STEIN'S METHOD FOR COMPOUND POISSON APPROXIMATION: THE LOCAL APPROACH[1]

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In the present paper, compound Poisson approximation by Stein's method is considered. A general theorem analogous to the local approach for Poisson approximation is proved. It is then applied to a reliability problem involving the number of isolated vertices in the rectangular lattice on the torus.

1. Introduction. Let Γ denote an arbitrary finite collection of indices, usually denoted by α, β and so on. Let I_α, $\alpha \in \Gamma$, be 0-1 valued, possibly dependent, random variables with $\mathbb{E}I_\alpha = \mathbb{P}[I_\alpha = 1] = 1 - \mathbb{P}[I_\alpha = 0]$ and let $W = \sum_{\alpha \in \Gamma} I_\alpha$. Assume that there is available a natural local dependence structure which allows us for each α to define an index set Γ_α^s designating those summands other than α which are more closely related to I_α. Such a dependence structure is appropriate if one assumes conditions such as stationary m-mixing [Leadbetter, Lindgren and Rootzén (1983)] or local dependence [Barbour, Chen and Loh (1992)].

In the Stein–Chen method for Poisson approximation, this structure is exploited to give a bound on the total variation distance between $\mathscr{L}(W)$ and the Poisson distribution with parameter $\mathbb{E}W$, denoted by $\mathrm{Po}(\mathbb{E}W)$, expressed in terms of joint moments of the random variable I_α and

$$(1.1) \qquad S_\alpha = \sum_{\beta \in \Gamma_\alpha^s} I_\beta,$$

as in the following theorem [Chen (1975), Arratia, Goldstein and Gordon (1989), Barbour, Holst and Janson (1992)].

THEOREM 1. *With the above definitions, for any choice of the index sets Γ_α^s,*

$$(1.2) \quad \begin{aligned} &d_{TV}(\mathscr{L}(W), \mathrm{Po}(\lambda)) \\ &\leq c_2(\lambda) \sum_{\alpha \in \Gamma} \left((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}S_\alpha + \mathbb{E}\{I_\alpha S_\alpha\} \right) + c_1(\lambda) \sum_{\alpha \in \Gamma} \eta_\alpha, \end{aligned}$$

where $\lambda = \mathbb{E}W$, $c_1(\lambda) = \sqrt{2/(e\lambda)}$, $c_2(\lambda) = \lambda^{-1}(1 - e^{-\lambda})$ and

$$(1.3) \qquad \eta_\alpha = \mathbb{E}\left| \mathbb{E}\{I_\alpha \mid (I_\beta : \beta \in \Gamma_\alpha^w)\} - \mathbb{E}I_\alpha \right|,$$

with $\Gamma_\alpha^w = \Gamma \setminus \{\{\alpha\} \cup \Gamma_\alpha^s\}$.

Note that the local approach gives good results only if the structure of dependence allows us to arrange η_α to be very small for each $\alpha \in \Gamma$. The contribution to estimate (1.2) arising from (1.3) is, for large λ, multiplied by a factor of order $\lambda^{-1/2}$, as compared with the factor of order λ^{-1} multiplying the remaining estimates.

Note that if there is a tendency for clustering, then the contribution to (1.2) coming from $\mathbb{E}\{I_\alpha S_\alpha\}$ can be substantial. In such cases, one is interested in finding better bounds on the total variation distance. One possibility for approximating the behaviour of W, when the tendency to clumping is great, is the compound Poisson family.

In Section 2, we prove an analogue of Theorem 1 for compound Poisson approximation. In Section 3, it is applied to a problem in reliability theory: k-out-of-n isolated vertices in the rectangular lattice on the torus.

2. The compound Poisson local approach.

The integer-valued compound Poisson distributions of concern here have the form $\mathrm{CP}(\boldsymbol{\lambda}) = \mathscr{L}(\Sigma_{i \geq 1} i N_i)$, where the $N_i \sim \mathrm{Po}(\lambda_i)$ are independent for $i = 1, 2, \ldots$ and $\Sigma_{i \geq 1} \lambda_i < \infty$; $\boldsymbol{\lambda}$ is used to denote $\Sigma_{i \geq 1} \lambda_i \delta_i$, where δ_i is the unit mass at i. If $\lambda_1 > 0$ and $\lambda_i = 0$ for $i \geq 2$, the usual Poisson distribution results: see also Aldous (1989).

Recently, Barbour, Chen and Loh (1992) introduced a Stein equation for compound Poisson approximation. On the lattice $\mathbb{Z}^+$, it takes the form

$$(2.1) \qquad jg(j) - \sum_{i=1}^{\infty} i\lambda_i g(j + i)$$

$$= f_{\boldsymbol{\lambda}, A}(j) := \mathrm{CP}(\boldsymbol{\lambda})\{A\} - I[j \in A], \qquad j \geq 0,$$

for $A \subseteq \mathbb{Z}^+$, where $\mathrm{CP}(\boldsymbol{\lambda})\{A\} = \mathbb{P}[X \in A]$ for $X \sim \mathrm{CP}(\boldsymbol{\lambda})$ and $f_{\boldsymbol{\lambda}, A} \colon \mathbb{Z}^+ \to \mathbb{R}$ is thus bounded in modulus by 1. Note that $\mathbb{E} f_{\boldsymbol{\lambda}, A}(Z) = \mathrm{CP}(\boldsymbol{\lambda})\{A\} - \mathbb{P}[Z \in A]$ for any integer-valued random variable Z and that $\mathbb{E} f_{\boldsymbol{\lambda}, A}(Z) = 0$ if $Z \sim \mathrm{CP}(\boldsymbol{\lambda})$. Note also that if $g \colon \mathbb{Z}^+ \to \mathbb{R}$ is any bounded function and $Z \sim \mathrm{CP}(\boldsymbol{\lambda})$ with $\mathbb{E}Z = \Sigma_{i \geq 1} i\lambda_i < \infty$, then $\mathbb{E}\{Zg(Z) - \Sigma_{i \geq 1} i\lambda_i g(Z + i)\} = 0$. By taking the expectation of the CP equation (2.1) for the random variable W,

$$(2.2) \qquad \mathbb{E}\left\{Wg(W) - \sum_{i=1}^{\infty} i\lambda_i g(W + i)\right\} = \mathrm{CP}(\boldsymbol{\lambda})\{A\} - \mathbb{P}[W \in A]$$

and

$$d_{TV}(\mathscr{L}(W), \mathrm{CP}(\boldsymbol{\lambda})) = \sup_{A \subseteq \mathbb{Z}^+} \{|\mathrm{CP}(\boldsymbol{\lambda})\{A\} - \mathbb{P}[W \in A]|\}$$

$$= \sup_{A \subseteq \mathbb{Z}^+} \{|\mathbb{E} f_{\boldsymbol{\lambda}, A}(W)|\}$$

$$= \sup_{A \subseteq \mathbb{Z}^+} \left\{\left|\mathbb{E}\left\{Wg_{\boldsymbol{\lambda}, A}(W) - \sum_{i=1}^{\infty} i\lambda_i g_{\boldsymbol{\lambda}, A}(W + i)\right\}\right|\right\},$$

where $g_{\boldsymbol{\lambda}, A}$ is the Stein–Chen transform of the function $I[\cdot \in A] - \mathrm{CP}(\boldsymbol{\lambda})\{A\}$, that is, the solution g of (2.1). It thus follows that if we can bound the

left-hand side of (2.2), then we will find the bound on the total variation distance between $\mathscr{L}(W)$ and $CP(\boldsymbol{\lambda})$.

The most important ingredient of the Poisson local approach is the definition of the sets of dependence, with Γ_α^s containing the indices of indicators which are strongly dependent on I_α and Γ_α^w containing the remaining indices of weakly dependent indicators. For the compound Poisson local approach, one has to treat the strongly dependent indicators more carefully to allow for α-clumps.

So, divide Γ into four subsets $\{\alpha\}$, Γ_α^{vs}, Γ_α^b and Γ_α^{vw}, where

$$\Gamma_\alpha^{vs} = \big\{ \beta \in \Gamma \setminus \{\alpha\} : I_\beta \text{ very strongly dependent on } I_\alpha \big\},$$

$$\Gamma_\alpha^{vw} = \big\{ \beta \in \Gamma \setminus \{\alpha\} : I_\beta \text{ very weakly dependent on } \{I_\gamma, \gamma \in \{\alpha\} \cup \Gamma_\alpha^{vs}\} \big\}$$

and

$$\Gamma_\alpha^b = \Gamma \setminus \big\{\{\alpha\} \cup \Gamma_\alpha^{vs} \cup \Gamma_\alpha^{vw}\big\}$$

is the set of boundary indicators. Then set

$$U_\alpha = \sum_{\beta \in \Gamma_\alpha^{vs}} I_\beta, \qquad Z_\alpha = I_\alpha + U_\alpha,$$

$$X_\alpha = \sum_{\beta \in \Gamma_\alpha^b} I_\beta,$$

(2.3)

$$Y_\alpha = W - I_\alpha - U_\alpha - X_\alpha = \sum_{\beta \in \Gamma_\alpha^{vw}} I_\beta,$$

$$W_\alpha = W - I_\alpha, \qquad W_{\alpha,U} = W_\alpha - U_\alpha = W - I_\alpha - U_\alpha = X_\alpha + Y_\alpha.$$

Thus Z_α can be thought of as the size of the α-clump and Y_α should only depend weakly on Z_α.

REMARK 1. Taking $\Gamma_\alpha^{vs} = \varnothing$, $\Gamma_\alpha^b = \Gamma_\alpha^s$ and $\Gamma_\alpha^{vw} = \Gamma_\alpha^w$ in Theorem 2 below turns out to give the same result as Theorem 1: For Poisson approximation to be good, the α-clump must be negligible, and then $\Gamma_\alpha^{vs} = \varnothing$ is an adequate choice.

Define

$$\|\Delta g\| = \sup_{j \geq 1} |g(j+1) - g(j)| \quad \text{and} \quad \|g\| = \sup_{j \geq 1} |g(j)|.$$

THEOREM 2. *With the above definitions, for any choice of the index sets* Γ_α^{vs}, Γ_α^{vw} *and any bounded function* g,

(2.4)

$$\left| \mathbb{E}\bigg\{ Wg(W) - \sum_{i=1}^\infty i\lambda_i g(W+i) \bigg\} \right|$$

$$\leq \|\Delta g\| \sum_{\alpha \in \Gamma} \big((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\} \big) + \|g\|\phi,$$

where $\boldsymbol{\lambda} = \sum_{i=1}^{D+1} \lambda_i \delta_i$, $\lambda_i = (1/i)\sum_{\alpha \in \Gamma} \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}$, $D = \max_{\alpha \in \Gamma}\{|\Gamma_\alpha^{vs}|\}$ *and* $\phi = \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \phi_{\alpha i}$, *with*

$$\phi_{\alpha i} = \mathbb{E}\big| \mathbb{E}\{I_\alpha I[Z_\alpha = i] | (I_\beta : \beta \in \Gamma_\alpha^{vw})\} - \mathbb{E}\{I_\alpha I[Z_\alpha = i]\} \big|.$$

PROOF.   In what follows, we use an appropriate expression for $\mathbb{E}\{I_\alpha g(W)\}$, which makes $\mathbb{E}\{Wg(W)\}$ close to $\sum_{i=1}^{\infty} i\lambda_i \mathbb{E}g(W+i)$ in (2.2). Observe that

$$
\mathbb{E}\{I_\alpha g(W)\} = \mathbb{E}\{I_\alpha g(W_\alpha + 1)\}
$$

$$
= \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i] g(W_{\alpha,U} + i)\}
$$

(2.5)
$$
= \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i] g(Y_\alpha + i)\}
$$

$$
+ \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i] (g(Y_\alpha + X_\alpha + i) - g(Y_\alpha + i))\}.
$$

Set

$$
\lambda_i = \frac{1}{i} \sum_{\alpha \in \Gamma} \mathbb{E}\{I_\alpha I[Z_\alpha = i]\} \quad \text{for } i = 1, \ldots, D+1,
$$

$$
\lambda_i = 0 \quad \text{for } i > D+1 \text{ where } D = \max_{\alpha \in \Gamma}\{|\Gamma_\alpha^{vs}|\}.
$$

The left-hand side of (2.5) can be rewritten as

$$
\mathbb{E}\left\{ Wg(W) - \sum_{i=1}^{D+1} i\lambda_i g(W+i) \right\}
$$

$$
= \sum_{\alpha \in \Gamma} \mathbb{E}\{I_\alpha g(W)\} - \sum_{i=1}^{D+1} i\lambda_i \mathbb{E}\{g(W+i)\}
$$

$$
= \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i](g(Y_\alpha + X_\alpha + i) - g(Y_\alpha + i))\}
$$

$$
+ \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \big[\mathbb{E}\{I_\alpha I[Z_\alpha = i] g(Y_\alpha + i)\}
$$

$$
- \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}\mathbb{E}\{g(Y_\alpha + i)\} \big]
$$

$$
+ \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}\mathbb{E}\{g(Y_\alpha + i) - g(W + i)\}.
$$

Note that

$$
\big| \mathbb{E}\{I_\alpha I[Z_\alpha = i](g(Y_\alpha + X_\alpha + i) - g(Y_\alpha + i))\} \big|
$$

$$
\leq \|\Delta g\|\mathbb{E}\{I_\alpha I[Z_\alpha = i] X_\alpha\}, \quad \big| \mathbb{E}\{g(Y_\alpha + i) - g(W + i)\} \big|
$$

$$
\leq \|\Delta g\|\mathbb{E}\{I_\alpha + U_\alpha + X_\alpha\},
$$

and taking $\phi_{\alpha i} = \mathbb{E}|\mathbb{E}\{I_\alpha I[Z_\alpha = i] | (I_\beta : \beta \in \Gamma_\alpha^{vw})\} - \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}|$ we get

$$
\big| \mathbb{E}\{I_\alpha I[Z_\alpha = i] g(Y_\alpha + i)\} - \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}\mathbb{E}\{g(Y_\alpha + i)\} \big| \leq \|g\|\phi_{\alpha i}.
$$

It thus follows that

$$\left| \mathbb{E}\left\{ Wg(W) - \sum_{i=1}^{D+1} i\lambda_i g(W+i) \right\} \right|$$

$$\leq \|\Delta g\| \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i] X_\alpha\}$$

$$+ \|\Delta g\| \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}\mathbb{E}\{I_\alpha + U_\alpha + X_\alpha\}$$

$$+ \|g\| \sum_{\alpha \in \Gamma} \sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \phi_{\alpha i}$$

$$= \|\Delta g\| \sum_{\alpha \in \Gamma} \left((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\} \right) + \|g\|\phi,$$

where $\phi = \sum_{\alpha \in \Gamma}\sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \phi_{\alpha i}$. This is what we wanted to prove. $\square$

REMARK 2. In the compound Poisson local approach, as compared to the Poisson local approach, there is the extra freedom to chose the $\Gamma_\alpha^{vs} \neq \varnothing$. The advantage is that in the compound Poisson error estimate (2.4), there is no term $\sum_{\alpha \in \Gamma}\sum_{\beta \in \Gamma_\alpha^{vs}}\mathbb{E}\{I_\alpha I_\beta\}$—local clumps do not worsen the approximation, but are taken care of in the specification of the compound Poisson distribution. For Poisson approximation, such a term would have to appear.

REMARK 3. As in the Poisson case, the compound Poisson local approach is easiest if $\phi = 0$.

Define

$$(2.6) \quad \begin{aligned} c_2'(\lambda) &= \sup_{A \subseteq \mathbb{Z}^+} \sup_{j \geq 1} |g_{\lambda,A}(j+1) - g_{\lambda,A}(j)| = \sup_{A \subseteq \mathbb{Z}^+} \|\Delta g_{\lambda,A}\|, \\ c_1'(\lambda) &= \sup_{A \subseteq \mathbb{Z}^+} \sup_{j \geq 1} |g_{\lambda,A}(j)| = \sup_{A \subseteq \mathbb{Z}^+} \|g_{\lambda,A}\|, \end{aligned}$$

where $g_{\lambda,A}$ is the solution of (2.1).

COROLLARY 1. *Under the assumptions of Theorem 2,*

$$(2.7) \quad \begin{aligned} &d_{TV}(\mathscr{L}(W), \mathrm{CP}(\lambda)) \\ &\quad \leq c_2'(\lambda) \sum_{\alpha \in \Gamma} \left((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\} \right) + c_1'(\lambda)\phi, \end{aligned}$$

where $\lambda = \sum_{i=1}^{D+1} \lambda_i \delta_i$, $\lambda_i = (1/i)\sum_{\alpha \in \Gamma}\mathbb{E}\{I_\alpha I[Z_\alpha = i]\}$, $D = \max_{\alpha \in \Gamma}\{|\Gamma_\alpha^{vs}|\}$ *and* $\phi = \sum_{\alpha \in \Gamma}\sum_{i=1}^{|\Gamma_\alpha^{vs}|+1} \phi_{\alpha i}$, *with*

$$\phi_{\alpha i} = \mathbb{E}\left| \mathbb{E}\left\{ I_\alpha I[Z_\alpha = i] \big| (I_\beta : \beta \in \Gamma_\alpha^{vw}) \right\} - \mathbb{E}\{I_\alpha I[Z_\alpha = i]\} \right|.$$

PROOF. Let $g = g_{\lambda, A}$ be the Stein–Chen transform of the function $I[\cdot \in A] - CP(\lambda)\{A\}$. Then it follows from Theorem 2 that

$$
\begin{aligned}
d_{TV}(\mathscr{L}(W), CP(\lambda)) &= \sup_{A \subseteq \mathbb{Z}^+} \left| \mathbb{P}[W \in A] - CP(\lambda)\{A\} \right| \\
&= \sup_{A \subseteq \mathbb{Z}^+} \left| \mathbb{E}\left\{ Wg(W) - \sum_{i=1}^{D+1} i\lambda_i g(W + i) \right\} \right| \\
&\leq c_2'(\lambda) \sum_{\alpha \in \Gamma} \left((\mathbb{E} I_\alpha)^2 + \mathbb{E} I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\} \right) \\
&\quad + c_1'(\lambda)\phi. \qquad \qquad \square
\end{aligned}
$$

REMARK 4. The bounds known for $c_1'(\lambda)$ and $c_2'(\lambda)$ in the compound Poisson case are less good than for the Poisson. For any $\lambda = \sum_{i=1}^\infty \lambda_i \delta_i$,

$$
c_2'(\lambda) \leq \left(1 \wedge \frac{C}{\lambda_1} \right) \exp\left\{ \sum_{i=1}^\infty \lambda_i \right\}
$$

and if $i\lambda_i \searrow 0$, then

$$
c_1'(\lambda) \leq \begin{cases} 1, & \text{if } \lambda_1 - 2\lambda_2 \leq 1, \\ \dfrac{1}{\sqrt{\lambda_1 - 2\lambda_2}} \left[2 - \dfrac{1}{\sqrt{\lambda_1 - 2\lambda_2}} \right], & \text{if } \lambda_1 - 2\lambda_2 > 1, \end{cases}
$$

and

$$
c_2'(\lambda) \leq \left\{ 1 \wedge \frac{1}{\lambda_1 - 2\lambda_2} \left[\frac{1}{4(\lambda_1 - 2\lambda_2)} + \log^+ 2(\lambda_1 - 2\lambda_2) \right] \right\}
$$

as proved by Barbour, Chen and Loh (1992).

Note that to determine the approximating compound Poisson distribution in Theorem 2, one should compute λ_i for $i = 1, \ldots, D+1$ and $D = \max_{\alpha \in \Gamma}\{|\Gamma_\alpha^{vs}|\}$. This is not always a simple matter. Sometimes it is possible to approximate the random variable W by a compound Poisson distribution determined by a smaller number of λ_i's, for which the computations are more tractable. The next theorem is sometimes of help in this respect.

THEOREM 3. For $\lambda = \sum_{i=1}^{D+1} \lambda_i \delta_i$ define $\lambda^* = \sum_{i=1}^l \lambda_i^* \delta_i$, $l < D+1$, so that $\sum_{i=1}^{D+1} i\lambda_i = \mathbb{E}W = \sum_{i=1}^l i\lambda_i^*$, $\lambda_1^* = \lambda_1 + \sum_{i=l+1}^{D+1} i\lambda_i$, $\lambda_j^* = \lambda_j$ for $j = 2, \ldots, l$ and $\lambda_j^* = 0$ for $j \geq l+1$. Then

$$
\begin{aligned}
d_{TV}(\mathscr{L}(W), CP(\lambda^*)) \leq c_2'(\lambda^*) &\left(\sum_{\alpha \in \Gamma} \left((\mathbb{E} I_\alpha)^2 + \mathbb{E} I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} \right. \right. \\
&\left. \left. + \mathbb{E}\{I_\alpha X_\alpha\} \right) + \sum_{i=l+1}^{D+1} i(i-1)\lambda_i \right) + c_1'(\lambda^*)\phi.
\end{aligned}
$$

PROOF. Note that, with $g^* = g_{\lambda^*, A}$,

$$|\mathbb{P}[W \in A] - \mathrm{CP}(\lambda^*)\{A\}|$$

$$= \left|\mathbb{E}\left\{Wg^*(W) - \sum_{i=1}^{l} i\lambda_i^* g^*(W+i)\right\}\right|$$

$$= \left|\mathbb{E}\left\{Wg^*(W) - \sum_{i=1}^{D+1} i\lambda_i g^*(W+i)\right\}\right.$$

$$\left. + \mathbb{E}\left\{\sum_{i=1}^{D+1} i\lambda_i g^*(W+i) - \sum_{i=1}^{l} i\lambda_i^* g^*(W+i)\right\}\right|$$

$$\leq \|\Delta g^*\|\left(\sum_{\alpha \in \Gamma} \left((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\}\right) + \sum_{i=l+1}^{D+1} i(i-1)\lambda_i\right)$$

$$+ \|g^*\|\phi$$

from Corollary 1 and from the observation that

$$\left|\mathbb{E}\left\{\sum_{i=1}^{D+1} i\lambda_i g^*(W+i) - \sum_{i=1}^{l} i\lambda_i^* g^*(W+i)\right\}\right|$$

$$= \left|\mathbb{E}\left\{\sum_{i=1}^{D+1} i\lambda_i g^*(W+i) - \left(\lambda_1 + \sum_{i=l+1}^{D+1} i\lambda_i\right)g^*(W+1)\right.\right.$$

$$\left.\left. - \sum_{i=2}^{l} i\lambda_i g^*(W+i)\right\}\right|$$

$$= \left|\mathbb{E}\left\{\sum_{i=l+1}^{D+1} i\lambda_i\big(g^*(W+i) - g^*(W+1)\big)\right\}\right|$$

$$\leq \|\Delta g^*\| \sum_{i=l+1}^{D+1} i(i-1)\lambda_i.$$

By taking the supremum over all $A \subseteq \mathbb{Z}^+$, we get the theorem. □

REMARK 5. As in Remark 4, there are bounds:

$$c_2'(\lambda^*) \leq \left\{1 \wedge \frac{C}{\lambda_1^*}\right\}\exp\left\{\sum_{i=1}^{\infty} \lambda_i^*\right\}$$

and if $i\lambda_i^* \searrow 0$, then

$$(2.8) \quad c_1'(\lambda^*) \leq \begin{cases} 1, & \text{if } \lambda_1^* - 2\lambda_2^* \leq 1, \\ \dfrac{1}{\sqrt{\lambda_1^* - 2\lambda_2^*}}\left[2 - \dfrac{1}{\sqrt{\lambda_1^* - 2\lambda_2^*}}\right], & \text{if } \lambda_1^* - 2\lambda_2^* > 1, \end{cases}$$

$$c_2'(\lambda^*) \leq \left\{1 \wedge \frac{1}{\lambda_1^* - 2\lambda_2^*}\left[\frac{1}{4(\lambda_1^* - 2\lambda_2^*)} + \log^+ 2(\lambda_1^* - 2\lambda_2^*)\right]\right\}.$$

3. The k-out-of-n isolated vertices in the rectangular lattice on the torus.

Consider a rectangular lattice on the torus with n vertices and $N = 2n$ edges. Note that it is a 4-regular graph without triangles. Assume that the edges can be deleted independently of each other with a constant probability $1 - p = q$.

This structure could represent a multiprocessor, where the vertices represent different processors and the edges represent connections between them. Each edge can be up (works, be present in the graph) or can be down (fails, be deleted in the graph) independently of each other. The system fails if there are k isolated vertices in the graph. The reliability of the system is equal to the probability that the system works, which is equal to the probability that there are less than k isolated vertices in the graph. Can we give a bound for this probability?

To attack the problem, define $\Gamma = \{1, \ldots, n\}$ to index the vertices of the graph. Let $\mathbb{P}[e_{\alpha\beta} \text{ deleted}] = q$ for all $\alpha, \beta \in \Gamma$ such that v_α and v_β are lattice neighbours; that is, when they are connected by an edge $e_{\alpha\beta}$. Let $I_\alpha = I[v_\alpha \text{ isolated}]$ for $\alpha \in \Gamma$ and $W = \sum_{\alpha \in \Gamma} I_\alpha$. With the above definitions $\mathbb{P}[v_\alpha \text{ isolated}] = q^4$ and $\mathbb{E}W = nq^4$. The random variable W counts the number of isolated vertices in the graph. The reliability of the system is equal to $\mathbb{P}[W < k]$.

In what follows, we find an appropriate approximation to this probability using the compound Poisson local approach.

Define

$$\Gamma_\alpha^{vs} = \{\beta \neq \alpha : v_\beta \text{ and } v_\alpha \text{ are lattice neighbours}\},$$

$$\Gamma_\alpha^b = \{\gamma \neq \alpha : v_\gamma \text{ and } v_\beta, \beta \in \Gamma_\alpha^{vs} \text{ are lattice neighbours}\}.$$

Note that having no triangles in the graph means that there are no edges between vertices in Γ_α^{vs}. Note also that $|\Gamma_\alpha^{vs}| = 4$ and $|\Gamma_\alpha^b| = 8$ are constant for all vertices v_α, $\alpha \in \Gamma$.

First we compute the upper bound stated in Theorem 2:

$$\sum_{\alpha \in \Gamma} (\mathbb{E}I_\alpha)^2 = nq^8,$$

$$\sum_{\alpha \in \Gamma} \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} = \sum_{\alpha \in \Gamma} \sum_{\beta \in \Gamma_\alpha^{vs} \cup \Gamma_\alpha^b} \mathbb{E}I_\alpha \mathbb{E}I_\beta = 12nq^8,$$

$$\sum_{\alpha \in \Gamma} \mathbb{E}\{I_\alpha X_\alpha\} = \sum_{\alpha \in \Gamma} \sum_{\beta \in \Gamma_\alpha^b} \mathbb{E}\{I_\alpha I_\beta\} = 8nq^8,$$

from the independence of I_α and I_β for $\beta \in \Gamma_\alpha^b$, and $\phi = 0$. It thus follows that

$$(3.1) \qquad \sum_{\alpha \in \Gamma} \left((\mathbb{E}I_\alpha)^2 + \mathbb{E}I_\alpha \mathbb{E}\{U_\alpha + X_\alpha\} + \mathbb{E}\{I_\alpha X_\alpha\}\right) = 21nq^8.$$

In the next step we have to determine the approximating compound Poisson distribution by computing the λ_i's. We have to compute

$$\lambda_i = \frac{1}{i} \sum_{\alpha \in \Gamma} \mathbb{E}\{I_\alpha I[Z_\alpha = i]\}$$

$$= \frac{q^4}{i} \sum_{\alpha \in \Gamma} \mathbb{P}\left[\sum_{\beta \in \Gamma_\alpha^{vs}} I_\beta = i - 1 | I_\alpha = 1\right] \quad \text{for } i = 1, \dots, 5.$$

For each $\alpha \in \Gamma$, because there are no triangles,

$$\sum_{\beta \in \Gamma_\alpha^{vs}} I_\beta \sim \text{Bi}(4, q^3) \quad \text{conditional on } I_\alpha = 1,$$

and we get

$$\lambda_i = \frac{n}{i} q^4 \binom{4}{i-1} q^{3(i-1)} (1 - q^3)^{5-i}$$

$$= \frac{n}{i} \binom{4}{i-1} q^{3i+1} (1 - q^3)^{5-i}.$$

Note that

$$\lambda_1 = nq^4(1 - q^3)^4, \qquad \lambda_2 = 2nq^7(1 - q^3)^3$$

and

$$\lambda_i = O(nq^{10}) \quad \text{for } 3 \le i \le 5.$$

Thus we can take $l = 2$ and set

(3.2)
$$\lambda_1^* = \lambda_1 + \sum_{i=3}^{5} i\lambda_i = \mathbb{E}W - 2\lambda_2,$$

$$\lambda_2^* = \lambda_2 = 2nq^7(1 - q^3)^3,$$

since $6\lambda_3 + 12\lambda_4 + 20\lambda_5$ is of smaller order than the bound in (3.1) if $q \searrow 0$ as $n \to \infty$.

In the next step we have to show that $i\lambda_i^* \searrow 0$, so as to be able to use the bound (2.8) on $c_2'(\boldsymbol{\lambda}^*)$. Note that for $0 \le q < 1$,

$$\frac{2\lambda_2^*}{\lambda_1^*} = \frac{4q^3(1 - q^3)^3}{1 - 4q^3(1 - q^3)^3} < 1$$

and

(3.3) $$\lambda_1^* - 2\lambda_2^* = \mathbb{E}W - 4\lambda_2 = nq^4\left(1 - 8q^3(1 - q^3)^3\right).$$

Note also that

(3.4) $$\sum_{i=3}^{5} i(i-1)\lambda_i = 12nq^{10}(1 + O(q^3)).$$

Now, by combining Theorem 3 and (3.1)–(3.4), we are ready to state the following theorem.

THEOREM 4. *For a lattice on the torus with n vertices, if $0 \le q < 1$ and λ_1^*, λ_2^* are defined as in (3.2), then*

$$d_{TV}(\mathscr{L}(W), \mathrm{CP}(\boldsymbol{\lambda}^*))$$

$$\le \frac{1}{nq^4\left(1 - 8q^3(1 - q^3)^3\right)}(1 + \log^+ 2nq^4)(21nq^8 + 12nq^{10}(1 + O(q^3)))$$

$$\le (1 + \log^+ 2nq^4)21q^4(1 + O(q^2)).$$

If $nq^4 \to \mu_\infty > 1$ is held constant as $n \to \infty$, then

$$d_{TV}(\mathscr{L}(W), \mathrm{CP}(\boldsymbol{\lambda}^*)) = O\left(\frac{1}{n}\right).$$

Note that, for Poisson approximation, if $nq^4 \to \mu_\infty$, the approximation is only accurate to order $O(n^{-3/4})$.

Theorem 4 allows us to approximate the reliability of the system by the compound Poisson distribution with λ_1^* and λ_2^*. Thus,

$$\mathbb{P}[W < k] \approx \sum_{i=0}^{k-1} \mathbb{P}[X = i] \quad \text{where } X \sim \mathrm{CP}(\boldsymbol{\lambda}^*)$$

and the λ_i^*'s are as in (3.2). Let $N_1 \sim \mathrm{Po}(\lambda_1^*)$ and $N_2 \sim \mathrm{Po}(\lambda_2^*)$ be independent of each other. Then

$$\mathbb{P}[X = i] = \mathbb{P}[N_1 + 2N_2 = i]$$

$$= \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{\lambda_1^{*\,i-2j}}{(i - 2j)!} \frac{\lambda_2^{*\,j}}{j!} \exp(-(\lambda_1^* + \lambda_2^*)).$$

This observation allows us to state the following theorem.

THEOREM 5. *Under the assumptions of Theorem 4, if q^4 is small, for any $k \ge 0$,*

$$\mathbb{P}[W < k] \approx \sum_{i=0}^{k-1} \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{\lambda_1^{*\,i-2j}}{(i - 2j)!} \frac{\lambda_2^{*\,j}}{j!} \exp(-(\lambda_1^* + \lambda_2^*)),$$

where $\lambda_1^* = nq^4(1 - 4q^3(1 - q^3)^3)$ and $\lambda_2^* = 2nq^7(1 - q^3)^3$.

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