (F2), while using no more fuel than would be required to leave us at our destination with a stipulated reserve (F3)–(F4).

There are cases where (F2) forces an infeasibility even for a ship proceeding at top speed at all times. Such cases are best reported as infeasible; however, we prefer an elastic solution where reserve fuel can be used at a large penalty if (F4) would otherwise force an infeasibility. Introduce $PENALTY\_FUEL_v$ as the amount of $\underline{reserve\_fuel}_v$ that we are forced to use:

Formulation (*ELASTIC_MINFUEL*) *for vessel* v:

$$\underset{\substack{FUEL,\\ PENALTY\_FUEL,\\ HOURS}}{\text{MIN}} \quad FUEL_v + 10PENALTY\_FUEL_v \tag{SF0}$$

$$\text{s.t.} \quad \sum_{s\in S} speed_s HOURS_s \geq distance, \tag{SF1}$$

$$\sum_{s\in S} HOURS_s \leq \overline{hours}, \tag{SF2}$$

$$FUEL_v + PENALTY\_FUEL_v \geq \sum_{s\in S} frate_{v,s} HOURS_s, \tag{SF3}$$

$$FUEL_v \leq \overline{fuel}_v - \underline{reserve\_fuel}_v, \tag{SF4}$$

$$HOURS_s \geq 0 \quad \forall\, s \in S, \tag{SF5}$$

$$\underline{reserve\_fuel}_v \geq PENALTY\_FUEL_v \geq 0. \tag{SF6}$$

(SF0) expresses a fuel cost that increases by a factor of 10 when $PENALTY\_FUEL_v$ is used. (Any factor exceeding 1 would produce the same solution.) Transits that would consume the entire fuel supply are still infeasible.

We can also formulate an equivalent problem for a group of ships in company, taking advantage of the vessel index v that we introduced earlier.

Formulation (*MINFUEL*) *for group transit*:

$$\underset{FUEL,\,HOURS}{\text{MIN}} \quad \sum_{v\in V} FUEL_v \tag{G0}$$

$$\text{s.t.} \quad \sum_{s\in S} speed_s HOURS_s \geq distance, \tag{G1}$$

$$\sum_{s\in S} HOURS_s \leq \overline{hours}, \tag{G2}$$

$$FUEL_v \geq \sum_{s\in S} frate_{v,s} HOURS_s \quad \forall\, v \in V, \tag{G3}$$

$$\overline{fuel}_v - \underline{reserve\_fuel}_v \geq FUEL_v \quad \forall\, v \in V, \tag{G4}$$

$$HOURS_s \geq 0 \quad \forall\, s \in S. \tag{G5}$$

If the group includes an oiler escort, we could modify this group transit model to reflect the capacity of the oiler to replenish ship bunkers as necessary. The most likely application of such a group transit planner would be to assess when and where to refuel the group (e.g., Borden 2001).

The group of ships will remain in physical proximity as long as all ships schedule the same speed at the same time. In principle, one could also formulate a similar problem where $HOURS_{v,s}$ replaces $HOURS_s$. This would save some fuel and still have everybody transit the required distance in the required number of hours; however, it would also have the ships spatially scattered at intermediate times. Because ships in company generally support each other, we forego this formulation.

Acknowledgments

The Office of Naval Research has supported Brown and Rosenthal.

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From the editor-in-chief: I have received a verification letter regarding this work. The author of the letter is knowledgeable of the situation and states that the authors of the paper "discovered a way to save a lot of fuel very easily, without ever having to buy new equipment or modify anything in a ship."

There are occasions where simple models and approaches can have big impact. Congress has been studying ways to save fuel in naval operations. See, for example, http://www.fas.org/sgp/crs/weapons/RL33360.pdf.

Perhaps they should consider a fleet-wide adoption of the very low-cost approach described in this paper.