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Steiner Minimal Trees for Regular Polygons

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Abstract. Fifty years ago Jarnik and Kössler showed that a Steiner minimal tree for the vertices of a regular n-gon contains Steiner points for $3 \leq n \leq 5$ and contains no Steiner point for $n = 6$ and $n \geq 13$. We complete the story by showing that the case for $7 \leq n \leq 12$ is the same as $n \geq 13$. We also show that the set of n equally spaced points yields the longest Steiner minimal tree among all sets of n cocircular points on a given circle.

1. Introduction

A Steiner minimal tree (SMT) for a set of points P in the plane is a shortest network interconnecting P. The construction of an SMT for a general set P is known [7] to be an NP-complete problem. Recently, SMTs have been constructed for special point sets P such as ladders [1], splitting trees [9], zigzag lines [5], cocircular points [6], and bar waves [4]. However, a special class of sets for which the study of SMTs was started a half century back has remained an unsolved problem. Let A_n denote the set of vertices of a regular n-gon. The SMT problem for A_n was first studied by Jarnik and Kössler [10] in 1934. They obtained SMTs for $n \leq 6$ and also proved a beautiful theorem which says that for $n \geq 13$ an SMT can be obtained by deleting an edge from the perimeter of the regular n-gon. Since an SMT can also be obtained in this manner for $n = 6$, an obvious conjecture is that an SMT can be so obtained for all $n \geq 6$. Kotzig [11] discussed some properties of the angles of an SMT for $n \leq 8$. In this article we will prove this conjecture in its entirety as our Theorem 1.

Theorem 1. *The perimeter of a regular n-gon minus any side is an SMT for* A_n *for* $n \geq 6$.

We also prove

Theorem 2. *For any n cocircular points on a given circle, the set of n equally spaced points yields a longest SMT.*

2. The Case $n \geq 11$

In this section we show that some recent results on the Steiner ratio (to be defined shortly) can be used to dispose of the conjecture for all $n \geq 11$.

A minimal spanning tree (MST) for a set of points P is a shortest tree interconnecting P such that the vertex-set of the tree is P. The *Steiner ratio* ρ is defined as

$$\inf_P \frac{\text{length of an SMT for } P}{\text{length of an MST for } P}.$$

Gilbert and Pollak [8] cojectured that $\rho = \sqrt{3}/2$ while Du and Hwang [3] proved that $\rho \geq 0.8$. Recently, Chung and Graham [2] announced a proof that $\rho \geq 0.8241$. The Steiner ratio was surprisingly used in [6] to prove a result about SMTs for cocircular points, via the following lemma:

Lemma 0. *Suppose that an n-gon circumscribed in a unit circle has at most one side longer than m with*

$$m = \min\{[\alpha\beta + \sqrt{\alpha^2 + (1-\beta^2)/4}]/(\alpha^2 + \tfrac{1}{4}),\ \gamma\},$$

where

$$\alpha = \sqrt{3} + 1 - 1/(2\bar{\rho}),$$
$$\beta = 1 - (1 - \bar{\rho})\pi/\bar{\rho}$$

($\bar{\rho}$ is a lower bound for ρ) and

$$\gamma = 2(\sqrt{3}+1)/[(\sqrt{3}+1)^2 + \tfrac{1}{4}] = 0.708\ldots.$$

Then its MST (which is the perimeter of the n-gon minus the logest side) is also its SMT.

Set $\bar{\rho} = 0.824$. We obtain $m > 0.6034$. On the other hand, the length of a side of the regular n-gon

$$l_n = \sqrt{2\left(1 - \cos\frac{2\pi}{n}\right)} = 2\sin\frac{\pi}{n}$$

is monotone decreasing in n for $n \geq 3$. Furthermore,

$$l_n \leq l_{11} < 0.5635 < 0.6034 < m \qquad \text{for} \quad n \leq 11.$$

By Lemma 0 we obtain

Theorem 1. *The MST of a regular n-gon is also its SMT for* $n \geq 11$.

3. Some Facts About SMTs

Consider any tree T interconnecting a set of points $P = \{p_1, \ldots, p_n\}$. We will refer to the p_i's as the *regular points* and any other points in T as *Steiner points.* T is called a *Steiner tree* if all subtending angles are at least 120° and each Steiner point has three incident edges (this implies that the subtending angles are exactly 120° for a Steiner point). It is well known [8] that a Steiner tree for n points has at most $n-2$ Steiner points and is called a *full Steiner tree* if it has $n-2$ points. It is also well known [8] that an SMT must be a Steiner tree and can always be decomposed into subtrees which are full Steiner trees. Finally, it is well known [8] that an SMT always lies within the convex hull of P.

A *topology* of a Steiner tree T is a specification of all edges in T. A Steiner tree for a given topology either exists uniquely or does not exist. When a full Steiner tree with a given topology exists, Melzak [12] gave a recursive construction for it which also yields a line segment, which we call the *axis*, whose length equals that of the Steiner tree.

Let C denote a unit circle with center o. Let R_n denote a regular n-gon inscribed in C with vertex set $A_n = \{a_1, \ldots, a_n\}$. Throughout the paper we denote the line segment between two points x and y by $[x, y]$ and its length by (xy).

Lemma 1. *Let T be an SMT for R_n. Then we may assume that no Steiner point s of T can have an incident edge as long as l_n.*

Proof. Suppose to the contrary that l is such an edge. Delete l and decompose T into two subtrees. Then there must exist a j such that a_j and a_{j+1} are not in the same subtree. Connect a_j, a_{j+1} and we obtain an interconnecting tree not longer than T. □

Lemma 2. *Let C be a unit circle with center o. Let p, q be two points such that $(po) \geq 1 \geq (qo)$ and $\measuredangle oqp \leq 60°$ (see Fig. 1). Then $(pq) \geq (po)$.*

Proof. In $\triangle opq$, $\measuredangle qpo \leq \measuredangle oqp \leq 60°$ since $(po) \geq (qo)$. Hence $\measuredangle poq \geq 60° \geq \measuredangle oqp$. It follows $(pq) \geq (po)$. □

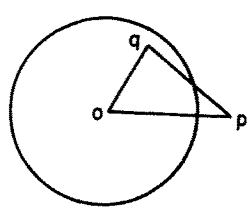

Fig. 1. $(pq) > (po)$.

The following Lemma is not directly related to SMTs but often facilitates an argument that a certain topology does not exist.

Lemma 3. *Let $CA_1 \dots A_m D$ be a polygon lying within another polygon $CB_1 \dots B_n D$. Then*

$$\sum_{i=1}^{m} \measuredangle A_i - \sum_{i=1}^{n} \measuredangle B_i \geq (m-n)180°.$$

Proof. Using the fact that an n-gon has total inner degrees $(n-2)180°$ and the fact that $\measuredangle A_1CD \leq \measuredangle B_1CD$ and $\measuredangle CDA_m \leq \measuredangle CDB_n$. □

A path $a_i s_1 \dots s_m a_j$ in an SMT T is called a *Steiner path* if $s_1, \dots, s_n$, are all Steiner points and $\measuredangle s_i = 120$ for $i = 1, \dots, m$ in the $(m+2)$-gon $a_i s_1 \dots s_m a_j$.

Lemma 4. *Suppose that T is an SMT for R_n. Let $P = a_i s_1 s_2 \dots s_m a_j$ be a Steiner path.*

(i) $m \leq 3$. *There are no regular points between a_i and a_j.*
(ii) $m = 4$. *There is at most one regular point between a_i and a_j but none if $n \leq 9$. No such P can exist for $n \leq 6$.*
(iii) $m = 5$. *No such P can exist for $n \leq 11$.*
(iv) $m \geq 6$. *No such P can exist.*

Proof. It is easily verified that $(a_i a_j) < 2l_n$ for all m. Thus at most one regular point can exist between a_i and a_j.

$m = 1$ *or* 2. Suppose to the contrary that a_i and a_j are not adjacent. Then $n \geq 4$. Let a_k be the regular point between a_i and a_j. For $m = 1$ consider the quadrilateral $a_i s_1 a_j a_k$. We have

$$\measuredangle a_i + \measuredangle a_j = 360° - \measuredangle s_1 - \measuredangle a_k \leq 360° - 120° - 90° = 150°.$$

For $m = 2$ consider the pentagon $a_i s_1 s_2 a_j a_k$. We have

$$\measuredangle a_i + \measuredangle a_j = 540° - \measuredangle s_1 - \measuredangle s_2 - \measuredangle a_k \leq 540° - 120° - 120° - 90° = 210°.$$

Hence either $\measuredangle a_i$ or $\measuredangle a_j$ is less than 120°. Suppose that $\measuredangle a_i < 120°$. Then a_k is connected to a_j in T. Let T' be obtained by substituting $[a_k, a_i]$ for $[a_k, a_j]$ in T. Then T' and T have the same length. Yet T' cannot be optimal since $\measuredangle a_k a_i s_1 < 120°$, a contradiction to the optimality of T.

$m = 3$. We have $n \geq m + 2 = 5$. If $n = 5$, then there is not other Steiner point and each of s_1, s_2, and s_3 must connect a distinct regular point outside of the pentagon $a_i s_1 s_2 s_3 a_j$. Thus there is no more regular point to fill between a_i and a_j. Therefore we may assume that $n \geq 6$.

Suppose to the contrary that a_k exists between a_i and a_j. Consider the hexagon $a_is_1s_2s_3a_ja_k$. Note that all angles except $\measuredangle a_i$ and $\measuredangle a_j$ are at least 120°. Hence either the hexagon is regular, or at least one of $\measuredangle a_i$ and $\measuredangle a_j < 120°$. The former case is impossible since we do not allow $(s_1s_2) = (a_ka_i) = l_n$. The latter case is also impossible by analogous argument as used in the case $m = 1$ or 2.

$m = 4$. $n \geq m + 2 = 6$. If $n = 6$, the hexagon $a_is_1s_2s_3s_4a_j$ has internal degree less than 720°, an absurdity. For $n \geq 7$ suppose that a_k exists between a_i and a_j. For $n \leq 9$, the heptagon $a_is_1s_2s_3s_4a_ja_k$ has internal degrees less than 900°, an absurdity.

$m = 5$. Consider the heptagon $a_1s_1s_2s_3s_4s_5a_j$,

$$\measuredangle a_i + \measuredangle a_j = 900° - 5 \cdot 120° = 300° > \measuredangle a_{i-1}a_ia_{i+1} + \measuredangle a_{j-1}a_ja_{j+1} \qquad \text{for} \quad n \leq 11,$$

an absurdity.

$m \geq 6$. Consider the $(m+2)$-gon $a_is_1 \ldots s_ma_j$,

$$\measuredangle a_i + \measuredangle a_j = m \cdot 180° - m \cdot 120° = m \cdot 60° \geq 360° \qquad \text{for} \quad m \geq 6, \text{ an absurdity.} \ \square$$

Lemma 5. *Let T be an SMT for A_n, $n \leq 10$, with a Steiner point s. Then T must be full.*

Proof. Let $T' \subseteq T$ be a full Steiner tree containing s. Then T' partitions the unit circle into convex regions each bounded by a Steiner path and an arc. By Lemma 4 such an arc can contain at most one additional regular point. In fact, the only case in which an additional regular point may exist is when $n = 10$ and the Steiner path bounding the region has $m = 4$. We now show that even for this case no additional regular point can exist on the arc, i.e., T is a full SMT.

Suppose to the contrary that a regular point a_k exists on the arc a_ia_j (see Fig. 2). In the heptagon $a_is_1s_2s_3s_4a_ja_k$

$$\measuredangle a_i + \measuredangle a_j = 900° - 4 \times 120° - 144° = 276°.$$

Therefore

$$\measuredangle s_1a_ia_{i+1} + \measuredangle a_{j-1}a_js_4 = 2 \times 144° - 276° = 12°$$

and

$$\min\{(s_1a_{i+1}), (s_4a_{j-1}) \geq l_n \sin 48°/\sin 120° > 0.858 l_n$$

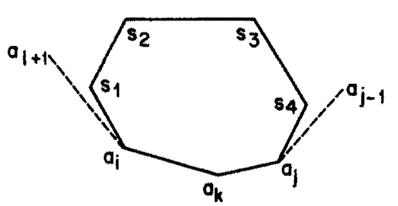

Fig. 2. A Steiner path with $m = 4$ and a regular point.

(it is easily verified that (s_1a_{i+1}) will be greater if s_1 connects a_{i+1} through another Steiner point). Now it is a simple matter to show that $(s_1s_4) > 2l_n$ which implies that one of (s_1s_2), (s_2s_3), and $(s_3s_4) > l_n$, a contradiction to Lemma 1. □

Define D to be the diameter (of the unit circle $c)\perp[a_1, a_n]$.

Lemma 6. *Suppose that a topology is symmetric with respect to an edge e. Then the Steiner tree it yields is symmetric with respect to D with e overlapping D for odd n and $e \perp D$ for even n. Suppose that a topology is symmetric with respect to a point p. Then the Steiner tree it yields is symmetric with respect to the center o with p being o.*

Proof. Clear from Melzak's construction for SMT. □

4. Proof of Theorem 1 for $8 \geq n \geq 10$

Suppose that T is an SMT for A_n with a Steiner point. By Lemma 5 we may assume that T is full. Let d be a point of T closest to the center o and let d lie on the edge e. Let q be an endpoint of e. Partition T into two trees T_1 and T_2 at q and without loss of generality assume that T_1 contains e and the k regular points $\{a_1, a_2, .., a_k\}$. By Melzak's construction of the full Steiner tree, there exists a line segment $[p, q]$ which is the axis of T_1 and overlaps e. Our goal is to show that for certain n T_1 cannot exist by proving $(pq) > kl_n$, so T_1 can be replaced by the path $a_1a_2 \dots a_k$ and some suitable $[a_j, a_{j+1}]$ to obtain a shorter connecting tree. However, since (po) is much easier to compute than (pq), we will prove $po > kl_n$ instead and use Lemma 2 to justify the replacement. One condition of Lemma 2 is that $\measuredangle pqo \leq 60$. The following lemma will take care of that condition.

Lemma 7. $\measuredangle dqo > 60°$.

Proof. Let e' be a second edge of T at q such that o lies in the. 120° angle enclosed by e and e' (possibly their extensions). Let (od') be the distance from o to e'. Since $(od) \leq (od')$, $\measuredangle dqo \leq \measuredangle d'qo$. But $\measuredangle dqo + \measuredangle d'qo = 120°$. Hence $\measuredangle dqo \leq 60°$. □

Lemma 8. $k \neq 1$ *for* $n > 6$.

Proof. From Lemmas 7$\measuredangle a_1qo \leq 60°$. From Lemma 2

$$(a_1q) \geq (a_1o) = 1 > l_n \qquad \text{for} \quad n > 6.$$ □

Lemma 9. $k \neq 2$ *for* $8 \leq n \leq 10$.

Proof. Suppose to the contrary that $k = 2$. Let pa_1a_2 be a regular triangle with p outside of the unit circle. Then by Melzak's construction $[p, q]$ is the axis of

T_1. We now prove that $(po) > 2l_n$, or $((po)/l_n)^2 > 4$:

$$\begin{aligned}(po)^2 &= (a_1o)^2 + (a_1p)^2 - 2(a_1o)(a_1p)\cos \measuredangle oa_1p \\ &= 1 + l_n^2 - 2l_n \cos(90° - 180°/n + 60°) \\ &= 1 + l_n^2 + \sqrt{3}l_n \cos(180°/n) - l_n \sin(180°/n) \\ &= 1 + l_n^2 + \sqrt{3}l_n\sqrt{1 - l_n^2/4} - l_n^2/2 \\ &= 1 + l_n^2/2 + \sqrt{3}l_n\sqrt{1 - l_n^2/4}.\end{aligned}$$

Clearly, $(po)^2/l_n^2$ is monotone decreasing in l_n^2. For $8 \le n \le 10$ l_n^2 is largest for $n = 8$ and $l_8^2 = 2(1 - \cos 45°) = 2 - \sqrt{2}$. Now

$$\left(\frac{(po)}{l_8}\right)^2 = \frac{1}{2-\sqrt{2}} + \frac{1}{2} + \sqrt{3} - \frac{1}{4} \ge 4.297 > 4. \qquad \square$$

Lemma 10. *$k \ne 3$ for $8 \le n \le 10$.*

Proof. Suppose to the contrary that $k = 3$. Let $a_3a_2p_1$ and $p_1a_1p_2$ be regular triangles such that p_1 and o are on different sides of $[a_2, a_3]$, and p_2 and o are on different sides of $[p_1a_1]$ (see Fig. 3). We now prove that $(p_2o) > 3l_n$ or $(p_2o)/l_n)^2 > 9$. Note that

$$(p_2o)^2 = (p_1o)^2 + (p_1p_2)^2 - 2(p_1o)(p_1p_2)\cos \measuredangle op_1p_2.$$

Now

$$\begin{aligned}(p_1p_2) = (p_1a_1) &= [(a_1a_2)^2 + (p_1a_2)^2 - 2(a_1a_2)(p_1a_2)\cos \measuredangle a_1a_2p_1]^{1/2} \\ &= 2l_n[(1 - \cos \measuredangle a_1a_2p_1)/2]^{1/2} \\ &= 2l_n \sin(\measuredangle a_1a_2p_1/2) \\ &= 2l_n \sin(60° + 180°/n).\end{aligned}$$

Hence

$$(p_1p_2)/l_n = 2\sin(60° + 180°/n) \ge 2\sin(60° + 18°) > 1.956$$

and $(p_1p_2)^2/l_n^2 > 3.827$ for $n \le 10$. Furthermore,

$$\begin{aligned}\measuredangle op_1p_2 &= \measuredangle op_1a_2 + \measuredangle a_2p_1a_1 + \measuredangle a_1p_1p_2 \\ &= 30° + 30° - 180°/n + 60° = 120° - 180°/n\end{aligned}$$

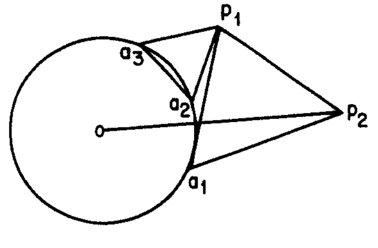

Fig. 3. $(p_2o) > 3l_n$ for $k = 3$.

and

$$-2\cos \measuredangle op_1p_2 = 2\cos(60^\circ + 180^\circ/n),$$

which is clearly monotone increasing in n for $n \geq 6$. Therefore, for $8 \leq n \leq 10$

$$-2\cos \measuredangle op_1p_2 \geq 2\cos 82.5^\circ = 0.261.$$

Therefore, for $8 \leq n \leq 10$ we have

$$\left(\frac{p_2o}{l_n}\right)^2 \geq 4.297 + 3.827 + (0.261)\sqrt{(4.297)(3.827)} = 9.182 > 9. \qquad \square$$

Lemma 11. *$k \neq 4$ for $8 \leq n \leq 10$.*

Proof. Suppose to the contrary that $k = 4$. There are three nonisomorphic topologies for T_1 which we will call topologies 4, 5, and 6 and their Melzak's constructions are shown in Figs. 4-6, respectively. We show that the axis of T_1 is too long for all three topologies.

Topology 4 (Fig. 4)
As shown in the proof of Lemma 10,

$$(a_2p_2) = 2l_n \sin\left(60^\circ + \frac{180^\circ}{n}\right),$$

$$\measuredangle a_3a_2p_1 = 90^\circ - \left(60^\circ + \frac{180^\circ}{n}\right)$$

$$= 30^\circ - \frac{180^\circ}{n}.$$

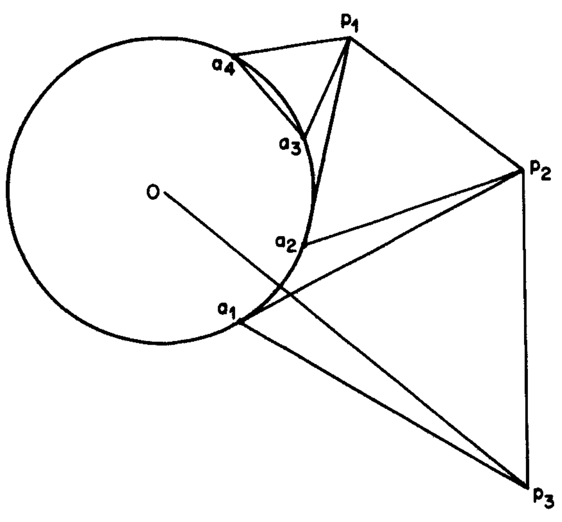

Fig. 4. Melzak's construction for topology 4.

Therefore,

$$\measuredangle a_1 a_2 p_2 = 90° + \frac{540°}{n},$$

$$(a_1 p_2)^2 = (a_1 a_2)^2 + (a_2 p_2)^2 - 2(a_1 a_2)(a_2 p_2)\cos \measuredangle a_1 a_2 p_2$$
$$= l_n^2 \left[1 + 4\sin^2\left(60° + \frac{180°}{n}\right) + 4\sin\left(60° + \frac{180°}{n}\right)\sin\frac{540°}{n} \right].$$

Furthermore,

$$\sin \measuredangle a_2 p_2 a_1 = \frac{l_n}{(a_1 p_2)} \sin \measuredangle a_1 a_2 p_2 = \frac{l_n}{(a_1 p_2)} \cos\frac{540°}{n}.$$

Hence

$$\measuredangle o a_1 p_3 = \left(90° - \frac{180°}{n}\right) + 60° + \measuredangle a_2 a_1 p_2$$
$$= 150° - \frac{180°}{n} + \left(180° - \left(90° + \frac{500°}{n}\right) - \measuredangle a_2 p_2 a_1\right)$$
$$= 240° - \frac{720°}{n} - \measuredangle a_2 p_2 a_1.$$

It follows

$$\left(\frac{(p_3 o)}{l_n}\right)^2 = \left(\frac{1}{l_n}\right)^2 + \frac{(a_1 p_2)^2}{l_n^2} + 2\cdot\frac{1}{l_n}\cdot\frac{(a_1 p_2)}{l_n}\cos\left(\frac{720°}{n} + \measuredangle a_2 p_2 a_1 - 60°\right).$$

We compute $((p_3 o)/l_n)^2$ for $n = 8, 9, 10$.

n	$(a_1 p_2)^2/l_n^2$	$(a_1 p_2)/l_n$	$\sin \measuredangle a_2 p_2 a_1$	$1/l_n$	$(p_3 o)^2/l_n^2$
8	>8.595	>2.937	<0.137	>1.306	>16.3
9	>8.290	>2.879	<0.175	>1.461	>17.7
10	>7.990	>2.826	<0.210	>1.618	>18.9

Topology 5 (Fig. 5)

Note that $\triangle p_1 p_2 a_3 = \triangle p_1 a_2 a_4$. Hence $(p_2 a_3) = (a_3 a_1)$. Also note that $(p_1 p_2) = (p_1 a_4) = (p_1 a_1)$. Hence $\triangle p_1 a_3 p_2 = \Delta a_1 a_3 p_1$. It follows

$$\measuredangle p_1 a_3 p_2 = \measuredangle a_1 a_3 p_1 = 60° + 180°/n$$

and

$$\measuredangle a_1 a_3 p_2 = 120° + 360°/n.$$

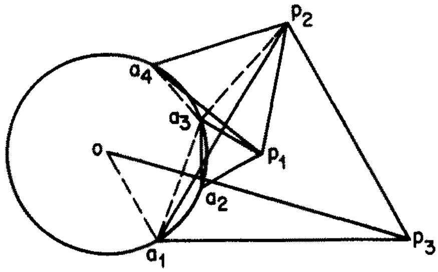

Fig. 5. Melzak's construction for topology 5.

Furthermore,

$$(p_2a_3)=(a_2a_4)=(a_1a_3)=2l_n\left[\frac{1-\cos(180°(n-2)/n)}{2}\right]^{1/2}$$

$$=2l_n\sin\left(180°-\frac{180°}{n}\right)=2l_n\cos\frac{180°}{n}.$$

Therefore,

$$(p_3a_1)=(a_1p_2)=[(a_1a_3)^2+(a_3p_2)^2-2(a_1a_3)(a_3p_2)\cos\measuredangle a_1a_3p_2]^{1/2}$$

$$=4l_n\cos\frac{180°}{n}\left[\frac{1-\cos(120°+360°/n)}{2}\right]^{1/2}$$

$$=4l_n\cos\frac{180°}{n}\sin\left(60°+\frac{180°}{n}\right)$$

$$=2l_n\left[\sin\left(60°+\frac{360°}{n}\right)+\sin 60°\right]\qquad \text{using}\quad 2\sin A\cos B$$

$$=\sin(A+B)+\sin(A-B)$$

$$\geq 2l_n\left[\sin\left(60°+\frac{360°}{8}\right)+\sin 60°\right]>3.663l_n\qquad \text{for}\quad n\geq 8.$$

Finally

$$\measuredangle oa_1p_3=60°+\left(90°-\frac{360°}{n}\right)-90°-\left(60°+\frac{160°}{n}\right)=180°-\frac{540°}{n},$$

$$\frac{(p_3o)^2}{l_n^2}=\frac{(a_1o)^2}{l_n^2}+\frac{(p_3a_1)^2}{l_n^2}+\frac{(p_3a_1)^2}{l_n^2}-2\frac{(a_1o)}{l_n}\frac{(p_2a_1)}{l_n}\cos\measuredangle oa_1p_3$$

$$>\frac{1}{l_8^2+}(3.663)^2+2\frac{1}{l_8}(3.663)\cos\frac{540°}{8}$$

$$=(1.306)^2+(3.063)^2+2(1.306)(3.663)(0.382)$$

$$=18.778>16.$$

Topology 6 (Fig. 6)

$$(p_1p_3)^2=(p_1p_2)^2=(p_1o)^2+(p_2o)^2-2(p_1o)(p_2o)\cos\measuredangle p_2op_1$$
$$=2(p_1o)^2\left(1-\cos\frac{720^\circ}{n}\right),$$
$$(p_3o)^2=(p_1o)^2+(p_1p_3)^2-(p_1o)(p_1p_3)\cos\measuredangle op_1p_3$$
$$=(p_1o)^2\left[1+2\left(1-\cos\frac{720^\circ}{n}\right)\right.$$
$$\left.-2\sqrt{2\left(1-\cos\frac{720^\circ}{n}\right)}\cos(60^\circ+90^\circ-\tfrac{1}{2}\measuredangle p_3op_1)\right)$$
$$=(p_1o)^2\left[3-\cos\frac{720^\circ}{n}+2\sqrt{2\left(1-\cos\frac{720^\circ}{n}\right)}\cos\left(30^\circ+\frac{360^\circ}{n}\right)\right],$$
$$((p_3o)/l_n)^2=((p_1o)/l_n)^2\left[3-2\cos\frac{720^\circ}{n}+2\sqrt{2\left(1-\cos\frac{720^\circ}{n}\right)}\cos\left(30^\circ+\frac{360^\circ}{n}\right)\right].$$

We compute $(p_3o/l_n)^2$ for $n=8, 9, 10$.

n	First term	Second term	Product
8	>4.297	>3.732	>16.039
9	>6.536	>3.056	>19.963
10	>8.073	>2.525	>20.384

We now prove Theorem 1 for $8\le n\le 10$. Suppose that T is an SMT for A_n with a Steiner point. Since q can be either endpoint of e, we may assume that the number of regular points in T_1 does not exceed that of T_2, i.e., $k\le n/2$. For $n=8$ and 9 Lemmas 8–11 say that T_1 is not optimal. For $n=10$ the only case that needs to be considered is when T_1 and T_2 cover five regular points each.

Consider the two Steiner paths P_1 and P_2 containing e. We may assume without loss of generality that a_1 and a_{10} are the endpoints for P_1, while a_5 and a_6 are the endpoints of P_2. Let m_1 and m_2 denote the number of Steiner points on P_1 and P_2. By Lemma 5 $m_1, m_2\le 4$. Since

$$(a_1a_5)=\sqrt{2(1-\cos 144^\circ)}>1.9>3l_{10}$$

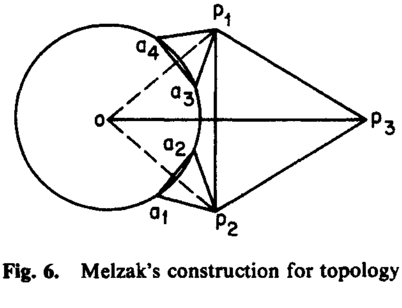

Fig. 6. Melzak's construction for topology 6.

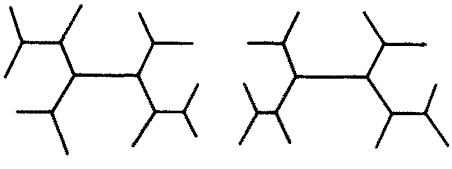

Fig. 7. Two topologies for $m_1 = m_2 = 4$.

and each edge in T is shorter than l_{10}, there must be at least four edges connecting a_1 and a_5. Therefore $m_1 = m_2 = 4$.

There exist two nonisomorphic topologies for $m_1 = m_2 = 4$ as shown in Fig. 7. If T has topology 7(a), then by Lemma 6 T must be asymmetric with respect to the center which is on e. Therefore we can turn the left half of the tree upside down and obtain a tree of the same length but having 7(b) as its topology (Fig. 8). Namely, it suffices to prove that T cannot have 7(b) as its topology.

In $\triangle s_4a_4a_5$, $\measuredangle s_4a_4a_5 = \frac{2}{10} \cdot 180° = 36°$:

$$(a_5a_4) = \frac{(a_4a_5)\sin 36°}{\sin 120°} = \frac{(0.618)(0.588)}{0.866} = 0.42.$$

Extend $[a_5, s_4]$ and $[a_1, s_1]$ to meet at b. Then $bs_1s_3s_4$ is a parallelogram. Hence $(s_4b) = (s_3s_1)$ and $(s_1b) = (s_3s_4)$. In $\triangle ba_5a_1$, $\measuredangle ba_5a_1 = \measuredangle a_5a_1b = 30$ and $\measuredangle a_1ba_5 = 120$. Furthermore,

$$(a_1b) = (a_5b) = (a_5s_4) + (s_3s_1) \le 0.4204 + 0.618 = 1.038.$$

Therefore

$$(a_1a_5) = \sqrt{3}a_1b \le (1.732)(1.038) = 1.798.$$

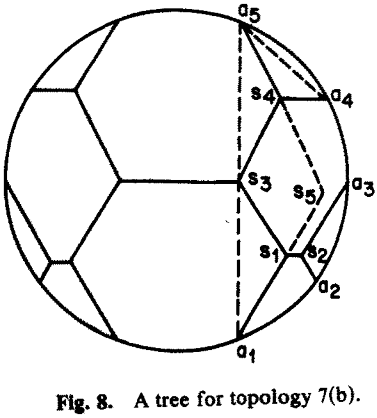

Fig. 8. A tree for topology 7(b).

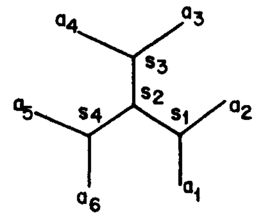

Fig. 9. A unique topology for $n=6$.

But from $\triangle a_1 a_5 o$

$$(a_1 a_5) = \sqrt{2(1-\cos 144°)} > 1.9, \qquad \text{a contradiction.}$$

5. Proof of Theorem 1 for $n=6, 7$

For $n=6$, Lemma 4 reduces the nonisomorphic topologies to the unique one shown in Fig. 9. Since this topology is symmetric with respect to s_2, s_2 must be the center o and T must be symmetric with respect to o. Therefore the length of T is $3\sqrt{3} > 5$ which is the length of an MST.

For $n=7$ Lemma 4 reduces the nonisomorphic topologies to the three shown in Fig. 10. Topology 10(a) can be quickly disposed of by comparing the angles of the polygonal path $a_1a_2a_3a_4a_5a_7$ and those of the Steiner path $a_1s_1s_2s_3s_4a_7$, using Lemma 3. The length of the tree yielded by topology 10(c) has been computed in [13] to be $5.6676 > 5.2068$ which is the length of an MST. We now show that the tree yielded by topology 10(b) is not an SMT as $(a_4 s_3) > l_7$.

Since the topology 10(b) is symmetric with respect to $[a_4, s_3]$, T must be symmetric to D and $[a_4, s_3]$ must overlap with D (see Fig. 11) Extend $[a_3, s_2]$ to b such that $[a_1, b] \| [s_1, s_2]$. Extend $[a_4, s_3]$ to c such that $[a_1, c] \| [s_1, s_3]$. Then

$$(a_4 s_3) = (a_4 c) - (a_3 b) + (a_3 s_2).$$

Now

$$\measuredangle a_4 a_1 c = \measuredangle a_3 a_2 s_2 = \measuredangle a_3 a_2 a_6 - \measuredangle s_2 a_2 a_6 = \tfrac{3}{7} \cdot 180° - 30° = \frac{330°}{7}.$$

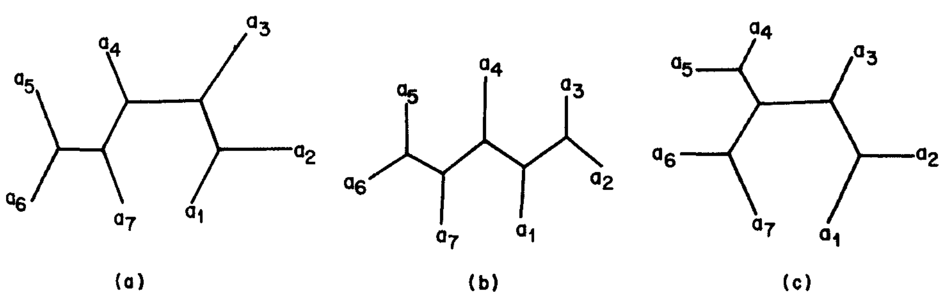

Fig. 10. Three topologies for $n=7$.

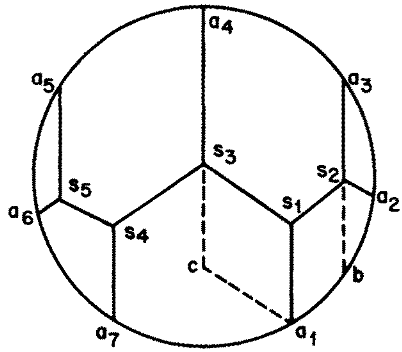

Fig. 11. Tree for 10(b).

Hence

$$(a_3s_2) = \frac{(a_2a_3)\sin(330°/7)}{\sin 120°} = \frac{\sqrt{2[1-\cos(360°/7)]}\sin(330°/7)}{\sin 120°},$$

$$(a_4c) = \frac{(a_1a_4)\sin(330°/7)}{\sin 120°} = \frac{\sqrt{2[1-\cos(1080°/7)]}\sin(330°/7)}{\sin 120°}.$$

Furthermore,

$$\measuredangle ba_1a_3 = \measuredangle s_5a_6a_5 = \measuredangle a_3a_2s_2 \qquad \text{by symmetry.}$$

Hence

$$(a_3b) = \frac{(a_1a_3)\sin(330°/7)}{\sin 120°} = \frac{\sqrt{2[1-\cos(720°)/7)]}\sin(330°/7)}{\sin 120°}.$$

Therefore,

$$\begin{aligned}(a_4s_3) &= \left[\sqrt{2\left(1-\cos\frac{360°}{7}\right)} + \sqrt{2\left(1-\cos\frac{1080°}{7}\right)}\right.\\ &\quad \left. - \sqrt{2\left(1-\cos\frac{720°}{7}\right)}\right]\frac{\sin(330°/7)}{\sin 120°}\\ &= (1.950+0.868-1.563)(0.733)/0.866 = 1.06 > l_7.\end{aligned}$$

6. The Longest Steiner Minimal Trees for n Cocircular Points

The MST for any n cocircular points is clearly longest when the n points are equally spaced. Now for any n given points, the length of an SMT never exceeds that of an MST. Furthermore, Theorem 1 tells us that an MST is an SMT for the equally spaced set if $n \geq 6$. Therefore Theorem 2 is proved for $n \geq 6$. The proof of Theorem 2 for $n = 3, 4, 5$ will each be given separately.

Let C_n denote a set of n points on the unit circle. Let P_n denote the enclosing polygon of C_n.

Lemma 12. *For $3 \le n \le 5$, if one of the angles of P_n is 120° or larger, than an SMT for C_n is shorter than that for A_n.*

Proof. We show that an MST for C_n is shorter than the SMT for A_n. □

Without loss of generality, assume $\measuredangle a_1a_2a_3 \ge 120°$. Then

$$\measuredangle a_1oa_2 + \measuredangle a_2oa_3 \le 120°.$$

By standard minimization techniques it is easily seen that the longest MST for n cocircular points satisfies the angle conditions

$$\measuredangle a_1oa_2 = \measuredangle a_2oa_3 = 60°,$$

and

$$\measuredangle a_3oa_4 = \cdots = \measuredangle a_noa_1 = 240°/(n-2).$$

The length of such an MST is

$$2\sqrt{2(1-\cos 60°)} + (n-3)\sqrt{2\left[1-\cos\left(\frac{240°}{n-2}\right)\right]}$$
$$= \begin{cases} 2 < 3 & \text{for} \quad n=3, \\ 2+\sqrt{3} < \sqrt{2}+\sqrt{6} & \text{for} \quad n=4, \\ 2+2.572 < 4.574 & \text{for} \quad n=5, \end{cases}$$

where the right side of the inequality is the length of an SMT for A_n.

Corollary. *If an SMT for C_n is not full, then its length is shorter than that of A_n.*

We now prove Theorem 2 for $n=3$. Consider C_3 such that all angles of P_3 are less than 120. Construct a regular $\triangle BCD$ such that A and D are on different sides of $[B, C]$. Then (AD) is the length of the SMT for C_3 (see Fig. 12). Let $\measuredangle oBD = \theta$:

$$(AD) \le (Ao) + (oD)$$
$$= 1 + \frac{(oB)\sin\theta}{\sin 30} \le 3.$$

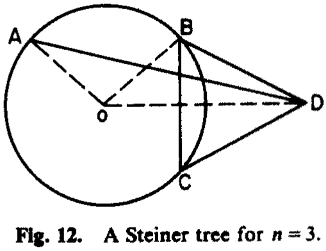

Fig. 12. A Steiner tree for $n = 3$.

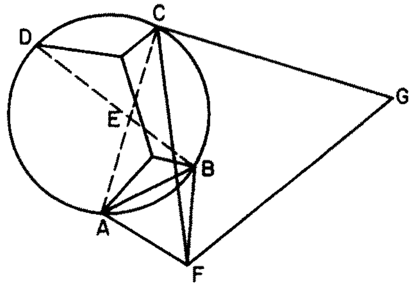

Fig. 13. A Steiner tree for $n=4$.

Next we prove Theorem 2 for $n=4$. Consider C_4 such that all angles of P_4 are less than 120°. Suppose that the diagonals $[A, C]$ and $[B, D]$ meet at E. Without loss of generality, assume that $\measuredangle AEB \leq 90$. Then the Steiner tree T as shown in Fig. 13 exists.

Construct a regular $\triangle ABF$ and a regular $\triangle FCG$. Then the length of T is (DG). But $\triangle GFB \equiv \triangle CFA$, hence $(GB)=(AC)$ and $\measuredangle FBG = \measuredangle FAC$. Furthermore,

$$\begin{aligned}\measuredangle GBD &= 360° - \measuredangle GBF - \measuredangle DBF \\ &= 360° - \measuredangle FAC - \measuredangle EBF \\ &= 360° - (360° - 60° - \measuredangle AEB) \\ &= 60° + \measuredangle AEB.\end{aligned}$$

In $\triangle GDB$

$$\begin{aligned}(GD)^2 &= [(GB)^2 + (BD)^2 - 2(GB)(BD)\cos\measuredangle GBD]^{1/2} \\ &= [(AC)^2 + (BD)^2 - 2(AC)(BD)\cos(60 + \measuredangle AEB)]^{1/2} \\ &\leq (2^2 + 2^2 - 2\cdot 2\cdot 2\cdot \cos 150°)^{1/2} \\ &= (8 + 4\sqrt{3})^{1/2} \\ &< \sqrt{2} + \sqrt{6}.\end{aligned}$$

Finally, we prove Theorem 2 for $n=5$. Without loss of generality, assume that the polygons under study are inscribed in a unit circle. Let M denote the length of an SMT for A_5. Then it is straightforward to calculate

$$\begin{aligned}M &= 4(\sin 36° + \sin 72°)\sin 96° \\ &= 4.574.\end{aligned}$$

Consider a C_5 with points A, B, C, D, and E. By Lemma 12 we may assume that $\measuredangle A$, $\measuredangle B$, $\measuredangle C$, $\measuredangle D$, and $\measuredangle E$ are all less than 120°. Therefore there exist five full Steiner trees where one of them is as shown in Fig. 14 and the other other four can be obtained by rotating the points.

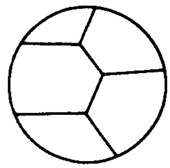

Fig. 14. A Steiner tree for $n=5$.

Let M_1, M_2, M_3, M_4, and M_5 denote the lengths of these five trees, respectively. We prove that

$$\sum_{i=1}^{5} M_i < 5M,$$

where M is the length of an SMT for A_5. Therefore the SMT for C_5, which is the shortest one among the five trees, must be shorter than M.

Construct equilateral triangles $\triangle ABD'$, $\triangle BCE'$, $\triangle CDA'$, $\triangle DEB'$, $\triangle EAC'$, $\triangle A'C'B''$, $\triangle B'D'C''$, $\triangle C'E'D''$, $\triangle D'A'E''$, and $\triangle E'B'A''$ (see Fig. 15). Then $[A, A'']$, $[B, B'']$, $[C, C'']$, $[D, D'']$, and $[E, E'']$ are the five axes. Since $(x'', x) \le (x'', o)+(o, x)=(x'', o)+1$ for $x=A, B, C, D, E$, it suffices to prove

$$S \equiv (A''o)+(B''o)+(C''o)+(D''o)+(E''o) \le 5(M-1).$$

Construct a circle through the three points A'', B', and E' and meet $[A'', o]$ (or its extension) at G. Then

$$(A''o)=(GB')+(GE')+(Go) \qquad (\text{or } -(Go)).$$

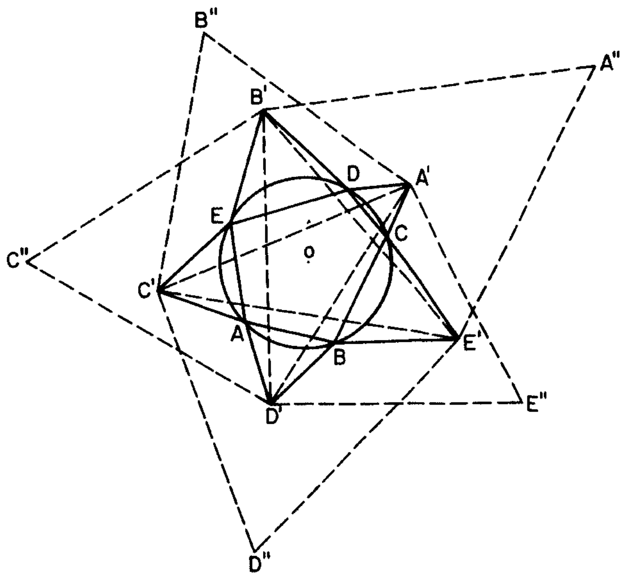

Fig. 15. Axes for the five full Steiner trees.

Define

$$\measuredangle A''oB' = \alpha_1, \quad \measuredangle B''oC' = \alpha_2, \quad \ldots, \quad \measuredangle E''oA' = \alpha_5,$$

$$\measuredangle E'oA'' = \beta_1, \quad \measuredangle A'oB'' = \beta_2, \quad \ldots, \quad \measuredangle D'oE'' = \beta_5,$$

$$\measuredangle CoD = 2\theta_1, \quad \measuredangle DoE = 2\theta_2, \quad \ldots, \quad \measuredangle BoC = 2\theta_5.$$

Since

$$\measuredangle A''GB' = \tfrac{1}{2}\measuredangle E'GB' = \tfrac{1}{2}(180° - \measuredangle B'A''E') = 60°,$$

we have

$$(A''o) = \frac{\sin\alpha_1}{\sin 60°}(oB') + \frac{\sin\beta_1}{\sin 60°}(oE') + \frac{\sin(60° - \alpha_1)}{\sin 60°}(oB').$$

Note that

$$(oB') = 2\sin(30° + \theta_2),$$

$$(oE') = 2\sin(30° + \theta_5),$$

$$\frac{\sin(60° - \alpha_1)}{\sin 60°}(oB') = \frac{\sin(60° - \beta_1)}{\sin 60°}(oE').$$

we have

$$\begin{aligned}
(A''o) &= \frac{1}{\sqrt{3}}\{[(oB')\sin\alpha_1 + (oE')\sin\beta_1] \\
&\qquad + (oB')[\sin\alpha_1 + \sin(60° - \alpha_1)] + (oE')[\sin\beta_1 + \sin(60° - \beta_1)]\} \\
&= \frac{1}{\sqrt{3}}\{[(oB')\sin\alpha_1 + (oE')\sin\beta_1] \\
&\qquad \times (oB')\cos(30° - \alpha_1) + (oE')\cos(30° - \beta_1)\} \\
&= \frac{1}{\sqrt{3}}\{(oB')[\sin\alpha_1 + \cos(30° - \alpha_1)] + (oE')[\sin\beta_1 + \cos(30° - \beta_1)]\} \\
&= \frac{1}{\sqrt{3}}\{(oB')[\sin\alpha_1 + \sin(60° + \alpha_1)] + (oE')[\sin\beta_1 + \sin(60° + \beta_1)]\} \\
&= \frac{1}{\sqrt{3}}\{(oB')\sin(30° + \alpha_1)\cos 30° + (oE')\sin(30° + \beta_1)\cos 30°\} \\
&= \sin(30° + \alpha_1)\sin(30° + \theta_2) + \sin(30° + \beta_1)\sin(30° + \theta_5) \\
&= \cos(\alpha_1 - \theta_2) - \cos(60 + \alpha_1 + \theta_2) + \cos(\beta_1 - \theta_5) - \cos(60 + \beta_1 + \theta_5) \\
&= \cos(\alpha_1 - \theta_2) + \sin(\alpha_1 + \theta_2 - 30°) + \cos(\beta_1 - \theta_5) + \sin(\beta_1 + \theta_5 - 30°).
\end{aligned}$$

Therefore we can write

$$S = S' + S'',$$

where

$$S' = \cos(\alpha_1 - \theta_2) + \cos(\beta_1 - \theta_5) + \cdots + \cos(\alpha_5 - \theta_1) + \cos(\beta_5 - \theta_4),$$

$$S'' = \sin(\alpha_1 + \theta_2 - 30°) + \sin(\beta_1 + \theta_5 - 30°) + \cdots + \sin(\alpha_5 + \theta_1 - 30°) + \cos(\beta_5 + \theta_4 - 30°).$$

To bound S' and S'' we need the following lemma.

Lemma 13. *Let* $\measuredangle XYZ = y$ *where* $60° < y < 180°$. *Construct equilateral triangle* $\triangle XZW$ *and define* $\measuredangle WYZ = w$ *(see Fig.* 16). Then

$$\min\{60°, y - 60°\} \le w \le \max\{60°, y - 60°\}.$$

Proof. Construct a circle circumscribing the three points X, Y, and Z. Then W lies outside of the circle if $y < 120°$, on the circle if $y = 120°$, and inside the circle if $y > 120°$. Consider the first case. When Y moves from Z to X along the arc ZX, clearly, w increases from $y - 60°$ to $60°$ since the angle of the arc it faces also increases. An analogous argument proves Lemma 13 for the other two cases. □

We may assume without loss of generality that $\theta_i \le 41.25°$ for $1 \le i \le 5$ since otherwise the MST for C_5 is already shorter than M.

Define $\theta_6 = \theta_1$ and $\theta_0 = \theta_5$. By Lemma 13

$$\alpha_i \ge \min\{60°, \theta_{i-1} + 2\theta_i + \theta_{i+1} - 60°\} > \theta_{i+1}.$$

Furthermore, $\alpha_i - \theta_{i+1} \le \max\{60°, \theta_{i-1} + 2\theta_i + \theta_{i+1} - 60°\} - \theta_{i+1} \le 63.75°$. Similarly, we can show $\theta < \beta_i - \theta_{i-1} \le 63.75°$. Since $\cos x$ is concave for $0° \le x \le 90°$ and

$$\sum_{i=1}^{4} (\alpha_i - \theta_{i+1}) + \alpha_5 - \theta_1 + \beta_1 - \theta_5 + \sum_{i=2}^{5} (\beta_i - \theta_{i-1}) = 360°,$$

S' achieves its maximum when

$$\alpha_1 - \theta_2 = a_2 - \theta_3 = \cdots = \alpha_5 - \theta_1 = \beta_1 - \theta_5 = \cdots = \beta_5 - \theta_1 = 360°/10 = 36°.$$

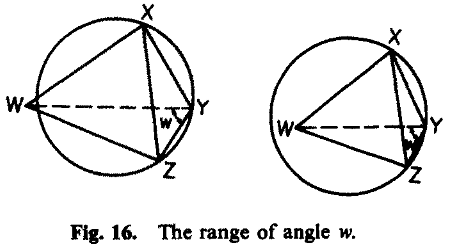

Fig. 16. The range of angle w.

Next note that $\theta_1+\theta_2=180°-\measuredangle CDE>60°$ and $\theta_1\le 41.25°$ implies $\theta_2>15°$. Hence

$$0°\le \alpha_i+\theta_{i+1}-30°\le \max\{30°+\theta_{i+1},\ \theta_{i-1}+2\theta_i+2\theta_{i+1}-90°\}$$
$$\le \max\{71.25°, 113.75°\}.$$

Since $\sin x$ is concave for $0°\le x\le 180°$ and

$$\alpha_1+\theta_2-30°+\beta_1+\theta_5-30°+\cdots+\alpha_5+\theta_1-30°+\beta_5+\theta_4-30°=780°,$$

S'' achieves its maximum when

$$\alpha_1+\theta_2-30°=\beta_1+\theta_5-30°=\cdots=\alpha_5+\theta_1-30°=\beta_5+\theta_4-30°=780°/10=78°.$$

It is easily verified that when $C_5=A_5$ the conditions on α_i, β_i, and θ_i to maximize S' and S'' are exactly fulfilled and $S=5(M-1)$.

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